On Cox processes and credit risky securities[1]

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[1]Previous title: On Cox processes and credit risky bonds. This paper is a revised and extended version of parts of Chapter 3 of my Ph.D. thesis. I have received useful comments from my committee members Robert Jarrow, Rick Durrett, Sid Resnick and Marty Wells, as well as from Michael Hauch, Brian Huge, Martin Jacobsen, Monique Jeanblanc, Kristian Miltersen, Lasse Heje Pedersen, Torben Skødeberg and Stuart Turnbull and from participants at seminars at the 1993 FMA Doctoral seminar in Toronto, Merrill Lynch, Aarhus University, Odense University, University of Copenhagen, Stockholm School of Economics, University of Lund, New York University, Erasmus University, the 1994 Risk conference in Frankfurt, IBC conference on Credit derivatives in London 1996, the 1996 conference on Mathematical Finance in Aarhus. All errors are of course my own.

1 Introduction

The aim of this paper is to illustrate how Cox processes - also known as doubly stochastic Poisson processes - provide a useful framework for modeling prices of financial instruments in which credit risk is a significant factor. Both the case where credit risk enters because of the risk of counterparty default and the case where some measure of credit risk such as a credit spread is used as an underlying variable in a derivative contract can be analyzed within our framework.

The key contribution of the approach is the simplicity by which it allows credit risk modeling when there is dependence between the default-free term structure and the default characteristics of firms. As an illustration of the method we present a generalized version of a Markovian model presented in Jarrow, Lando and Turnbull [13]. This generalization allows the intensities which control the default intensities and the transitions between ratings to depend on state variables which simultaneously govern the evolution of the term structure and possibly other economic variables of interest. This model can be used to analyze a variety of contracts in which credit ratings and default are part of the contractual setup. Two important examples are default protection contracts which provide insurance against default of a firm and so-called 'credit triggers' in swap contracts which typically demand the settlement of a contract in the event of downgrading of one of the parties to the contract. By imposing some additional structure on the model we derive a new class of models which in a special setup become 'sums of affine term structure models'. These models simultaneously model term structures for different ratings and allow spreads to fluctuate randomly even in periods where the rating of the defaultable firm does not change. Such models are useful for managers of corporate bond portfolios (and risky government debt) who are constrained by credit lines setting upper limits on the amount of bonds held below a certain rating category. Given such lines the event of a downgrading may force the manager to either sell off part of the position or try to off-load some credit exposure in credit derivatives markets.

The modeling framework is similar to that of Jarrow and Turnbull [14], Jarrow, Lando and Turnbull [13], Lando [16], Madan and Unal [19], Artzner and Delbaen [1], Duffie and Singleton [7], [8], and Duffie, Schroder and Skiadas [9] in that default is modeled through a random intensity of the default time. However, this paper dispenses with the independence assumption employed in Jarrow and Turnbull [14], Jarrow, Lando and Turnbull [13], Madan and Unal [19]. In this respect our approach resembles that of the works of Duffie and Singleton [7],[8], and Duffie, Schroder and Skiadas [9]. The approach in this paper is not based on the recursive methods employed there. This sacrifices some generality but the structure of Cox processes produces a class of models which are simple to derive yet seems rich enough to handle important applications when only one-sided credit risk is considered. For an analysis of two sided default risk in an intensity-based framework, see Duffie and Huang [6].

The pros and cons of modeling default in an intensity-based framework as opposed to using the classical approach originating in Black and Scholes [2] and Merton [20] are discussed elsewhere, see for example Duffie and Singleton [7],[8], and the review papers by Cooper and Martin [3] and Lando [17]. Here we simply note that this paper and the work by Duffie and Singleton [8] point to one important advantage of the intensity-based framework, namely the fact that the intensity based framework is capable of reducing the technical difficulties of modeling defaultable claims to the same one faced when modeling the term structure of non-defaultable bonds and related derivatives.

The structure of the paper is as follows: In section 2 we outline the basic construction of a Cox process, concentrating on the first jump time of such a process. Knowing how to construct a process means knowing how to simulate it. Hence this section is important if one wants to implement the models. Furthermore, the explicit construction gives the key to proving the results later on by - essentially - using properties of conditional expectations.

Section 3 then shows how to calculate prices of three 'building blocks' from which credit risky bonds and derivatives can be constructed. Several examples of claims are given which are constructed from these building blocks. Within this framework we have great flexibility in how we specify the behavior of the state variables which simultaneously determine the default free term structure and the likelihood of default. When diffusions are used as state variables, the Feynman-Kac framework handles pricing of credit risky derivative securities in virtually the same way as it handles derivatives in which credit risk is not present. That particular setup is one of 'diffusion with killing' - a special case of the general framework. We also outline how the notion of fractional recovery, as introduced in Duffie and Singleton [8], may be viewed in terms of thinning of a point process.

Section 4 is an applications of the methodology. It is shown how a Markovian model proposed by Jarrow, Lando and Turnbull [13] may be generalized to include transition rates between credit ratings which depend on the state variables. One important aspect of this generalization is that credit spreads are allowed to fluctuate randomly even between ratings transitions. Section 5 gives a concrete implementation of the generalized Markovian model by considering an affine model for both the interest rate and the transition intensities. It is shown how one can calibrate the model across all credit classes simultaneously to fit initial spot rate spreads and the sensitivities of these spreads to changes in the short treasury rate. Section 6 concludes.

2 Construction of a Cox process

Before setting up our model of credit risky bonds and derivatives, we give both an intuitive and a formal description of how the default process is modeled. The formal description is extremely useful when performing calculations and simulations of the model, whereas the intuitive description is helpful when the economic content of the models are to be understood.

Recall that an inhomogeneous Poisson process N with (non-negative) intensity function $l(\cdot)$ satisfies

$$P(N_t - N_s = k) = \frac{\left(\int_s^t l(u)du\right)^k}{k!} \exp\left(-\int_s^t l(u)du\right) \qquad k = 0, 1, \ldots.$$

In particular, assuming $N_0 = 0$, we have

$$P(N_t = 0) = \exp\left(-\int_0^t l(u)du\right).$$

A way of simulating the first jump τ of N is to let E_1 be a unit exponential random variable and define

$$\tau = \inf\{t : \int_0^t l(u)du \geq E_1\} \tag{2.1}$$

A Cox process is a generalization of the Poisson process in which the intensity is allowed to be random but in such a way that if we condition on a particular realization $l(\cdot, \omega)$ of the intensity, the jump process becomes an inhomogeneous Poisson process with intensity $l(s, \omega)$.

In this paper we will write the random intensity on the form

$$l(s, \omega) = \lambda(X_s)$$

where X is an R^d-valued stochastic process and $\lambda : R^d \to [0, \infty)$ is a non-negative, continuous function. The assumption that the intensity is a function of the current level of the state variables, and not the whole history, is convenient in applications, but it is not necessary from a mathematical point of view.

The state variables will include interest rates on riskless debt and may include time, stock prices, credit ratings and other variables deemed relevant for predicting the likelihood of default. Intuitively, given that a firm has survived up to time t, and given the history of X up to time t, the probability of defaulting within the next small time interval Δt is equal to $\lambda(X_t)\Delta t + o(\Delta t)$.

Formally, we have a probability space $(\Omega, \mathcal{F}, P)$ large enough to support an R^d-valued stochastic process $X = \{X_t : 0 \leq t \leq T_f\}$ which is right-continuous with left limits and a unit exponential random variable E_1 which is independent of X. Given also is a function $\lambda : R^d \to R$ which we assume is non-negative and continuous. From these two ingredients we define the default time τ as follows:

$$\tau = \inf\{t : \int_0^t \lambda(X_s) ds \geq E_1\}. \tag{2.2}$$

This default time can be thought of as the first jump time of a Cox process with intensity process $\lambda(X_s)$. Note that this is an exact analogue to equation (2.1) with a random intensity replacing the deterministic intensity function.

When $\lambda(X_s)$ is large, the integrated hazard grows faster and reaches the level of the independent exponential variable faster, and therefore the probability that τ is small becomes higher. Sample paths for which the integral of the intensity is infinite over the interval $[0, T_f]$ are paths for which default always occurs.

From the definition we get the following key relationships:

$$P\left(\tau > t | \left(X_s\right)_{0 \leq s \leq t}\right) = \exp\left(-\int_0^t \lambda\left(X_s\right) ds\right) \qquad t \in [0, T_f] \tag{2.3}$$

$$P\left(\tau > t\right) = E \exp\left(-\int_0^t \lambda\left(X_s\right) ds\right) \qquad t \in [0, T_f] \tag{2.4}$$

What we have modeled above, is only the first jump of a Cox process. When modeling a Cox process past the first jump - something we will need in Section 4 - one has to insure that the integrated intensity stays finite on finite intervals if explosions are to be avoided. One then proceeds as follows: Given a probability space $(\Omega, \mathcal{F}, P)$ large enough to support a standard unit rate Poisson process N with $N_0 = 0$ and a non-negative stochastic process $\lambda(t)$ which is independent of N and assumed to be right-continuous and integrable on finite intervals, i.e.

$$\Lambda(t) := \int_0^t \lambda(s) ds < \infty \qquad t \in [0, T]$$

Then $\Lambda(0) = 0$ and Λ has non-decreasing realizations. Now defining

$$\tilde{N}_t := N(\Lambda(t))$$

we have a Cox process with intensity measure Λ. For the technical conditions we need to check to see that this definition makes sense, see Grandell [12] pp. 9-16. This approach is also relevant when extending the models presented in this paper to cases of repeated defaults by the same firm.

3 Three basic building blocks

The model for the default-free term structure of interest rates is given by a spot rate process, a money market account and an equivalent martingale measure to which all expectations in this paper refer:

$$\begin{aligned} B(t) &= \exp\left(\int_0^t r_s ds\right) \\ p(t,T) &= E\left(\left.\frac{B(t)}{B(T)}\right| \mathcal{F}_t\right) \end{aligned}$$

The default time τ is modeled using the framework described in the previous section. The intensity process is denoted λ. Hence we may write the information at time t as

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$$

where processes relevant in determining values of the spot rate and the hazard rate of default are adapted to $\mathcal{G}$ and

$$\mathcal{H}_t = \sigma\{1_{\{\tau \leq s\}} : 0 \leq s \leq t\}$$

holds the information of whether there has been a default at time t. Most applications will be built around a state variable process X, in which case we write $R(X_t)$ for the spot rate r_t at time t and the informational setup may be stated as

$$
\begin{aligned}
\mathcal{G}_t &= \sigma\{X_s : 0 \leq s \leq t\} \\
\mathcal{H}_t &= \sigma\{1_{\{\tau \leq s\}} : 0 \leq s \leq t\} \\
\mathcal{F}_t &= \mathcal{G}_t \vee \mathcal{H}_t
\end{aligned}
$$

$\mathcal{F}_t$ then corresponds to knowing the evolution of the state variables up to time t and whether default has occurred or not.

Apart from the default characteristics and the default-free term structure of interest rates what we need to know to price a defaultable contingent claim are its promised payments and its payment in the event of default (i.e. its recovery). The basic building blocks in constructing such claims are

1. $X1_{\{\tau>T\}}$: A payment $X \in \mathcal{G}_T$ at a fixed date which occurs if there has been no default before time T. As an example think of a vulnerable European option, i.e. an option whose issuer may default before the option's expiration. A possible partial recovery can be captured in two ways: Let δ_T be a $\mathcal{G}_T-$measurable random variable, and think of partial recovery either as a recovery at the time of maturity of the form $\delta_T 1_{\{\tau \leq T\}} = \delta_T - \delta_T 1_{\{\tau>T\}}$, which consists of a non-defaultable claim and our first basic building block, or a recovery at the time of default as in the third building block below.

2. $Y_s 1_{\{\tau>s\}}$: A stream of payments at a rate specified by the $\mathcal{G}_t-$adapted process Y which stops when default occurs. This may be used to model idealized swaps in which we think of payments taking place continually in time. A possible recovery is captured by the next building block.

3. A recovery payment at the time of default of the form Z_τ where Z is a $\mathcal{G}_t-$adapted stochastic process and where $Z_\tau = Z_{\tau(\omega)}(\omega)$. For a contingent claim expiring at date T, think of Z as being 0 after time T. This building block models a resettlement payment at the time of default.

Note that the random variable X and the process Y can be thought of as promised payments, whereas the payment captured by Z is an actual payment at the time of default.

The key advantage of the setup is that the calculation of prices becomes similar to ordinary term structure modeling:

Proposition 3.1 *Assume that the expectations*

$$
\begin{aligned}
& E\left(\exp\left(-\int_t^T r_s ds\right)|X|\right) \\
& E\int_t^T |Y_s| \exp\left(-\int_t^s r_u du\right) ds \qquad \textit{and} \\
& E\int_t^T |Z_s\lambda_s| \exp\left(-\int_t^s (r_u+\lambda_u) du\right) ds
\end{aligned}
$$

are all finite. The following identities hold for the three claims listed above:

$$
\begin{aligned}
& E\left(\exp\left(-\int_t^T r_s ds\right) X 1_{\{\tau > T\}} \middle| \mathcal{F}_t\right) \\
= \; & 1_{\{\tau>t\}} E\left(\exp\left(-\int_t^T (r_s+\lambda_s) ds\right) X \middle| \mathcal{G}_t\right)
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
& E\left(\int_t^T Y_s 1_{\{\tau > s\}} \exp\left(-\int_t^s r_u du\right) ds \middle| \mathcal{F}_t\right) \\
= \; & 1_{\{\tau>t\}} E\left(\int_t^T Y_s \exp\left(-\int_t^s (r_u+\lambda_u) du\right) ds \middle| \mathcal{G}_t\right)
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
& E\left(\exp\left(-\int_t^\tau r_s ds\right) Z_\tau \middle| \mathcal{F}_t\right) \\
= \; & 1_{\{\tau>t\}} E\left(\int_t^T Z_s\lambda_s \exp\left(-\int_t^s (r_u+\lambda_u) du\right) ds \middle| \mathcal{G}_t\right)
\end{aligned} \tag{3.3}
$$

Proof. First we show that

$$
E\left(1_{\{\tau \geq T\}} \middle| \mathcal{G}_T \vee \mathcal{H}_t\right) = 1_{\{\tau>t\}} \exp\left(-\int_t^T \lambda_s ds\right) \tag{3.4}
$$

In the following computations we use the fact that the conditional expectation is clearly 0 on the set $\{\tau \leq t\}$ and that the set $\{\tau > t\}$ is an atom of

$\mathcal{H}_t$. Therefore

$$
\begin{aligned}
& E\left(1_{\{\tau\geq T\}}|\mathcal{G}_T\vee\mathcal{H}_t\right) \\
= \;& 1_{\{\tau>t\}}E\left(1_{\{\tau\geq T\}}|\mathcal{G}_T\vee\mathcal{H}_t\right) \\
= \;& 1_{\{\tau>t\}}\frac{P\left(\{\tau\geq T\}\cap\{\tau>t\}|\mathcal{G}_T\right)}{P\left(\tau>t|\mathcal{G}_T\right)} \\
= \;& 1_{\{\tau>t\}}\frac{P\left(\tau\geq T|\mathcal{G}_T\right)}{P\left(\tau>t|\mathcal{G}_T\right)} \\
= \;& 1_{\{\tau>t\}}\frac{\exp\left(-\int_0^T\lambda_s ds\right)}{\exp\left(-\int_0^t\lambda_s ds\right)} \\
= \;& 1_{\{\tau>t\}}\exp\left(-\int_t^T\lambda_s ds\right)
\end{aligned}
$$

which proves (3.4).

Now consider (3.1).

$$
\begin{aligned}
& E\left(\exp\left(-\int_t^T r_s ds\right)X1_{\{\tau>T\}}\middle|\mathcal{F}_t\right) \\
= \;& E\left(E\left(\exp\left(-\int_t^T r_s ds\right)X1_{\{\tau>T\}}\middle|\mathcal{G}_T\vee\mathcal{H}_t\right)|\mathcal{F}_t\right) \\
= \;& E\left(\exp\left(-\int_t^T r_s ds\right)XE\left(1_{\{\tau>T\}}\middle|\mathcal{G}_T\vee\mathcal{H}_t\right)\middle|\mathcal{F}_t\right) \\
= \;& 1_{\{\tau>t\}}E\left(\exp\left(-\int_t^T r_s+\lambda_s ds\right)X\middle|\mathcal{F}_t\right)
\end{aligned}
$$

Now we want to replace the conditioning on $\mathcal{F}_t$ with conditioning on $\mathcal{G}$. Recall that E_1 is a unit exponential random variable which is independent of the sigma field $\mathcal{G}_T$. In particular, E_1 is independent of the sigma field $\sigma\left(\exp\left(-\int_t^T(r_s+\lambda_s)\,ds\right)X\right)\vee\mathcal{G}_t$ and therefore (see Williams [21] 9.7(k))

$$
\begin{aligned}
& E\left(\exp\left(-\int_t^T r_s+\lambda_s ds\right)X\middle|\mathcal{G}_t\vee\sigma(E_1)\right) \\
= \;& E\left(\exp\left(-\int_t^T r_s+\lambda_s ds\right)X\middle|\mathcal{G}_t\right).
\end{aligned}
\tag{3.5}
$$

But we also have the following inclusion among sigma fields:

$$
\mathcal{G}_t\subset\mathcal{G}_t\vee\mathcal{H}_t\subset\mathcal{G}_t\vee\sigma(E_1). \tag{3.6}
$$

Combining (3.5) and (3.6) we conclude that

$$\begin{aligned} & E\left(\exp\left(-\int_t^T r_s+\lambda_s ds\right)X\middle|\mathcal{G}_t\vee\mathcal{H}_t\right) \\ = \;& E\left(\exp\left(-\int_t^T r_s+\lambda_s ds\right)X\middle|\mathcal{G}_t\right). \end{aligned}$$

which is what we wanted to show.

The proof of (3.2) proceeds exactly as above by first conditioning on $\mathcal{G}_T\vee\mathcal{H}_t$.

For the proof of (3.3) note that conditionally on $\mathcal{G}_T$ and for $s>t$ the density of the default time is given by

$$\frac{\partial}{\partial s}P(\tau\le s|\,\tau>t,\mathcal{G}_T)=\lambda_s\exp\left(-\int_t^s\lambda_u du\right).$$

Hence we find

$$\begin{aligned} & E\left(\exp\left(-\int_t^\tau r_s ds\right)Z_\tau\middle|\mathcal{F}_t\right) \\ = \;& E\left(E\left(\exp\left(-\int_t^\tau r_s ds\right)Z_\tau\middle|\mathcal{G}_T\vee\mathcal{H}_t\right)|\mathcal{F}_t\right) \\ = \;& 1_{\{\tau>t\}}E\left(\int_t^T Z_s\lambda_s\exp\left(-\int_t^s(r_u+\lambda_u)du\right)ds\middle|\mathcal{F}_t\right) \\ = \;& 1_{\{\tau>t\}}E\left(\int_t^T Z_s\lambda_s\exp\left(-\int_t^s(r_u+\lambda_u)du\right)ds\middle|\mathcal{G}_t\right) \end{aligned}$$

where in the last line we have used the same argument as above to replace $\mathcal{F}_t$ by $\mathcal{G}_t$ ■

We now turn to examples and illustrations of this framework.

Example 3.2 *Option on credit spread.*

In the examples considered above the primary focus is securities with credit risk. Note, however, that a model of the default intensity λ produces a model for the yield spread for bonds of all maturities. Hence our framework will easily produce models for options on yield spreads as well.

Example 3.3 *The independence case.*

A primary focus of this paper is to allow for dependence between default intensities and state variables. Note that the independence case is easily captured as well. In the state variable formulation we would simply let $X = (X^1, X^2)$ with X^1 and X^2 independent processes and define the spot rate as the process $r(X^1)$ and the default intensity process as $\lambda(X^2)$. Of course, either of these processes may be included as a state variable themselves.

Example 3.4 *(Feynman-Kac representation).*

An important special case is when

$$\mathcal{G}_t = \sigma\{X_s : 0 \leq s \leq t\} \tag{3.7}$$

where X is an R^d-valued diffusion process with diffusion matrix $a(x,t)$ and drift vector $b(x,t)$ and infinitesimal generator

$$\mathcal{A}_t F(x) := \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(t,x)\frac{\partial^2 F(x)}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial F(x)}{\partial x_i} \qquad F \in C^2(R^d)$$

Let $T = T_f$ and let $f : R^d \to R$, $g : [0,T] \times R^d \to R$ and $k : [0,T] \times R^d \to [0,\infty)$ be continuous functions. Note that when (3.7) holds, the relations (3.1),(3.2) and (3.3) are all of the form[1]

$$\begin{aligned} v(t,x) &= E^{t,x}\left(f\left(X_T\right)\exp\left(-\int_t^T k(u,X_u)du\right)\right) \\ &\quad + E^{t,x}\left(\int_t^T g(s,X_s)\exp\left(-\int_t^s k(u,X_u)du\right)ds\right) \end{aligned} \tag{3.8}$$

Hence, under suitable regularity conditions[2] on the diffusion coefficients and on the functions f, g, k, the expectation in (3.8) is also a solution (satisfying an exponential growth condition) to the Cauchy problem

$$\begin{aligned} -\frac{\partial v}{\partial t} + kv &= \mathcal{A}_t v + g \quad [0,T] \times R^d \\ v(T,x) &= f(x) \quad x \in R^d \end{aligned}$$

and we see that the credit risky securities can be modeled using exactly the same techniques as default-free securities.

[1]Note that in this section we will explicitly write time dependence of payments, spot rates and hazard rates. In previous sections time dependence is possible simply by letting one of the state variables be time - here we prefer to separate it from the diffusion components.

[2]See Karatzas and Shreve [15] and Duffie [5] for results and references to some of the literature on which combinations of conditions one can impose to obtain this correspondence.

Example 3.5 Fractional recovery and thinning.

In Duffie and Singleton [8], and Duffie, Schroder and Skiadas [9] a notion of fractional recovery is considered which allows not only the intensity but the recovery rate to enter into the expression for the default adjusted spot rate process. Here we sketch how this may be given a simple interpretation in the Cox process framework through the notion of thinning. Receiving a fraction ϕ_t of pre-default value in the event of default of a contract is equivalent, from a pricing perspective, to receiving the outcome of a lottery in which the full pre-default value is received with probability (under the martingale measure) ϕ_t and 0 is received with probability $1-\phi_t$, i.e the event of default has been retained with probability $1-\phi_t$. This in turn may be viewed as a default process in which there is 0 recovery but where the default intensity has been thinned using the process ϕ, producing a new default intensity of $\lambda(1-\phi)$. In the case of a claim with promised payment X_T and fractional recovery at the default date of ϕ_τ one then obtains that with $Z_\tau = \phi_\tau c_{\tau-}$, the price of this claim at time t is given as

$$\begin{aligned} c_t &= E\left(\exp\left(-\int_t^T r_s ds\right) X 1_{\{\tau>T\}} + \exp\left(-\int_t^\tau r_s ds\right) Z_\tau 1_{\{t<\tau\leq T\}} \middle| \mathcal{F}_t\right) \\ &= 1_{\{\tau>t\}} E\left(\exp\left(-\int_t^T (r_s+\lambda_s(1-\phi_s))ds\right) X \middle| \mathcal{G}_t\right). \end{aligned}$$

4 A generalized Markovian model

In this section, as a concrete application of the framework presented above. we show how to generalize the model described in Jarrow, Lando and Turnbull [13] for pricing defaultable bonds. It is shown how the model can be generalized to incorporate state dependence in transition rates and risk premia allowing for stochastic changes in credit spreads even between ratings transitions.

There, the time to default is modeled as the absorption time in state K of a Markov chain with generator matrix

$$A = \begin{pmatrix} -\lambda_1 & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1K} \\ \lambda_{21} & -\lambda_2 & \lambda_{23} & \cdots & \lambda_{2K} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & -\lambda_{K-1} & \lambda_{K-1,K} \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{4.1}$$

$$\lambda_{ij} \geq 0 \text{ for } i \neq j$$

and

$$\lambda_i = \sum_{j=1, j \neq i}^{K} \lambda_{ij}, \ i = 1, \ldots, K-1.$$

In the study of Fons and Kimball [10] it is documented that default rates of lower rated firms show significant time variation and even for the top rated firms for which default is extremely rare the standard deviations of default rates may still be significant in relative terms. Also, Duffee [4] shows that noncallable bond yield spreads fall when the Treasury term structure yield increases. Whether this variation can be captured through variation in the state variables or whether a parametrization in terms of constant intensities as in Jarrow, Lando and Turnbull [13] performs equally well is an interesting empirical issue which we are currently checking. However, it is necessary with an extension if one is to take into account the random fluctuations on credit spreads for constant time to maturity bonds.

Here, we will show how the Markovian model can be generalized to incorporate random transition rates and random default intensities. We will start with a fairly general model and then impose special structure to gain computational advantages.

First, define the matrix

$$A_X(s) = \begin{pmatrix} -\lambda_1(X_s) & \lambda_{12}(X_s) & \lambda_{13}(X_s) & \cdots & \lambda_{1K}(X_s) \\ \lambda_{21}(X_s) & -\lambda_2(X_s) & \lambda_{23}(X_s) & \cdots & \lambda_{2K}(X_s) \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \lambda_{K-1,1}(X_s) & \lambda_{K-1,2}(X_s) & \cdots & -\lambda_{K-1}(X_s) & \lambda_{K-1,K}(X_s) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

and assume that

$$\lambda_i(X_s) = \sum_{j=1, j \neq i}^{K} \lambda_{ij}(X_s), \ i = 1, \ldots, K-1.$$

It is not essential that the random entries $A_X(s)$ are written as functions of state variables, but we choose this formulation since this we believe will be the interesting one to work with in practice. Also, we could of course include time as one of the state variables. Heuristically, we think of, say, $\lambda_1(X_s)\Delta t$ as the probability that a firm in rating class 1 will jump to a different class or default within the (small) time interval Δt. We now give the precise mathematical construction and note again that this is essential if one wants to run simulations with the model:

Let $E_{11}, \ldots, E_{1K}, E_{21}, \ldots, E_{2K}, \ldots$ be a sequence of independent unit exponential random variables.[3]

Assume that a firm starts out in rating class η_0. Define

$$\tau_{\eta_0,i} = \inf\{t : \int_0^t \lambda_{\eta_0,i}(X_s)ds \geq E_{1i}\} \qquad i = 1, \ldots, K$$

and let

$$\tau_{\eta_0} = \min_{i \neq \eta_0} \tau_{\eta_0,i}.$$

Here, τ_{η_0} models the time the rating changes from the initial level η_0 and this occurs the first time a set of 'competing risks' which tries to pull the rating away from η_0 experiences a jump. The new rating level may then be written as

$$\eta_1 = \arg\min_{i \neq \eta_0} \tau_{\eta_0,i}.$$

and if the value of η_1 is K this corresponds to default. Now let

$$\tau_{\eta_1,i} = \inf\{t : \int_{\tau_{\eta_0}}^t \lambda_{\eta_1,i}(X_s)ds \geq E_{2i}\} \qquad i = 1, \ldots, K$$

and

$$\tau_{\eta_1} = \min_{i \neq \eta_1} \tau_{\eta_1,i}.$$

Hence the next jump occurs as time τ_{η_1}, the new rating level is

$$\eta_2 = \arg\min_{i \neq \eta_1} \tau_{\eta_1,i}$$

and so forth. We use τ to denote the time of default, i.e.

$$\tau = \inf\{t : \eta_t = K\}$$

With this construction we obtain a continuous time process which conditionally on the evolution of the state variables is a non-homogeneous Markov

[3] Strictly speaking, we will only need K-1 exponentials for each jump but the notation will be simpler by ignoring this for now.

chain. Conditionally on the evolution of the state variables, the transition probabilities of this Markov chain satisfy

$$\frac{\partial P_X(s,t)}{\partial s} = -A_X(s)P_X(s,t).$$

It is worth noting that the solution of this equation in the time-inhomogeneous case is typically *not* equal to

$$P_X(s,t) = \exp\left(\int_s^t A_X(u)du\right)$$

which is most easily seen by recalling that when A and B are square matrices, we only have

$$\exp(A+B) = \exp(A)\exp(B)$$

when A and B commute.

For a way of writing the solution using product integral notation, see for example of Gill and Johansen [11].

5 Pricing bonds and derivatives in the generalized Markov model

A general bond pricing formula for defaultable bonds with zero recovery is then given in the following

Proposition 5.1 *Consider a zero coupon bond maturing at time T issued by a firm whose rating at time t is i and whose credit rating and default intensity evolves as described by the process η above. Then given zero recovery in default, the price of the bond is given as*

$$v^i(t,T) = E\left(\exp\left(-\int_t^T R(X_s)ds\right)(1 - P_X(t,T)_{i,K})\middle|\mathcal{G}_t\right)$$

where $P_X(t,T)_{i,K}$ is the (i,K)'th element of the matrix $P_X(t,T)$.

Proof.

$$v^i(t,T) \;=\; B(t)E^i\left(\frac{1_{\{\tau>T\}}}{B(T)}\middle|\mathcal{F}_t\right)$$

$$
\begin{aligned}
&= E^i\left(1_{\{\tau>T\}}\exp\left(-\int_t^T R(X_s)\,ds\right)\bigg|\,\mathcal{F}_t\right)\\
&= E\left(\exp\left(-\int_t^T R(X_s)\,ds\right)E^i\left(1_{\{\tau>T\}}\Big|\,\mathcal{G}_T\vee\mathcal{F}_t\right)\bigg|\,\mathcal{F}_t\right)\\
&= E\left(\exp\left(-\int_t^T R(X_s)ds\right)(1-P_X(t,T)_{i,K})\bigg|\,\mathcal{G}_t\right)
\end{aligned}
$$

where we use the same argument as earlier to replace $\mathcal{F}_t$ by $\mathcal{G}_t$. ∎

While this approach produces models that can easily be simulated using the construction of the process η given above, it is of interest to derive models which, although more restrictive in their assumptions, offer computational tractability and possibly even analytical solutions. One way to do this is to impose more structure on the conditional intensity matrix such that the expression for $P_X(s,t)$ simplifies:

Lemma 5.2 *For $i = 1,\ldots,K-1$, let $\mu_i : R^d \to [0,\infty)$ be a non-negative function defined on the state space of X and assume that for almost every sample path of X, we have*

$$\int_0^{T_f}\mu_i(X_s)ds < \infty.$$

Let $\mu(X_s)$ denote the $K\times K$ diagonal matrix $diag(\mu_1(X_s),\ldots,\mu_{K-1}(X_s),0)$. For each path of X assume that the time dependent generator matrix has the representation

$$A_X(s) = B\mu(X_s)B^{-1},$$

where B is a $K\times K$ matrix whose columns consist of K eigenvectors of $A_X(s)$. Also, define the diagonal matrix

$$E_X(s,t) = \begin{pmatrix} \exp\left(\int_s^t \mu_1(X_u)du\right) & 0 & \cdots & 0\\ 0 & \ddots & \cdots & 0\\ \vdots & \cdots & \exp\left(\int_s^t \mu_{K-1}(X_u)du\right) & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}$$

Then with $P_X(s,t) = BE_X(s,t)B^{-1}$, $P_X(s,t)$ satisfies Kolmogorov's backward equation

$$\frac{\partial P_X(s,t)}{\partial s} = -A_X(s)P_X(s,t)$$

and $P_X(s,t)$ is the transition probability of an (inhomogeneous) Markov chain on $\{1,\ldots,K\}$.

Proof.

Since

$$A_X(s)B = B\mu_X(s)$$

we see that

$$\begin{aligned}\frac{\partial P_X(s,t)}{\partial s} &= B(-\mu_X(s))E_X(s,t)B^{-1} \\ &= -A_X(s)BE_X(s,t)B^{-1} \\ &= -A_X(s)P_X(s,t)\end{aligned}$$

which shows that $P_X(s,t)$ is indeed a solution to the backward equation. From section 4.4 in Gill and Johansen [11] we know that under our assumptions on $A_X(s)$ this equation has a unique solution, and the solution is a transition matrix for an inhomogeneous Markov chain with the density of the 'intensity measure' given by $A_X(s)$. Hence $P_X(s,t)$ defines a Markov chain, as was to be shown. ■

The assumption that the eigenvectors of $A_X(s)$ exist and are not time varying is a strong assumption. The simplification is, however, profound as well as we see in the next result which delivers a bond pricing model which models bonds of all credit categories simultaneously:

Proposition 5.3 *Assume that transitions between rating classes conditionally on X are governed by the generator matrix*

$$A_X(s) = B\mu(X_s)B^{-1}$$

where B and $\mu(X_s)$ are defined as in Lemma 5.2. Assume in addition, that the K'th row of $A_X(s)$ is 0 (corresponding to the state of default being an absorbing state). The spot rate process for default free debt is given by $R(X)$. The price at time t of a zero coupon bond with maturity T, issued by a firm in rating class i and (for simplicity) with zero recovery in default is given by

$$v^i(t,T) = \sum_{j=1}^{K-1} \beta_{ij} E\left(\exp\left(\int_t^T \left(\mu_j\left(X_u\right) - R\left(X_u\right) du\right)\right)\middle| \mathcal{G}_t\right)$$

where the coefficients β_{ij} are defined below.

Proof. Conditionally on X the probability of defaulting before T given a credit rating of i at time t (and hence no previous default since default in this model is an absorbing state) is equal to the $(i,K)'$th entry of the matrix $P_X(t,T)$. This entry has the form

$$P_X(t,T)_{i,K} = \sum_{j=1}^{K} b_{ij} \exp\Big(\int_t^T \mu_j(X_u)du\Big) b_{jK}^{-1},$$

where b_{jK}^{-1} denotes the $(j,K)'th$ entry of B^{-1} and using the fact that the rows of A_X sum to 0 and that the last row of A_X is 0, one may check that $b_{iK}b_{KK}^{-1} = 1$. Defining $\beta_{ij} = -b_{ij}b_{jK}^{-1}$ we may therefore write the probability of no default as

$$1 - P_X(t,T)_{i,K} = \sum_{j=1}^{K-1} \beta_{ij} \exp\Big(\int_t^T \mu_j(X_u)du\Big) \tag{5.1}$$

Let superscript i on the expectation operator signify a time t rating of the issuer of i. The superscript is removed when the expectation does not depend on this information. Now combine Proposition 5.1 and (5.1) to obtain

$$v^i(t,T) = \sum_{j=1}^{K-1} \beta_{ij} E\left(\exp\left(\int_t^T \left(\mu_j\left(X_u\right) - R\left(X_u\right) du\right)\right) \Bigg| \mathcal{G}_t\right) \qquad \blacksquare$$

We have expressed the price of bond in credit class i as a linear combination of functionals of the form

$$E \exp\left(\int_0^t \left(\mu_j\left(X_u\right) - R\left(X_u\right) du\right)\right).$$

Note that by using the results on affine term structure models (letting μ and R be affine functions of diffusions with affine drift and volatility) we obtain here a class of models whose bond prices are expressed as sums of affine models.

To keep the exposition simple we have looked at zero coupon bonds with zero recovery. The models above, however, also allow us to price more complicated contracts whose contractual payoff are linked to the rating of the issuer. Examples include credit options and swaps with credit triggers. This exercise can be carried out for all the three building blocks that we priced in Section 3. Consider an example linked to credit insurance in which the holder of the contingent claim receives a payment which depends on state variables and on the rating of some company at a fixed time T. Here, the contract itself need not have any credit risk in that it may easily pay out an amount in the event of default of the firm whose rating determines the payout.

Proposition 5.4 *Let the model be as in the previous proposition. Consider a contingent claim whose payoff at time T is given as*

$$C_T = C(X_T, \eta_T).$$

Assume that the rating at time t of the firm under consideration is i. The price of this claim at time t is given by

$$C_t^i = \sum_{m=1}^{K} \sum_{k=1}^{K} E\left(b_{ik} b_{km}^{-1} C(X_T, m) \exp\left(\int_t^T \left(\mu_k\left(X_u\right) - R\left(X_u\right) du\right)\right) \middle| \mathcal{G}_t \right).$$

Proof. First, note that

$$P_X(t,T)_{i,m} = \sum_{k=1}^{K} b_{ik} b_{km}^{-1} \exp(\int_t^T \mu_k(X_u) du),$$

and therefore

$$\begin{aligned}
C_t^i &= E^i\left(\sum_{m=1}^{K} C(X_T, m) P_X(t,T)_{i,m} \exp\left(-\int_t^T R\left(X_u\right) du \right) \middle| \mathcal{G}_t \right) \\
&= E\left(\sum_{m=1}^{K} C(X_T, m) P_X(t,T)_{i,m} \exp\left(-\int_t^T R\left(X_u\right) du \right) \middle| \mathcal{G}_t \right) \\
&= E\left(\sum_{m=1}^{K} C(X_T, m) \sum_{k=1}^{K} b_{ik} b_{km}^{-1} \exp(\int_t^T \mu_k(X_u) du) \exp\left(-\int_t^T R\left(X_u\right) du \right) \middle| \mathcal{G}_t \right) \\
&= \sum_{m=1}^{K} \sum_{k=1}^{K} E\left(b_{ik} b_{km}^{-1} C(X_T, m) \exp\left(\int_t^T \left(\mu_k\left(X_u\right) - R\left(X_u\right) du\right)\right) \middle| \mathcal{G}_t \right)
\end{aligned}$$

which is the desired result. ■

Note that this pricing formula allows payments to be directly linked to the rating status of a certain risky firm.

Clearly, the parametric assumptions we impose to get the very tractable model above are strong. By stochastic scaling of the eigenvectors we allow stochastic changes the $(K-1)^2$ transition and default intensities to vary in a $K-1$ dimensional subspace and this will allow stochastic fluctuations to affect higher rated and lower rated firms differently. Empirical analysis of this form is currently being carried out in separate work.

Allowing the generator to be specified more freely will complicate computation but simulation is by no means the only way out: Consider again the equation

$$\frac{\partial P_X(s,t)}{\partial s} = -A_X(s)P_X(s,t).$$

To compute the matrix $P_X(s,t)$, use the approximation based on a partition $s = s_0 < s_1 < \ldots < s_n = t$. Then define recursively

$$\begin{aligned} P_X(t,t) &= I \qquad \text{and} \\ P_X(s_i,t) &= P_X(s_{i+1},t) - A_X(s_{i+1})P_X(s_{i+1},t)(s_i - s_{i+1}) \quad i = 0,\ldots,n-1. \end{aligned}$$

The entries in this approximation will all be functions of values of the process X at $n+1$ distinct values, which again brings us back to a standard numerical problem involving only the diffusion X.

6 Fitting spreads and spread sensitivities

We consider in this section an illustrative implementation of the generalized Markovian model presented above. A full implementation and testing of the models will be the topic of subsequent work and therefore the chosen parameters and prices should be seen primarily as illustrative and as providing a way for the reader to check the calibration procedure.

The goal is to show how the generalized model building on a Vasicek model of the riskless term structure can be adapted in the approach using modified eigenvalues of the generator matrix to produce a model which fits short credit spreads and *changes* in short spreads as a function of *changes* in the short rate for all credit classes simultaneously. The generalization to multifactor models with affine structure will be apparent.

Assume that the spot rate process for riskless debt is given as

$$dr_s = b(a - r_t)dt + \sigma dB_t.$$

Now calibrate the model to data by performing the following steps:

1. Estimate an empirical generator matrix A and assume that there exists a matrix B whose columns are the eigenvectors of A.

2. Let
$$\mu_j(r_s) = \gamma_j + \kappa_j r_s,\ j = 1, \ldots, K-1$$
where γ_j, κ_j are constants.

3. Let $\mu(r_s)$ denote a $K \times K$ diagonal matrix whose first $K-1$ diagonal elements are given as above and with the last diagonal element equal to zero. Define
$$A_X(s) = B\mu(r_s)B^{-1}.$$

4. Choose the parameters $(\gamma_j, \kappa_j)_{j=1,\ldots,K-1}$ such that 'two characteristics' of each yield curve are satisfied simultaneously across credit classes. An example follows shortly.

By this specification of the generator we may encounter negative transition intensities but just as we may choose to work with Gaussian interest rates as an approximation, despite the fact that this means negative interest rates with positive probability, we will allow the approximation of the default intensity under the risk neutral measure to be negative with positive probability. The constants will be chosen in practice such that the intensities are negative with very small probability.

In our example we work with spot spreads and sensitivities in the spot spreads to changes in r as the characteristics we wish to match. This choice has the advantage of producing a simple linear equation to determine γ and μ. To see this, consider the expression for the prices at time 0 of defaultable bonds with zero recovery in this special setup:

$$\begin{aligned} v^i(t,T) &= \sum_{j=1}^{K-1} \beta_{ij} E\left(\exp\left(\int_t^T \gamma_j + \kappa_j r_s - r_s ds\right)\middle| \mathcal{G}_t\right) \\ &= \sum_{j=1}^{K-1} \beta_{ij} E\left(\exp\left(\int_t^T \gamma_j - (1-\kappa_j) r_s ds\right)\middle| \mathcal{G}_t\right). \end{aligned}$$

Define the spot spread for class i as

$$s^i(r_t) = \lim_{T\to t}\left(-\frac{\partial}{\partial T}\log v^i(t,T)\right) - r_t.$$

Now we find

Proposition 6.1

$$
\begin{aligned}
s^i(r_t) &= -\sum_{j=1}^{K-1} \beta_{ij}\mu_j(r_t) \\
\frac{\partial}{\partial r_t} s^i(r_t) &= -\sum_{j=1}^{K-1} \beta_{ij}\kappa_j.
\end{aligned}
$$

Proof. Note that

$$
\begin{aligned}
& -\frac{\partial}{\partial T}\log v^i(t,T) \\
= \quad & \frac{-1}{v^i(t,T)}\frac{\partial}{\partial T}v^i(t,T) \\
= \quad & \frac{-1}{v^i(t,T)}\sum_{j=1}^{K-1}\beta_{ij}E\left(\left(\gamma_j + (\kappa_j - 1)r_T\right)\exp\left(\int_t^T \gamma_j + \kappa_j r_s - r_s ds\right)\Bigg| \mathcal{G}_t\right) \\
\underset{T\to t}{\to} \quad & \sum_{j=1}^{K-1} -\beta_{ij}\left(\gamma_j + (\kappa_j - 1)r_t\right)
\end{aligned}
$$

and using the fact that $\sum_{j=1}^{K-1}\beta_{ij} = 1$ we find after subtracting the spot rate that the spread is

$$
s^i(r_t) = \sum_{j=1}^{K-1} -\beta_{ij}\left(\gamma_j + \kappa_j r_t\right)
$$

where in the last limit argument we use dominated convergence for conditional expectations. Now differentiate this expression to obtain

$$
\frac{\partial}{\partial r_t}s^i(r_t) = -\sum_{j=1}^{K-1}\beta_{ij}\kappa_j
$$

so κ determines the spread sensitivity to changes in the spot rate. ■

Assume that we observe at time 0 when the level of interest rates is r_0, a vector of spot yield spreads given by the vector

$$
\widehat{s}_0 = \left(\widehat{s}^1(r_0), \ldots, \widehat{s}^{K-1}(r_0)\right)
$$

and that we have estimated a vector of sensitivities given by

$$
\widehat{ds}_0 = \left(\widehat{ds}^1(r_0), \ldots, \widehat{ds}^{K-1}(r_0)\right).
$$

As a consequence of the proposition, if we can calibrate our model to having the correct spread initially and the correct sensitivities by solving the systems

$$\begin{aligned} -\boldsymbol{\beta}(\gamma + \kappa r_0) &= \widehat{s}_0 \\ -\boldsymbol{\beta}\kappa &= \widehat{ds}_0 \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\beta} &= \left(\beta_{ij}\right)_{i,j=1,\ldots,K-1} \\ \kappa &= (\kappa_1, \ldots, \kappa_{K-1}) \text{ and} \\ \gamma &= (\gamma_1, \ldots, \gamma_{K-1}). \end{aligned}$$

We give an illustrative example:
In Jarrow, Lando and Turnbull [13] the generator matrix is given by

$$A = \begin{pmatrix} -0.1153 & 0.1019 & 0.0083 & 0.0020 & 0.0031 & 0.0000 & 0.0000 & 0.0000 \\ 0.0091 & -0.1043 & 0.0787 & 0.0105 & 0.0030 & 0.0030 & 0.0000 & 0.0000 \\ 0.0010 & 0.0309 & -0.1172 & 0.0688 & 0.0107 & 0.0048 & 0.0000 & 0.0010 \\ 0.0007 & 0.0047 & 0.0713 & -0.1711 & 0.0701 & 0.0174 & 0.0020 & 0.0049 \\ 0.0005 & 0.0025 & 0.0089 & 0.0813 & -0.2530 & 0.1181 & 0.0144 & 0.0273 \\ 0.0000 & 0.0021 & 0.0034 & 0.0073 & 0.0568 & -0.1928 & 0.0479 & 0.0753 \\ 0.0000 & 0.0000 & 0.0142 & 0.0142 & 0.0250 & 0.0928 & -0.4318 & 0.2856 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}$$

Assuming a Vasicek model under the risk neutral measure with parameters

$$\begin{aligned} a &= 0.05 \\ b &= 0.01 \\ \sigma &= 0.015 \\ r_0 &= 0.05 \end{aligned}$$

Let the spreads in basis points for classes AAA down to CCC be given as[4]

$$\widehat{s}_0 = (16, 20, 27, 44, 89, 150, 255)$$

[4]These spreads are consistent with JP Morgan's CreditMetrics estimates for spreads of Industrial bonds with maturity one year as of July 11, 1997. The spot rates will of course slightly different

and for the purpose of illustration let the spread sensitivities be given as[5]

$$\widehat{ds}_0 = (-0.2, -0.3, -0.4, -0.5, -0.6, -1.0, -2.0)$$

With these we can calibrate the model and get estimates of (κ, γ):

$$\begin{aligned}
\gamma &= (-0.1745, -0.1687, -0.1175, -0.0934, -0.0831, -0.0721, -0.0325) \\
\kappa &= (2.8004, 2.7181, 1.8026, 1.4139, 1.2640, 1.1200, 0.5348).
\end{aligned}$$

The resulting term structures of yield spreads are shown in Figure 1 and Figure 2.

It is important to stress that this method of adjusting the eigenvalues, while convenient because of preserving closed-form solutions, is problematic in the sense that one cannot be sure that desirable stochastic monotonicity properties of the rating and default process are preserved. Therefore, when inserting the estimated parameters into the pricing relation, the yield curves of different rating categories may cross. If one allows all entries of the random generator to vary freely (and not only in a space of dimension equal to set number of rating classes) this can be controlled. It is also possible to randomize only the default intensities associated with each class in a way which leaves the transition intensities between non-default states fixed. This method will give a natural way of incorporating fractional recovery rates into the intensities. In both cases several numerical issues arise which will be treated in separate work.

7 Conclusion

We have laid out a framework which is convenient for analyzing financial instruments subject to credit risk through counterparty default and derivatives whose underlying is a credit risk variable such as a credit spread. We take into account the possible correlation between default free bonds and default probabilities by letting interest rates and hazard rates be governed by common state variables. The main feature of the framework is that it reduces the technical issues of modeling credit risk to the same issues we face when modeling the ordinary term structure of interest rates. As a main application

[5]Both Longstaff and Schwartz [18] and Duffee [4] find that spreads are negatively correlated with the level of interest rates and that the sensitivity is stronger for lower rated bonds.

we show how to generalize a model of Jarrow, Lando and Turnbull [13] to allow for stochastic transition intensities between rating categories. A main application of this model is pricing of claims in which the credit rating of the defaultable party enters explicitly. An implementation is given in a simple one factor model in which the affine structure gives closed form solutions.

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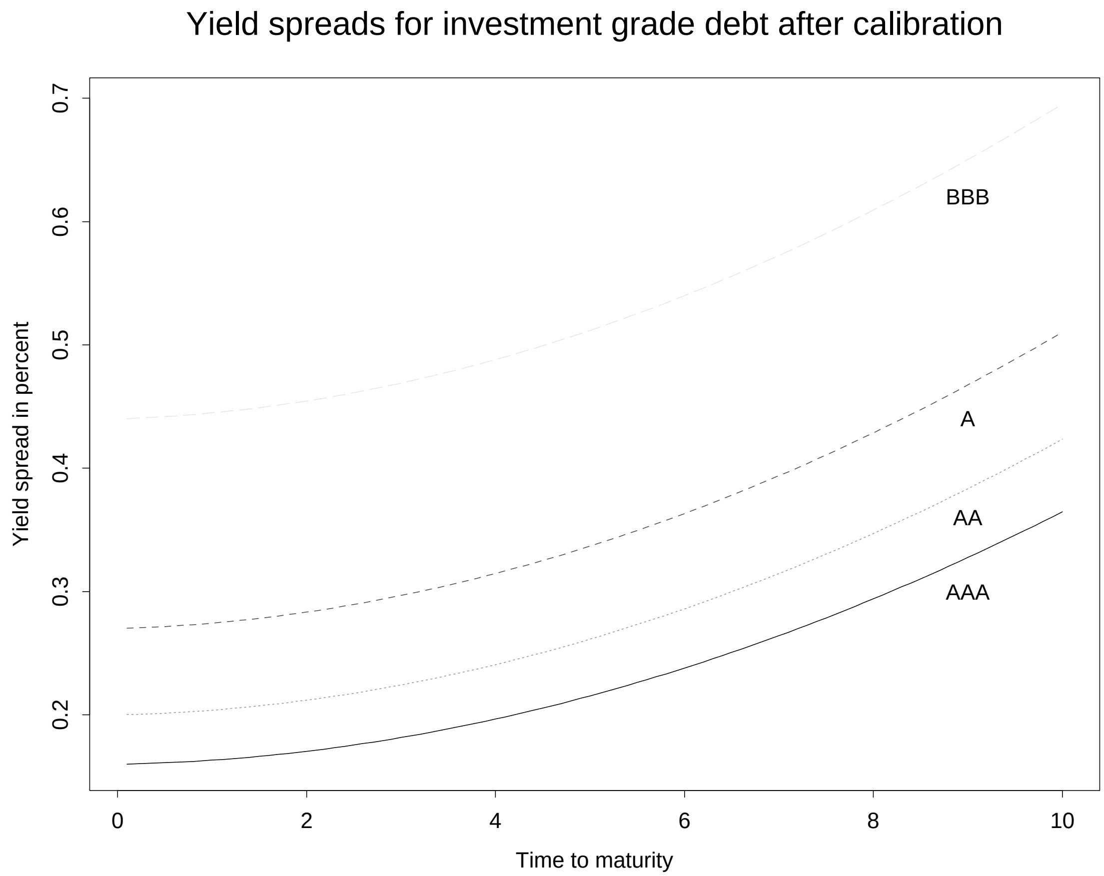

Figure 1: Yield spreads on defaultable zero coupon bonds as a function of time to maturity for investment grade ratings. The yields have been calibrated using an empirically observed generator and a risk-adjustment to match both observed spot spreads and changes in spot spreads due to changes in short treasury rates.

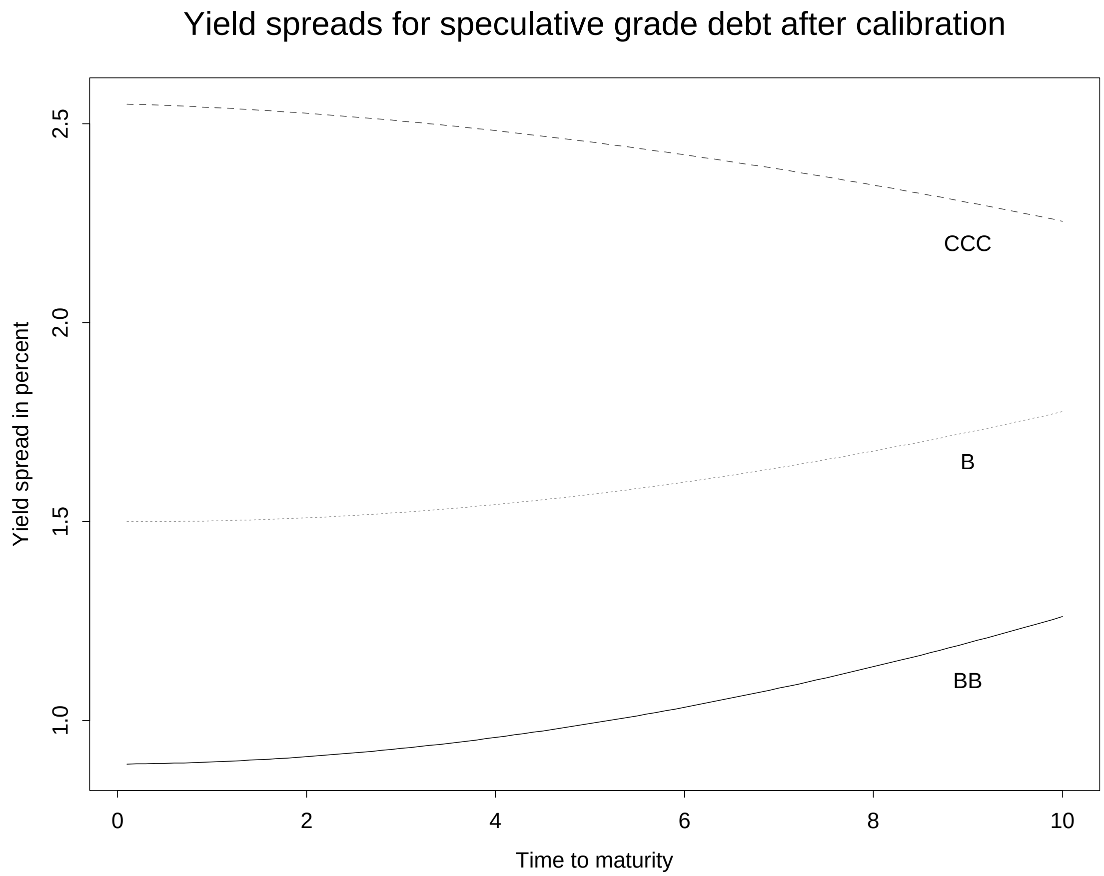

Figure 2: Yield spreads on defaultable zero coupon bonds as a function of time to maturity for speculative grade ratings. The yields have been calibrated using an empirically observed generator and a risk-adjustment to match both observed spot spreads and changes in spot spreads due to changes in short treasury rates.