Calibration of stochastic models for interest rate derivatives*

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Abstract

For the pricing of interest rate derivatives, various stochastic interest rate models are used. The shape of such a model can take very different forms, such as direct modeling of the probability distribution (e.g. a generalized beta function of second kind), a short rate model (e.g. a Hull-White model), or a forward rate model (e.g. a LIBOR market model).

This paper describes the general structure of optimization in the context of interest rate derivatives. Optimization in finance finds its particular application within the context of calibration problems. In this case, calibration of the (vector-valued) state of a given stochastic model to some target state, which is determined by available relevant market data, implies a continuous optimization of the model parameters such that, a global minimum of the distance between the target state and the model state is achieved.

In this paper, a novel numerical algorithm for the optimization of parameters of stochastic interest rate models is presented. The optimization algorithm operates within the model parameter space on an adaptive lattice with a number of lattice points per dimension which

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is both, low and fixed. In this way, a considerable performance gain is achieved, as compared to algorithms working with non-adaptive lattices requiring increasing and/or large numbers of lattice points. As compared to standard algorithms, e.g. those of Levenberg-Marquardt type, the presented adaptive lattice algorithm reduces also the danger to get trapped near a wrong local minimum.

As a numerical example, its application is demonstrated by optimizing volatility and mean reversion parameters of the Hull White model, such that the latter becomes calibrated to the swaption volatility market relevant for a given OTC bond option.

1 Calibration problems in interest rate models

The most widely used classes of valuation models for interest rate derivatives are either short rate models, with the (normally distributed) Hull White (extended Vasicek) model [1] and the (log-normal) Black Karasinski model [2] as their most prominent representatives, or forward rate models, such as the LIBOR market (rate) model [3, 4]. In both classes, calibration of the model parameters to specific market data is required. Typical financial instruments that are priced by short rate models are Bermudan swaptions or bond options. The LIBOR market model is typically applied to evaluate (often via a simulation) structured rate legs with caps, floors, barriers, triggers or other exotic payoff structures. Sometimes, alternatively to the underlying normal or log-normal processes, also more general Levy processes are modeled directly via a modified probability distribution.

Traders of interest derivatives and risk controllers usually have some expectation about the market of the products they are dealing with. Although with different emphasis, both traders and risk controllers, require their pricing models to reflect the realities of the market segments underlying to their products. This implies that the free parameters of the models have to be calibrated in such a manner as to be as much as possible consistent with the market. Note that, the necessity for calibration here is similar to that with models in fundamental science, particularly physics. There usually exist several model parameters which must be calibrated such that certain related observable "constants" really take their actually observed values. E.g. a recent problem, in the context of quantum gravity models, is the appropriate calibration of the fundamental length of quantum geometry (see e.g. [5]). In general, all models have to be calibrated, in order to be consistent with the relevant fundamental constants of nature. This fine-tuning of fundamental theories (like the standard model of particle physics), such that the fundamental constants match their observed values (of our present universe), is nothing but a special type of calibration. Here, the present universe is playing a role analogously to the current markets in finance.

Requirements of traders and risk controllers however may differ on their

choice of priorities, between the ability of the model to match the market at any instant and its ability to make a fair prediction of the market from a given instant to the near future. This choice usually depends on the actual requirements. E.g. frequent *recalibration* of the model parameters to the prices quoted in the market may be perfectly suited in order to satisfy directly an explicit mark-to-market pricing view (e.g. inherent in most prescriptions of risk reporting). Nevertheless, it may be unsatisfactory for a trader who needs his model to provide a volatility surface which is as time-stable as possible, or (similarly) for a risk controller trying to hedge the vega-parameter, i.e. the sensitivity of prices with respect to volatility changes. The recalibration method is usually also rejected for all serious fundamental model of nature. Fundamental models are usually supposed to make reasonable predictions during the typical time scale of their applicability *without* any recalibration of their parameters, which are in fact determined once-for-all. Some fundamental cosmological theories may involve running coupling constants, where some parameters change during time. But this change happens through an a priori predetermined functional relation for time dependence, whence there is no space for recalibration of the parameters. Similarly to finance, where a model strictly needs to match at a given time only actual market values, a cosmological model is required only at present time to match the fine-tuned values of the actual universe. Nevertheless, in both cases it is desirable to calibrate not only just the current values, but also to some extent the very functional relation of time dependence which then provides a relation to past and/or future. The prominent example for calibration of time dependence in finance is the volatility surface (see [6]). The afore mentioned situation of practitioners requiring time-stability of the volatility surface is in fact very close to the fundamental view, although in finance applications often a compromise between frequent recalibration and an once-for-all calibration has to be taken. For fundamental cosmological models one would require a one-for-all calibration of time-dependence being direct and exact at all times, while with interest rate models, a calibrated volatility surface is usually only obtained via optimization, and calibration usually can not avoid some increasing deviation from markets as time passes, unless the model is recalibrated.

Furthermore, depending on the requirements, the calibration can be done either once for a whole portfolio of interest rate derivatives, or it can be done for each trade separately. The choice of calibration may also depend on whether the trades are standardized (exchange traded), or whether trades are not standardized (OTC trades). In the former case, the trades may be considered as a part of the market to which they are linked. In the portfolio case, the relevant (market) instruments together with their quotes are taken directly from market data providers. In the single trade case, the structure of the derivative itself (i.e. its relevant events like exercise

dates, underlying maturity) determines the relevant instruments, for which relevant target values then have to be interpolated from directly observable market instruments.

Sometimes calibration to the target values of the market can be achieved exactly, by a direct method, i.e. by solving directly the relevant equations. This is e.g. the case, when a volatility parameter of some model is tuned to a single European option price quoted in the market.

However, in many cases a direct and exact calibration to the target values is not possible. This is the situation, where optimization techniques become important in mathematical finance. An optimization method generally consists of certain search algorithms, which are targeted to optimize certain model parameters, such that the deviation between model and target values of certain state variables becomes minimal. The typical difference of an optimization as compared to a direct calibration is that, in general, a zero deviation can not be enforced.

Calibration problems requiring optimization can vary considerably in their details. Some very different calibration problems in the context of interest rate models are e.g. given by the following three examples:

- Example 1: calibration of the parameters of a model's probability distribution, such that, for any given European option maturity, predetermined market values (determined by caplet/floorlet volatilities) quoted for different strikes (cap/floor rates) are matched in an optimal manner;

- Example 2: calibration of volatility and mean reversion of an (arbitrage-free) short rate model applied to Bermudan options (swaptions or bond options), such that the implied market values of the relevant European options (determined by swaption volatilities) are matched in an optimal manner;

- Example 3: calibration of volatilities and correlations of the forward rates of a LIBOR market model such that the volatility term structure matches exactly to European caplet/floorlet prices, while the forward rate correlation matrix matches a target correlation function in an optimal manner.

Nevertheless, a common feature of the above examples for typical calibration problems is the optimization of some special model parameters θ with respect to some distance function

$$d(\theta) \quad := \quad f(\mathbf{Q}_{\text{model}}(\theta) - \mathbf{Q}_{\text{target}}) \tag{1}$$

between a target (market determined) state variable $\mathbf{Q}_{\text{target}}$ and the corresponding state $\mathbf{Q}_{\text{model}}(\theta)$ of the model depending on its parameters θ.

Here both $\mathbf{Q}_{\text{target}}$ and $\mathbf{Q}_{\text{model}}(\theta)$ are vector states within a state space $V \subset \mathbb{R}^n$ of finite dimension n, and f is a non-negative function on V such that $f(x) = 0 \Rightarrow x = 0$. Generally f should depend on the individual distance functions f_i on $V_i \subset \mathbb{R}$, which are measuring the distance $f_i(Q_{\text{model},i}(\theta) - Q_{\text{target},i})$ between the individual model and target states components, $Q_{\text{model},i}(\theta), Q_{\text{target},i} \in V_i$, $i = 1, \ldots, n$.

A standard choice on finite-dimensional real vector spaces is the Euclidean distance, specified by setting $f := ||\cdot||_2$, the Euclidean l_2-norm [7] on $V \subset \mathbb{R}^n$, and correspondingly also $f_i := |\cdot|$, the Euclidean norm on $V_i \subset \mathbb{R}$.

In some cases (Example 1 and 2), particular state components corresponds to particular individual benchmark instruments. But this does not need always to be the case. E.g. for the calibration of the correlation matrix of example 3, an individual state component corresponds to a matrix element, and an individual matrix element is related to a symmetric *pair* of benchmark instruments with different maturity. In this case it is important to distinguish between the components of the target states (i.e. the correlation matrix elements) and the benchmark instruments (i.e. the interest forwards of different maturities).

Equivalently to (1), one can also consider its square

$$\begin{aligned} d^2(\theta) &:= ||\mathbf{Q}_{\text{model}}(\theta) - \mathbf{Q}_{\text{target}}||_2^2 \\ &= \sum_{i=1}^{n} (Q_{\text{model},i}(\theta) - Q_{\text{target},i})^2 \\ &= \int_{\chi \in P(M)} (\mathbf{Q}_{\text{model},\chi}(\theta) - \mathbf{Q}_{\text{target},\chi})^2 d\chi \quad . \end{aligned} \tag{2}$$

In the last line of (2), $P(M)$ is a parameter space labeling all theoretically possible different components of a state, and $d\chi$ is the measure on $P(M)$ selecting just those components, for which the target quote $\mathbf{Q}_{\text{target},\chi}$ is actually provided. For specific values of parameters of χ the index $i = i(\chi)$ of a particular state space components is obtained.

Here and below it is always assumed that $P(M)$ indeed can be derived by a suitable construction (e.g. powers) from a more elementary space M parametrizing the different benchmark instruments. (Nevertheless, this does *not* mean that $P(M)$ is isomorphic to M. E.g. the term structure correlation components are indexed by symmetric pairs of maturities $(T_1, T_2) \in M \times M$, while M is the space of maturities of the underlying instruments of the term structure.) Since indices of state space components are given as points in $P(M)$, states $\mathbf{Q}_{\text{model/target}}$ are also considered as functions $\mathbf{Q}_{\text{model/target},\cdot}$ on (a discrete subspace of) $P(M)$.

Alternatively to the absolute distance (2), one might rather want to use a relative distance measure which does not depend on the absolute size of

the vector-states themselves. The symmetric relative deviation

$$\delta(\theta) \quad := \quad \frac{\sqrt{2}d(\theta)}{\sqrt{||\mathbf{Q}_{\text{target}}||^2 + ||\mathbf{Q}_{\text{model}}(\theta)||^2}} \tag{3}$$

is a choice which neither prefers the state of the model nor that of the market. (It implies a neutral view on the question of whether the target market is more right or the prediction of the model.)

Below, the algorithm presented in Section (3) will be based on the latter choice (3), and likewise also the example calibration of the Hull White model (see Section 4).

In general, the dimension $n := \dim V$ with $V \ni \mathbf{Q}_{\text{target}}$ is determined by the total number n_{market} of relevant market values.

In practice, the most important calibration problems are related to market instrument parameter space M which is (at most) 3-dimensional, with its dimensions specified by option maturity T, underlying maturity t, and strike X as parameters. Let n_T, n_t, n_X be the number of different option maturities, underlying maturities, and strikes, respectively, quoted in the market. Let us assume that the $n_T \times n_t \times n_X$ lattice cube is complete (without degeneracies), then

$$n_{\text{market}} \quad = \quad n_T \cdot n_t \cdot n_X \quad .$$

If the states $\mathbf{Q}_{\text{model/target}}$ are given via functions on $P(M) = M$, then $n = n_{\text{market}}$. Since the benchmark instruments quoted in the market form a discrete set, here

$$d\chi = \sum_{j=1}^{n_T} \sum_{k=1}^{n_t} \sum_{l=1}^{n_X} \delta_{(T_j,t_k,X_l)}(\chi) dT dt dX \quad .$$

However, the states $\mathbf{Q}_{\text{model/target}}$ may also be given as function on a (subspace of a) higher-dimensional power $P(M)$ of M (see Example 3 below). For example, if the target space is given by square matrices, then the different state components are labeled by a space $P(M) \subset M \times M$, where M is the space of market parameters. With $\Psi_1, \Psi_2 \in M$, the parameters of $P(M)$ are given as $\chi = \chi(\Psi_1, \Psi_2)$. If M contains n_{market} relevant indices, i.e. $I \subset M$ is a discrete subset labeling $|I| = n_{\text{market}}$ different instruments, then $P(M)$ contains at most n^2_{market} relevant component indices, since $|\chi(I, I)| \leq n^2_{\text{market}}$. Hence the number of state components is $n \leq n^2_{\text{market}}$. For example, correlation matrices are a maximal-dimensional subspace within the symmetrical matrices. Hence $P(M) \subset \text{Sym}(M \times M)$ where $M \subset \mathbb{R}$. The measure $d\chi$ here has nonvanishing support only at indices $\chi(I, I) \in \text{Sym}(I \times I)$. Therefore, here the state space has dimension $n = \dim V = \frac{n_{\text{market}} \cdot (n_{\text{market}} - 1)}{2}$.

For the examples above the optimization specializes as follows:

- Example 1: At a single fixed maturity T, the comparing state variables $\mathbf{Q}_{\text{target/model}} \in \mathbb{R}^{2n}$ are given by the caplet values $\text{PV}^c_{\text{target/model}}(T, X_i)$ and floorlet values $\text{PV}^c_{\text{target/model}}(T, X_i)$ for different strikes X_i, $i = 1, \ldots, n$. The components of θ are the free parameters of the probability distribution. E.g. for the generalized beta of second kind in Section 2.1 below, $\theta := (z, p, q)$ with $z \in \mathbb{R}$ and $p, q \in \mathbb{R}^+$. Here, $P(M) = M$. Since for any $(X, T) \in [\mathbb{R}^+]^2$ there is both a caplet value and a floorlet value, the instrument parameter space is $M = \mathbb{Z}_2 \times M_0 = M_0 \oplus M_0$, the direct double of the common parameter space M_0 of caplets and floorlets. Calibrating a European option (i.e. a single maturity T), then $\dim M = \dim M_0 = 1$ and $n = n_X$. More generally, when calibrating a Bermudan option (i.e. considering simultaneously different exercise times $T_1 < \cdots < T_{n_T} = T$), then $\dim M = 2$ and $n = n_T \cdot n_X$.

- Example 2: The comparing state variables $\mathbf{Q}_{\text{target/model}} \in \mathbb{R}^n$ are given by the European option values $\text{PV}_{\text{target/model,i}} := \text{PV}_{\text{target/model}}(T_i)$ for different maturities T_i, $i = 1, \ldots, n$. The relevant short rate model are arbitrage-free and hence mean reverting. Hence, $\theta := (a, \sigma)$ with a the mean reversion and σ the short rate volatility parameter (see Section 2.2 below). Here $P(M) = M$ with $\dim M = 1$ and $n = n_T$.

- Example 3: Here the comparing state variables are correlation matrices $\mathbf{Q}_{\text{target/model}} := \rho_{\text{target/model}} \in \mathbb{R}^n$, for forward rates f_i of different maturities T_i, $i = 1, \ldots, n_T$. For a m-factor model ($1 < m \leq n_T$), $\theta := (\theta_{ij})$ is the symmetric matrix of angular coordinates on $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ (see Section 2.3 below). Here the state component parameter space is $P(M) = \text{Sym}(M \times M)$ with an instrument parameter space M with $\dim M = 1$. With $n_{\text{market}} = n_T$, then $n = \frac{n_T \cdot (n_T - 1)}{2}$.

Once the distance function (1) is specified, the optimization problem amounts to finding (an approximation to) a global minimum

$$d_{\min} \quad := \quad \min_{\theta \in S} d(\theta) \tag{4}$$

over the space of admissible model parameters S. Within a calibration problem, all parameter values θ should a priori be equally plausible. There should not be any distinguished values or critical points within S. Hence S will be an open space.

Nevertheless, in general there may be a boundary ∂S constraining the variation of θ. However, in some situations S may be reparametrized $S' \to S$ by a parameter transformation $\theta' \mapsto \theta$, such that $\partial S'$ is located at infinity w.r.t S'. Thus the constrained optimization over S implied by (4) can be replaced by an unconstrained optimization over S'.

If some model parameter $s \in S$ (e.g. a volatility parameter σ) is constrained to $s > 0$, by a re parametrization $s = e^r$ with $r \in \mathbb{R}$, the constrained calibration over $\mathbb{R}^+$ is replaced by an unconstrained calibration over $\mathbb{R}$.

The global optimization problem, of finding θ such that $d(\theta) = d_{\min}$, has no unique solution in general. Some (local) minimum of a function over its parameter space may be detected by a standard Levenberg-Marquardt algorithm [8, 9] or some modification involving Tikhonov regularization [10]. This requires to compare at any point θ in parameter space not only the state $\mathbf{Q}(\theta)$, but also its Jacobi matrix $\nabla_\theta^T \mathbf{Q}(\theta)$. The corresponding Sobolev norm might then be considered instead of the distance function (1). However, a well-known problem with the search for a global minimum is the possibility of getting trapped within a wrong local minimum. In order to solve this problem, a simulated annealing algorithm has been proposed in [11]. Further deterministic approaches are described in [12]. Below we will present another algorithm, which is based on an adaptive refinement of a lattice $L \subset S$. This algorithm is particularly useful in the case when $p := \dim S$ is not too large. The method is described in detail in Section 3.

In the following sections, three example cases for the calibration of interest models are discussed.

2 Example cases for the calibration of interest rate models

2.1 Calibrating the probability distribution of the model

A log-normal probability distribution is completely determined by its first and second momentum. Therefore with a log-normal probability distribution, there are no remaining degrees of freedom for an additional calibration beyond ATM prices (i.e. normalized strike equal to 1).

Therefore, an extension of the space of admissible probability distributions is required (within in the range of Levy processes), in order to calibrate to options prices quoted at different strike values. One possible extension (proposed by [13]) enlarges the space of probability density distributions from the log-normal density to the generalized beta density of the second kind, defined as the following 4-parameter family on $\mathbb{R}^+$:

$$x \mapsto \rho_{a,b,p,q}(x) := \frac{|a| x^{ap-1}}{b^{ap} B(p,q)[1 + (\frac{x}{b})^a]^{p+q}} \quad , \tag{5}$$

with real parameters a, b, p, q with $p, q > 0$.

With (5) the expected payoff of a call/put at maturity T is

$$\begin{aligned} P_{T,X} &= E_{a,b,p,q}(Z_{T,X}) = \int_0^\infty (\pm(y - X))^+ \rho_{a,b,p,q}(y) dy \qquad (6) \\ &= \pm \int_X^\infty y \rho_{a,b,p,q}(y) dy \mp \int_X^\infty X \rho_{a,b,p,q}(y) dy \quad . \end{aligned}$$

The arbitrage-free condition for the forward rate implies

$$F = E_{a,b,p,q}[S_T] = b \cdot \underbrace{\frac{B(p+\frac{1}{a}, q-\frac{1}{a})}{B(p,q)}}_{=:\lambda^{-1}} \quad ,$$

admitting to define a new parameter as

$$z \quad := \quad \frac{\left(\lambda \frac{F}{X}\right)^a}{1+\left(\lambda \frac{F}{X}\right)^a} \tag{7}$$

From (5), (6) and (7) the value of a caplet/floorlet (i.e. a call/put) at maturity T is obtained as

$$P^{c,p}_{z,p,q}(T,X) \quad = \quad \pm F\left(\frac{1}{2} \mp \frac{1}{2} \pm I_z(q-\frac{1}{a}, p+\frac{1}{a})\right) \mp X\left(\frac{1}{2} \mp \frac{1}{2} \pm I_z(q,p)\right) \quad .$$

With $\theta = (z,p,q) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, the corresponding present value then is

$$PV^{c,p}_{\theta}(T,X) = PV^{c,p}_{z,p,q}(T,X) \quad = \quad df_T P^{c,p}_{z,p,q}(T,X) \quad , \tag{8}$$

where df_T is the discount factor up to maturity T.

For a given time-to-maturity T, let the available market data include the implied present values of caplets $PV^c_m(T,X_i)$ and floorlets $PV^p_m(T,X_i)$ for different strike rates X_i, $i = 1\ldots,n$. A weight factor $w_i \geq 0$ may be assigned to each strike in order to account for variations in the quality of the quoted prices. Hence, using (8) for $n > 1$, the relevant calibration then amounts to minimizing a (square of the Euclidean) distance

$$\begin{aligned} d^2_{\mathbf{w};T}(\theta) = d^2_{\mathbf{w};T}(z,p,q) := \sum_{i=1}^{n} & \quad w^c_i [PV^c_{z,p,q}(T,X_i) - PV^c_m(T,X_i)]^2 \\ & + \quad w^p_i [PV^p_{z,p,q}(T,X_i) - PV^p_m(T,X_i)]^2 \end{aligned} \tag{9}$$

over parameters z,p,q.

Note that minimization of (9) amounts to a calibration for a fixed time-to-maturity. More generally, if the interest rate derivative implies several relevant maturities T_k, $k = 1\ldots,m$, then a linear combination

$$d^2_{\mathbf{v},\mathbf{w}}(z,p,q) \quad := \quad \sum_{k=1}^{m} v_k \cdot d^2_{\mathbf{w};T_k}(z,p,q) \quad , \tag{10}$$

with additional linear weights $v_k \geq 0$ for different maturities T_k, may provide a suitable distance function for optimization.

2.2 Calibration of a short rate model

Any stochastic short rate model depends on the percentage volatility σ of the short rate. The most competitive current short rate models are all no-arbitrage models of mean-reverting type. The short rate follows a process

$$d(f(r)) \quad = \quad (b(t) - af(r))dt + \sigma dz \quad , \tag{11}$$

where a is the mean reversion, σ is the volatility of the short rate, and $b(t)$ is determined by the current term-structure. The short-rate process for Hull-White model is normally distributed, i.e. $f(r) = r$, and for the Black-Karasinski model log-normal, i.e. $f(r) = \ln r$. In the former case, σ is the volatility rate of dr, i.e. of instantaneous changes of the short rate, in the latter case of $\frac{dr}{r}$, i.e. of the instantaneous return of the short rate.

For European options (depending only on a single exercise date), it is market standard to quote the implied Black volatility derived from market prices by the Black price model. Within this model the arbitrage-free assumption is met by the condition that the forward price (for a Bond option) or the forward rate (for a swaption) is equal to the expected value of the spot price or spot rate at option maturity T. The implied Black volatility σ_{Black} of the spot underlying is the finally relevant parameter of the Black model. Given maturity and forward value, the implied Black volatility corresponds one-to-one to the present value PV_{Black}.

For European options, such as bond options or swaptions, market prices or corresponding implied Black volatilities are available. These market prices should also be matched by the short rate model.

This poses the following calibration problem:

Given the market prices $PV^i_{\text{target}} = PV^i_{\text{Black}}$ for a series of European options, with different maturities T_i, $i = 1, \ldots, n$, on the same underlying, volatility σ and mean reversion a of the short rate model should be chosen such that all n market prices are matched with $PV^i_{\text{target}} = PV^i_{\text{Black}}$. Here $\theta := (a, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$ and

$$\begin{aligned} d^2(\theta) \quad &:= \quad ||\mathbf{Q}_{\text{model}}(\theta) - \mathbf{Q}_{\text{target}}||_2^2 \\ &= \quad \sum_{i=1}^{n} (PV_{\text{HW},i}(a, \sigma) - Q_{\text{Black},i})^2 \quad . \end{aligned} \tag{12}$$

Hence, the calibration involves optimizing a and σ such that $d(\theta) = d_{\min}$.

2.3 Calibration of a LIBOR market model

A LIBOR market model is a model for evolving a given set of forward rates $f_i(t) := L(t, T_i, T_{i+1})$, $i = 1, \ldots, n$. Here T_{i-1} is the beginning of the i-th LIBOR period, T_i is its end. Given m independent stochastic factors (i.e.

Wiener processes dz_k, $k = 1, \ldots, m$), the general evolution is described by the stochastic vector process

$$\frac{df_i}{f_i} \quad = \quad \mu_i dt + \sigma_i \sum_{k=1}^{m} b_{ik} dz_k \quad . \tag{13}$$

In (13), μ_i are deterministic drift factors, σ_i is the volatility rate of the instantaneous return of the forward rate f_i, satisfying

$$\sigma^2_{Black}(T_i) \quad = \quad \int_0^{T_i} \sigma_i^2(u) du \quad , \tag{14}$$

and b_{ik} are the components of an $n \times k$ matrix $\mathbf{B}$ of Brownian shocks satisfying the condition

$$\sum_{k=1}^{m} b_{ik}^2 \quad = \quad 1 \quad , \tag{15}$$

ensuring that caplets have the correct prices.

The correlation matrix of the forward rates is given as

$$\rho_{ij} \quad = \quad [\mathbf{b}\mathbf{b}^{\mathbf{T}}]_{ij} = \sum_{k=1}^{m} b_{ik} b_{kj} \quad , i, j = 1, \ldots, n. \tag{16}$$

The assumption of stochastically independent forward rates amounts to $b_{ik} = \delta_{ik}$. Clearly this assumption is in general not satisfied. If however it is imposed, after fitting their instantaneous volatilities σ_i via (14) to terminal Black prices (of caplets), no degrees of freedom are left, and hence no further calibration is necessary.

Therefore, the calibration problem arises due to the correlation of the forward rates, that is in the case of some $b_{ik} \neq \delta_{ik}$. On the one hand, it is only in the case of nonvanishing correlation that the number of stochastic factors m can be lower than the number of forward rates n. On the other hand, the full $n \times n$ covariance matrix introduces more degrees of freedom than one can deal with in practice, and often also more than the number of available additional market instruments (e.g. swaptions) for calibration. Therefore, the case of nonvanishing correlation requires to reduce drastically the number of stochastic factors m and the degrees of freedom of the correlation matrix by a parametrization and calibration to a suitably chosen target correlation ρ_{target}. The latter is parametrized in such a form that the market situation is represented. One choice which takes into account the decay of correlation with rate β, for increasing distance between the option maturities T_i and T_j, is

$$\rho_{\text{target},ij} \quad := \quad \rho_{\text{long}} + (1 - \rho_{\text{long}}) e^{-\beta |T_i - T_j|} \quad , i, j = 1, \ldots, n. \tag{17}$$

Here ρ_{long} is the limit correlation for $|T_i - T_j| \to \infty$.

Using (16) and (17), the calibration task for the LIBOR market model implies minimizing

$$\begin{aligned} d^2(\mathbf{B}) &:= \sum_{i,j=1}^{n} (\rho_{\mathbf{B},ij} - \rho_{\text{target},ij})^2 \\ &= \sum_{i,j=1}^{n} ([\mathbf{b}\mathbf{b}^{\mathbf{T}}]_{ij} - \rho_{\text{target},ij})^2 \quad . \end{aligned} \tag{18}$$

Due to the constraint (15), the optimization of $d(\mathbf{B})$ appears quite difficult. However the constraint can be solved explicitly by reparametrizing the matrix $\mathbf{B}$. Condition (15) implies that the vectors $\mathbf{b_i}$, $i = 1 \ldots, n$ all lie on a hypersphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$. Hence polar coordinates provide the appropriate reparametrization:

$$\begin{aligned} b_{ik} &= \cos\theta_{ik} \prod_{j=1}^{k-1} \sin\theta_{ij}, \quad k = 1, \ldots, m-1 \\ b_{im} &= \prod_{j=1}^{m-1} \sin\theta_{ij} \quad . \end{aligned} \tag{19}$$

Hence the constraint optimization of (18) becomes equivalent to an unconstrained optimization of

$$d^2(\mathbf{\Theta}) := \sum_{i,j=1}^{n} (\rho_{\mathbf{\Theta},ij} - \rho_{\text{target},ij})^2 \tag{20}$$

over the symmetric matrix $\mathbf{\Theta}$ with elements θ_{ik}.

3 Adaptive lattice calibration algorithm

In this section we describe an adaptive lattice calibration algorithm for solving the unconstrained optimization of the relative (symmetric) deviation between model and target space towards a global minimum over parameters $\theta \in S$.

In the following algorithm, the parameter space S is assumed to only consist of positive parameters, i.e. $S \subset [\mathbb{R}^+]^p$. Let $p = \dim S$. The algorithm uses the following calibration parameters:

- l_i prescribed log span of parameter s_i, $i = 1, \ldots, p$
- n_i (odd) number grid values for parameter s_i, $i = 1, \ldots, p$

The start value $\mathrm{s}_0 \in L \subset S$ with components $s_{i,0}$, $i = 1, \ldots, p$, has been fixed in advanced. The following steps (i) - (iv) are repeated until the accuracy δ or the maximal number of iteration $I_{\max}$ is reached:

1. Generate a lattice $L \subset S$ of $n_1 \times \cdots \times n_p$ grid points, in which each dimension is log-equidistant. For $k = 1, \ldots, p$ the grid values are $s_{i,-\frac{n_i-1}{2}}, \cdots, s_{i,0}, \cdots, s_{i,\frac{n_i-1}{2}}$ with a logarithmic distance $\log_{10} s_{i,k+1} - \log_{10} s_{i,k} = \frac{l_i}{n_i-1}$.

2. For each point $\theta \in L$ the symmetric distance between target and model state is calculated as follows:

   - Determine the n-dimensional vector $\mathbf{Q}_{\text{target}}$ of target values.
   - Determine the n-dimensional vector $\mathbf{Q}_{\text{model}}(\theta)$ of model values.
   - Compute the l_2-distance (according to (2) between the Black and Hull White vector as $d(\theta) := ||\mathbf{Q}_{\text{model}}(\theta) - \mathbf{Q}_{\text{target}}||_2$.
   - Compute the symmetric (relative) deviation (3) between the states,

$$\delta(\theta) \quad := \quad \frac{\sqrt{2}d(\theta)}{\sqrt{||\mathbf{Q}_{\text{target}}||^2 + ||\mathrm{Q}_{\mathbf{model}}(\theta)||^2}} \quad .$$

3. Determine θ_0 such that $\delta(\theta_{\mathbf{0}})$ is minimal.

4. Modify the grid values of all parameters as follows:

   - For $\theta_0 \in L - \partial L$: Set the new start point to $\mathbf{s}_0 := \theta_0$, and for $i = 1, \ldots, p$ the new log span $l_i := l_i \cdot \frac{2f}{n_i-1}$.
   - For $\theta_0 \in \partial L$: For $i = 1, \ldots, p$:
     - If $\theta_{i,0} = s_{i,\pm\frac{n_i-1}{2}}$, reset the start value to $s_{i,0} := s_{i,0} \cdot 10^{\pm\frac{n_i-2}{n_i-1}l_i}$ respectively and leave the log span l_i unchanged.
     - If $s_{i,-\frac{n_i-1}{2}} < \theta_{i,0} < s_{i,\frac{n_i-1}{2}}$ leave the start value $s_{i,0}$ unchanged and set $l_i := l_i \cdot \frac{2f}{n_i-1}$.

The factor f above is included in order to admit a tuning of the rescaling factor $\frac{2f}{n_i-1}$ of l_i. An appropriate rescaling factor is essential, in order to ensure that after each iterations step the optimal point is contained in the new lattice, and in order to obtain a sufficient fast down scaling of the lattice size. In order to enable sufficient capability of lattice moves to escape from a wrong local minima, the rescaling factors $\frac{2f}{n_i-1}$ should not be too small as compared to the number of lattice points. On the one hand, this escape capability is increasing with the number of neighbour generations around the optimal point that are also contained in the lattice of the next iteration, when the lattice spacing is reduced. On the other hand, the speed of the optimization is decreasing with the number of lattice points taken into account. For odd n_i, a convenient choice is a rescaling factor of 0.5, which is obtained for $f = \frac{n_i-1}{4}$.

Care is also necessary in order to set an appropriate stopping criterion for the optimization: If the calibration yields a deviation less than an externally specified value δ, this would mean that all target values are met by the model exactly up to accuracy δ. However, within a non-trivial calibration problem, an externally given accuracy δ can not be enforced. In this case, which should be considered as the generic one, the deviation (3) does not provide a safe stopping criterion. Alternative, additional stopping criteria could be added, such as bounds for the size of the lattice spacing. However, as a final safe stopping criteria, the calibration should stop after a pre-specified number $I_{\max}$ of iterations. The next section gives a practical example demonstrating how this algorithm works.

4 Numerical example: calibration of the Hull-White model

The Hull White model is a very common short rate model with 2 parameters, the volatility, $\theta_1 := \sigma \in \mathbb{R}^+$, and the mean reversion rate, $\theta_2 := a \in \mathbb{R}^+$. As a mean reverting short rate model, it allows to reduce the dissipation of the volatility term structure (related to forward rates of different maturities). Mean reversion is commonly considered as a necessary ingredient of a reasonable short rate model. In order to keep a short rate model (in particular the Hull White model) mean reverting, the mean reversion a either should be calibrated separately, on the basis of additional input from market data (as to obtain maximal homogeneity of vola surfaces with time), or (as this additional input often is not available) a should be bounded from below $a > a_{\min}$. This point being taken care about, mean reversion and volatility can be calibrated simultaneously by the adaptive lattice algorithm described in Section 3.

As a numerical example, the Hull White model is calibrated as to fit in an optimal manner to a volatility term structure, with synthetic calibration instruments derived according to the structure of a given (OTC) Bermudan bond option. The calibration is done at end-of-day for a given evaluation date, which is fixed for this example to 2007-08-29.

The underlying bond is assumed to pay on a notional of 1000000 EUR a semiannual coupon of 5% with day count convention 30/360, for interest periods starting at 2005-04-01, and ending at 2015-04-01.[1] Corresponding to typical termination rights at 1 to 5 years before maturity of the bond, the exercise dates and times of the Bermudan option are given as in Table 1. Furthermore, the strike-price of the option is set to 100%.

[1] Although for calibration itself such details about the underlying bond are in fact irrelevant, some rough description is included here, in order to give a feeling for the context of a typical application.

Table 1: Exercise date and time (in years), corresponding swaption volatility σ_{swaption} and related bond price volatility σ_{price}, and components (in EUR) of target state Q_{Black} and calibrated model state $Q_{\text{HW}}(\theta_c)$ after 15 iterations.

exercise date	exercise time	σ_{swaption}	σ_{price}	$Q_{\text{Black},i}$	$Q_{\text{HW},i}(\theta_c)$
2010-04-01	2.5890411	0.1342176	0.02632104	26785.51	26683.10
2011-04-01	3.5890411	0.1322203	0.02139749	22003.80	21971.07
2012-04-01	4.5938244	0.1312164	0.01644428	16742.50	16797.64
2013-04-01	5.5890411	0.1296219	0.01120726	11242.18	11329.75
2014-04-01	6.5890411	0.1278247	0.00571640	5614.99	5743.81

The set of calibration instruments is derived from the Bermudan option. It consists of corresponding European bond options, each with a maturity corresponding to one of its exercise dates. All European options have the same strike and same underlying bond as the Bermudan option.

The interest term structure is a standard EUR curve for the valuation date (2007-08-29). Since OTC bond options are not a standardized market segment by themselves, their volatilities are derived from the related swaption market. Hence, the fundamental market reference for volatilities is given by the implied swaption volatilities quoted for the given valuation date. For each European option, with its given option and bond maturity, first a corresponding swaption volatility $\sigma_{\text{swaption},i}$ is interpolated from available market quotes. Then, $\sigma_{\text{swaption},i}$ is converted into a corresponding price volatility $\sigma_{\text{price},i}$ for the bond (see Table 1).

The target values $Q_{\text{Black},i}$ of the corresponding benchmark options are their present values according to the standard Black model, computed with the volatility $\sigma_{\text{price},i}$ derived from interpolated implied swaption volatilities. The resulting components of the target state $Q_{\text{Black},i}$ are listed in Table 1.

The calibration algorithm of Section 3 is applied with 15 iterations, starting from $(\sigma_0, a_0) = (0.15, 0.03)$. During computation, the algorithm was configured to a number of lattice points per dimension $n = n_\sigma = n_a = 5$, i.e. for each iteration the lattice just consists from its center point (σ_0, a_0) and its surrounding neighbours up to second degree. Hence, the log-span of the lattice is $l_\sigma = \log_{10} \frac{\sigma_2}{\sigma_{-2}}$ for volatility, and $l_a = \log_{10} \frac{a_2}{a_{-2}}$ for mean reversion. The starting value for both is equal to $0.6 \cdot (n-1) = 2.4$. Furthermore, with $f = 1$, for both log-spans, the rescaling factor is set to $\frac{2f}{n-1} = 0.5$. This choice assures that, the optimal lattice point and its nearest neighbours are still points of the lattice also during the next iteration.

For each iteration step, Table 2 shows the actual lattice values in volatility and mean reversion, plus the actual log-spans of the lattice. In each step, for all actual lattice points, the Hull White model state is computed and compared with the target state, using the relative symmetric deviation (3). The index $[i_0, j_0]$ of the optimal lattice point, and the corresponding value

Table 2: Numerical values during the example calibration.

Iteration	1	2	3	4	5
σ_{-2}	0.00946436	0.00015000	0.00237734	0.00474342	0.00670025
σ_{-1}	0.03767830	0.00059716	0.00474342	0.00670025	0.00796327
σ_0	0.15000000	0.00237734	0.00946436	0.00946436	0.00946436
σ_1	0.59716076	0.00946436	0.01888388	0.01336876	0.01124841
σ_2	2.37733979	0.03767830	0.03767830	0.01888388	0.01336876
$\log_{10} \frac{\sigma_2}{\sigma_{-2}}$	2.40000000	2.40000000	1.20000000	0.60000000	0.30000000
a_{-2}	0.00189287	0.03000000	0.05985787	0.08455149	0.11943215
a_{-1}	0.00753566	0.05985787	0.08455149	0.10048963	0.13020308
a_0	0.03000000	0.11943215	0.11943215	0.11943215	0.14194538
a_1	0.11943215	0.23829847	0.16870240	0.14194538	0.15474665
a_2	0.47546796	0.47546796	0.23829847	0.16870240	0.16870240
$\log_{10} \frac{a_2}{a_{-2}}$	2.40000000	1.20000000	0.60000000	0.30000000	0.15000000
$[i_0, j_0]$	$[-2, 1]$	$[1, 0]$	$[0, 0]$	$[0, 1]$	$[0, 0]$
$\delta(a_{i_0}, \sigma_{j_0})$	0.03982438	0.03982438	0.03982438	0.00602651	0.00602651
iteration	6	7	8	9	10
σ_{-2}	0.00796327	0.00868143	0.00906445	0.00926224	0.00936276
σ_{-1}	0.00868143	0.00906445	0.00926224	0.00936276	0.00941342
σ_0	0.00946436	0.00946436	0.00946436	0.00946436	0.00946436
σ_1	0.01031790	0.00988192	0.00967089	0.00956707	0.00951557
σ_2	0.01124841	0.01031790	0.00988192	0.00967089	0.00956707
$\log_{10} \frac{\sigma_2}{\sigma_{-2}}$	0.15000000	0.07500000	0.03750000	0.01875000	0.00937500
a_{-2}	0.13020308	0.13594751	0.13594751	0.13891408	0.13966578
a_{-1}	0.13594751	0.13891408	0.13742279	0.13966578	0.14004315
a_0	0.14194538	0.14194538	0.13891408	0.14042155	0.14042155
a_1	0.14820787	0.14504283	0.14042155	0.14118141	0.14080096
a_2	0.15474665	0.14820787	0.14194538	0.14194538	0.14118141
$\log_{10} \frac{a_2}{a_{-2}}$	0.07500000	0.03750000	0.01875000	0.00937500	0.00468750
$[i_0, j_0]$	$[0, 0]$	$[0, -1]$	$[0, 1]$	$[0, 0]$	$[0, -1]$
$\delta(a_{i_0}, \sigma_{j_0})$	0.00602651	0.00550853	0.00504732	0.00504732	0.00502304
iteration	11	12	13	14	15
σ_{-2}	0.00941342	0.00941342	0.00941342	0.00939439	0.00937540
σ_{-1}	0.00943886	0.00942613	0.00941977	0.00940073	0.00938172
σ_0	0.00946436	0.00943886	0.00942613	0.00940707	0.00938806
σ_1	0.00948993	0.00945160	0.00943249	0.00941342	0.00939439
σ_2	0.00951557	0.00946436	0.00943886	0.00941977	0.00940073
$\log_{10} \frac{\sigma_2}{\sigma_{-2}}$	0.00468750	0.00234375	0.00117188	0.00117187	0.00117188
a_{-2}	0.13966578	0.13910162	0.13853974	0.13798014	0.13742279
a_{-1}	0.13985434	0.13928942	0.13872678	0.13816642	0.13760832
a_0	0.14004315	0.13947747	0.13891408	0.13835296	0.13779410
a_1	0.14023222	0.13966578	0.13910162	0.13853974	0.13798014
a_2	0.14042155	0.13985434	0.13928942	0.13872678	0.13816642
$\log_{10} \frac{a_2}{a_{-2}}$	0.00234375	0.00234375	0.00234375	0.00234375	0.00234375
$[i_0, j_0]$	$[-1, -2]$	$[-1, -2]$	$[-2, -2]$	$[-2, -2]$	$[-1, -2]$
$\delta(a_{i_0}, \sigma_{j_0})$	0.00502039	0.00496791	0.00492933	0.00490104	0.00486930

$\delta(\sigma_{i_0}, a_{j_0})$ of the deviation are then listed, too.

After each iteration, the point (σ_{i_0}, a_{j_0}) is becoming the center point for the lattice of next iteration. In general both, the center point of the lattice and its log-spans, l_σ and l_a, can change. The freedom to change the lattice after each iteration is essential to the adaptive flexibility of the optimization. Furthermore, considerable performance can be saved by using small (here 5×5) flexible lattices.

In the present example, after iterations 2, 4, 7, 8, and 10, the center point and both log-spans are changed, after iteration 11 and 12 only l_σ is changed, after iteration 1 only l_a is changed, and after iterations 13 and 14 none of them is changed. After iterations 3, 5, 6, and 9 the center point is not changing, while the lattice is contracting in size (decreasing both log-spans). After $I_{\max} = 15$ iterations the optimization is stopped with calibrated $\theta_c = (0.00938172, 0.13742279)$, yielding a Hull White present value of 30097.21 EUR for the Bermudan bond option, which is (as expected) still more than the most valuable European option from Table 1.

5 Conclusions and further studies

In Section 1, the general structure of mathematical finance calibration problems involving optimization was described and discussed. For three examples it was demonstrated how they fit into the general scheme. Each example represents a fundamental and typical class of stochastic models as applied in the valuation of interest rate derivatives:

- direct models for the probability density (see Example 1 and Section 2.1)
- short rate models (see Example 2 and Section 2.2)
- forward market rate models (see Example 3 and Section 2.3)

There exist different equivalent approaches to describe a stochastic model. In the first case, a direct ansatz (5) was made for the probability distribution. In the other two cases, it was convenient to start initially from a stochastic differential equation for the process, as in (11) or (13). Finally however, the latter approach also implies a definite probability distribution for the model.

All three examples require the optimization, i.e. minimization, of some Euclidean distance function (2) on a state space. For Example 1, this distance is explicitly given by (9) in the European case (single event at maturity T), or by (10) in the Bermudan case (several events at $T_1 < \cdots < T_{n_T} = T$). With Example 2, the distance is given through (12), and in Example 3, via (20). In the latter example, constrained optimization could be replaced by an unconstrained optimization, thanks to a reparametrization (19) of the space of calibration parameters, which in fact forms a hypersphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$.

In Section 3 a new adaptive lattice algorithm was presented as some example for an optimization algorithm. Unlike the standard Levenberg-Marquardt algorithm [8, 9], it operates on a lattice which adapts after in each iteration step both, its location and its lattice spacing. The algorithm explores on a double-logarithmic lattice not only the nearest neighbours of a minimum, but also further lattice points beyond. With appropriately chosen rescaling factor, one can assure that the lattice points of the next iteration contain the optimal point from the last iteration. This ensures that in each iteration, the deviation is not increasing. Since the lattice is moving, the algorithm has furthermore the ability to hop out of a current (true or apparent) sink around a wrong local minimum to another sink around another, more optimal, local minimum. Therefore, the risk to get trapped in a wrong local minimum, although theoretically still present, is reduced considerably in practical applications.

The numerical example of Section 4 demonstrates the application of this algorithm for a calibration of the Hull White 1-factor short rate model. For a given Bermudan (OTC) bond option, the mean reversion and the volatility are optimized, such as to minimize the symmetric relative Euclidean deviation (3) between the Hull White model state and the target state. Both states are determined with the European instruments present values as components. For the former, these values depend on the model parameters, for the latter, they are derived from interpolated market quotes via the Black model. The latter is used in the market to imply volatilities one-to-one from prices and vice versa.

In practice, with complicated optimization problems and larger portfolios of trades, it is inevitable to increase performance of calibration as much as possible. This can be partially achieved by an economic design of optimization algorithms, such as using small adaptive lattices (as e.g. in the algorithm presented above) rather than the traditional large and rigid ones. Another strategy is to reduce the application of optimization (and other expensive algorithms) just to the extent they are really needed. For example, it may be recommended, first to exploit direct calibration methods as much as possible in advance, before starting with the optimization itself. In the best case, then just a fine-tuning is left for optimization.

Effective variants of the algorithm of Section 3 have been implemented and tested for both, C++ and Java, within several applications. The Java implementation is part of the current commercial software library of Value & Risk [14]. It is applied both, within the Value & Risk Valuation Engine (for pricing), and at the heart of further other applications using the same pricing kernel. The Value & Risk applications are used productively at several major banks for more than 10 years.

The general structure of calibration with both, optimization methods and direct methods, is also at the heart of a modularly structured calibration engine, used for the calibration of different interest models of the three

typical classes described above. In particular advanced calibration methods for market forward rate models are a topic of current and further investigations.

In practice, very fast calibration can often be achieved by appropriately combining different algorithms of both types. For this purpose however, a strict modularization of the calibration engine is a necessary condition. Currently a modular calibration engine [15] is developed at Value & Risk as a separate commercial application, which is extending its calibration functionality both, to new interest rate models, and from interest rate derivatives to other asset classes, including inflation-linked products, credits and credit derivatives.

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