The t Copula and Related Copulas

Stefano Demarta & Alexander J. McNeil
Department of Mathematics
Federal Institute of Technology
ETH Zentrum
CH-8092 Zurich
mcneil@math.ethz.ch

May 2004

Abstract

The t copula and its properties are described with a focus on issues related to the dependence of extreme values. The Gaussian mixture representation of a multivariate t distribution is used as a starting point to construct two new copulas, the skewed t copula and the grouped t copula, which allow more heterogeneity in the modelling of dependent observations. Extreme value considerations are used to derive two further new copulas: the t extreme value copula is the limiting copula of componentwise maxima of t distributed random vectors; the t lower tail copula is the limiting copula of bivariate observations from a t distribution that are conditioned to lie below some joint threshold that is progressively lowered. Both these copulas may be approximated for practical purposes by simpler, better-known copulas, these being the Gumbel and Clayton copulas respectively.

1 Introduction

The t copula (see for example Embrechts, McNeil & Straumann (2001) or Fang & Fang (2002)) can be thought of as representing the dependence structure implicit in a multivariate t distribution. It is a model which has received much recent attention, particularly in the context of modelling multivariate financial return data (for example daily relative or logarithmic price changes on a number stocks). A number of recent papers such as Mashal & Zeevi (2002) and Breymann et al. (2003) have shown that the empirical fit of the t copula is generally superior to that of the so-called Gaussian copula, the dependence structure of the multivariate normal distribution. One reason for this is the ability of the t copula to capture better the phenomenon of dependent extreme values, which is often observed in financial return data.

The objective of this paper is to bring together what is known about the t copula, particularly with regard to its extremal properties, to present some extensions of the t copula that follow from the representation of the multivariate t distribution as a mixture of multivariate normals, and to describe copulas that are related to the t copula through extreme value theory. For example, if random vectors have the t copula we would like to know the limiting copula of componentwise maxima of such random vectors, and also the limiting copula of observations that are conditioned to lie below or above extreme thresholds.

The paper is organized as follows. In the next section we describe the multivariate t distribution and its copula, the so-called t copula. In Section 3 we describe properties of the t copula, with a focus on coefficients of tail dependence and joint quantile exceedance probabilities. Brief notes on the statistical estimation of the t copula are given in Section 4.

The final sections of the paper contain the four new copulas. The skewed t copula and the grouped t copula are introduced in Section 5. The t-EV copula and its derivation as the copula of the limiting distribution of multivariate componentwise maxima of iid t-distributed random vectors are described in Section 6. The t tail limit copulas, which provide the limiting copulas for observations from the bivariate t copula that are conditioned to lie above or below extreme thresholds, are described in Section 7. Comments are made on the usefulness of all of these new copulas for practical data analysis.

2 The Multivariate t Distribution and its Copula

2.1 The multivariate t distribution

The d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)'$ is said to have a (non-singular) multivariate t distribution with ν degrees of freedom, mean vector $\boldsymbol{\mu}$ and positive-definite dispersion or scatter matrix Σ, denoted $\mathbf{X} \sim t_d(\nu, \boldsymbol{\mu}, \Sigma)$, if its density is given by

$$f(\mathbf{x}) = \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d|\Sigma|}} \left(1 + \frac{(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right)^{-\frac{\nu+d}{2}}. \tag{1}$$

Note that in this standard parameterization $\operatorname{cov}(\mathbf{X}) = \frac{\nu}{\nu-2}\Sigma$ so that the covariance matrix is not equal to Σ and is in fact only defined if $\nu > 2$. Useful references for the multivariate t are Johnson & Kotz (1972) (Chapter 37) and Kotz et al. (2000).

It is well-known that the multivariate t belongs to the class of multivariate normal variance mixtures and has the representation

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \sqrt{W}\mathbf{Z}, \tag{2}$$

where $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$ and W is independent of $\mathbf{Z}$ and satisfies $\nu/W \sim \chi^2_\nu$; equivalently W has an inverse gamma distribution $W \sim \operatorname{Ig}(\nu/2, \nu/2)$. The normal variance mixtures in turn belong to the larger class of elliptically symmetric distributions. See Fang, Kotz & Ng (1990) or Kelker (1970).

2.2 The t copula

A d-dimensional copula C is a d-dimensional distribution function on $[0,1]^d$ with standard uniform marginal distributions. Sklar's Theorem (see for example Nelsen (1999), Theorem 2.10.9) states that every df F with margins $F_1, \ldots, F_d$ can be written as

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)), \tag{3}$$

for some copula C, which is uniquely determined on $[0,1]^d$ for distributions F with absolutely continuous margins. Conversely any copula C may be used to join any collection of univariate dfs $F_1, \ldots, F_d$ using (3) to create a multivariate df F with margins $F_1, \ldots, F_d$.

For the purposes of this paper we concentrate exclusively on random vectors $\mathbf{X} = (X_1, \ldots, X_d)'$ whose marginal dfs are continuous and strictly increasing. In this case the so-called copula C of their joint df may be extracted from (3) by evaluating

$$C(\mathbf{u}) := C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \tag{4}$$

where the F_i^{-1} are the quantile functions of the margins. The copula C can be thought of as the df of the componentwise probability transformed random vector $(F_1(X_1), \ldots, F_d(X_d))'$.

The copula remains invariant under a standardization of the marginal distributions (in fact it remains invariant under any series of strictly increasing transformations of the components of the random vector $\mathbf{X}$). This means that the copula of a $t_d(\nu, \boldsymbol{\mu}, \Sigma)$ is identical to that of a $t_d(\nu, \mathbf{0}, P)$ distribution where P is the correlation matrix implied by the dispersion matrix Σ. The unique copula is thus given by

$$C^t_{\nu,P}(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d|P|}} \left(1 + \frac{\mathbf{x}'P^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+d}{2}} d\mathbf{x}, \quad (5)$$

where t_ν^{-1} denotes the quantile function of a standard univariate t_ν distribution. In the bivariate case we simplify the notation to $C^t_{\nu,\rho}$ where ρ is the off-diagonal element of P.

In what follows we will often contrast the t copula with the unique copula of a multivariate Gaussian distribution, which is extracted from the df of multivariate normal by the same technique and will be denoted C^{Ga}_P (see Embrechts et al. (2001)). It may be thought of as a limiting case of the t copula as $\nu \to \infty$.

Simulation of the t copula is particularly easy: we generate a multivariate t-distributed random vector $\mathbf{X} \sim t_d(\nu, \mathbf{0}, P)$ using the normal mixture construction (2) and then return a vector $\mathbf{U} = (t_\nu(X_1), \ldots, t_\nu(X_d))'$, where t_ν denotes the df of a standard univariate t. For estimation purposes it is useful to note that the density of the t copula may be easily calculated from (4) and has the form

$$c^t_{\nu,P}(\mathbf{u}) = \frac{f_{\nu,P}\left(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\right)}{\prod_{i=1}^d f_\nu(t_\nu^{-1}(u_i))}, \quad \mathbf{u} \in (0,1)^d, \quad (6)$$

where $f_{\nu,P}$ is the joint density of a $t_d(\nu, \mathbf{0}, P)$-distributed random vector and f_ν is the density of the univariate standard t-distribution with ν degrees of freedom.

2.3 Meta t distributions

If a random vector $\mathbf{X}$ has the t copula $C^t_{\nu,P}$ and univariate t margins with the same degree of freedom parameter ν, then it has a multivariate t distribution with ν degrees of freedom. If, however, we use (3) to combine any other set of univariate distribution functions using the t copula we obtain multivariate dfs F which have been termed meta-t_ν distribution functions (see Embrechts et al. (2001) or Fang & Fang (2002)). This includes, for example, the case where $F_1, \ldots, F_d$ are univariate t distributions with different degree of freedom parameters $\nu_1, \ldots, \nu_d$.

3 Properties of the t Copula

For this section it suffices to consider a bivariate random vector (X_1, X_2) with continuous and strictly increasing marginal dfs and unique copula C.

3.1 Kendall's τ Rank Correlation

Kendall's tau is a well-known measure of concordance for bivariate random vectors (see, for example, (Kruskal, 1958)). In general the measure is calculated as

$$\rho_\tau(X_1, X_2) = E\left(\text{sign}(X_1 - \widetilde{X}_1)(X_2 - \widetilde{X}_2)\right), \quad (7)$$

where $(\widetilde{X_1}, \widetilde{X_2})$ is a second independent pair with the same distribution as (X_1, X_2).

However, it can be shown (see Nelsen (1999), page 127, or Embrechts et al. (2001)) that the Kendall's tau rank correlation ρ_τ depends only on the copula C (and not on the marginal distributions of X_1 and X_2) and is given by

$$\rho_\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1. \quad (8)$$

Remarkably Kendall's tau takes the same elegant form for the Gauss copula C_ρ^{Ga}, the t copula $C_{\nu,\rho}^t$ or the copula of essentially all useful distributions in the elliptical class, this form being

$$\rho_\tau(X_1, X_1) = \frac{2}{\pi}\arcsin\rho. \tag{9}$$

A proof of this result can be found in Fang & Fang (2002); a proof of a slightly more general result applying to all elliptical distributions has been derived independently in Lindskog et al. (2003).

3.2 Tail Dependence Coefficients

The coefficients of tail dependence provide asymptotic measures of the dependence in the tails of the bivariate distribution of (X_1, X_2). The *coefficient of upper tail dependence* of X_1 and X_2 is

$$\lim_{q\to 1} P\left(X_2 > F_2^{-1}(q) \mid X_1 > F_1^{-1}(q)\right) = \lambda_u, \tag{10}$$

provided a limit $\lambda_u \in [0,1]$ exists, and the coefficient of lower tail dependence is

$$\lim_{q\to 0} P\left(X_2 \leq F_2^{-1}(q) \mid X_1 \leq F_1^{-1}(q)\right) = \lambda_\ell, \tag{11}$$

provided a limit $\lambda_l \in [0,1]$ exists. Thus these coefficients are limiting conditional probabilities that both margins exceed a certain quantile level given that one margin does.

These measures again depend only on the copula C of (X_1, X_2) and we may easily derive the copula-based expressions used by Joe (1997) from (10) and (11) using basic conditional probability and (4). The copula-based forms are

$$\lambda_u = \lim_{q\to 1^-}\frac{\overline{C}(q,q)}{1-q}, \quad \lambda_\ell = \lim_{q\to 0^+}\frac{C(q,q)}{q}, \tag{12}$$

where $\overline{C}(u,u) = 1 - 2u + C(u,u)$ is known as the survivor function of the copula. The interesting cases occur when these coefficient are strictly greater zero as this indicates a tendency for the copula to generate joint extreme events. If $\lambda_\ell > 0$, for example, we talk of tail dependence in the lower tail; if $\lambda_\ell = 0$ we talk of asymptotic independence in the lower tail.

For the copula of an elliptically symmetric distribution like the t the two measures λ_u and λ_ℓ coincide, and are denoted simply by λ. For the Gaussian copula the value is zero and for the t copula it is positive; a simple formula was calculated by Embrechts et al. (2001) using an argument that we reproduce here.

Proposition 1. *For continuously distributed random variables with the t copula $C_{\nu,P}^t$ the coefficient of tail dependence is given by*

$$\lambda \;=\; 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right), \tag{13}$$

where ρ is the off-diagonal element of P.

Proof. Applying l'Hôspital's rule to the expression for $\lambda = \lambda_\ell$ in (12) we obtain

$$\lambda = \lim_{u\to 0^+}\frac{dC(u,u)}{du} = \lim_{u\to 0^+} P(U_2 \leq u \mid U_1 = u) + \lim_{u\to 0^+} P(U_1 \leq u \mid U_2 = u),$$

where (U_1, U_2) is a random pair whose df is C and the second equality follows from an easily established property of the derivative of copulas (see Nelsen (1999), pages 11, 36). Suppose we now define $Y_1 = t_\nu^{-1}(U_1)$ and $Y_2 = t_\nu^{-1}(U_2)$ so that $(Y_1, Y_2) \sim t_2(\nu, \mathbf{0}, P)$. We have, using the exchangeability of (Y_1, Y_2), that

$$\lambda = 2\lim_{y\to -\infty^+} P(Y_2 \leq y \mid Y_2 = y).$$

Since, conditionally on $Y_1 = y$ we have

$$\left(\frac{\nu+1}{\nu+y^2}\right)^{1/2} \frac{Y_2 - \rho y}{\sqrt{1-\rho^2}} \sim t_1(\nu+1, 0, 1) \tag{14}$$

this limit may now be easily evaluated and shown to be (13). $\square$

Using an identical approach we can show that the Gaussian copula has no tail dependence, provided $\rho < 1$. This fact is much more widely known and has been demonstrated in a variety of different ways (see Sibuya (1961) or Resnick (1987), Chapter 5). Coefficients of tail dependence for the t copula are tabulated in Table 1. Perhaps surprisingly, even for negative and zero correlations, the t-copula gives asymptotic dependence in the tail.

ν/ρ	-0.5	0	0.5	0.9	1
2	0.06	0.18	0.39	0.72	1
4	0.01	0.08	0.25	0.63	1
10	0.0	0.01	0.08	0.46	1
∞	0	0	0	0	1

Table 1: Coefficient of tail dependence of the t copula $C^t_{\nu,P}$ for various values of ν and ρ.

Hult & Lindskog (2001) have given a general result for tail dependence in elliptical distribution, and hence its copula. It is well known (see Fang et al. (1987)) that a random vector $\mathbf{X}$ is elliptically distributed if and only if $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + RA\mathbf{S}$ where R is a scalar random variable independent of $\mathbf{S}$, a random vector distributed uniformly on the unit hypersphere, $\boldsymbol{\mu}$ is the location vector of the distribution and A is related to the dispersion matrix by $\Sigma = AA'$. Hult and Lindskog show that a sufficient condition for tail dependence is that R has a distribution with a so-called regularly varying or power tail (see, for example, Embrechts et al. (1997)). In this case they give the alternative formula

$$\lambda = \frac{\int_{(\pi/2-\arcsin\rho)/2}^{\pi/2} \cos^\alpha t dt}{\int_0^{\pi/2} \cos^\alpha t dt}, \tag{15}$$

where α is the so-called tail index of the distribution of R. For the multivariate t it may be shown that $R^2/d \sim F(d,\nu)$ (the usual F distribution) and the tail index of the distribution of R turns out to be $\alpha = \nu$. The formulas (15) and (13) then coincide. Hult and Lindskog conjecture that the regular variation of the tail of R is a necessary condition.

3.3 Joint Quantile Exceedance Probabilities

While tail dependence as presented in the previous section is an asymptotic concept, the practical implications can be seen by comparing joint quantile exceedance probabilities. To motivate this section we consider Figure 1 which shows 5000 simulated points from four bivariate distributions. The distributions in the top row are meta-Gaussian distributions; they share the same copula C^{Ga}_ρ. The distributions in the bottom row are meta-t distributions; they share the same copula $C^{\mathrm{t}}_{\nu,\rho}$. The values of ν and ρ in all pictures are 4 and 0.5 respectively. The distributions in the left column share the same margins, namely standard normal margins. The distributions in the right column both have standard t_4 margins. The distributions on the diagonal are of course elliptical, being standard bivariate normal and standard bivariate t_4; they both have linear correlation $\rho = 0.5$. The other distributions are not elliptical and do not necessarily have linear correlation 50%, since altering the margins alters the linear correlation. All four distributions have identical Kendall's tau values given by (9).

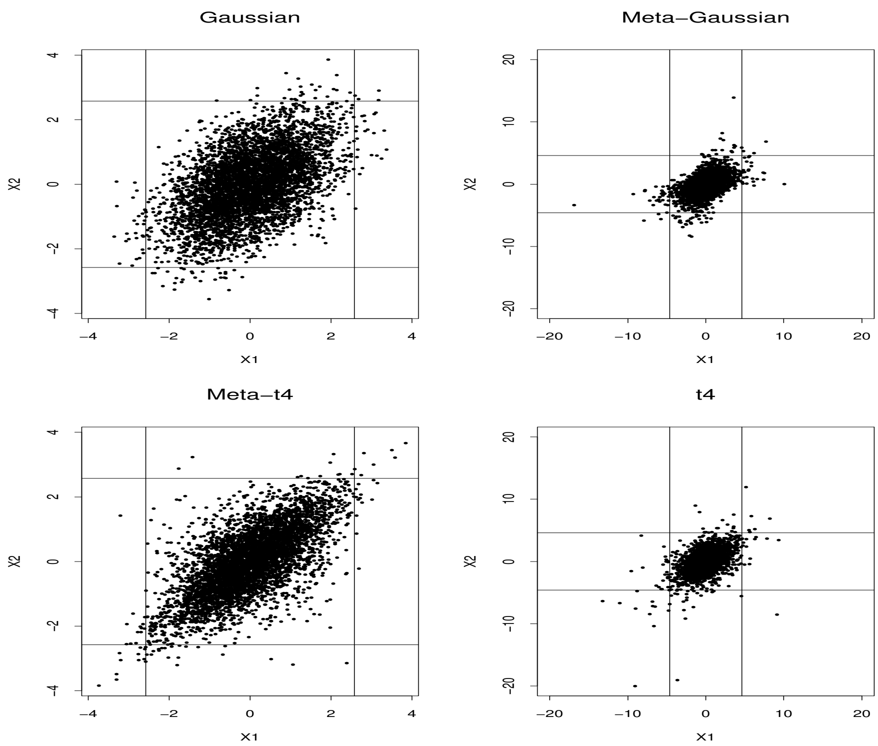

Figure 1: 5000 simulated points from 4 distributions. Top left: standard bivariate normal with correlation parameter $\rho = 0.5$. Top right: meta-Gaussian distribution with copula C_ρ^{Ga} and t_4 margins. Bottom left: meta-t_4 distribution with copula $C_{4,\rho}^{\mathrm{t}}$ and standard normal margins. Bottom right: standard bivariate t_4 distribution with correlation parameter $\rho = 0.5$. Horizontal and vertical lines mark the 0.005 and 0.995 quantiles.

The vertical and horizontal limes mark the true theoretical 0.005 and 0.995 quantiles for all distributions. Note that for the meta-t distributions the number of points that lie below both 0.005 quantiles or exceed both 0.995 quantiles is clearly greater than for the meta-Gaussian distributions, and this can be thought of as a consequence of the tail dependence of the t copula. The true theoretical ratio by which the number of these joint exceedances in the t models should exceed the number in the Gaussian models is 2.79 which may be read from Table 2, whose interpretation we now discuss.

ρ	Copula	Quantile			
		0.05	0.01	0.005	0.001
0.5	Gauss	1.21×10^{-2}	1.29×10^{-3}	4.96×10^{-4}	5.42×10^{-5}
0.5	t8	1.20	1.65	1.94	3.01
0.5	t4	1.39	2.22	2.79	4.86
0.5	t3	1.50	2.55	3.26	5.83
0.7	Gauss	1.95×10^{-2}	2.67×10^{-3}	1.14×10^{-3}	1.60×10^{-4}
0.7	t8	1.11	1.33	1.46	1.86
0.7	t4	1.21	1.60	1.82	2.52
0.7	t3	1.27	1.74	2.01	2.83

Table 2: Joint quantile exceedance probabilities for bivariate Gaussian and t copulas with correlation parameter values of 0.5 and 0.7. For Gaussian copula the probability of joint quantile exceedance is given; for the t copulas the factors by which the Gaussian probability must be multiplied are given.

In Table 2 we have calculated values of $C^{\mathrm{Ga}}_{\rho}(u,u)/C^{\mathrm{t}}_{\nu,\rho}(u,u)$ for various ρ and ν and $u = 0.05, 0.01, 0.005, 0.001$. For notes on the method we have used to calculate these values see Appendix A.1. The rows marked *Gauss* contain values of $C^{\mathrm{Ga}}_{\rho}(u,u)$, which is the probability that two random variables with this copula lie below their respective u-quantiles; we term this event a joint quantile exceedance. Obviously it is identical to the probability that both rvs lie above their $(1-u)$-quantiles. The remaining rows give the values of the ratio and thus express the amount by which the joint quantile exceedance probabilities must be inflated when we move from models with a Gaussian copula to models with a t copula.

ρ	Copula	Dimension d			
		2	3	4	5
0.5	Gauss	1.29×10^{-3}	3.66×10^{-4}	1.49×10^{-4}	7.48×10^{-5}
0.5	t8	1.65	2.36	3.09	3.82
0.5	t4	2.22	3.82	5.66	7.68
0.5	t3	2.55	4.72	7.35	10.34
0.7	Gauss	2.67×10^{-3}	1.28×10^{-3}	7.77×10^{-4}	5.35×10^{-4}
0.7	t8	1.33	1.58	1.78	1.95
0.7	t4	1.60	2.10	2.53	2.91
0.7	t3	1.74	2.39	2.97	3.45

Table 3: Joint 1% quantile exceedance probabilities for multivariate Gaussian and t equicorrelation copulas with correlation parameter values of 0.5 and 0.7. For Gaussian copula the probability of joint quantile exceedance is given; for the t copulas the factors by which the Gaussian probability must be multiplied are given.

In Table 3 we extend Table 2 to higher dimensions. We now focus only on joint exceedances of the 1% (or 99% quantiles). We tabulate values of the ratio

$$C^{\mathrm{Ga}}_{P}(u,\ldots,u)/C^{\mathrm{t}}_{\nu,P}(u,\ldots,u),$$

where P is an equicorrelation matrix with all correlations equal ρ. It is noticeable that not only do these values grow as the correlation parameter or degrees of freedom falls, they also grow with the dimension of the copula.

Consider the following example of the implications of the tabulated numbers. We study daily returns on five stocks which are roughly equicorrelated with a correlation of 50%. In reality they are generated by a multivariate t distribution with four degrees of freedom. If we erroneously assumed a multivariate Gaussian distribution we would calculate that the probability that on any day all returns would drop below the 1% quantiles of their marginal distributions is 7.48×10^{-5}. In the long run such an event will happen once every 13369 days on average, that is roughly once every 51 years (assuming 260 days in the stock market year). In the true model the event actually occurs with a probability that is 7.68 times higher, making it more of a seven year event.

4 Estimation of the t Copula

When estimation of a parametric copula is the primary objective, the unknown marginal distributions of the data enter the problem as nuisance parameters. The first step is usually to transform the data onto the "copula scale" by estimating the unknown margins and then using the probability-integral transform. Denote the data vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and write the jth component of the ith vector as $X_{i,j}$. We assume in the following that these are from a meta t distribution and the parameters of the copula $C^t_{\nu,P}$ are to be determined.

Broadly speaking the marginal modelling can be done in three ways: fitting parametric distributions to each margin; modelling the margins nonparametrically using a version of the empirical distribution functions; using a hybrid of the parametric and nonparametric methods.

The first method has been termed the IFM or inference-functions-for-margins method by Joe (1997) following terminology used by McLeish & Small (1988). Asymptotic theory has been worked out for this approach (Joe (1997)) but in practice the success of the method is obviously dependent upon finding appropriate parametric models for the margins, which may not always be so straightforward when these show evidence of heavy tails and/or skewness.

The second method involving estimation of the margins by the empirical df has been termed the pseudo-likelihood method and extensively investigated by Genest et al. (1995); consistency and asymptotic normality of the resulting copula parameter estimates are shown in the situation when $\mathbf{X}_1, \ldots, \mathbf{X}_n$ form an iid data sample. Writing $\mathbf{X}_i = (X_{i,1}, \ldots, X_{i,d})'$ for the ith data vector, the method involves estimating the jth marginal df F_j by

$$\widehat{F}_j(x) = \frac{1}{n+1} \sum_{i=1}^{n} 1_{\{X_{i,j} \leq x\}}. \tag{16}$$

The pseudo-sample from the copula is then constructed by forming vectors $\widehat{\mathbf{U}}_1, \ldots, \widehat{\mathbf{U}}_n$ where

$$\widehat{\mathbf{U}}_i = (U_{i,1}, \ldots, U_{i,d})' = \left(\widehat{F}_1(X_{i,1}), \ldots, \widehat{F}_d(X_{i,d})\right)'. \tag{17}$$

Observe that, even if the original data vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are iid, the pseudo-sample data are dependent, because the marginal estimates $\widehat{F}_j$ are constructed from all of the original data vectors through the univariate samples $X_{1,j}, \ldots, X_{n,j}$. Note also that division by $n+1$ in (16) keeps transformed points away from the boundary of the unit cube.

A hybrid of the parametric and nonparametric methods could be developed by modelling the tails of the marginal distributions using a generalized Pareto distribution as suggested by extreme value theory (Davison & Smith (1990)) and approximating the body of the distribution using the empirical distribution function (16).

4.1 Maximum likelihood

Assuming the marginal dfs have been estimated by one of the methods described above and that pseudo-copula data (17) have been obtained, we can use ML to estimate the parameters ν and P of the t copula. The estimates are obtained by maximizing

$$\log L(\nu, P; \widehat{\mathbf{U}}_1, \ldots, \widehat{\mathbf{U}}_n) = \sum_{i=1}^{n} \log c_{\nu,P}(\widehat{\mathbf{U}}_i) \tag{18}$$

with respect to ν and P, where $c^t_{\nu,P}$ denotes the density of the t copula in (6).

This maximization is not particularly easy in higher dimensions due to the necessity of maximizing over the space of correlation matrices P. For this reason, the method described in the next section is of practical interest.

4.2 Method-of-Moments using Kendall's tau

A simple method based on Kendall's tau for estimating the correlation matrix P which partly parameterizes the t copula was suggested in Lindskog (2000) and Lindskog et al. (2003). The method consists of constructing an empirical estimate of Kendall's tau for each bivariate margin of the copula and then using relationship (9) to infer an estimate of the relevant element of P. More specifically we estimate $\rho_\tau(X_j, X_k)$ by calculating the standard sample Kendall's tau coefficient

$$\widehat{\rho_\tau}(X_j, X_k) = \binom{n}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq n} \operatorname{sign}(X_{i_1,j} - X_{i_2,j})(X_{i_1,k} - X_{i_2,k}), \tag{19}$$

from the original data vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$; this yields an unbiased and consistent estimator of (7). An estimator of P_{jk} is then given by $\sin\left(\frac{\pi}{2}\widehat{\rho_\tau}(X_j, X_k)\right)$. Note that this amounts to a method-of-moments estimate because the true moment (7) is replaced by its empirical analogue to turn (9) into an estimating equation for the parameter ρ.

In order to obtain an estimator of the entire matrix P we can collect all pairwise estimates in an empirical Kendall's tau matrix R^τ defined by $R^\tau_{jk} = \widehat{\rho_\tau}(X_j, X_k)$ and then construct the estimator $P^* = \sin\left(\frac{\pi}{2}R^\tau\right)$. However, there is no guarantee that this componentwise transformation of the empirical Kendall's tau matrix will be positive definite (although in our experience it mostly is). In this case P^* can be adjusted to obtain a positive definite matrix using a procedure such as the eigenvalue method of Rousseeuw & Molenberghs (1993).

The easiest way to estimate the remaining parameter ν is by maximum likelihood with the P matrix held fixed, which is a special case of the general ML method discussed in the previous section. This method has been implemented in practice in the work of Mashal & Zeevi (2002) and found to give very similar estimates to the full maximum likelihood procedure.

5 Generalizations of t Copula Via Mixture Constructions

The t copula has been found in empirical studies, such as those of Mashal & Zeevi (2002) and Breymann et al. (2003), to be a better model than the Gauss copula for the dependence structure of multivariate financial returns, which often seem to show empirical evidence of tail dependence.

However a drawback of the t copula is its strong symmetry. The t copula is the df of a radially symmetric distribution; if $(U_1, \ldots, u_d)$ is a vector distributed according to $C^t_{\nu,P}$ then

$$(U_1, \ldots, U_d) \stackrel{d}{=} (1 - U_1, \ldots, 1 - U_d),$$

which means, for example, that the level of tail dependence in any corner of the copula is the same as that in the "opposite" corner.

Moreover, whenever P is an equicorrelation matrix the t copula is an exchangeable copula, i.e. the df of a random vector whose distribution is invariant under permutations. In the bivariate case, this means that $(U_1, U_2) \stackrel{d}{=} (U_2, U_1)$ so that the diagonal $u_1 = u_1$ is an axis of symmetry of the copula. We now look at extensions of the t copula that attempt to introduce more asymmetry.

5.1 Skewed t copula

A larger class of multivariate normal mixture distributions, known as mean-variance mixtures, may be obtained by generalizing the construction (2) to get

$$\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\gamma} g(W) + \sqrt{W}\mathbf{Z}, \tag{20}$$

for some function $g : [0, \infty) \to [0, \infty)$ and a d-dimensional parameter vector $\boldsymbol{\gamma}$. When $\boldsymbol{\gamma} \neq \mathbf{0}$ this gives a family of skewed, non-elliptically-symmetric distributions. Much attention has been received by the family obtained when $g(W) = W$ and W has a so-called generalized inverse Gaussian (GIG) distribution. In this case $\mathbf{X}$ is said to have a multivariate generalized hyperbolic distribution; see, for example, Barndorff-Nielsen & Blæsild (1981) or Blæsild & Jensen (1981).

A special, but little-studied, case of this family is encountered when $W \sim \text{Ig}(\nu/2, \nu/2)$ (since inverse gamma is a special case of the GIG distribution). The resulting mixture distribution could be referred to as a skewed multivariate t (although there are a number of other multivariate distributions sharing this name) and has density

$$f(\mathbf{x}) = c\frac{K_{\frac{\nu+d}{2}}\left(\sqrt{(\nu+(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}))\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma}}\right)\exp((\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}\boldsymbol{\gamma})}{\left(\sqrt{(\nu+(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}))\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma}}\right)^{-\frac{\nu+d}{2}}\left(1+\frac{(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right)^{\frac{\nu+d}{d}}}, \tag{21}$$

where K_λ denotes a modified Bessel function of the third kind (see Abramowitz & Stegun (1965), Chapters 9 and 10) and the normalizing constant is

$$c = \frac{2^{\frac{2-(\nu+d)}{2}}}{\Gamma(\frac{\nu}{2})(\pi\nu)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}}.$$

We denote this distribution by $\mathbf{X} \sim t_d(\nu, \boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma})$. Properties of the modified Bessel function of the third kind may be used to show that as $\boldsymbol{\gamma} \to \mathbf{0}$ the skewed t density converges to the usual multivariate t density in (1).

Moments of this distribution are easy to calculate because of the normal mixture structure of the distribution and are given by

$$\begin{aligned} E(\mathbf{X}) &= E(E(\mathbf{X} \mid W)) = \boldsymbol{\mu} + E(W)\boldsymbol{\gamma} = \boldsymbol{\mu} + \frac{\nu}{\nu-2}\boldsymbol{\gamma}, \\ \operatorname{cov}(\mathbf{X}) &= E(\operatorname{var}(\mathbf{X} \mid W)) + \operatorname{var}(E(\mathbf{X} \mid W)) = \frac{\nu}{\nu-2}\Sigma + \frac{2\nu^2}{(\nu-2)^2(\nu-4)}\gamma\gamma'. \end{aligned}$$

The covariance matrix is only finite when $\nu > 4$, which contrasts with the symmetric t distribution where we only require $\nu > 2$. In other words, using a mean-variance mixture construction of the form (20) with $g(w) = w$ results in a skewed distribution which has heavier marginal tails than the non-skewed special case obtained when $\boldsymbol{\gamma} = \mathbf{0}$. (The tail of $|X_1|$ will have tails that decay like $x^{-\nu/2}$ rather than $x^{-\nu}$ in the symmetric case.) This possibly undesirable feature could be avoided by setting $g(w) = w^{1/2}$ which would give a skewed kind of distribution whose tails behaved in the same way in both the skewed and symmetric cases. However this distribution would not reside in the class of generalized hyperbolic distribution and would be somewhat less analytically tractable.

We persist with the model described by (21) and refer to its copula as a skewed t copula. In particular we denote by $C^t_{\nu,P,\boldsymbol{\gamma}}$ the copula of a $t_d(\nu, \mathbf{0}, P, \boldsymbol{\gamma})$ distribution, where P is a correlation matrix. For simulation purposes it is useful to note that the univariate margins of this distribution are $t_1(\nu, 0, 1, \gamma_i)$ distributions for $i = 1, \ldots, d$.

To appreciate the flexibility of the skewed t copula it suffices to consider the bivariate case $C^t_{\nu,\rho,\gamma_1,\gamma_2}$. In Figure we have plotted simulated points from nine different examples of this copula; the centre picture corresponds to the case when $\gamma_1 = \gamma_2 = 0$ and is thus the ordinary t copula; all other pictures show copulas which are non-radially symmetric copulas, as is obvious by rotating each picture 180 degrees about the point (1/2,1/2); the three pictures on the diagonal show exchangeable copulas while the remaining six are non-exchangeable.

5.2 Grouped t copula

The grouped t copula has been suggested by Daul et al. (2003) and the basic idea is to construct a copula closely related to the t copula where different subvectors of the vector $\mathbf{X}$ can have quite different levels of tail dependence. To this end we build a distribution using a generalization of the mixing construction in (2) where instead of multiplying all components of a correlated Gaussian vector with the root of a single inverse-gamma-distributed variate W we multiply different subgroups with different variates W_j where $W_j \sim \operatorname{Ig}(\nu_j/2, \nu_j/2)$ and the W_j are perfectly positively dependent.

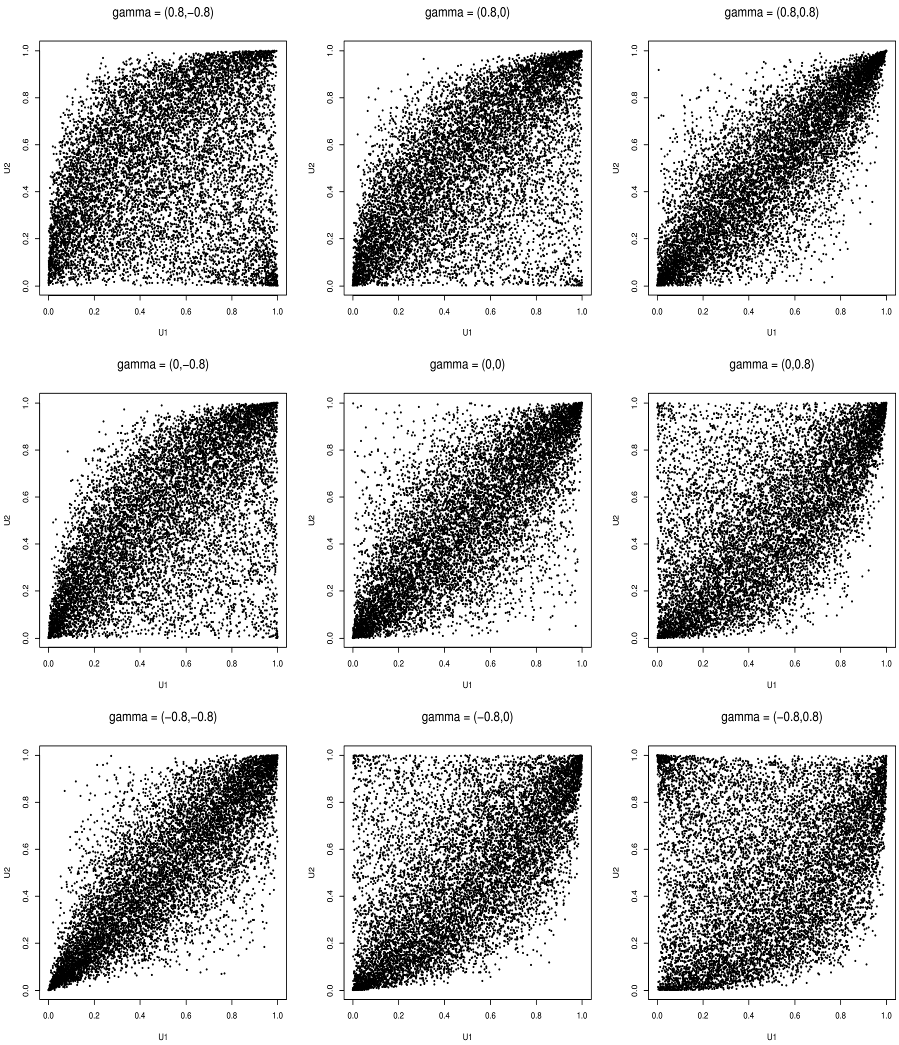

Figure 2: 10000 simulated points from the bivariate skewed t copula $C^t_{\nu,\rho,\gamma_1,\gamma_2}$ for $\nu = 5$, $\rho = 0.8$ and various values of the parameters (γ_1, γ_2) as shown above each picture.

The rationale is to create groups whose dependence properties are described by the same ν_j parameter, which dictates in particular the extremal dependence properties of the group, whilst using perfectly dependent mixing variables to create a distribution and copula whose calibration may be achieved by the kind of rank-correlation-based methods we discussed in Section 4.2.

Let G_ν denote the df of a univariate $\mathrm{Ig}(\nu/2, \nu/2)$ distribution. Let $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$ and let $U \sim U(0,1)$ be a uniform variate independent of $\mathbf{Z}$. Partition $\{1, \ldots, d\}$ into m subsets of sizes $s_1, \ldots, s_m$ and for $k = 1, \ldots, m$ let ν_k be the degree of freedom parameter associated with group k. Let $W_k = G^{-1}_{\nu_k}(U)$ so that $W_1, \ldots, W_m$ are perfectly dependent (in the sense that they have a Kendall's tau value of one). Finally define

$$\mathbf{X} = (\sqrt{W_1} Z_1, \ldots, \sqrt{W_1} Z_{s_1}, \sqrt{W_2} Z_{s_1+1}, \ldots, \sqrt{W_2} Z_{s_1+s_2}, \ldots, \sqrt{W_m} Z_d)'.$$

From (2) it follows that $(X_1, \ldots, X_{s_1})$ has a s_1-dimensional t-distribution with ν_1 degrees of freedom and, for $k = 1, \ldots, m-1$, the vector $(X_{s_1+\cdots+s_k+1}, \ldots, X_{s_1+\cdots+s_k+s_{k+1}})$ has a s_{k+1}-dimensional t-distribution with ν_{k+1} degrees of freedom. The grouped t copula is the unique copula of the multivariate df of $\mathbf{X}$. Note that like the t copula, the skewed t copula and anything based on a mixture of multivariate normals, it is very easy to simulate, which has been a further motivation for its use in financial modelling where Monte Carlo methods are popular.

Moreover the parameter estimation method based on Kendall's tau described in Section 4.2 may be applied. Daul et al. (2003) show that when $\nu_1 \neq \nu_2$, $X_i = \sqrt{W_1} Z_i$ and $X_j = \sqrt{W_2} Z_j$ with $i \neq j$ then the approximate identity

$$\rho_\tau(X_1, X_2) \approx \frac{2}{\pi} \arcsin \rho$$

holds, where ρ is the correlation between Z_i and Z_j. The approximation error is shown to be extremely small. Thus estimates of correlation parameters of the grouped t copula may be inferred from inverting this relationship and degree of freedom parameters may be estimated by applying maximum likelihood methods to subgroups which are considered a priori to have different tail dependence characteristics.

6 The t-EV Copula

In this section we derive a new extreme value copula, known as the t-EV copula or t limit copula, which can be thought of as the limiting dependence structure of componentwise maxima of iid random vectors having a multivariate t distribution or meta-t distribution. The derivation requires a brief summary of relevant information from multivariate extreme value theory.

6.1 Limit copulas for multivariate maxima

Consider *iid* random vectors $\mathbf{X}_1, \dots, \mathbf{X}_n$ with distribution function F (assumed to have continuous margins) and define $\mathbf{M}_n$ to be the vector of componentwise maxima (i.e. the jth component of $\mathbf{M}_n$ is the maximum of the jth component over all n observations). We say that F is in the maximum domain of attraction of the distribution function H, if there exist sequences sequences of vectors $\mathbf{a}_n > \mathbf{0}$ and $\mathbf{b}_n \in \mathbb{R}^n$ such that

$$\lim_{n\to\infty} P\left(\frac{M_{n,1} - b_{n,1}}{a_{n,1}} \leq x_1, \dots, \frac{M_{n,d} - b_{n,d}}{a_{n,d}} \leq x_d\right) = \lim_{n\to\infty} F^n(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n) = H(\mathbf{x}). \quad (22)$$

A non-degenerate limiting distribution H in (22) is known as a multivariate extreme value distribution (MEVD). Its margins must be of extreme value type, that is either Gumbel, Fréchet or Weibull. This is dictated by standard univariate EVT; see, for example, Embrechts et al. (1997). The unique copula C_0 of the limit H must satisfy the scaling property

$$C_0(u_1^t, \dots, u_d^t) = C_0^t(u_1, \dots, u_d), \quad \forall t > 0, \quad (23)$$

as is shown in Galambos (1987) (where the copula is referred to as a stable dependence function) or Joe (1997), page 173. Any copula with the property (23) is known as an extreme value copula (EV copula) and can arise as the copula in a limiting MEVD.

A number of characterizations of the EV copulas are known. In particular, the bivariate EV copulas are characterized as being copulas of the form

$$C_0(u_1, u_2) = \exp\left(\log(u_1 u_2)\, A\left(\frac{\log(u_1)}{\log(u_1, u_2)}\right)\right), \quad (24)$$

for some function $A : [0,1] \to [0,1]$ known as the dependence function, which must be convex and satisfy $\max(w, 1-w) \leq A(w) \leq 1$ for $0 \leq w \leq 1$. See, for example, Joe (1997), page 175, or Pickands (1981), Genest et al. (1995) or Tiago de Oliveira (1975).

If we have convergence in distribution as in (22) then the margins of the underlying df F determine the margins of H, but are irrelevant to the limiting copula C_0. The copula C of F determines C_0. One may thus define the concept of a copula domain of attraction and speak of certain underlying copulas C being attracted to certain EV copula limits C_0. See again Galambos (1987).

In this context we note an interesting property of upper tail dependence coefficients. The set of upper tail dependence coefficients for the bivariate margins of C can be shown to be identical to those of C_0, the limiting copula; see Joe (1997), page 178. If the upper tail dependence coefficients of C are all identically zero then the limit C_0 must be $C_0(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i$, which is the so-called independence copula, since this is the only EV copula with upper tail dependence coefficients identically zero. Multivariate maxima from distributions without tail dependence, such as the Gaussian distribution, are independent in the limit.

These facts motivate us to search for the limit for maxima of random vectors whose dependence is described by the multivariate t copula; we know that the limit cannot be the independence copula in this case. We require a workable characterization of a copula domain of attraction and use the following.

Theorem 2. *Let C be a copula and C_0 an extreme value copula. Then C is attracted to the EV copula limit C_0 if and only if for all $\mathbf{x} \in [0, \infty)^d$*

$$\lim_{s \to 0} \frac{1 - C_(1 - sx_1, \ldots, 1 - sx_d)}{s} = -\log C_0(\exp(-x_1), \ldots, \exp(-x_d)). \tag{25}$$

For a proof see Demarta (2001). Note also that this result follows easily from a series of very similar characterizations given byTakahashi (1994) which are listed in Kotz & Nadarajah (2000), page.

6.2 Derivation of the t-EV Copula

We use Proposition 2 and calculate a limit directly from (25). The techniques of calculation are very similar to those used in Proposition 1. We restrict our attention to the bivariate case $d = 2$; in fact, it is possible although notationally cumbersome to derive a limit in the general case.

We begin with a useful lemma which shows how extreme quantiles of the univariate t distribution scale.

Lemma 3.

$$\lim_{s \to 0} \frac{t_\nu^{-1}(1 - sx)}{t_\nu^{-1}(1 - s)} = \lim_{s \to 0} \frac{t_\nu^{-1}(sx)}{t_\nu^{-1}(s)} = x^{-1/\nu}. \tag{26}$$

This is proved using the so-called regular variation property of the tail of the univariate t distribution in Appendix A.2.

Proposition 4. *The bivariate t copula $C_{\nu,\rho}^t$ is attracted to the EV limit given by*

$$C_{\nu,\rho}^{tEV}(u_1, u_2) = \exp\left(\log(u_1 u_2)\, A_{\nu,\rho}\left(\frac{\log(u_1)}{\log(u_1 u_2)}\right)\right), \tag{27}$$

where

$$A_{\nu,\rho}^{tEV}(w) = w t_{\nu+1}\left(\frac{\left(\frac{w}{1-w}\right)^{1/\nu} - \rho}{\sqrt{1-\rho^2}}\sqrt{\nu+1}\right) + (1-w) t_{\nu+1}\left(\frac{\left(\frac{1-w}{w}\right)^{1/\nu} - \rho}{\sqrt{1-\rho^2}}\sqrt{\nu+1}\right). \tag{28}$$

Proof. We first evaluate the limit in the lhs of (25), which we call $\ell(x_1, x_2)$, for fixed $x_1 \geq 0$ and $x_2 \geq 0$. Clearly, for boundary values we have $\ell(x_1, 0) = x_1$, $\ell(0, x_2) = x_2$ and $\ell(0, 0) = 0$. To evaluate the limit when $x_1 > 0$ and $x_2 > 0$ we introduce a random pair (U_1, U_2) with df

$C^t_{\nu,\rho}$ and calculate

$$
\begin{aligned}
&\lim_{s\to 0+} \frac{1 - C^t_{\nu,\rho}(1-sx_1, 1-sx_2)}{s} \\
&= \lim_{s\to 0+} x_1 \frac{\partial}{\partial u_1} C^t_{\nu,\rho}(u_1,u_2)\big|_{1-sx_1,1-sx_2} + x_2 \frac{\partial}{\partial u_2} C^t_{\nu,\rho}(u_1,u_2)\big|_{1-sx_1,1-sx_2} \\
&= \lim_{s\to 0+} x_1 \cdot \underbrace{P(U_2 \le 1 - sx_2 \mid U_1 = 1 - sx_1)}_{\mathrm{P}_1} + x_2 \cdot \underbrace{P(U_1 \le 1 - sx_1 \mid U_2 = 1 - sx_2)}_{\mathrm{P}_2}.
\end{aligned}
$$

Let $Y_1 = t_\nu^{-1}(U_1)$ and $Y_2 = t_\nu^{-1}(U_2)$ and introduce the notation $q(s,x) = t_\nu^{-1}(1-sx)$. The bracketed conditional probability term P_1 can be evaluated easily using (14) and is

$$
\begin{aligned}
\mathrm{P}_1 &= P(Y_2 \le q(s,x_2) \mid Y_1 = q(s,x_1)) \\
&= t_{\nu+1}\left(\frac{q(s,x_2) - \rho q(s,x_1)}{\sqrt{1-\rho^2}} \left(\frac{\nu + q(s,x_1)^2}{\nu+1} \right)^{-1/2} \right) \\
&= t_{\nu+1}\left(\frac{q(s,x_2)/q(s,x_1) - \rho}{\sqrt{1-\rho^2}} \frac{\sqrt{\nu+1}}{\sqrt{1+\nu/q(s,x_1)^2}} \right).
\end{aligned}
$$

A similar expression holds for P_2. Since $\sqrt{1+\nu/q(s,x_1)^2} \to 1$ as $s \to 0$ and the only remaining term depending on s is $q(s,x_2)/q(s,x_1)$ the limit can be obtained using Lemma 3 and is

$$
\ell(x_1,x_2) = x_1 \cdot t_{\nu+1}\left(\frac{\left(\frac{x_1}{x_2}\right)^{1/\nu} - \rho}{\sqrt{1-\rho^2}} \sqrt{\nu+1} \right) + x_2 \cdot t_{\nu+1}\left(\frac{\left(\frac{x_2}{x_1}\right)^{1/\nu} - \rho}{\sqrt{1-\rho^2}} \sqrt{\nu+1} \right). \tag{29}
$$

Using (25) the limiting copula must be of the form

$$
C^{tEV}_{\nu,\rho}(u_1,u_2) = \exp\left(-\ell(-\log u_1, -\log u_2)\right),
$$

and by observing that $\ell(x_1,x_2) = (x_1+x_2)\ell(x_1/(x_1+x_2), x_2/(x_1+x_2))$ we see that this can be rewritten as

$$
C^{tEV}_{\nu,\rho}(u_1,u_2) = \exp\left(\log(u_1u_2)\ell\left(\frac{\log u_1}{\log(u_1u_2)}, 1 - \frac{\log u_1}{\log(u_1u_2)} \right) \right).
$$

Setting $A_{\nu,\rho}(w) = \ell(w, 1-w)$ on $[0,1]$ we obtain the form given in (27) and (28). It remains to be verified that this is an EV copula; this can be done by checking that $A_{\nu,\rho}(w)$ defined by (28) is a convex function satisfying $\max(w, 1-w) \le A(w) \le 1$ for $0 \le w \le 1$. □

6.3 Using the bivariate t-EVcopula in practice

The bivariate t-EV copula of proposition 4 is not particularly convenient for practical purposes. The copula density that is required for maximum likelihood inference is quite cumbersome and our experience also suggests that the parameters ν and ρ are not well identified.

However it can be shown that for any choice of the parameters ν and ρ, the A-function of the t-EV copula given in (28) has a functional form which is almost identical to the A-functions of the Gumbel and Galambos EV copulas. The Gumbel copula in particular has been widely used in applied work. The A-functions of these copulas are respectively

$$
A^{\mathrm{Gu}}_\theta(w) = \left(w^\theta + (1-w)^\theta \right)^{1/\theta}, \tag{30}
$$

$$
A^{\mathrm{Ga}}_\theta(w) = 1 - \left(w^{-\theta} + (1-w)^{-\theta} \right)^{-1/\theta}, \tag{31}
$$

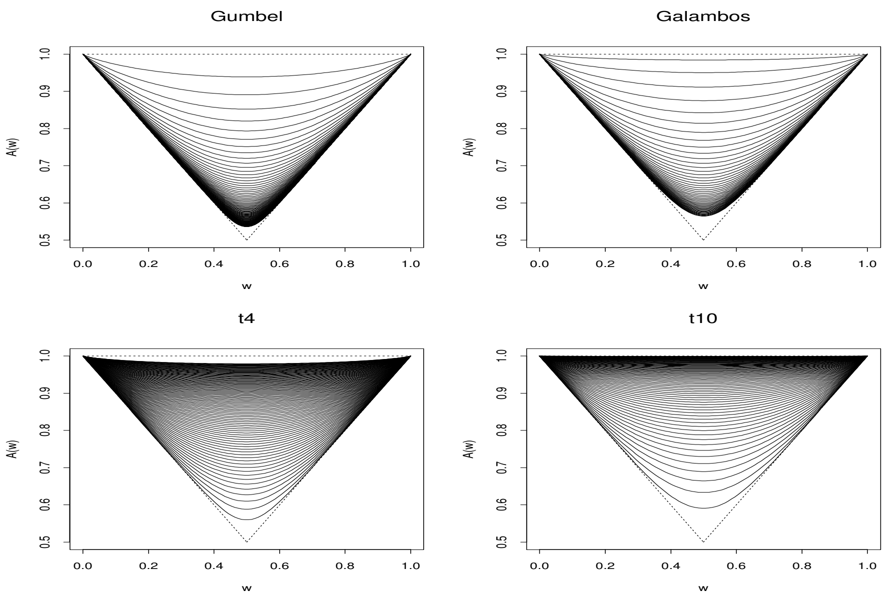

Figure 3: Plot of the A functions for four copulas; in Gumbel case θ runs from 1.1 to 10 in steps of size 0.1; in Galambos case θ runs from 0.2 to 5 in steps of size 0.1; in t_4 and t_{10} cases ρ runs from -0.2 to 0.9 in 100 equally spaced steps.

and the expressions for the copulas are obtained by inserting these in (24). All three A-functions are shown in Figure 3 for a variety of values of the parameters.

The parameter θ of the Gumbel or Galambos A-functions can always be chosen so that the curve is extremely close to that of the t-EV A-function for any values of ν and ρ. We have comfirmed empirically that if we fix (ν, ρ) for the t-EV model and minimize the sum of squared errors $(A_{\nu,\rho}(w_i) - A_\theta(w_i))^2$ at $n = 100$ equally spaced points $(w_i)_{i=1,\ldots,100}$ on $[0, 1]$, with respect to θ then the resulting curve in the Gumbel or Galambos models is indistinguishable from the t-EV curve. The implication is that in all situations where the t-EV copula might be deemed an appropriate model we can work instead with the simpler Gumbel or Galambos copulas.

7 The t Tail Copulas

7.1 Limits for lower and upper tail copulas

Consider a random vector (X_1, X_2) with continuous margins F_1 and F_2 whose copula C is exchangeable. We consider the distribution of (X_1, X_2) conditional on both being being below their v-quantiles, an event we denote by

$$A_v = \left\{X_1 \le F_1^{-1}(v), X_2 \le F_2^{-1}(v)\right\}, \quad 0 < v \le 1.$$

Assuming $P(A_v) = C(v, v) \neq 0$, the probability that X_1 lies below its x_1-quantile and X_2 lies below its x_2-quantile conditional on this event is

$$P\left(X_1 \le F_1^{-1}(x_1), X_2 \le F_2^{-1}(x_2) \mid A_v\right) = \frac{C(x_1, x_2)}{C(v, v)}, \quad x_1, x_2 \in [0, v].$$

Considered as a function of x_1 and x_2 this defines a bivariate df on $[0, v]^2$ and by Sklar's theorem we can write

$$\frac{C(x_1, x_2)}{C(v, v)} = C_v^{\mathrm{lo}}(F_{(v)}(x_1), F_{(v)}(x_2)), \quad x_1, x_2 \in [0, v],$$

for a unique copula C_v^{lo} and continuous marginal distribution functions

$$F_{(v)}(x) = P\left(X_1 \le F_1^{-1}(x) \mid A_v\right) = \frac{C(x,v)}{C(v,v)}, \quad 0 \le x \le v. \tag{32}$$

This unique copula may be written as

$$C_v^{\mathrm{lo}}(u_1,u_2) = \frac{C(F_{(v)}^{-1}(u_1), F_{(v)}^{-1}(u_2))}{C(v,v)}, \tag{33}$$

and will be referred to as the *lower tail copula* of C at level v. Juri and Wüthrich Juri & Wuethrich (2002), who developed the approach we describe in this section, refer to it as a lower tail dependence copula or LTDC. It is of interest to attempt to evaluate limits for this copula as $v \to 0$; such a limit will be known as a *limiting lower tail copula* and denoted C_0^{lo}. Upper tail copulas can be defined in an analogous way if we condition on variables being above their v-quantiles for $0 \le v < 1$. Similarly upper tail limit copulas are the limits as $v \to 1$.

Limiting lower and upper tail copulas must possess a stability property under the kind of conditioning operations discussed above. For example, a limiting lower tail copula must be stable under the operation of calculating lower tail copulas as in (33). It must satisfy the relation

$$C_{0,v}^{\mathrm{lo}}(u_1,u_2) := \frac{C_0^{\mathrm{lo}}(F_{(v)}^{-1}(u_1), F_{(v)}^{-1}(u_2))}{C_0^{\mathrm{lo}}(v,v)} = C_0^{\mathrm{lo}}(u_1,u_2). \tag{34}$$

An example of a limiting lower tail copula is the Clayton copula

$$C_\alpha^{Cl}(u_1,u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \tag{35}$$

which is a limit for many underlying copulas, including many members of the Archimedean family. It may be easily verified that this copula has the stability property in (34).

7.2 Derivation of the t tail copulas

We wish to find upper and lower tail copulas for the t copula. The general result we use is expressed in terms of survival copulas; if C is a bivariate copula then its survival copula is given by

$$\widehat{C}(u_1,u_2) = u_1 + u_2 - 1 + C(1-u_1, 1-u_2).$$

If C is the df of (U_1,U_2) then $\widehat{C}$ is the df of $(1-U_1, 1-U_2)$. Thus for a radially symmetric copula, like the t copula, we have $\widehat{C} = C$, but this is not generally the case.

An elegant general result follows directly from a theorem in Juri & Wuethrich (2002); this shows how to find tail limit copulas for any bivariate copula that is attracted to an EV limiting copula.

Theorem 5. *If C is attracted to the EV copula C_0 with upper tail dependence coefficient $\lambda_u > 0$ then its survival copula $\widehat{C}$ has a limiting lower tail copula which is the copula of the df*

$$G(x_1,x_2) = \frac{(x_1+x_2)\left(1 - A\left(\frac{x_1}{x_1+x_2}\right)\right)}{2\left(1 - A\left(\frac{1}{2}\right)\right)}, \tag{36}$$

where $A(\omega)$ is the A-function of C_0. Also C has a limiting upper tail copula which is the survival copula of the copula of the df G.

We conclude from this result and the radial symmetry of the t copula that the lower tail limit copula of the t copula is the copula of the df G in (36) in the case when $A(w)$ is the A-function of the t-EV copula given in (28). The upper tail limit copula is the survival copula of this limit.

7.3 Use of the bivariate t-LTLcopula in practice

The t lower tail limit copula is concealed in a somewhat complex bivariate df and cannot be easily extracted in a closed form and used for practical modelling purposes. Our philosophy once again is to look for alternative models that can play the role of the true limiting copula without any loss of flexibility. Since the A-function of the t-EV copula can be effectively substituted by that of the Gumbel or Galambos copulas we can investigate the df G that is obtained when these alternative A-functions are inserted in (36).

It turns out that a tractable choice is the Galambos copula, which yields the G function

$$G(x_1, x_2) = \left(\frac{x_1^{-\theta} + x_2^{-\theta}}{2} \right)^{-1/\theta}, \quad (x_1, x_2) \in (0, 1]^2.$$

It is easily verified using (4) that the copula of this bivariate df is the Clayton copula (35). Thus we conclude that the t lower tail limit copula may effectively be replaced by the simple, well-known Clayton copula for any practical work. This finding underscores an empirical observation by Breymann et al. (2003) that for bivariate financial return data where the t copula seemed to be the best overall copula model for the dependence, the Clayton copula seemed to be the best model for the most extreme observations in the joint lower tail and the survival copula of Clayton to be the best model for the most extreme observations in the joint upper tail.

A Appendix

A.1 Evaluation of Joint Quantile Exceedance Probabilities

We consider in turn the Gaussian copula C_P^{Ga} and t copula $C_{\nu,P}^t$ in the case when P is an equicorrelation matrix with non-negative elements, i.e. all diagonal elements equal to ρ where $\rho \geq 0$. We recall that if $\mathbf{X} \sim N_d(\mathbf{0}, P)$ then

$$X_i \stackrel{d}{=} \sqrt{\rho} Z + \sqrt{1-\rho}\,\epsilon_i, \quad i = 1, \ldots, d, \tag{37}$$

where $\epsilon_1, \ldots, \epsilon_d, Z$ are iid standard normal variates. This allows us to calculate

$$\begin{aligned} C_P^{\mathrm{Ga}}(\mathbf{u}) &= P(X_1 \leq \Phi^{-1}(u), \ldots, X_d \leq \Phi^{-1}(u)) \\ &= E\left(P\left(\epsilon_1 \leq \frac{\Phi^{-1}(u) - \sqrt{\rho} z}{\sqrt{1-\rho}}, \ldots, \epsilon_d \leq \frac{\Phi^{-1}(u) - \sqrt{\rho} z}{\sqrt{1-\rho}} \mid Z = z \right) \right) \\ &= E\left((\Phi(Y))^d \right), \end{aligned}$$

where $Y \sim N(\mu, \sigma^2)$ with $\mu = \Phi^{-1}(u)/\sqrt{1-\rho}$ and $\sigma^2 = \rho/(1-\rho)$. The final expectation may be calculated easily using numerical integration.

For the t copula we recall the mixture representation (2). We will calculate the copula of the random vector $\sqrt{W}\mathbf{X}$ where $\mathbf{X} \sim N_d(\mathbf{0}, P)$ as above and W is an independent variate with an inverse gamma distribution ($W \sim \mathrm{Ig}(\nu/2, \nu/2)$). This allows us to calculate that

$$\begin{aligned} C_{\nu,P}^t(\mathbf{u}) &= P(\sqrt{W} X_1 \leq t^{-1}(u), \ldots, \sqrt{W} X_d \leq t^{-1}(u)) \\ &= E\left(P\left(\epsilon_1 \leq \frac{t^{-1}(u) - \sqrt{\rho w} z}{\sqrt{(1-\rho) w}}, \ldots, \epsilon_d \leq \frac{t^{-1}(u) - \sqrt{\rho w} z}{\sqrt{(1-\rho) w}} \mid Z = z, W = w \right) \right) \\ &= E\left((\Phi(Y))^d \right), \end{aligned}$$

where $Y \stackrel{d}{=} a/\sqrt{W} + bZ$ with $a = t^{-1}(u)/\sqrt{1-\rho}$ and $b = \sqrt{\rho/(1-\rho)}$. In this case the evaluation of the expectation requires a numerical double integration; in the inner integral the density of Y is evaluated by applying the convolution formula to $a/\sqrt{W} + bZ$. Results may be obtained using standard mathematical software.

A.2 Proof of Lemma 3

It is well known, and can be easily shown exploiting the rule of Bernoulli-L'Hôpital, that the tail of a t distribution function, $\bar{t}_\nu(x) = 1 - t_\nu(x)$, is regularly varying at ∞ with index $-\nu$. This means that $\bar{t}_\nu(x) = x^{-\nu}L(x)$, where $L(x)$ is a slowly-varying function satisfying

$$\lim_{s\to\infty} \frac{L(sx)}{L(s)} = 1.$$

For more on regular and slow variation see, for example, Resnick (1987).

Proof. Since, for any x we have $x = -t_\nu^{-1}(\bar{t}_\nu(x))$ the identity

$$x = \frac{t_\nu^{-1}(\bar{t}_\nu(sx))}{t_\nu^{-1}(\bar{t}_\nu(s))}$$

must also hold for all s. Hence taking limits we obtain

$$x \quad = \quad \lim_{s\to\infty} \frac{t_\nu^{-1}(\bar{t}_\nu(sx))}{t_\nu^{-1}(\bar{t}_\nu(s))} = \lim_{s\to\infty} \frac{t_\nu^{-1}(x^{-\nu}s^{-\nu}L(sx))}{t_\nu^{-1}(s^{-\nu}L(s))} = \lim_{v\to 0} \frac{t_\nu^{-1}(x^{-\nu}v)}{t_\nu^{-1}(v)} = \lim_{v\to 0} \frac{t_\nu^{-1}(1 - x^{-\nu}v)}{t_\nu^{-1}(1-v)}$$

where we use the fact that $L(sx)/(Ls) \to 1$ and $s^{-\nu}L(s) \to 0$ as $s \to \infty$. The identities (26) follow. $\square$

References

Abramowitz, M. & Stegun, I., eds. (1965). *Handbook of Mathematical Functions.* Dover Publications, Inc., New York.

Barndorff-Nielsen, O. & Blæsild, P. (1981). Hyperbolic distributions and ramifications: contributions to theory and application. In *Statistical Distributions in Scientific Work*, vol. 4.

Blæsild, P. & Jensen, J. (1981). Multivariate distributions of hyperbolic type. In *Statistical Distributions in Scientific Work*, vol. 4.

Breymann, W., Dias, A. & Embrechts, P. (2003). Dependence structures for multivariate high-frequency data in finance. *Quant. Finance* **3**, 1–14.

Daul, S., De Giorgi, E., Lindskog, F. & McNeil, A. (2003). The grouped t–copula with an application to credit risk. *RISK* **16**, 73–76.

Davison, A. & Smith, R. (1990). Models for exceedances over high thresholds (with discussion). *J. R. Stat. Soc. Ser. B Stat. Methodol.* **52**, 393–442.

Demarta, S. (2001). *Extreme Value Theory and Copulas.* Master's thesis, ETH Zürich, `http://www.math.ethz.ch/~demarta`.

Embrechts, P., Klüppelberg, C. & Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin.

Embrechts, P., McNeil, A. & Straumann, D. (2001). Correlation and dependency in risk management: properties and pitfalls. In *Risk Management: Value at Risk and Beyond*, M. Dempster & H. Moffatt, eds. `http://www.math.ethz.ch/~mcneil`: Cambridge University Press, pp. 176–223.

Fang, H. & Fang, K. (2002). The meta–elliptical distributions with given marginals. *J. Multivariate Anal.* **82**, 1–16.

Fang, K., Kotz, S. & Ng, W. (1990). *Symmetric multivariate and related distributions.* Chapman & Hall.

Fang, K.-T., Kotz, S. & Ng, K.-W. (1987). *Symmetric Multivariate and Related Distributions.* Chapman & Hall, London.

Galambos, J. (1987). *The Asymptotic Theory of Extreme Order Statistics.* Kreiger Publishing Co., Melbourne, FL.

Genest, C., Ghoudi, K. & Rivest, L. (1995). A semi–parametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika* **82**, 543–552.

Hult, H. & Lindskog, F. (2001). Multivariate extremes, aggregation and dependence in elliptical distributions. Preprint.

Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman & Hall, London.

Johnson, N. & Kotz, S. (1972). *Continuous Multivariate Distributions.* Wiley, New York.

Juri, A. & Wuethrich, M. (2002). Tail dependence from a distributional point of view. Preprint.

Kelker, D. (1970). Distribution theory of spherical distributions and a location–scale parameter generalization. *Sankhiă A* **32**, 419–430.

Kotz, S., Balakrishnan, N. & Johnson, N. (2000). *Continuous Multivariate Distributions.* New York: Wiley.

Kotz, S. & Nadarajah, S. (2000). *Extreme Value Distributions, Theory and Applications.* Imperial College Press, London.

Kruskal, W. (1958). Ordinal measures of association. *J. Amer. Statist. Assoc.* **53**, 814–861.

Lindskog, F. (2000). Modelling dependence with copulas. RiskLab Report, ETH Zurich.

Lindskog, F., McNeil, A. & Schmock, U. (2003). Kendall's tau for elliptical distributions. In *Credit Risk – measurement, evaluation and management*, Bol, Nakhaeizade et al., eds. Physica–Verlag, Heidelberg, pp. 149–156.

Mashal, R. & Zeevi, A. (2002). Beyond correlation: extreme co–movements between financial assets. Unpublished, Columbia University.

McLeish, D. & Small, C. (1988). *The Theory and Applications of Statistical Inference Functions*, vol. 44 of *Lecture Notes in Statistics.* Springer-Verlag, New York.

Nelsen, R. B. (1999). *An Introduction to Copulas.* Springer, New York.

Pickands, J. (1981). Multivariate extreme value distribution. *Proc. 43rd. Session Int. Stat. Inst Buenos Aires* **Book 2**, 859–878.

Resnick, S. (1987). *Extreme Values, Regular Variation and Point Processes.* Springer, New York.

Rousseeuw, P. & Molenberghs, G. (1993). Transformation of non positive semidefinite correlation matrices. *Comm. Statist. Theory Methods* **22**, 965–984.

Sibuya, M. (1961). Bivariate extreme statistics. *Annals of Mathematical Statistics* **11**, 195–210. REMOVE THIS REFERENCE.

Takahashi, R. (1994). Domains of attraction of multivariate extreme value distributions. *J. Res. Natl. Inst. Stand. Technol.* **99**, 551–554.

Tiago de Oliveira, J. (1975). Bivariate and multivariate extremal distributions. *Statistical Distributions in Scientific Work* **1**, 355–361. G. Patil et al., Dr. Reidel Publ. Co.