The Staple Inn Actuarial Society

Modern Valuation Techniques

Stuart Jarvis
Frances Southall FIA
Elliot Varnell

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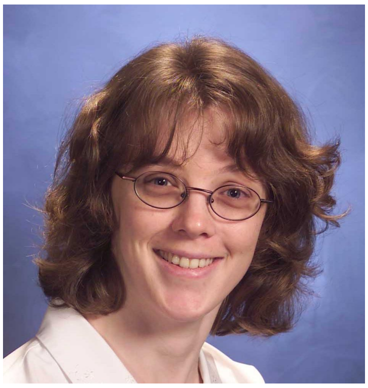
About the Authors

Stuart Jarvis

Stuart received a first class degree in mathematics from Cambridge University, and a doctorate in mathematics from Oxford University.

After some years of postdoctoral research he joined the actuarial profession, and has worked in Bacon and Woodrow's Birmingham office, in its employee benefits department since 1997.



Frances Southall

She has a first class honours degree in mathematics from Oxford University.

She joined Bacon & Woodrow in 1995 and qualified as an actuary in 1999.

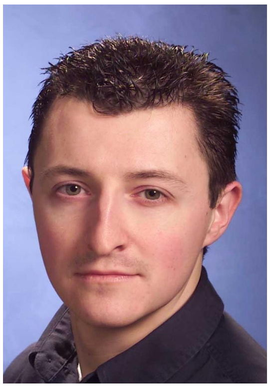
Her actuarial work has included: pensions research, pensions consultancy and research and development.

Elliot Varnell

He has a first class honours degree in Physics from Nottingham and Masters in Microwave Systems from Portsmouth.

He worked as a scientist before joining Bacon & Woodrow in 1996. He is currently studying to become an actuary.

His actuarial work has included Prophet development, stochastic model development and software design.



Contents

1 Introduction

1.1 Purpose of the paper

This paper describes techniques for valuing cashflows generated within a stochastic projection model, on a market consistent basis.

There has been a recent trend in accounting standards towards disclosures that are marked to market. For example, this is a stated intention of the ASB in the UK. We therefore believe these techniques provide the actuarial profession with a valuable tool.

The key tool we introduce is the state-price deflator (or, more simply, deflator). State-price deflators form a bedrock for modern finance, but are more often applied through the better-known method of risk-neutral valuation.

Although risk-neutral valuation is successfully applied in a wide range of market contexts, it can be confusing, and we believe that it has not been well received by the actuarial community.

Traditionally actuaries value cashflows using deterministic calculations or projection of the cashflows stochastically and discounting at a risk-adjusted rate. We advocate using deflators to value cashflows. Deflators are consistent with economic principles and, where the model is calibrated to the market, will produce market-consistent valuations.

1.2 Actuarial valuation methodology

Actuarial literature (e.g. Wilkie (1995); Ranne (1998); Duval et al (1999)) contains many stochastic models that project indices and asset prices into the future, but which do not place a market value on other cashflows. This is because the valuation first requires a suitable risk-adjusted rate at which to discount any set of cashflows. This means that the job of a stochastic model is only half-done.

For example, a model's results may indicate a high probability that one investment outperforms another. But before a value judgement can be made, risk-discount rates have to be chosen. There is no good way of choosing these such that, where these investments are marketable, the model's values are consistent with those of the market.

The difficulty in placing market values on cashflows generated stochastically has led to some actuaries concluding that projection models are necessarily inconsistent with valuation models. We disagree with this view and aim to show how the two model types can be consistent. We illustrate how a projection model should be constructed in order that any cashflows can be easily valued.

1.3 Modern finance valuation methodology

Modern financial theory is able to reconcile projection and valuation through the use of the deflator. The risk associated with an asset used for calibration is reflected in the model in such a way that the market value of this risk automatically allows for differences in return.

We can consider a deflator as a stochastic discount function.
Consider C_t as a stochastic cashflow at time t. The deflator technique calculates its present value using the following equation, where D_t is the stochastic deflator for the model:

$$Value = E[D_t C_t]$$

This is similar in principle to (but much more powerful than) the actuarial methodology of multiplying the expected value of the cashflow by a deterministic discount function, v^t:

$$Value = v^t E[C_t]$$

1.4 Benefits of the modern finance approach

- The deflator is associated only with the model and does not depend on the cashflow being valued. This makes valuations of stochastic cashflows considerably simpler. This is illustrated in the life insurance example in section 12.
- The deflator technique produces market consistent valuations, and so reflects the fact that changing an institutional investment strategy cannot (to first order) create or destroy economic value. However, changing the strategy may change the distribution of future cashflows, so transferring value between the stakeholders. The deflator technique enables actuaries to analyse these transfers. This is illustrated in the pensions example in section 11.
- Similarly the market consistency of the deflator technique is useful when considering the capital structure of a firm. Changing the capital structure of a firm cannot (to first order) change the market value of the firm. The capital structure can however affect the market value through second order effects such as tax and the impact of credit risk upon business dealings. The deflator technique enables actuaries to focus on these so-called frictional costs which can influence the market value of the firm. This is illustrated in our general insurance example in section 10.

1.5 Structure of the paper

In sections 2 to 6 we introduce and illustrate the basic principles of modern finance theory and valuation using a simple model.

Sections 7 to 8 extend the previous sections into more complicated examples and consider the differences between the deflator technique and traditional valuation methods.

Sections 9 to 12 introduce and illustrate the use of the deflator technique with examples from different actuarial work areas.

The appendices contain more mathematical and economic ideas, proofs and examples. These are not necessary for the understanding of the main paper, but are included for completeness and may be of interest to readers.

1.6 Originality

This paper is almost entirely an exposition of standard material from elsewhere. The basic technique has even appeared under various guises in papers presented to the Institute and Faculty (and their student bodies) in recent years.

However we believe there is a need to clarify the ideas and have sought to present the material again here as simply as possible.

A list of references is provided for the reader who wishes to study the source material.

1.7 Acknowledgements

We are indebted to Paul Coulthard, Shirleen Stibbe, and especially Andrew Smith for their invaluable contributions. We are also grateful to Alam Arbaney, Stavros Christofides, Gareth Collard, Tim Gordon, Cliff Speed, and Brian Wilson for their comments and to Ralph Maciejewski for his technical support. Finally we would like to thank Richard Chapman for his continued support for our work.

2 Arbitrage

2.1 Introduction

In this chapter we shall be considering a powerful pricing mechanism known as arbitrage-free pricing that forms the bedrock of valuation within modern finance.

In this chapter we state the definition of arbitrage and illustrate the arbitrage concept using a well-known forward stock price example. We continue this with a discussion about arbitrage in real markets and finish with the importance of using an arbitrage-free model for valuation and projection.

2.2 Definition of arbitrage

Colloquially, an arbitrage is a “free lunch" for investors; something for nothing.

More formally, an arbitrage is a portfolio of assets, with zero initial value, that provides a non-negative value in all possible future states and positive value in at least one state.

An arbitrage opportunity exists when an investor can construct two different portfolios of differing price, which provide the same cash flows. Selling the more expensive portfolio and buying the cheaper portfolio with the proceeds produces an unlimited return without capital expenditure on the part of the investor.

2.3 Example – stock forward price

The classic place to start in any exposition of financial mathematics is often with the question:

"What is the forward price of a stock?"

The forward price of a stock is the price (agreed today) at which two investors will trade the stock at some future time (specified in the contract).

Suppose we have the following information for a share:

Current share price	£1.20
Expected growth in share price	10% pa
Guaranteed return on cash over 1 year	5%

We assume that bid-offer spreads (including dealing costs) are negligible and that the stock pays no dividend. All investors are able to lend and borrow cash on the same terms, 5% over one year.

Given the above parameters, what value should we put on the forward or equivalently what value should we agree to trade the share in one year from now?

2.3.1 First attempt

A first guess at the forward price might be the expected price of the share in one year $1.10 \times £1.20 = £1.32$.

This seems a reasonable answer at first sight. To see the problem with this price, consider an investor offering to buy or sell a forward at this price. A second investor may then choose to adopt the following strategy:

Second investor’s actions now:

- Sells the forward thereby agreeing to sell the share for £1.32 in one year from now.

- Borrows £1.20 at 5% to buy one share now.
- Buys the share.

Second investor's actions one year from now:

- Completes the forward by selling the share to the first investor for the agreed £1.32.
- Repays his creditors, 1.05 × £1.20=£1.26.

The second investor's profit can be determined by considering his cash flows.

	Strategy 1
Contract	Sell a Forward
Buy one share	(120)
Borrow cash in the market	120
Net cash flow now	*0*
Forward price receipts	132
Repay borrowings	(126)
Net cash flow at expiry	*6*

The cash flows we have shown in the table are entirely deterministic. The cash flow table is the same regardless of the final share price. The second investor would make a certain profit of 6 pence per contract, at the expense of the first investor.

Furthermore the second investor would no doubt sell as many forward contracts as possible to the first investor until the first investor realised his mistake. This unlimited profit opportunity is an arbitrage because it creates a profit from nothing. The second investor has not had to use any of his own capital.

2.3.2 Second attempt

Suppose the first investor realises his mistake and decides to reduce the price of the forward contract to £1.24. This time the second investor adopts a different strategy, buying the forward, as shown in the following cash flow table.

	Strategy 2
Contract	Buy a Forward
Sell one share	120
Invest cash in the market	(120)
Net cash flow now	*0*
Forward price payment	(124)
Investment proceeds	126
Net cash flow at expiry	*2*

Now the second investor is guaranteed a profit of 2p per contract and will once again make unlimited profit until the first investor realises his mistake.

It appears that investors are able to find an arbitrage even when the forward price is low.

2.3.3 Final attempt

It turns out that the only price that avoids an arbitrage is the current share price with interest at the guaranteed return on cash, $120 \times 1.05 = 126$p.

The reason is that a hedge portfolio can replicate the forward contract. The second investor constructed the hedge portfolio. In the first attempt he sold the forward, and his hedge portfolio strategy was to borrow sufficient money to buy the share. The forward and the hedge portfolio have the same initial cashflows; zero.

At the termination of the forward contract, the forward contract results in the exchange of a share for the price of the forward. The hedge portfolio contains a share and an obligation to repay the borrowing. The cashflows of the forward contract and the hedge portfolio will be equal if the price of the forward meets the obligation to repay the money borrowed.

To avoid an arbitrage opportunity, the cashflows from the hedge-portfolio must be the same as the cashflows from the forward contract.

Below we show the investor following the above strategy with the arbitrage-free price and the first two strategies for comparison.

	Strategy 3	**Strategy 2**	**Strategy 1**
Contract	Hedge a Forward	Buy a Forward	Sell a Forward
Sell (buy) one share	(120)	120	(120)
Borrow (invest) cash in the market	120	(120)	120
Net cash flow now	*0*	*0*	*0*
Forward price receipts (payment)	126	(124)	132
Investment proceeds (repay borrowings)	(126)	126	(126)
Net cash flow at expiry	*0*	*2*	*6*

2.4 Arbitrage in real markets

Arbitrage opportunities are not available to most investors. Specialist arbitrage players operate in most markets, seeking out arbitrage opportunities and taking advantage until the opportunity ceases to exist. Unless an investor has large amounts of capital, a fast reaction time, very low dealing costs and few regulatory constraints, the opportunities will have gone before they become cost-effective.

The forward stock example is a simple demonstration of arbitrage-free pricing. Arbitrage-free pricing can also be demonstrated in several other financial markets.

- Foreign exchange – spot & forward
- Equity markets – equity options & futures
- Debt markets – Government bonds

Many institutional investors lack the flexibility to be specialist arbitrage players and as such are well advised to approach these markets as if they were arbitrage-free.

2.5 Arbitrage and valuation

Although the discussion of this chapter has been phrased in terms of market prices the underlying concept has as much relevance for the actuary calculating present values.

A model must provide a unique value for a unique set of future cashflows regardless of the portfolio used to produce them. If this were not the case, the model would not produce consistent valuations. A valuation model which claims to be able to value cashflows must therefore be arbitrage-free if it is to make any sense.

2.6 Summary

Expected returns on assets may be valid, but are misleading when it comes to valuation.

In markets where a hedge portfolio can be constructed for a contract, the value of the contract is the value of the hedge portfolio. This is the arbitrage-free pricing method.

Although short-lived arbitrage opportunities might exist, actuarial models should respect the fact that markets are effectively arbitrage-free if the models are to produce meaningful valuations.

3 State prices

3.1 Introduction

In this chapter we will build on the arbitrage-free concept by considering how an arbitrage-free model can be constructed using the concept of a state-price security.

A state-price security (or Arrow-Debreu Security) is a contract that agrees to pay one unit of currency if a particular state occurs at a particular time in the future. The prices of these securities, state prices, can be used as the building blocks for an arbitrage-free model.

We shall be using a simple model to illustrate the state price concept. We start by calibrating the model and calculating state prices, then we use the state prices to value certain assets. We show how the simple model can be extended before finishing the chapter with a discussion of state prices.

3.2 Example – feast and famine

3.2.1 Introduction

In this section we will use a small example, called the feast and famine example, to demonstrate how valuation in an arbitrage-free model leads naturally to state prices.

The model is a two-state example over one time-period that enables us to concentrate on how a product can be valued in a very simple world. It will enable us to gain a good understanding of the basic concepts.

3.2.2 Calibration of state prices

All models that hope to replicate market dynamics must be calibrated to market data. In our simple model there are only two possible future states (feast and famine), so only two assets are required to calibrate the model.

The two assets shown in the table below therefore determine the model.

	Asset A	Asset B
Market Price	1.65	1.00
Feast state payout	3	2
Famine state payout	1	0.5

Each asset is completely specified by the cashflow it produces in each state and its current market price.

We will have fully calibrated the model if we knew the market value of the state-price securities. We need to fill in the unknowns of the following table:

	Feast state price security	Famine state price security
Market price	?	?
Feast state payout	1	0
Famine state payout	0	1

To calculate the prices of these assets, we make use of the arbitrage-free pricing method illustrated in section 2. That is, we find a hedge portfolio that will replicate the payouts of the state price securities.

A little linear algebra can be used to show that portfolios containing the following units of Asset A and B will hedge the state price securities.

	Units of asset A	**Units of asset B**
Feast state price security	-1	2
Famine state price security	4	-6

A negative number of units indicates that the hedge portfolio should be short in that asset, a positive number indicates that it should be long in that asset.

To be long in an asset means to buy, and so hold, the asset.

To be short in an asset means to sell an asset that you do not own. This is possible as long as the buyer does not require physical delivery of the product, which is often the case in financial markets. If your portfolio is short in a particular share you would make a profit if its price fell. You can do this because you sold the share earlier at a high price and can close your position by buying the same share at a lower price.

The hedge portfolios replicate the cashflows of the securities. The principle of arbitrage-free pricing indicates that the securities must have the same value as their hedge portfolio. By valuing the hedge portfolios using current market prices, we get the following prices for the state price securities.

	Feast-state security	**Famine-state security**
Market price now	0.35	0.6

The calculations are readily checked by calculating market prices for Assets A and B using the state-price securities.

Considering the payouts in each state, we see that the price of Asset A should be equivalent to 3 feast-state securities plus 1 famine-state security. As expected, the calculations, shown in the table below, generate the original Asset A price.

	Asset A payouts	**State security prices**	**Hedge portfolio price calculations**
Feast state	3	0.35	3 x 0.35 = 1.05
Famine state	1	0.6	1 x 0.6 = 0.6
			Market price 1.05+0.6 = **1.65**

3.2.3 Valuation of a new asset using state prices

State price securities greatly simplify the valuation of arbitrary products. Simple linear algebra is all that is required.

The next table illustrates how a new risky asset, with known payouts in each state, can be valued using the state prices in the feast and famine example. The calculations are very similar to those used to check the price of asset A at the end of the calibration.

	New risky asset
Market price now	0.825
Feast state payout	1.5
Famine state payout	0.5

3.2.4 Risk-free asset

A risk-free asset provides a guaranteed return. Its cashflows are therefore the same regardless of which state occurs in the future. We calculate the price using the same methodology as above. The calculation reduces to the sum of the state prices.

	Risk free asset
Market price now	0.95
Feast state payout	1
Famine state payout	1
Guaranteed payout	1
Guaranteed return	5.26%

This calculation shows that the market price of the risk free asset is 0.95 which indicates a guaranteed return of 5.26%. The guaranteed return is also known as the risk-free rate.

3.3 Multiple-state model

The two-state model over one-period can be extended to a multiple-state model.

The same calibration and valuation approach can be used as in the two-state model although this becomes a tedious exercise in matrix algebra for a model with a large number of discrete states.

The valuation is undertaken using the following equation:

$$C = \sum_{AllStates} \psi(s) C(s)$$

where:

Market Price of Asset :	C
Payoff in State s :	$C(s)$
State Price for State s :	$\psi(s)$

State prices become unwieldy in a multiple-period setting. Fortunately there are better techniques for dealing with a multiple-period model. These are introduced later in this paper.

3.4 Constraints on State Prices

There are some constraints on the state prices which need to be observed if the model is to remain arbitrage-free.

- All state prices must be positive. If this were not the case, then buying a state-price security would present an arbitrage within the model.

- Since we know that the risk-free rate cannot be negative, the market price of the risk-free asset must always be less than or equal to one, hence the sum of the state prices should be less than or equal to one.

3.5 Summary

A state-price security pays out 1 if state s occurs and 0 otherwise.

A state price $\psi(s)$ is the price of the state-price security and is positive.

An n-state model is arbitrage-free if and only if state prices exist, and in that case, for any asset C, defined by its payouts $C(s)$ in each state s,

$$\textit{Value now of asset } C = \sum_{AllStates} C(s)\psi(s)$$

The risk-free asset is defined as the asset whose payout at some future time is independent of the state that occurs at that time.

The sum over all states of the state prices is less than or equal to 1.

$$\sum_{AllStates} \psi(s) \leq 1$$

4 Introduction to state-price deflators

4.1 Introduction

In section 3, state prices were defined and constructed for a one-period model with two states: the feast and famine example. Now we extend this example to consider the probability of the two states occurring. Introducing the probability of states occurring allows us to calculate state-price deflators for this example and we illustrate basic results using state-price deflators.

These results will be extended to more general cases, for example multiple-period models, in later sections.

4.2 State-price deflators for the feast and famine example

We return to the feast and famine example and introduce the probability of each state, feast and famine, occurring. We assume the state-probabilities are equal.

We also introduce a new concept called the state-price deflator, $D(s)$, as the ratio of the state price, $\psi(s)$, and the state-probability, $p(s)$:

$$D(s)=\frac{\psi(s)}{p(s)}$$

In the table below we have calculated the state-price deflator for the feast and famine example.

	Probability	State price	Deflator
Feast state	0.5	0.35	0.70
Famine state	0.5	0.6	1.20

4.3 Multiple-state model

We consider a one-period model with a finite number of possible states s at a particular future time t. Then, as has been described in previous sections, the arbitrage-free model has state prices $\psi(s)$ at time 0 for an asset paying 1 unit if state s occurs at the future time t.

The state-probability at time 0 is 1 and the value of a state-price security paying 1 at time 0 is 1. So the deflator at time 0 is 1.

We can use the state-price deflator and state probability, instead of the state-price, when putting a value on a cashflow or an asset. The introduction of probabilities means that valuation formulae are written in terms of expectations.

Using the deflator D in the valuation of any asset C gives:

$$\text{Value of asset } C = \sum_{AllStates} \psi(s)C(s) = \sum_{AllStates} p(s)D(s)C(s) = E[DC]$$

where $C(s)$ is the cashflow from C in state s.

We can consider the price of the risk-free asset paying 1 unit in all future states.

$$\text{Value of risk free asset} = \sum_{AllStates} p(s)D(s) = E[D]$$

So the expected value of the state-price deflator at a future time t is less than 1.

There are a couple of important points worth emphasising.

- We note that as state prices are always positive, state-price deflators are always positive.
- The state prices $\psi(s)$ are known at time 0. The deflator, on the other hand, is a random variable; it takes the value 1 at time 0, but after time 0 its value depends on the particular state generated by the model.

4.4 Summary

In a single-period model, the state-price deflator $D(s)$ is dependent on the future state s at time t and is defined as

$$D(s) = \frac{\psi(s)}{p(s)}$$

where $\psi(s)$ are the state prices and $p(s)$ is the probability of state s occurring.

The state-price deflator at time 0, takes the value 1.

The state-price deflator $D(s)$ is always positive.

$E[D]$ is the value of the risk-free asset paying 1 at time t and so $E[D]$ is less than 1.

The present value of cashflows $C(s)$ in states s occurring at future time t is

$$\textit{Value of asset } C = E[DC]$$

5 Utility

5.1 Introduction

In section 4, we introduced the concept of state-price deflators for the feast and famine example. In this section we introduce the utility function and we return to the feast and famine example to illustrate how a utility function can generate state-price deflators. We state the use of a utility function for the generation of deflators in general terms; the formal mathematical proof is contained in Appendix B.

5.2 Introduction to utility

Some practitioners assume that each investor has a utility function U that enables them to rank different portfolios. U is a function of the investor's wealth.

Economic arguments suggest that

- U is an increasing function (more wealth is always better).
- U has a decreasing gradient (the rate of increase in utility decreases with increasing wealth). The gradient of U is the marginal utility, so U has decreasing marginal utility.

For simplicity, we assume that U is differentiable. This enables results to be stated more cleanly below.

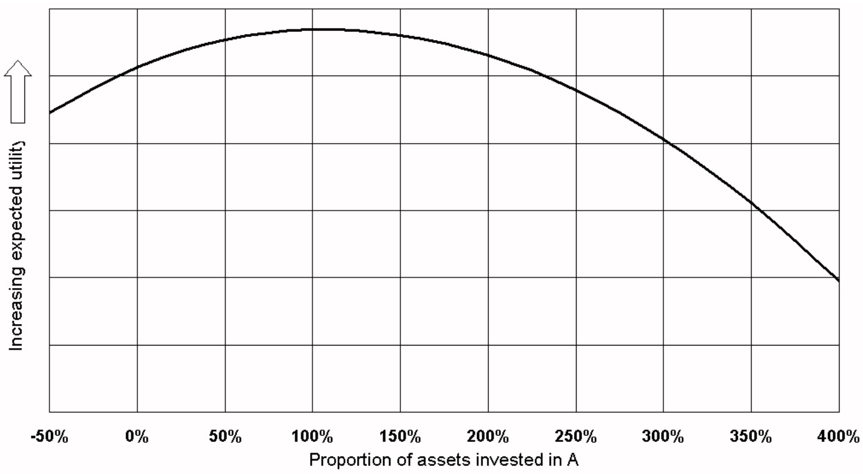
An investor seeks to invest so as to maximise his expected utility by choosing an optimal portfolio. In order for an optimal portfolio to exist, the model must be free of arbitrage. An arbitrage would enable the investor to increase his expected utility at nil cost without bound.

5.3 Feast and famine example

5.3.1 Utility function and optimal portfolio

Let the utility function be taken as $U(x) = \sqrt{x}$, where x is the level of wealth. This satisfies the above requirements for utility functions.

Using the feast and famine example described in section 3, an investor can use this utility function to establish what investment mix of assets A and B he should choose.



The graph shows how expected utility varies with the mix: the maximum is achieved by investing 108% of his assets in Asset A and –8% in Asset B. Equivalently this means purchasing 0.655 units of Asset A and selling 0.08 units of Asset B, costing a total of 1 unit.

5.3.2 Marginal utility of optimal portfolio

If $U(x)$ is the utility function of wealth x, then $U'(x)$ measures the *marginal utility* of 1 extra unit of wealth when the wealth level is x.

In the previous example, the marginal utility of the optimal portfolio (108% of A, -8% of B) can be calculated. Using $U'(x) = \frac{0.5}{\sqrt{x}}$ gives:

	Marginal utility of optimal portfolio	Wealth of optimal portfolio
Market price now		1
Feast state	0.372	1.804
Famine state	0.638	0.615

5.3.3 From the marginal utility to a deflator

We can compare the marginal utilities to the previously calculated deflators:

	Marginal utility of optimal portfolio	Deflator
Feast state	0.372	0.70
Famine state	0.638	1.20
Ratio of feast to famine	0.583	0.583

We notice that, up to a multiple, $U'(w_s^*)$ is a deflator for the model, where w_s^* is the wealth in state s for the optimal portfolio. This is not a coincidence, but in fact a general property of deflators.

5.4 The general case

We have seen in the famine & feast example that deflator values can be interpreted in terms of marginal utility.

It can be proved (see Appendix B) that the deflator for state s is related to the marginal utility for the optimal portfolio in the following way:

$$D(s) = \frac{w_0^*}{E\left[w_s^* U'\left(w_s^*\right)\right]} U'\left(w_s^*\right)$$

where w_0^* is the initial wealth of the optimal portfolio and w_s^* is the wealth of the same portfolio in state s.

If a deflator takes a higher value in one state than another, then the marginal utility of that state relative to the other is greater for an investor who is investing optimally. This is why deflators normally take higher values in poor times ('famine') than good ('feast').

Utility optimisation can be used to generate deflators for existing stochastic models which do not currently produce deflators.

5.5 Utility of the individual investor and the market's deflators

The utility function in the feast and famine example was for an individual investor, yet it produced deflators for the market as a whole and these deflators can be used to reproduce market prices. Our utility function was also an arbitrary function satisfying the conditions for a utility function.

We would not assume that every individual investor in the market had the same utility function and indeed it is not necessary. The method above produces the same deflators whatever utility function is selected for an individual investor, provided the investors agree on the probabilities for each state.

This is possible because each investor has a different optimal portfolio.

5.6 Summary

The individual's utility function is a function of wealth and enables the investor to rank different portfolios.

Marginal utility is the gradient of U.

The optimal portfolio is the portfolio that maximises expected utility.

The state-price deflator for a particular state is a constant multiple of the marginal utility of the optimal portfolio in that state.

The utility function and optimal portfolio relate to an individual investor whereas the state-price deflator relates to the market as a whole.

6 Risk-neutral valuation

6.1 Introduction

To define a state-price deflator in section 4 required the use of probabilities.

In this section, we use a simple example to illustrate the concept of changing the probabilities from the set of real-world probabilities. The state prices are fixed by the market, so changing the probabilities changes the state-price deflators maintaining the same market values.

We introduce a particular choice of probabilities; the risk-neutral probabilities, under which the mathematics of valuation becomes simpler. We consider why using deflators under the real-world probabilities may be preferable, although both methods will give the same values.

6.2 Feast and famine example

The deflator for the feast and famine example depends on the probability measure as shown in the table below:

	State prices	Probability	Deflator values
Feast state	0.35	p	$0.35 \div p$
Famine state	0.60	$1\text{-}p$	$0.60 \div (1\text{-}p)$

When $p = 0.368$, the deflator values become equal to each other: they take the common value 0.95, the sum of the original state prices.

We showed in section 4 that $E[D]$ is the price of the risk-free asset, for any probability distribution for the states considered. So if we chose the probabilities so that the deflator is constant, as we have done here, $E[D]$ is precisely the constant deflator, and so the constant deflator must be the price of the risk-free asset.

Under these 'risk-neutral' probabilities, A and B have the same return, equal to the return on the risk-free asset:

	Risk-free asset	Asset A	Asset B
Market price now	0.95	1.65	1
Feast state payout	1	3	2
Famine state payout	1	1	0.5
Expected future value	1	1.737	1.053
Expected return	5.3%	5.3%	5.3%

We can show this mathematically.

Consider an asset now priced X, with future cashflows at time 1 of X_1. We know that $X = E[DX_1]$ and the expected return can be written as $E[X_1]/X$.

If Q denotes the risk-neutral probabilities and r is the risk-free rate, then $D^Q=(1+r)^{-1}$.

So we can write the following:

$$X = E\left[DX_1\right] = E^{Q}\left[D^{Q}X_1\right] = \frac{1}{1+r}E^{Q}\left[X_1\right]$$

So the expected return under risk-neutral probabilities is

$$\frac{E^{Q}\left[X_1\right]}{X} = 1 + r$$

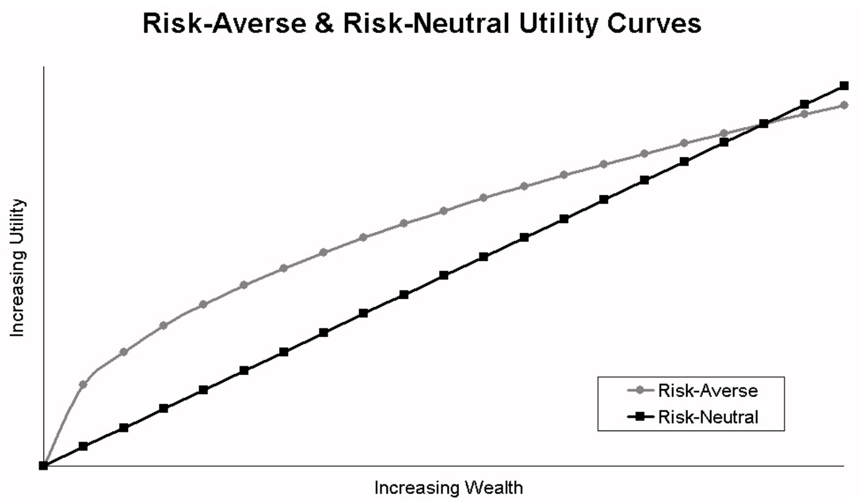
i.e. the risk-free rate.

Once the risk-neutral probabilities are known, assets can be priced using the formula $E[DX] = DE[X]$. This is similar to the traditional actuarial method (calculate an expected value then discount), except that the discount rate is not risk-adjusted: the probabilities are.

The numerical example illustrates how changing the probabilities can result in constant deflators and expected returns of any asset being equal to the risk-free rate. This change of probabilities is known as a change of measure. Appendix D provides the mathematical explanation of changing the measure to generate this risk neutrality.

6.3 Utility for a risk-neutral investor

As mentioned in section 5, an investor's utility curve is normally concave, which implies that he is risk-averse (i.e. for a given expected level of wealth, prefers certainty over uncertainty):



Under the change of measure above, the new probabilities were chosen so that the deflators were constant in any future state. Section 5 described how deflators were proportional to the marginal utility of the optimal portfolio. So we can consider the investor to have constant marginal utility under these probabilities. An investor having constant marginal utility is known as a risk-neutral investor, hence the name risk-neutral probabilities.

6.4 Advantages of a risk-neutral valuation

A risk-neutral valuation is a valuation carried out using the risk-neutral probabilities as described above.

It has the advantages of

- State independent discount factor, using the risk-free rate
- Expected returns at the risk-free rate on any asset

This eliminates the need for a subjective choice of discount rate.

It is commonly used in investment banks for pricing derivatives, for example, the Black Scholes call option formula in Appendix C.

6.5 Risk-neutral valuation versus valuation using deflators

We have shown that there are two ways of calculating values of cashflows

- Using real-world probabilities and deflators
- Using risk-neutral probabilities and discounting at the risk-free rate.

Whichever calculation method is used, the same value will be placed on the cashflows. This is illustrated in Appendix C, where deflators calculated using real-world probabilities are shown to reproduce a formula for the value of a call option developed using risk-neutral probabilities.

When deciding which method of valuation to use, the following should be considered:

- The main disadvantage of the risk-neutral valuation method is that the concept of risk-neutral probabilities is difficult to understand.
- Another disadvantage is that the risk-free rate and risk-neutral law change with currency, so multiple-currency models are harder to generate. Deflators can cope with multiple-currency models.

6.6 Summary

We have shown that we can change the probabilities (or measure) to the risk-neutral probabilities (or risk-neutral measure), to generate:

- Discount factor at the risk-free rate
- Expected returns on risky assets equal to the risk-free return

A risk-neutral valuation uses these changed probabilities and places exactly the same value on a set of cashflows as a real-world valuation using deflators.

7 Extending the deflator model

7.1 Introduction

In earlier sections we calculated state prices and state-price deflators for a single-period model; in this section we extend the ideas over many future periods.

We also consider the use of deflators in the modelling of multiple currencies and the term structure of interest rates.

7.2 Deflators in a multiple-period model

We assume a model with possible future states s at a particular future time $t \in \{t_1, t_2, .., T\}$. As with the single-period model, we have state prices $\psi_t(s)$ at time 0 for 1 unit in state s (at future time t) and the probability of this state occurring at time t, $p_t(s)$. As before we define a deflator process, D_t, which takes the following value if state s occurs:

$$D_t(s) = \frac{\psi_t(s)}{p_t(s)}$$

It is natural, particularly in multiple-period models, to consider deflators as a (positive) stochastic process, taking different values in each future state.

We can use deflators to value an asset C at time 0 with cashflows $C_t(s)$ at future times t:

$$\text{Value of Asset } C = \sum_{AllTimes} \sum_{AllStates} p_t(s) D_t(s) C_t(s) = \sum_{AllTimes} E[D_t C_t]$$

We can also value future cashflows at intermediate times: the value at time t of cashflow C_T payable at time T can be calculated using the same deflator; *a new one is not required*. The expectation operator now becomes a conditional expectation, E_t, conditional on information available at t:

$$\text{Value of } C_T \text{ at time } t = \frac{E_t[D_T C_T]}{D_t}$$

We note that in earlier sections, we valued cashflows at time 0. D_0 was shown to be unity. Hence the formula used in these earlier sections is a special case of this formula. A numerical example can be found in Appendix A.

In a multiple-period context, the ability to use the conditional form of the expectation operator means that the kind of equations which will occur are easy to state, manipulate and understand.

Consider an asset with uncertain market value A_t at future times t. An investor decides to purchase this asset at future time t and sell it at a later time T. Provided that the asset pays no dividends, the price that the investor pays at time t is the market value at that time, A_t. But the value must also be considered as the value at t of the future cashflow at time T (A_T). So using the formulae above, we can write

$$A_t = \text{Value of } A_T \text{ at time } t = \frac{E_t[D_T A_T]}{D_t}$$

Hence we have the relationship

$$D_t A_t = E_t[D_T A_T]$$

We have shown that for any asset with value A, DA is a martingale because its expected value at a future time T, conditional on information known at time t, is its value at time t.

In a model that contains a continuum of future states a form of state price density and a probability density are required. The ratio of the state price density to the probability density is the deflator.

7.3 Deflators in a multiple currency model

A state-price deflator depends upon a currency being chosen as the unit of account; all cashflows are assumed to be denominated in this currency. A different currency choice could be chosen. Fortunately there is a simple relationship between the resulting deflator and the original one, as this section will show.

Suppose that $D_t^{£}$ is a deflator relating to cashflows denominated in £. Suppose that $Y_t^{£\$}$ is the exchange rate between sterling and dollars, ie $X_t^{£}$ pounds is the same as $X_t^{£}Y_t^{£\$}$ dollars. It can be shown that $D_t^{£} \div Y_t^{£\$}$ is a deflator for dollar-denominated cashflows.

This relationship can be turned on its head. Rather than modelling one deflator and an exchange rate within a stochastic model, the two deflators can be modelled instead; the exchange rate then falls out as their ratio. This approach has been advocated for example by Rogers (1997), Smith & Speed (1998).

7.4 Deflators in a term structure model

In the context of single-period models, we observed that the sum of state prices is just the price of the risk-free asset. Within a multiple-period model, this becomes the statement that $E_t[D_T]/D_t$ is the price at time t of the risk-free asset which pays out one unit at time T. This asset is a 'zero-coupon bond'.

The zero-coupon bonds are the basic building blocks of yield curve models. Their prices, $P(t,T) = E_t[D_T]/D_t$, tell us everything we need to know about the term structure of interest rates at time t. For example, the 'short rate', r_t, is the theoretical rate available on very short money-market instruments at time t, so for small δ:

$$P(t, t+\delta) \cong e^{-r_t \delta}$$

Hence the short rate can be calculated as:

$$r_t = -\frac{\partial}{\partial T} \log P(t,T) \bigg|_{T=t}$$

There are many interest-rate models widely used within financial markets. Several of these are expressed as short-rate models, which means that the behaviour of the short rate r_t is usually defined in the form of a stochastic differential equation. Integration of this equation is normally required before the interest rate can be used within a model.

We suggest that defining the behaviour of the deflator may be a more efficient way to proceed. This deflator technique has been applied to many common interest rate models; see for example Vasicek (1977), Cox et al (1985), Rogers (1997), Leippold & Wu (1999). As would be expected from the previous section, this approach is particularly efficient when a modeller wishes to consider yield curves in different currencies simultaneously.

7.5 Summary

The deflator, D_t, is a positive process with the property that a stochastic cashflow X_T payable at time T has value at time $t<T$ given by the formula

$$\frac{E_t\left[D_T X_T\right]}{D_t}$$

Deflators have the martingale property; where A_t is the value of an asset and D_t is the deflator at time t, then for $T>t$

$$D_t A_t = E\left[D_T A_T\right]$$

The deflator technique is useful when modelling multiple currencies and the term structure of interest rates.

8 Valuation using deflators

8.1 Introduction

In this section we look at traditional actuarial valuation method and illustrate the difficulties of using this method to produce a market consistent valuation. We then explore the advantages of the deflator technique and we consider when the use of deflators is appropriate.

8.2 Traditional actuarial valuation method

Proposed accounting standards are forcing the profession to consider how traditional actuarial valuation methodology can be reconciled to market values.

The traditional valuation methodology would place a value, C_t, at time t of an uncertain future cashflow, C_T, at time $T>t$ using a discount rate d.

$$C_t = \left(\frac{1}{1+d}\right)^{T-t} E_t\left[C_T\right]$$

One of the actuary's tasks is to find a suitable value for d.

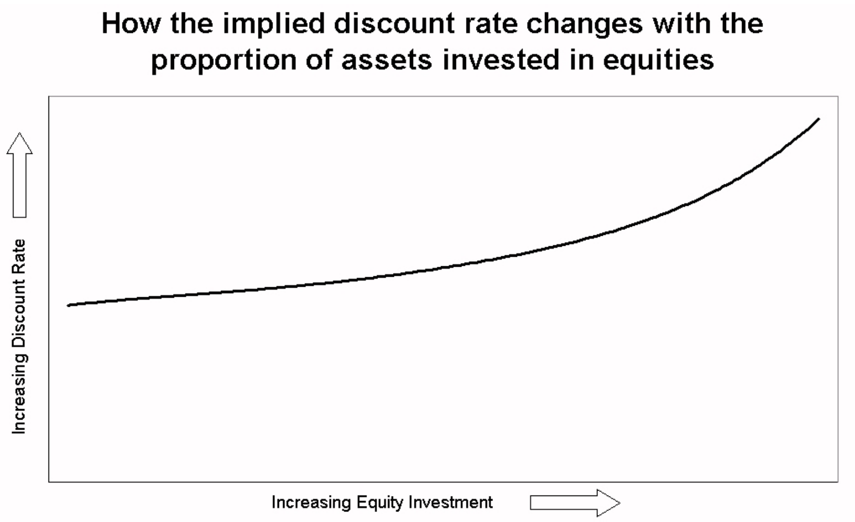
It has been shown that d is the risk-free rate if C_T is the same in all future states, i.e. the cashflows are guaranteed. The discount rate, d, should be changed to value riskier cashflows. However there is no objective method for determining the market-consistent risk discount rate for an arbitrary set of cash flows.

8.3 Illustration of the difficulties

The difficulty in determining a market-consistent discount rate to value a series of cashflows is illustrated by the use of deflators in a simple, hypothetical, life insurance model.

The simple model projects an annual premium with-profits policy and calculates the payments made to shareholders as a result of bonus declarations and the payments required, if any, at the termination of the policy to meet benefits in excess of the asset share. These payments were valued using deflators. An implied discount rate was then determined by equating the value of the payments using deflators to the discounted expected value of the payments at each future time.

The changes in the implied discount rate when the proportion of assets invested in equities increases are illustrated in the following graph. More details are in section 12.



The traditional actuarial valuation methodology requires the actuary to choose a discount rate using his judgement. The difficulty of this choice, especially where complex contracts are concerned, lead to inconsistent advice being given. The use of a deflator-based model removes this source of error.

8.4 The advantages of the deflator valuation method

Deflators can be used to place a market value on any set of cashflows generated by the model. The deflators are model dependent, but are independent of the cashflows being valued. No consideration of the riskiness of the cashflows needs to be made, the same deflators are always used.

In the possible future states, different values may be attached to an increase in wealth of a unit of currency. For example, an extra pound would have greater value in a recession than in a boom. These values are the marginal utility, and in section 5, we saw that deflators are proportional to the marginal utility of the optimal portfolio in each state. This means that the value to the investor of the cashflows in each future state is quantified in the valuation using deflators.

8.5 Appropriate use of deflators

Systematic risk in financial theory is risk correlated with financial markets. The deflator technique can only value risk associated with systematic risk.

Deflators are an appropriate technique when the value of the cash flows is largely influenced by other economic entities. For example, the value of a pension increasing in line with Limited Price Inflation (LPI) is completely dependent on future value of Retail Price Inflation (RPI).

Unsystematic risks such as the rate of improving mortality are not shown in the deflator. They are taken at their expected value: there is a nil risk premium for these risks. Therefore they may have to be allowed for separately in the liability projection.

Some hybrid risks, such as lapse rates, should have their unsystematic parts allowed for separately. In theory these do not affect value, but they may affect frictional costs. See the general insurance example, section 10.

8.6 Summary

Traditional actuarial methodology does not provide a method for determining a market-consistent risk discount rate to use in a valuation.

The deflator technique places a market value on any cashflow generated within the deflator model because it applies state-dependent discounting.

9 Introduction to the examples

In section 4 we showed that an arbitrary cash flow could be valued using the following equation:

$$C_t = \frac{E_t\left[C_T D_T\right]}{D_t}$$

This method could just as easily be applied to any function, f, of economic outputs, U_T using the following equation:

$$f\left(U_t\right) = \frac{E_t\left[f\left(U_T\right) D_T\right]}{D_t}$$

This permits the valuation of a very large range of products.

In this paper so far we have been using simple models. The intention was to give the reader a feeling for the underlying concepts and build confidence in the deflator technique.

To be useful in real-world valuations deflators need to be a part of a more complex model with outcomes generated by a continuous distribution. A continuous distribution is needed to investigate a wide range of future states and for convergence of valuation equations.

The mathematics required to build and calibrate continuous distribution deflator models can be involved. For those interested in the details we recommend Duffie (1996) or Shiryaev (1999).

Though difficult, proprietary models with continuous distribution outcomes and deflators have been built and calibrated. The next three chapters use continuous distribution models to illustrate issues in pensions, general insurance and life insurance.

10 General insurance example

10.1 Introduction

10.1.1 Deflators and systematic risk

As we have seen deflators are adept at valuing future cash flows. They work particularly well for valuing financial instruments, where the cash flows are contractually linked to indices for which we have observable histories.

Such cash flows consist principally of risks correlated to the market as a whole, that is, systematic (or non-diversifiable) risk. If we use deflators to value a firm, we likewise obtain a good measure of how investors are likely to value the firm allowing for its systematic risk. Modern finance theory tells us that the market only rewards investors for exposure to systematic risk.

10.1.2 Deflators and unsystematic risk

Deflator valuations are not affected by variability uncorrelated to capital markets. These risks are sometimes called *unsystematic*. The market does not reward unsystematic risk because of the investor's ability to diversify away the risk by holding a portfolio of assets. There is a current debate as to whether these unsystematic risks should affect valuations, for example in reserving for general insurance liabilities (see, for example, Hindley *et al* (2000)).

In their ground-breaking paper of 1958, Modigliani & Miller demonstrated that the way a firm was financed, either using debt or equity, made no fundamental difference to its market value.

Their argument showed that swapping equity capital for bond capital just increased the gearing of the firm and hence the return required by equity holders. They concluded that the capital structure of the firm was irrelevant to the firm's valuation. Deflators respect this conclusion.

The management usually have the objective of maximising the stock value. The Modigliani & Miller paper conclusions suggested that paying managers to choose a capital structure reduced the value of the firm by the combined value of the salaries involved.

Modigliani and Miller had considered a simple model of a company, which assumed that costs were linear with profit.

Because firms do still devote resources to finding an optimal capital structure, we assume this is not the whole story. Second order effects, also known as frictional costs need to be considered. They are the non-linear expenses incurred as a result of doing business. Examples include:

- Future business dealings being sensitive to credit risk.
- Project disruption and wastage of unbudgeted flows.
- Optimistic plans survive longer in uncertain world.
- Convex tax formulas – not able to use tax losses.
- Extra back office or processing expense.
- Capital raising, distribution, restructuring costs.
- Double taxation of income on invested risk capital.
- Operational risk of cash misuse.
- Management time opportunity cost.

10.1.3 What this means for managing a firm

In so far as they can minimize the frictional costs the management can then influence the market value of the firm.

Most actuarial models do not allow explicitly for frictional costs. Instead frictional costs are implicitly modelled using crude techniques, such as adjustment to a liability or embedded value discount rate.

Can we use deflators to value other cash flows such as the profit streams to shareholders, to come up with an estimate of the value of a company?

We can, but the cash flow model needs to be good enough. Financial products such as stock options have a contractual formula linking their value to an observable market value. There is no contractual formula linking profit streams to capital market inputs. The links are via actuarial formulas containing all sorts of estimated parameters and approximations.

An efficient stock market will value the firm on its ***true*** cash flow. A ***true*** cash flow model is realistic enough to reconcile to the market value of the firm.

A management wishing to maximize their shareholder value must allow for frictional costs when setting their strategy. In other words their cash flow model must bridge the gap between the outputs of an actuarial model and those of a ***true*** cash flow model by modelling frictional costs.

10.2 Theory

10.2.1 Introduction

In this section we will use a simple model of frictional cost and examine the results when applied to a simplified general insurer.

The example illustrates what is possible and it is understood that a practical analysis requires a more sophisticated frictional cost model. Such models exist but go beyond the scope of this paper. Further information can be found in Christofides (1998) and Witcraft (2001).

10.2.2 A frictional cost function

One way of modelling frictional costs is to consider them as a function of the profit generated. We would usually choose a positive, convex function. We choose a quadratic function for simplicity.

We propose that the cost function for the firm be defined as follows:

$$\textbf{\textit{Frictional Cost}} = K\left(\alpha + \frac{1}{\alpha}\left(P - \beta\right)^2\right)$$

We assume that the strategy chosen by the management can be encapsulated in the parameters of the function:

- α

  This parameter controls the variability of frictional costs as a function of profit. If α is large then the frictional cost does not depend strongly on profit. It also affects the minimal frictional cost.

- β

  This parameter sets the level of profit at which the minimum cost is attained. The value of this parameter would typically be chosen close to the expected profit level

The following parameter represents the inherent costs associated with the market, beyond management control:

- *K*

  A higher value indicates higher market frictional costs. This figure would need to be derived empirically.

We define the ***pure*** profit of the company as the profit according to the stochastic model. It does not take account of frictional costs. It is represented by the *stochastic* variable:

- *P*

This is distinct from ***true*** profit which is the ***pure*** profit less frictional costs.

Let us consider two firm strategies called Confident & Cautious.

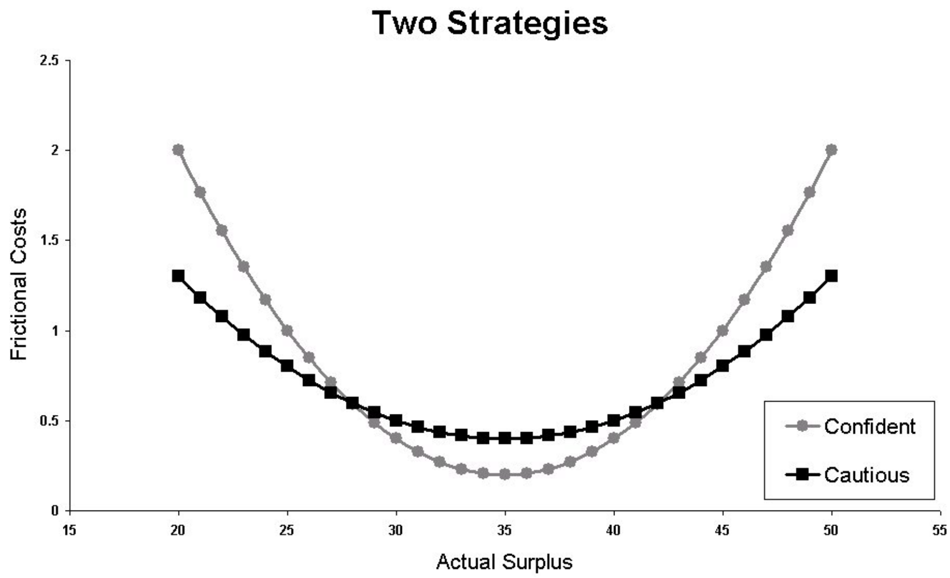
The parameters used for the two strategies are given in the following table.

	Alpha	Beta	K
Confident	5	35	0.4
Cautious	10	35	0.4

The *Confident* strategy is not forgiving if profits are far from the expected value. Its advantage is that it has lower costs if profitability is close to the expected value.

The *Cautious* strategy makes more provision for volatile profits at the expense of a higher frictional costs close to the expected profit.

In the figure below we show how the frictional cost curves look for two possible management strategies. It is clear that the Confident strategy is cheaper if the realized profit does not deviate too far from the expected profit of 35.



By selecting appropriate parameters the function we have shown that our simple function can be used to approximate profit-dependent frictional costs.

The functions are convex because there are frictional costs associated with doing better than expected as well as doing worse.

10.2.3 Finding the optimum strategy

The management of a general insurer will want to know what strategy they should pursue to maximize the market value of the firm for the shareholders.

This would mean finding the right balance between confidence about the expected profit and contingency in case the target is not met.

In our simple model this is the same as finding the parameter values, which minimize the market value of the frictional costs, given by the equation below. We seek parameter values, α and β, that will solve this equation.

$$min\left(E\left[DK\left(\alpha+\frac{1}{\alpha}(P-\beta)^2\right)\right]\right)$$

After some intermediate working we get the following expressions for optimum parameters α and β, which will minimize the expected frictional cost:

$$\beta=\frac{E[DP]}{E[D]}$$

$$\alpha=\frac{1}{E[D]}\sqrt{E\left[DP^2\right]E[D]-E[DP]^2}$$

If the profit is purely deterministic we find that the β parameter reduces to the deterministic pure profit. The α parameter reduces to zero as shown in the expression below.

$$\alpha=\frac{P}{E[D]}\sqrt{E[D]E[D]-E[D]^2}=0$$

We can show that as the pure profit becomes deterministic the frictional costs in this model come down to zero.

10.3 In Practice

10.3.1 Introduction

We have used output from a Dynamic Financial Analysis (DFA) model of a general insurer. DFA models can, of course, produce huge volumes of output, but in our example we focus particularly on the surplus and its value to shareholders. The profit is just the increase in surplus attributable to shareholders.

The assumptions used by our model are given in the table below. The combined ratio is defined as the ratio of claims and expenses to the premium.

Assumption	Value
Premium income	100
Initial investment portfolio value	30
Combined ratio mean	102%
Combined ratio standard deviation	8%
Tax rate	30% on Profit. Equity gains deferrable until realized.

We used the model to illustrate how adjustment of the asset mix of equity and bonds can maximize the value of a firm through the second order effects of frictional costs and taxation.

We considered five different investment strategies, each investing a different proportion of the portfolio in equity. The balance was invested in 10-year bonds. The model was run for one year and used 10000 simulations.

Strategy	1	2	3	4	5
% Equity	0%	10%	20%	30%	40%
% Bonds	100%	90%	80%	70%	60%

We analysed three outputs from the model:

- Expected Value of Gross Surplus
- Present Value of Gross Surplus
- Cost of Capital

For each of above we considered the breakdown between:

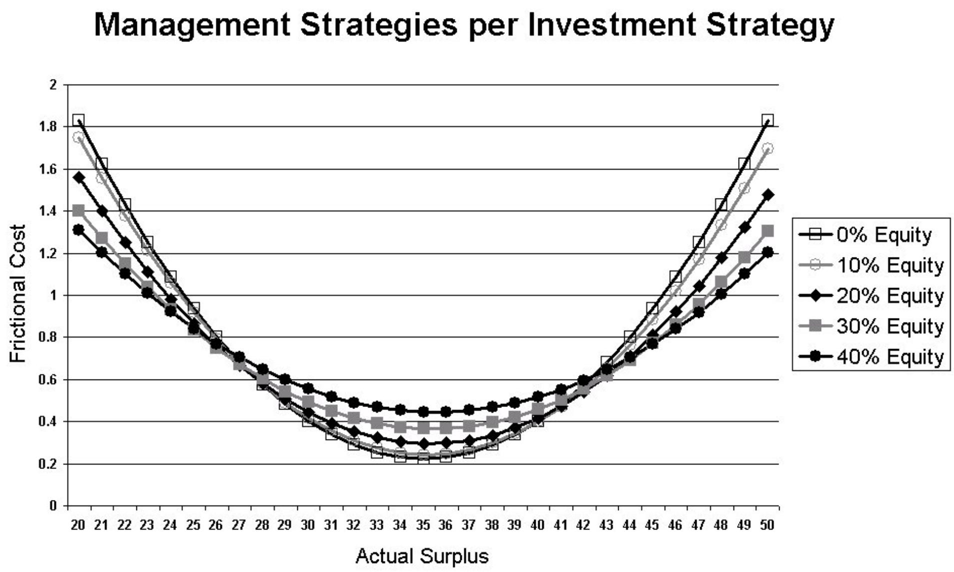
- Net Surplus attributable to shareholders.
- Tax
- Frictional Costs

We assumed that the firm's management strategy could be described by the quadratic cost function given earlier in this chapter.

We also assumed that the firm's management optimised their management strategy for each of the asset management strategies considered.

For each investment strategy we first ran a set of simulations to calculate the expectations used in the optimal management strategy formulae given earlier.

The different optimal management strategies required for each investment strategy are plotted below. We assumed the market-wide frictional costs, K, were 4%.

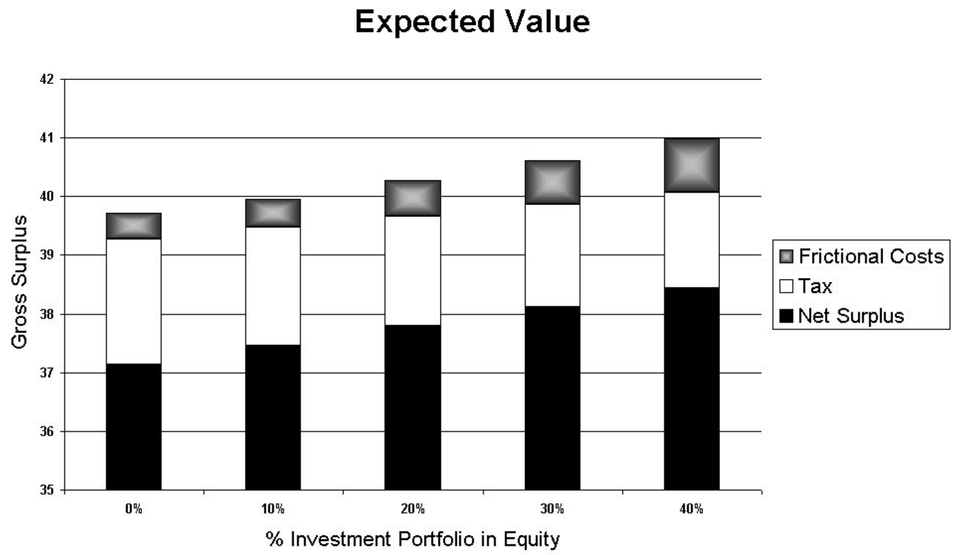


Using the terminology of the *Confident* and *Cautious* strategies earlier we see that the 0% equity investment strategy produces the most confident optimal management strategy. Meanwhile the 40% equity investment strategy produces the most cautious optimal management strategy.

These results make sense to our intuition, as more equity investment is likely to decrease the confidence in achieving a particular surplus figure.

10.3.2 Expected values

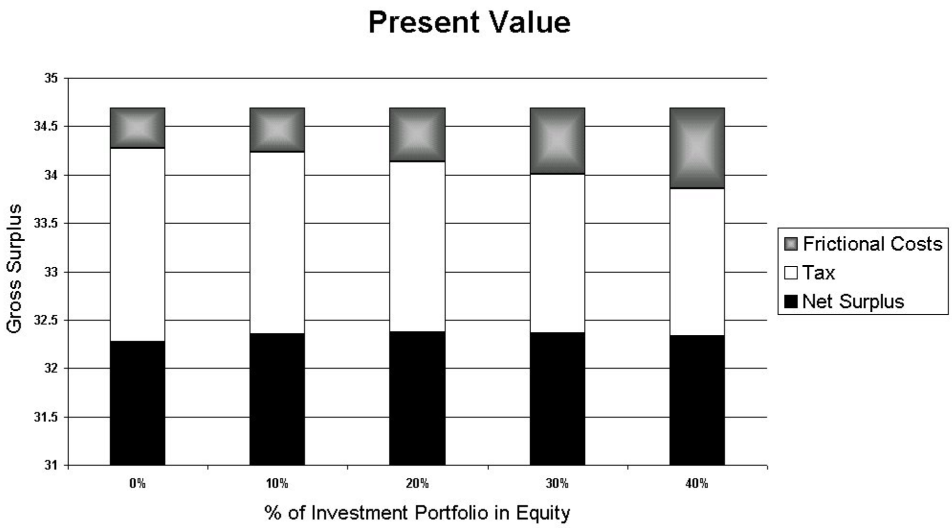
We investigated the breakdown of the expected value of the gross surplus. The breakdown is plotted below for each of the investment strategies.



We see that the expected value of the net surplus increases with the equity investment. If we were only to consider expected return as the criteria for our investment, most of our funds would be invested in equity. Clearly this would be unwise, as this would take no account of risk.

10.3.3 Present values

Next we investigated the breakdown of the present value of the gross surplus. The present values were calculated using the deflator technique discussed in Chapter 9. The breakdown is plotted below for each of the investment strategies.

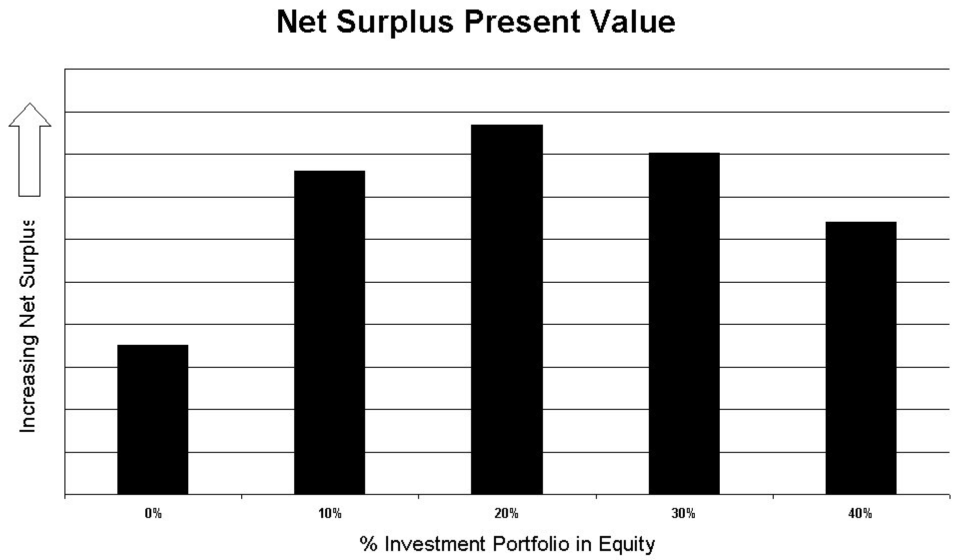


As we would expect, the present value of the gross surplus is unaffected by the different investment strategies. This demonstrates clearly the result of Modigliani & Miller. The present value of the gross surplus is unaffected by the investment strategy.

The stakeholders are the tax authority, the shareholders and a third group who benefit from frictional costs. The third group would consist of employees, recruitment agencies, suppliers etc.

The investment strategy does change the split between the stakeholders, and in particular the net surplus attributable to the shareholders, which will affect the profitability and consequently the market value of the firm. These are the second order effects discussed earlier in the paper.

If we zoom in on the net surplus figures, we can see the effect of the investment strategy on the present value of the surplus attributable to the shareholders.



It is clear that there is an optimum level of equity investment that balances the extra frictional costs of equity investment against the tax advantage of holding unrealised equity gains.

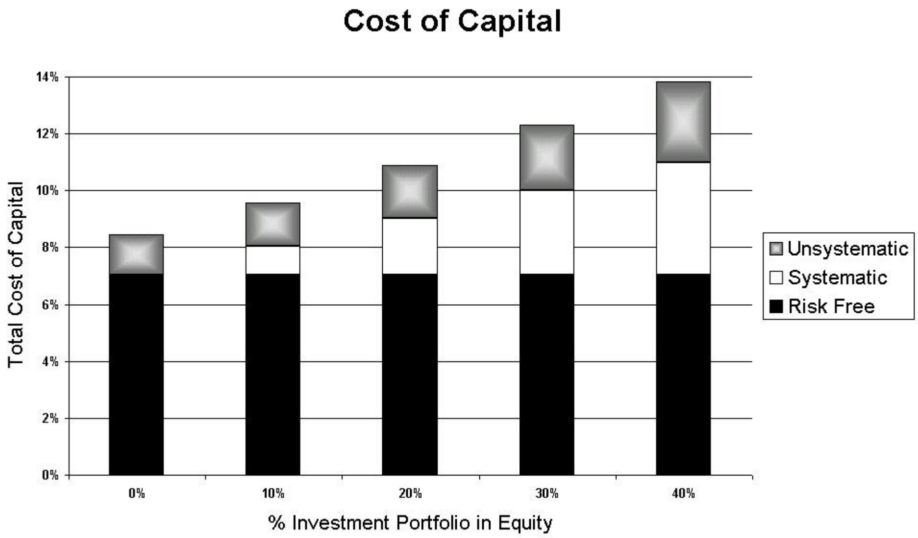
10.3.4 Cost of capital

Finally we looked at the cost of capital. Cost of capital is a measure used by many firms to communicate the rate of return required on new projects if they are to be allocated capital.

Frictional costs are usually ignored when the present value is calculated by applying cost of capital to cash flow projections. If the present values are to be comparable with market value the frictional costs need to be reflected in the cost of capital rate.

We thought it would be of interest to see how the cost of capital breaks down into three components:

- Risk free component.
- Systematic risk component.
- Unsystematic risk component.



Clearly the risk-free rate remains unchanged for all investment strategies. The systematic risk component representing risk correlated with financial markets increases with equity exposure. Systematic risk is nil when there is no equity exposure. The unsystematic risk is always present but increases with equity exposure because of the larger surplus variance and convex frictional cost function.

10.3.5 Review

We have shown, albeit with a simplified frictional cost model, that investment strategy can affect shareholder value through frictional costs. Furthermore we were only able to select the optimal investment strategy, and consequently the optimal management strategy, by using deflators and a suitable frictional cost function.

We think that a management with a remit to maximize shareholder value could make use of a deflator model combined with a suitable frictional cost function to make a decision on how to maximize shareholder value.

11 Pensions example

11.1 Outline approach

This section examines a simplified model of a company pension scheme. The purpose of this is not to give a realistic worked example that could be used for practical purposes, but instead to show the kind of considerations that arise when building such a model. Additional work is clearly required, but a forthcoming paper (Chapman et al (2001)) will address the main issues and carry these ideas much further forward than here.

The investment strategy of the scheme is the focus of the investigation. Traditionally, an 'asset-liability' model of the scheme is built, and 'optimal' strategies are suggested based on various criteria, including probabilities of:

- the scheme funding ratio (perhaps on a number of valuation measures) falling below a certain level
- the sponsor's contribution rate having to rise beyond some level

Our approach is to model the whole company (including the pension scheme), and look at the positions of the various stakeholders (shareholders, debtholders, scheme members). Because the model includes a deflator, we are able to show how the values of their stakes might change in response to the risks associated with different strategies for investing the scheme assets.

11.2 The scheme

The pension scheme is assumed to be the simplest possible: it has just one liability, of 100 (pounds, euros, whatever) payable in one year's time. The scheme has its own separate pot of assets to help cover this liability, but it can also call upon the assets of the employer to make up any shortfall.

The employer has assets outside the scheme, over which it has more direct control. These assets have been raised through a combination of equity and debt, and the outstanding debt is also due in one year's time. The debtholders will have first call on the company's assets if these assets are insufficient to cover both the debt and any shortfall within the scheme. This insolvency risk provides an incentive for the scheme's trustees to invest cautiously (in bonds).

Finally, if the scheme is in sufficient surplus in one year's time, the scheme members' benefits may be increased. This could be modelled in a number of ways: we assume a simple formula, that liabilities will increase by x% of any surplus above y% (for example, 20% of any assets in excess of 150% of the liabilities) This benefit-leakage risk provides an incentive for the scheme's trustees to invest in risky assets (in the equity market, in this model).

11.3 Stakeholders and strategy

There are therefore three stakeholders within this model: the scheme members (with whose interests the scheme's trustees are assumed to be aligned), the company's shareholders, and its debtholders.

The question we seek to answer is: given particular parameters of the model, what might the best investment strategy be for the scheme's trustees to adopt? Alternatively, what might be better from the shareholders' perspective or the debtholders'? As described above, the main risks, insolvency and benefit leakage, will benefit one party at the expense of another. Which is more important will depend on the particular situation.

11.4 The model

Since only a one-year time period is being considered, the model can be rather simple. We assume 1-year interest rates are 6%, and assume that the scheme invests in a combination of 1-year bonds and an equity index, which has a fixed lognormal return. The company's non-scheme assets are also assumed to give a lognormal return, with the volatility made up of two components: one correlated to the equity index (the systematic risk), measured by the company's beta, and one uncorrelated (the company-specific risk). As in the CAPM, we assume that only systematic risk is rewarded with a higher expected return.

The precise parameters we use are set out at the end of this appendix, with the case-specific ones shown in a table in the next section.

11.5 The results

The table below shows some examples that we have put through the model:

	Case A	Case B	Case C	Case D
Brief description	Large company, trustees powerful, 100% funded	New volatile company, trustees some discretion	Declining company with big scheme	Large company, no room for extra benefits
Beta	Low (0.5)	Very high (2)	High (1.5)	Low (0.5)
Specific risk	10%	30%	10%	10%
Nominal scheme liabilities	100	100	100	100
Face value of debt	100	100	20	100
Company value	1000	200	150	1000
Funding ratio	100%	110%	90%	90%
Benefit increases	40% of surplus over 120%	25% of surplus over 140%	None	None

Here 'company value' is the sum of the scheme's assets and the non-scheme assets over which the company has more direct control.

The output from the model consists of the values of the stakes held by the three parties, if a given investment strategy is followed by the scheme. Before showing these, however, we can imagine what the results would be had we used a discounted cashflow method. For example, in case B, the scheme starts in surplus and has a prospect of additional benefits if sufficient surplus is produced. The *expected* value of members' benefits in a year's time will therefore exceed 100 if a risky investment strategy is followed, and be exactly 100 if a bond strategy is followed. A typical asset-liability approach would recommend a risky strategy in this case; with the insolvency risk effectively being given nil value. The (unlikely) scenarios where the equity investment doesn't pay off are effectively ignored. Of course the deflator will give higher weight to these scenarios in order to properly account for the risks, and so a different picture emerges, as the following table shows.

		Member value	Debtholders' value	Owners' value	Total
A	**bond strategy**	94	94	812	1000
	equity strategy	95	94	811	1000
B	**bond strategy**	94	82	24	200
	equity strategy	91	78	31	200
C	**bond strategy**	94	19	37	150
	equity strategy	92	19	39	150
D	**bond strategy**	94	94	812	1000
	equity strategy	94	94	812	1000

We can explain these figures as follows:

(A) Company A is very large compared to the scheme and the outstanding debt, and so the value of the insolvency risk is nil for all practical purposes. But by investing in riskier assets, additional benefits may be given to members, and so the members' value is increased (slightly) at shareholders' expense. Debtholders are completely unaffected

(B) In this case, there is a significant downside (insolvency) risk and also, due to the initial surplus and prospect of increased benefits, an upside risk to equity investment for the members. The limited liability for the owners and the prospect for investment profits means that their share increases in value with riskier investment. The low values for the debt payments (the face value is discounted by around 20%) show the importance of the insolvency risk. Contrary to what we may have initially expected, it is this insolvency risk which should be highest in the trustees' minds: the value of members' benefits decreases if a 100% equity investment is chosen instead of 100% bonds

(C) The balance of risks is slightly different in this case: the scheme starts with a 10% deficit, and there is no prospect of additional benefits. Unsurprisingly, bonds offer the members the best protection, at the expense of the owners

(D) This is similar to case A: the company's size and low risk offers members protection for their benefits however the scheme is funded.

In three of these cases, the result could have been predicted with some confidence in advance, but in the fourth (case B) the balance between the upside and downside risks is difficult to judge. A *quantitative* tool such as a deflator is absolutely necessary if the answer is to be figured out. Using other methods such as discounted cashflow can make equities seem spuriously attractive.

11.6 What was ignored (and why it shouldn't have been)

This example has sought to briefly indicate how quantitative analysis can be brought to bear on a typical pension scheme problem. The idea of seeing a decision in terms of how value is transferred between stakeholders is not new, but it is only recently coming into actuarial advice.

The tentative conclusions we have advanced in this simplistic model will not necessarily hold in a more realistic implementation. For example, we believe that the following areas will require detailed investigation along the same lines before advice can be given:

- The projection period will of course need to be extended beyond one year, and dynamic strategies allowed for
- Future accrual of benefits and the sponsor's contribution levels need to be modelled. These will be material for many schemes and they may alter over the projection period in response to changes in the funding ratio or other indicators
- Additional stakeholders can be included. For example, the tax authorities; some assets are sheltered from tax by being held within the scheme, some not. In addition, the sponsor may pay dividends, whose tax treatment is different from that of changes in share value. This will tend to make equity investment less attractive for company owners. Additionally, the company management could be treated as a separate stakeholder. The interests of these 'agents' of the company's owners may not be aligned with the interests of the owners themselves, which gives rise to so-called 'agency costs'
- Inflation is an important risk for most pension schemes; this will need to be modelled, perhaps in tandem with a stochastic model of the interest rate term structure
- At a lower level, salary and withdrawal experience will need to be assumed; some dependence on the company's performance and that of the economy as a whole would be desirable; whether this is an important factor will depend on the company being considered

A forthcoming paper (Chapman et al (2001)) will take up some of these issues in greater depth.

11.7 The model: further details

For the mathematically inclined, the model used in this investigation was as follows.

- Constant interest rate of r = 6%, giving a bond price process $B_t = \exp(rt)$
- Equity market total return index: $M_t = \exp((r + \sigma\lambda - \sigma^2/2)t + \sigma W_t)$ where λ is the 'market price of risk' and W is a Brownian motion process
- Company's free assets form a process $S_t = \exp\left(\left(r + \beta\sigma\lambda - \xi^2/2\right)t + \xi\widetilde{W}_t\right)$, where $\xi\widetilde{W}_t = \sqrt{\delta^2 + \beta^2\sigma^2}\,\widetilde{W}_t = \beta\sigma W_t + \delta\hat{W}_t$ and $\hat{W}_t$ is a Brownian motion independent of W. Here, β is the company's beta, a measure of its correlation with the market as a whole, and δ is what we called above the company's 'specific' risk.
- Deflator $D_t = \exp\left(-\left(r + \lambda^2/2\right)t - \lambda W_t\right)$

The parameters used were $\lambda = 75\%, \sigma = 20\%, \sigma\lambda = 15\%$. (There is nothing significant about any of these: they just provide *some* parameters to enable the model to be run.)

12 Life insurance example

12.1 Introduction

Currently accepted valuation methodology is deterministic. However, for products where guarantees exist, stochastic models are more appropriate- in the deterministic world guarantees either bite or don't bite.

This example illustrates, with a simple (stochastic) model, how we can use deflators to value cashflows from a life insurance contract offering guarantees. We can use the results to derive an implied market consistent discount rate. This rate can be used as a measure of the risk involved in particular management strategies. We do not work deterministically, but note that the use of deflators avoids the subjective decision faced by actuaries when determining discount rates for valuations.

Shareholders' payouts from with profits policies depend on the bonuses added to the policies. The bonuses declared will depend on the return on the underlying funds and the office's strategy for determining bonuses. Hence shareholders' payouts can be very different in different economic scenarios.

Deflators allow the present value, on a market consistent basis, to be determined for any series of cash flows. No assumption about discount rate need be made. Essentially we can have a model that projects future cashflows and places a market consistent value on them.

12.2 The model

The model uses an economic scenario generator that produces returns on different asset types and deflators for each time period and simulation. As discussed earlier in the paper, the expected present value using deflators for each asset type is the current market value.

We consider a UK with-profits style contract. A single, annual premium policy is projected from inception to maturity.

Valuing a single with-profits policy in isolation is almost impossible. This is because of the role of the estate; it is very hard to separate the impact of a single policy or cohort of policies. We have followed the example of Hare et al (2001) in considering a single policy in which, somewhat artificially, the shareholders receive a share of bonuses but also explicitly underwrite the guarantees inherent in the bonus policy. This approach is consistent with some recent attempts to restructure with-profits funds making the role of shareholder capital more explicit.

We assume that a specified percentage of assets are invested in equities; this is the equity backing ratio. The asset share of the policy is projected using the returns generated by the economic scenario generator.

A bonus declaration policy is built into the model. Reversionary bonuses are set so that the guaranteed benefits plus reversionary bonuses target a specified proportion of asset share. Constraints are placed on the annual reversionary bonus rate that can be declared to allow for policyholders' reasonable expectations and smoothing (for example, a minimum bonus rate and a maximum increase from one year to the next).

We note that the full model illustrates the flow of value between the various stakeholders: policyholders, shareholders and the estate. Here we concentrate on the value of the shareholders' interest in the fund. However the model shows that changes in investment policy and bonus strategy have no effect on the total value of the fund to all stakeholders. Management decisions can only affect the apportioning of the value between the stakeholders.

The model is only concerned with allowing for investment risk; it is an illustrative tool. Other risks such as mortality have not been considered. Deflators ensure that there are no economic inconsistencies, such as arbitrage opportunities, and so the results are easier to interpret.

12.3 Calculations using the model

We value the cashflows payable to the shareholders as projected by the model and the cashflow required to meet guaranteed benefits in simulations where the asset share is too low at maturity. We note the cashflow at the end of the policy would be considered as a transfer from the estate in a full model of a fund.

We put a market consistent value on the cashflows using deflators as described earlier in the paper:

$$\sum_{AllTimes} E\left[D_t C_t\right]$$

where C_t is the scenario dependent cashflow and D_t is the scenario dependent deflator at time t.

We then calculated an implied constant discount rate d, by solving the following equation for d

$$\sum_{AllTimes} E\left[D_t C_t\right] = \sum_{AllTimes} \frac{E\left[C_t\right]}{\left(1+d\right)^t}$$

12.4 Results

Changing the equity backing ratio and bonus strategy affects the amount and value of the shareholders' payments under each simulation. We now investigate how such changes affect the implied discount rate. We have fitted curves to reduce sampling error.

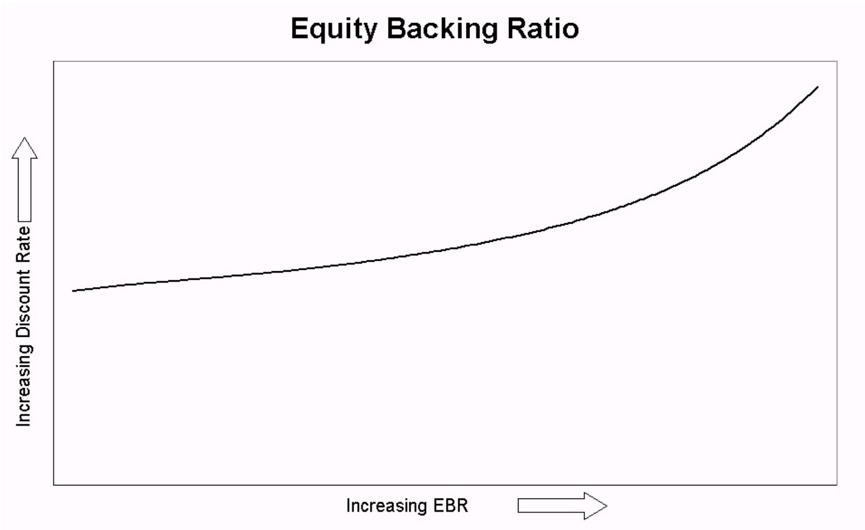
12.4.1 Increasing the equity backing ratio

Investing a greater proportion of the assets in equities results in a greater volatility of investment returns. This volatility is reflected in volatile bonus declarations and so more uncertain shareholders' payments.

The increased volatility also results in larger sudden falls in asset values. If this occurs at the end of the term of the policy, the guaranteed sum assured and declared reversionary bonuses may exceed the value of the asset share, and hence require a payment from the shareholders to meet this guarantee.

The model shows that the implied discount rate used to value shareholders' payments increases as the equity proportion increases, as illustrated below. This can be interpreted as deflators implicitly allowing for the risk of equity investment.

It is interesting to note that this is not a linear relationship. A small change in the equity backing ratio at high levels of the ratio can significantly affect the implied discount rate. The negative cashflows at maturity can be considered as the results of the shareholders' writing a put option on the asset share at maturity. The put option increases shareholder gearing and so increases the appropriate shareholder discount rate. The option is more significant for high ratios.

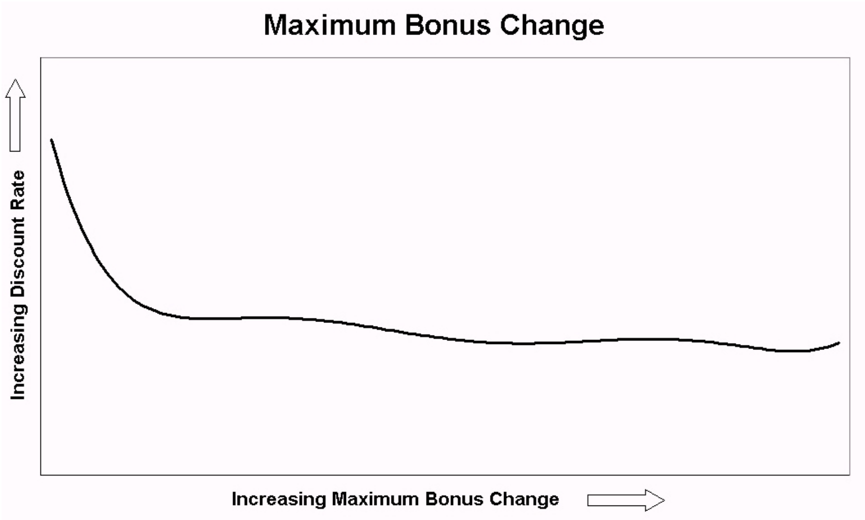


12.4.2 Increasing the minimum bonus rate

Increasing the minimum bonus rate reduces the uncertainty of shareholders' payments from reversionary bonuses. Shareholders have to meet any deficit in the asset share at the expiry of the policy. So increasing the minimum bonus rate increases the probability of having to meet this deficit and increases the uncertainty of these shareholders' cashflows at the expiry of the policy. The second of these factors dominates, as illustrated by the increase in implied discount rate with increasing minimum bonus rate.

12.4.3 Increasing the maximum bonus rise

A tight restriction on the allowed change in reversionary bonus from one year to the next, means that the shareholders' share of the cost of the bonus will be relatively stable. But when the asset shares are lower than the guarantees at maturity, a payment from the shareholders is required. These payments are more volatile when a tight restriction is in place. So the implied discount rate is relatively high for tight restrictions. Relaxing the restriction increases the volatility of the earlier payments relating to reversionary bonuses, but reduces the volatility of the shortfall payment. This reduction in volatility is reflected in the decrease in the discount rate as the restriction is reduced.



12.5 Conclusions

The basic model has illustrated a number of points:

- Stochastic projections of shareholder income are required in order to model the volatility of these cashflows from a with-profits policy.
- The implied discount rate required to put a market consistent value on the shareholders' cashflows is greatly affected by the underlying assets held and differences in the method of determining bonuses.
- The use of an economic model including deflators enables a market consistent value to be determined. It removes the subjectivity of choosing a discount rate. The results will be affected by the model used and the assumptions in the model.

The model used was based on Smith (2000).

13 Conclusions

- Many current actuarial stochastic projection models cannot place values on cashflows generated within the model, and therefore only perform half of the role they should.
- Putting values on stochastic cashflows amounts to requiring that the model contains a state-price deflator, i.e. a stochastic discount function.
- If a model produces a deflator there is no need for actuarial judgement in choosing discount rates for each cashflow. The same deflator can be used to value any cashflow.
- To first order, capital structure has no influence on aggregate value. That is, a change in investment or financing strategy does not create value by itself. This property is inherent in the deflator framework.
- Second order effects (e.g. non-linear impact of taxation and business costs) do influence value. These effects, the *main* determinants of adding value, can also be modelled within a deflator framework by using a frictional cost function.
- Risk-neutral valuation works well in some circumstances. However the presentation can mislead and the technique becomes impractical when more complicated situations are considered. In contrast state-price deflators can be used in both the financial markets where risk-neutrality is the dominant technique and in the actuarial world which has yet to achieve consensus in valuation methodology.

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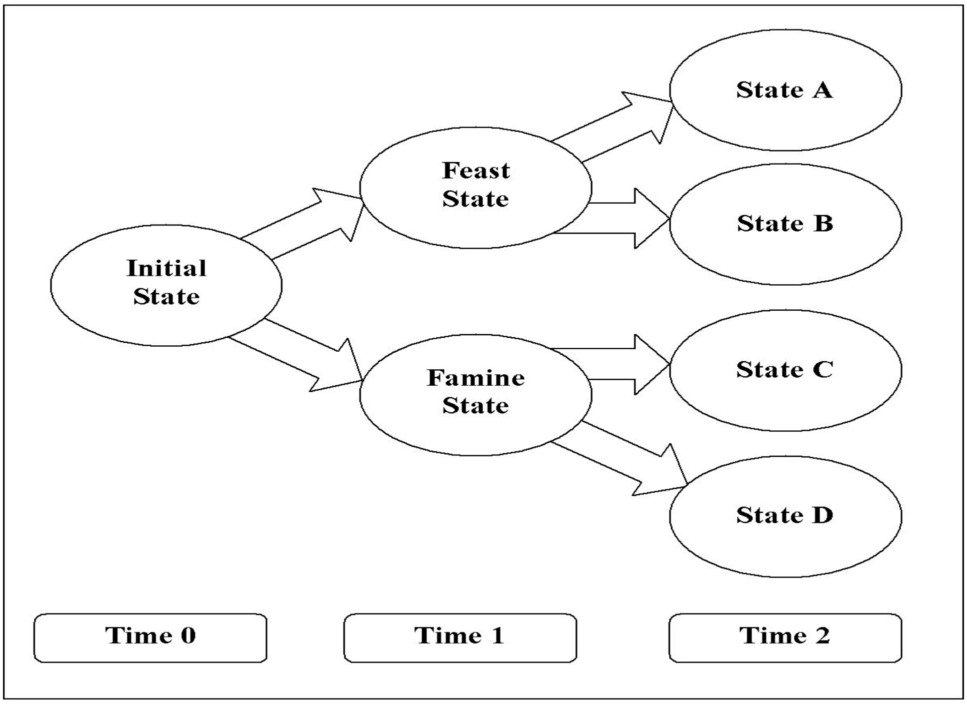
Appendix A Multiple-period model

A.1 Introduction

This appendix illustrates numerically the use of deflators in multiple-period models and proves the necessary formulae to show how deflators can be used in multiple-period models.

A.2 Numerical example

We extend our feast and famine example, introduced in section 3, into another time period. This multiple-period model can be considered as the combination of three single period models: the feast and famine example, the model from the feast state to states A and B and the model from the famine state to states C and D.



We know the figures for the time period 0 to 1:

	Probability	Feast-state Security	Famine-state Security	Deflator (time 1)
Market price now		0.35	0.6	1
Feast state payout	0.5	1	0	0.7
Famine state payout	0.5	0	1	1.2

Assume that at the famine state and the feast state, time 1, the prices of the state-price securities for states A and B, and states C and D respectively are known. Assume also that at the feast and famine states, the probabilities of the states at time 2 are known.

These values are given in the tables below.

FEAST STATE	Probability (conditional probability, given that feast state has occurred)	State-price security	State-price security
Market Price Feast State		0.4	0.5
State A Payout	0.6	1	0
State B Payout	0.4	0	1

FAMINE STATE	Probability (conditional probability, given that famine state has occurred)	State-Price Security	State-Price Security
Market price famine state		0.3	0.55
State C payout	0.8	1	0
State D payout	0.2	0	1

In order to calculate deflators, we need to calculate the state prices- the value at time 0 of an asset returning 1 in a state at time 2. We also need to calculate the probabilities of these states occurring, measured from time 0. The results are shown in the table below.

Once we have the state prices and probabilities measured from the initial state we can calculate the deflators as in section 4:

$$Deflator = \frac{StatePrice}{Probability}$$

	Probability	A-State Security	B-State Security	C-State Security	D-State Security	Deflator time 2
Market price now (state price)		0.4*0.35 = 0.14	0.5*0.35 = 0.175	0.18	0.33	
State A payout	0.5*0.6 = 0.3	1	0	0	0	0.467
State B payout	0.5*0.4 = 0.2	0	1	0	0	0.875
State C payout	0.4	0	0	1	0	0.450
State D payout	0.1	0	0	0	1	3.300

Let us consider how to value a cashflow at time 1 that is paid at time 2. Let the cashflow take values X_A, X_B etc. in the 4 possible states at time 2. Then we can value this cashflow at time 1 by considering the state prices for states A to D at time 1.

State time 1	State time 2	Cashflow time 2	State Price given state at time 1	Value of cashflow at time 1
Feast State	**A**	X_A	0.40	$0.40*X_A+0.50*X_B$
	B	X_B	0.50	
Famine State	**C**	X_C	0.30	$0.30*X_C+0.55*X_D$
	D	X_D	0.55	

Now consider:

$$E_1[D_2X]/D_1$$

State time 1	State time 2	Probability given state at time 1	Cashflow time 2	Deflator time 2	Deflator time 1	$E_1[D_2X]/D_1$
Feast State	**A**	0.6	X_A	0.467	0.7	$(0.6*0.467*X_A+ 0.4*0.875*X_B)/0.7 = 0.4*X_A+0.5*X_B$
	B	0.4	X_B	0.875		
Famine State	**C**	0.8	X_C	0.450	1.2	$(0.8*0.45*X_C+ 0.2*3.3*X_D)/1.2 = 0.3*X_C+0.55*X_D$
	D	0.2	X_D	3.300		

So we have shown that the value of cashflows payable at time T, at an intermediate time $t<T$, is

$$E_t[D_TX_T]/D_t$$

A.3 Conclusion

Deflators can be generated for multiple-period models and satisfy similar conditions as for single-period models.

Appendix B Marginal utility and deflators

B.1 Introduction

We saw in section 5 that deflators are proportional to the marginal utility of the optimal portfolio. We shall prove that relationship in this Appendix.

B.2 From optimal strategy to deflator

Consider the optimal strategy w^* that maximises expected utility at time T=1. The wealth distribution w_1^* (at T=1) can be compared with w_1 arising from an alternative investment strategy with the same initial outlay w_0^* at time 0. The optimality condition tells us that the expected utility of all mixed strategies (producing wealth $\theta w_1^* + (1-\theta)w_1$, for any θ) must be less than that of the optimal strategy.

Considering the Taylor series expansion of $f(\theta)= E\left[U(\theta w_1 + (1-\theta)w_1^*)\right]$ we have

$$E\left[U(\theta w_1 + (1-\theta)w_1^*)\right] = E\left[U(w_1^*)\right] + E\left[(w_1 - w_1^*)U'(w_1^*)\right] + o\left(\theta^2\right)$$

By the optimality of w_1^*, the maximum expected utility is attained at w_1^*, and so the function f has a maximum at θ=0. From the expansion above, the condition $f'(0)$=0 gives the condition

$$E\left[(w_1 - w_1^*)U'(w_1^*)\right] = 0$$

Equation B. 1

We can show that the optimal strategy produces a deflator at time T=1 as

$$D_1 = \frac{U'(w_1^*)w_0^*}{E[w_1^* U'(w_1^*)]} D_0$$

To prove that this formula for D_1 is a deflator, we must show that for any portfolio y

$$D_0\, y_0 = E[D_1\, y_1]$$

We define a constant k, known at time T=0, and write D_1 in terms of k

$$k = \frac{w_0^*}{E[w_1^* U'(w_1^*)]}, \quad D_1 = kU'(w_1^*)D_0$$

So for any alternative investment strategy w with the same initial outlay w_0^* at time 0

$$E\left[D_1 w_1\right] = E\left[U'(w_1^*)w_1\right]kD_0$$

Using Equation B. 1 and substituting for k we find

$$E\left[D_1 w_1\right] = E\left[U'(w_1^*)w_1^*\right]kD_0 = w_0^* D_0$$

Since any portfolio y can be written $Y{*}w$, for some portfolio w as above, where Y is constant, we have, for any portfolio y, proved the condition for D_1 to be a deflator.

Thus the optimal strategy produces, as a by-product, a deflator for the model. This can be used for pricing if desired.

For an investor investing optimally, his relative preference between two states ($s1$ and $s2$) at time 1 is the ratio of the deflators for these states. Using the definition of D_1 above, and because the wealth at time 1 depends on the state at that time, this ratio becomes:

$$\frac{D_1(s1)}{D_1(s2)} = \frac{U'(w_1^*(s1))}{U'(w_1^*(s2))}$$

So the relative preference between two states is the same as the ratio of the marginal utilities for the two states.

Appendix C Black-Scholes' model deflators

C.1 Introduction

There are a number of standard models which appear in textbooks and courses on financial mathematics. In this appendix we consider one of these models- the Black Scholes Model. We calculate deflators for the model and use these deflators to calculate an option price.

C.2 The Black Scholes model

The Black Scholes model is used to price options, by considering a bond B and stock S.

The bond *B* has a guaranteed return at the risk-free rate r, so satisfies the following equation for times *T*>*t*

$$B_T = B_t e^{r(T-t)}$$

The stock S satisfies the following equation for times T>t

$$S_T = S_t e^{(X_T - X_t)}$$

X is a Brownian motion with drift μ and standard deviation σ

$$X_T - X_t \sim N\big(\mu(T-t), \sigma^2(T-t)\big) \text{ or equivalently } X_T | X_t \sim N\big(\mu(T-t) + X_t, \sigma^2(T-t)\big)$$

This is often called a lognormal model for the stock price since S is lognormally distributed.

C.3 Form of the deflator

By considering the risk-neutral change of measure, we see that the deflator can be of the same form as S.

So there is a deflator of the form

$$D_t = D_0 e^{at + bX_t}$$

with a and b constants.

From the properties of a deflator we have

$$D_t B_t = E_t[D_T B_T] \text{ and } D_t S_t = E_t[D_T S_T]$$

C.4 Solving the equations

Substituting the equation for the deflator into the equations for bonds and stock prices, we can determine a and b.

$$b = \frac{1}{\sigma^2}\left(r - \mu - \frac{\sigma^2}{2}\right) \qquad a = \frac{1}{2\sigma^2}\left(\mu^2 - \left(r + \frac{\sigma^2}{2}\right)^2\right)$$

C.5 Equation for the deflator

So the equation for the deflator is

$$D_t = D_0 \exp\left\{\frac{1}{2\sigma^2}\left(\mu^2 - \left(r + \frac{\sigma^2}{2}\right)^2\right)t + \frac{1}{\sigma^2}\left(r - \mu - \frac{\sigma^2}{2}\right)X_t\right\}$$

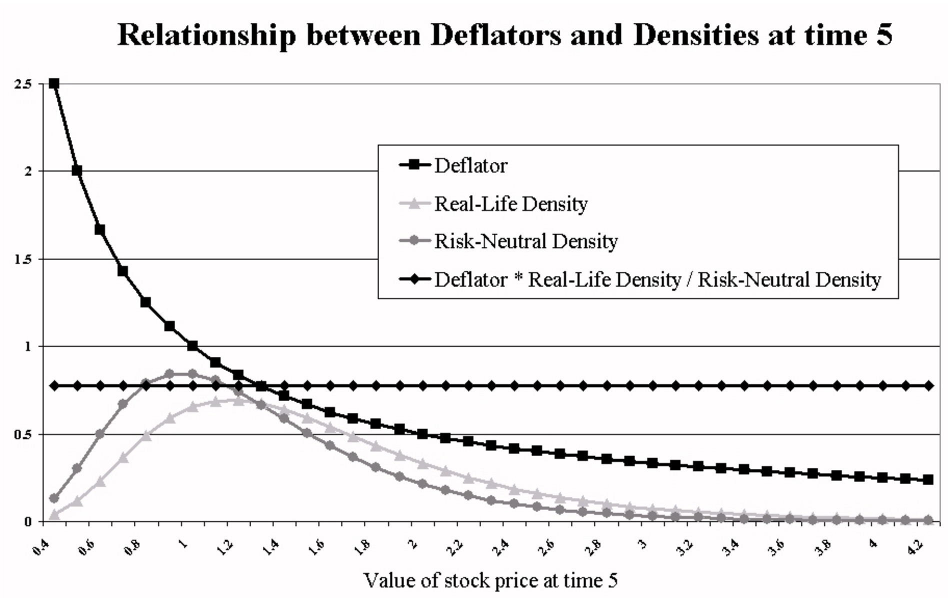
The equation for the deflator can also be written in terms of the stock price, using

$\frac{S_t}{S_0} = e^{X_t}$, taking $X_0=0$, $\alpha = \frac{1}{\sigma^2}\left(\mu + \frac{\sigma^2}{2} - r\right)$

$$D_t = D_0\left(\frac{S_t}{S_0}\right)^{-\alpha} \exp\left\{-r(1-\alpha)t + \frac{1}{2}\sigma^2\alpha(\alpha-1)t\right\}$$

C.6 Illustration of deflator

The graph below shows the value of the deflator at time 5 in comparison to the real-life density function of the stock price and the risk-neutral density of the stock price at this time.



The parameters used are

r=0.05

μ=0.07

σ=0.2

S_0=1

The risk-neutral density of the stock price is given by

$$S_t = S_0 e^{X_t},\ X_t \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$$

We can see that the dotted line is precisely the discount factor over the five years at the risk free rate (0.779). This shows that the deflators are equivalent to a change of measure to the risk-neutral measure.

Now we shall value a call option to show mathematically that these two methods are equivalent.

C.7 Call option

Let us consider a call option on stock S, with strike price K, at time T, where time now is t. Payout at time T is:

$$\max(S_T - K, 0)$$

C.8 Valuing the call option using deflators

Using the properties of a deflator, we can price a call option by considering the present value of the cashflow from the option. If the value of the call option now is C, then C must satisfy

$$D_t C = E_t \left[D_T \max(S_T - K, 0) \right]$$

Using the deflator for the Black Scholes Model, to generate the value of the call option under Black Scholes assumptions, we can write the equation above as

$$C = e^{aT - at - bX_t} E\left[e^{bX_T} \max\left(S_t e^{(X_T - X_t)} - K, 0 \right) X_t \right]$$

Evaluating this equation and substituting for a and b gives

$$C = S_t \Phi\left(\frac{\ln\left(\frac{S_t}{K}\right) + (T-t)\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \right) - K \exp\{-r(T-t)\} \Phi\left(\frac{\ln\left(\frac{S_t}{K}\right) + (T-t)\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \right)$$

This is precisely the formula that is derived from the Black Scholes Model using the risk-free density, without the use of deflators.

Appendix D Risk-neutral measure

This paper has focussed on the use of state-price deflators in actuarial modelling. This is related to the common technique of 'risk-neutral' valuation used by financial economists to price financial derivatives (for example). This appendix describes that technique, as a specific example of the use of deflators; but one that requires a 'change of measure'.

Being several mathematical steps too far for the intended readership of this paper, this topic has therefore been relegated to an appendix, despite its obvious importance. No part of the paper relies on the material presented here, which can all be gleaned from several standard texts, for example Baxter and Rennie (1996). The first three sections of this appendix present some brief mathematical preliminaries for what follows.

D.1 Change of measure

Investors have been assumed, in this paper, to know in advance some distribution of possible prices for the assets in which they are interested. The no-arbitrage assumption then led to these distributions often being redundant when it came to pricing assets.

It is then natural to wonder if the original distribution may not have been the best choice. Since the probabilities of future outcomes are always to some extent arbitrary, would a different choice have made the calculations easier? The answer is often an overwhelming *yes*.

Denote the original ('real-world' or 'best-estimate') measure by **P**. A 'measure' assigns probabilities to all possible events, (for example an 'event' might be equity returns being at least 10%pa over the next 3 years). A measure **Q** is *equivalent* to **P** if it agrees with **P** about which events have nil probability (for example the possibility of a share index having negative value). If this condition holds, then **Q** is related to **P** via a random variable R that measures the difference in weight the two measures give to each event: for any event A, with indicator function 1_A (ie a function which takes the value 1 if A occurs and 0 otherwise),

$$\mathbf{Q}(A) = \mathbf{E}_{\mathbf{Q}}(1_A) = \mathbf{E}_{\mathbf{P}}(R1_A).$$

R, which is called the Radon-Nikodym derivative, is usually denoted $d\mathbf{Q}/d\mathbf{P}$. When a time parameter is introduced, R becomes a process (via $R_t = \mathbf{E}_{\mathbf{P},t}(R)$), which satisfies, for any random variable X_T known by time T:

$$R_t \mathbf{E}_{\mathbf{Q},t}(X_T) = \mathbf{E}_{\mathbf{P},t}(R_T X_T).$$

R is a positive process, whose expected value under **P** is always one (since this is just the total probability, when measured under **Q**). Conversely, given any positive, unit-expectation martingale process, a new measure **Q** is obtained through the above construction (subject to technical constraints, see Duffie (1996) for details).

D.2 Change of numeraire

A numeraire is just a unit of account. The numeraire normally used (implicitly) is cash in a particular currency. Prices could equally well be expressed relative to any 'currency', as long as it is always positive. If C_t is a suitable (positive) process then an asset price process X_t in the original (cash) numeraire becomes X_t/C_t under the new numeraire. If D_t is a state-price deflator for the model in the original numeraire, then $\hat{D}_t = D_t C_t$ is a deflator in the new one:

$$\hat{D}_t(X_t / C_t) = D_t X_t = \mathbf{E}_t(D_T X_T) = \mathbf{E}_t(\hat{D}_T(X_T / C_T))$$

D.3 Equivalent martingale measure

A measure **Q**, equivalent to the real-world measure **P**, is an 'equivalent martingale measure' (EMM) for the asset *X* if all assets become martingales under **Q** when *X* is used as the numeraire. That is:

$$\mathbf{E}_{\mathbf{Q},t}(Y_T / X_T) = Y_t / X_t$$

for any (non-dividend paying) asset *Y*.

The asset *X* normally used is the 'market account' *B*, which increases at the risk-free rate of interest: $B_t = \exp\left(\int_0^t r_u du\right)$. This is often called the risk-free asset.

An equivalent description of the EMM for *X* is that it is a measure in which X^{-1} becomes a state-price deflator for the model.

D.4 Using deflators to change measure

The key property of a Radon-Nikodym derivative *R* described above is reminiscent of the defining property of a deflator *D*:

$$R_t \mathbf{E}_{\mathbf{Q},t}(Y_T) = \mathbf{E}_{\mathbf{P},t}(R_T Y_T)\,, \quad D_t X_t = \mathbf{E}_t(D_T X_T)\,.$$

These can be brought together by a choice of positive price process as numeraire: suppose *X* is a non dividend-paying asset. Then $R_t = D_t X_t / D_0 X_0$ is a unit-expectation positive martingale process, which can be used to produce a new measure, **Q** from the original **P**.

Using the two equations above, it is now easy to check that **Q** is an equivalent martingale measure for the asset *X*:

$$\mathbf{E}_{\mathbf{Q},t}(Y_T / X_T) = \mathbf{E}_{\mathbf{P},t}(D_T Y_T) / D_t X_t = Y_t / X_t\,.$$

D.5 Risk-neutral measure

The 'risk-neutral' measure is obtained by applying this procedure when the chosen numeraire is the risk-free asset B_t. A measure is obtained in which X_t/B_t is a martingale for all non dividend-paying assets X. In particular, the expected return on *X/B* is nil, which gives rise to the commonly known feature that, under this measure, all assets have the same expected return (namely that of *B*, the risk-free rate).

The really useful thing about this is that the valuation/pricing problem is now very much simplified: the model's deflator is B^{-1}, and *B* determines the returns of the asset distributions required for the model. For example, in a model that depends on (normal) diffusions, it is therefore only the instantaneous variances that need to be estimated, which is a good deal easier than the instantaneous means. (In principle at least, the instantaneous variance can be estimated from the recent history of the price process whereas to estimate the mean requires a long time-period, over which the implicit assumption of a constant mean may be less plausible. For further details, see for example Campbell et al (1997).)

These simplifications make risk-neutral valuation the method of choice for financial economists. The risk-neutral measure is often assumed without a second thought.

D.6 Harrison and Pliska

The key theorem (on which our whole paper relies) states that a deflator exists if and only if there is no arbitrage. This is alternatively stated as the existence of an equivalent martingale measure (with respect to the risk-free asset *B*) if and only if there is no arbitrage. (An EMM produces a state-price deflator, since, as described above, the Radon-Nikodym derivative with respect to the original measure is, after dividing by B, the same as a deflator.) Harrison and Pliska (1981) first stated this, for discrete models. Duffie (1996) gives the proof and references for numerous cases.

D.7 Real-world valuation

The problem with risk-neutral valuation (and in many applications, this is a mild problem) is that the symmetry of a model can be broken. In a model with multiple currencies, a particular currency, or numeraire, has to be chosen as the one in which risk-neutral distributions will be calculated. A different currency choice will give rise to a different set of distributions (though all related through a change of measure process related to the exchange rate).

Flesaker and Hughston (1997) argues against the symmetry-breaking in this situation, and recommends the deflator formalism, in the real-world measure, which this paper outlines. A new generation of models is now being built which have deflators at their heart, and which keep the risk-neutral measure at one remove (but, because of the presence of the deflators, readily available for particular pricing problems if required). We express the hope the actuarial profession will be able to engage in this process too. For example, Smith and Speed (1998) described one line of attack, which has since produced a full-scale model along these lines.

Appendix E Complete and incomplete markets

E.1 Introduction

A financial model is often described by specifying distributions for some basic set of assets. For example, risk premia and covariance matrices might be given for a number of equity and bond markets. In such a model, cashflows can be considered which are not the payoffs from these basic assets, but which depend in some way on these assets (for example a cashflow which pays the lower of £100 or the value of some specified portfolio might represent the value of a pension promise).

When all such arbitrary cashflows can be replicated by constructing a suitable (possibly dynamically-varying) portfolio from the initial assets, this produces a valuation of the cashflows, and the model (or the market generated by the basic assets) is called *complete*. In real life, markets are not usually complete.

In the case of a complete market, state prices can be calculated (since the payoffs from the state-price securities can be replicated) and so a (unique) deflator produced. In the incomplete case, the construction of a deflator is much less clear: does a deflator exist at all, and if so are some deflators better than others? This appendix briefly discusses these issues.

E.2 Deflator existence

The existence of a deflator for a model relies intimately on the model being free of arbitrage. Necessity of this condition was mentioned before: if an arbitrage existed, then unique values could not exist, and a deflator could not make sense. Sufficiency of the condition is a trickier matter, involving some mathematics, and in any generality requires the introduction of technical conditions on the model. This section discusses this mathematics.

Since a deflator for a model can be constructed from deflators for each time period, it suffices to focus on a single-period model, at least in the discrete case. (In the continuous-time case, an infinitesimal time period is the equivalent viewpoint.)

Duffie (1996) produces a geometric argument for existence which illustrates how the arbitrage-free condition plays a role: the following gives a flavour of this. Suppose the single-period model has N future states; then a cashflow we wish to value is specified by an N-vector of payoffs. Consider two subsets of this N-dimensional vector space:

(A) the payoffs from portfolios with zero current value

(B) cashflows whose payoffs are always non-negative

A is a vector subspace which, if the model contains no free lunches, meets the cone B only at the zero cashflow. There is therefore a vector within B, which is perpendicular to A. (For example, this vector is a multiple of the normal to a 'separating hyperplane', containing A and tangent to B, whose existence is the subject of a standard theorem.) The components of this vector are then (up to a multiple) state prices for the model. Different hyperplane choices correspond to different state price choices. In the complete case, A is already a hyperplane, and so the state prices are unique.

Extending A to a hyperplane translates into an alternative (algebraic) approach to constructing a deflator: the space of cashflows which can be replicated using the existing assets can be extended one dimension at a time by assigning values to assets outside this space. The arbitrage-free condition on the existing model is then just sufficient to show that values can be assigned which make the resulting extension also arbitrage-free.

E.3 Deflator construction

The constructive method of the last paragraph quickly becomes impractical: consider a typical discrete-time model that requires 1000 simulations for each time period to give a decent level of accuracy. Constructing a deflator for a 10 period projection would require producing simulations for 1000 states at each time and starting state at that time, and then sewing each of these into a full deflator process. This is not reasonable, so an alternative is needed.

One answer is to impose conditions on the functional form of the deflator: the most obvious is to use some form of time-homogeneity. For example, the form of the deflator used in Appendix C could be motivated by the need to have the model behave the same way in each time period.

Alternatively, or in addition, a deflator can be derived via the utility approach described in section 5. After specifying a utility function, an optimal portfolio can be derived, and then the deflator produced as the marginal utility of this optimal strategy.

In either case, the best approach must be to do some analysis of the model up-front, and then produce deflator values at the same time as the asset simulations. The question of incompleteness is then irrelevant: all cashflows can be valued, and no set of assets need play a distinguished role. In the 10-period projection above, 10,000 simulations can be generated rather than 10^{30}, and the model remains tractable.