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Applied Section

Solvency II: stability problems with the SCR aggregation formula

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One of the central issues in the Solvency II process will be an appropriate calculation of the Solvency Capital Requirement (SCR). This is the economic capital that an insurance company must hold in order to guarantee a one-year ruin probability of at most 0.5%. In the so-called standard formula, the overall SCR is calculated from individual SCRs in a particular way that imitates the calculation of the standard deviation for a sum of normally distributed risks (SCR aggregation formula). However, in order to cope with skewness in the individual risk distributions, this formula must be calibrated accordingly in order to maintain the prescribed level of confidence. In this paper, we want to show that the methods proposed and discussed thus far still show stability problems within the general setup.

Keywords: Solvency II; SCR; Skewness; Symmetry; Calibration; Copulas

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1. Introduction

In the European Solvency II project, one of the major topics is the appropriate determination of the so-called Solvency Capital Requirement (SCR).

> The SCR corresponds to the economic capital a (re)insurance undertaking needs to hold in order to limit the probability of ruin to 0.5%, i.e. ruin would occur once every 200 years… The SCR is calculated using Value-at-Risk techniques, either in accordance with the standard formula, or using an internal model: all potential losses, including adverse revaluation of assets and liabilities, over the next 12 months are to be assessed. The SCR reflects the true risk profile of the undertaking, taking account of all quantifiable risks, as well as the net impact of risk mitigation techniques.[1]

Further comments on this topic can be found in Ronkainen *et al*. (2007) and Sandström (2007). A suggestion for the 'standard formula' (and, in part, also for internal models) is to aggregate the capital requirements SCR_i of n different lines of business (lobs) to an overall SCR by the so called 'square root formula'[2]

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[1] Quoted from Proposal for a Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance – Solvency II, Commission Of The European Communities, Brussels, 10 July 2007.

[2] The notation used here and in the sequel makes it clearer that the SCRs also depend on the ruin probability α that is presently set to $\alpha = 0.005$ by the European Commission.

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$$\mathrm{SCR}(\alpha)=\sqrt{\sum_{i=1}^{n}\mathrm{SCR}_i^2(\alpha)+2\sum_{i<j}\rho_{ij}\mathrm{SCR}_i(\alpha)\ \mathrm{SCR}_j(\alpha)} \tag{1}$$

where ρ_{ij} denote the linear correlation coefficients between the risks of the different lobs (see e.g. Sandström (2007) p. 127, formula (1b), or Sandström (2006) Chapter 9). This formula is correct in the sense that the prescribed overall confidence level of 99.5% (or, more generally, $1-\alpha$ for a given ruin probability $0<\alpha<1$) is maintained within the world of *normal* risk distributions, for the Value-at-Risk (VaR) as well as for the Tail-Value-at-Risk (TVaR) as underlying risk measures (see e.g. Koryciorz (2004) Chapter 2). In particular, we have:

$$\mathrm{VaR}_i(\alpha)=\mu_i+\kappa(\alpha)\cdot\sigma_i,\quad \mathrm{TVaR}_i(\alpha)=\mu_i+\frac{e^{-((\kappa(\alpha))^2/2)}}{\sqrt{2\pi}\alpha}\sigma_i=\mu_i+\tau(\alpha)\cdot\sigma_i \tag{2}$$

where $\kappa(\alpha)=\Phi^{-1}(1-\alpha)$ denotes the $1-\alpha$-quantile of the standard normal distribution with cumulative distribution function (cdf)

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{u^2}{2}\right)du,\quad x\in\mathbb{R}. \tag{3}$$

Here, $\mu_i\in\mathbb{R}$ denotes the expectation of the risk of lob i, and $\sigma_i\geq 0$ its corresponding standard deviation. From the above formulas it follows, according to Sandström (2006, p. 214), that in the world of normal distributions, the capital requirements SCR_i are given by appropriate multiples of the standard deviations, as differences of the risk measure and the individual expectation:

$$\mathrm{SCR}_i(\alpha)=\delta(\alpha)\cdot\sigma_i\quad\text{with}\quad \delta(\alpha)=\begin{cases}\kappa(\alpha) & \text{for VaR}(\alpha)\\ \tau(\alpha) & \text{for TVaR}(\alpha)\end{cases}. \tag{4}$$

Note that the two types of SCR discussed in Sandström (2006), based on the standard deviation principle as well as on the VaR/TVaR, coincide in the normal world. There is, however, a major problem arising if the risks of the individual lobs are not normally distributed. This affects the square root formula in two ways: firstly, in a general misspecification of the overall SCR even if the risks are independent (and hence all ρ_{ij} are zero); secondly, in a misspecification of the overall SCR if the risks are uncorrelated but dependent. The first point has already been addressed in several publications before (see e.g. Sandström (2007) and references therein, or Sandström (2006) Chapter 9), considering certain calibration techniques that are based on the Cornish–Fisher expansion for the risk measures above and the skewness of the underlying risks. Thus far, the second point, to our knowledge, has not found that kind of attention.

In this paper, we firstly want to demonstrate that even if the individual SCRs (of the second type, based on VaR as the underlying risk measure) are exactly known, and the resulting aggregate risk distribution is symmetric (and hence no calibrations are necessary), the square root formula can severely underestimate the true SCR. Second, we show that under a certain kind of dependence structure (so called grid type copulas), it

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is easy to construct cases of uncorrelated risks for which the square root formula fails in a similar manner.

2. Aggregated SCRs for independent Beta distributed risks

In this section, we investigate the behaviour of the square root formula for certain independent Beta distributed risks. This class of risk distributions, for example, is used in certain geophysical modelling software tools in connection with ‘secondary uncertainties’ (for a survey of this topic, see Grossi & Kunreuther (2005) or Straßburger (2006)). Beta distributions are an appropriate modelling tool if the possible damages from the risk under consideration are bounded above, for example, by the sum insured in a windstorm portfolio. A further advantage of this family is the possibility of calculating explicitly the convolution density and cdf for integer values of the parameters that make a mathematical analysis easier.

In what follows, we consider Beta distributed risks X with densities

$$f_X(x;\ n,m) = (n+m+1)\binom{n+m}{n}x^n(1-x)^m, \quad 0 \le x \le 1; \quad n,m \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}. \tag{5}$$

$F_X(x;n,m)$ denotes the corresponding cdf. Since the densities in Eq. (5) are polynomials, the convolution density for the aggregated risk $S = X + Y$ for independent summands with parameters n_1, m_1, n_2, m_2 is piecewise polynomial, and can easily be calculated via the following formula:

$$f_S(x;\ n_1,m_1,n_2,m_2) = \begin{cases} \int_0^x f_X(y;\ n_1,m_1)\cdot f_Y(x-y;\ n_2,m_2)dy, & 0 \le x \le 1 \\ \int_{x-1}^1 f_X(y;\ n_1,m_1)\cdot f_Y(x-y;\ n_2,m_2)dy, & 1 \le x \le 2 \end{cases}. \tag{6}$$

Likewise, the cdf F_S for the aggregated risk S is also a piecewise polynomial and can be calculated via

$$F_S(x;\ n_1,m_1,n_2,m_2) = \begin{cases} \int_0^x f_S(u;\ n_1,m_1,n_2,m_2)du, & 0 \le x \le 1 \\ F_S(1) + \int_1^x f_S(u;\ n_1,m_1,n_2,m_2)du, & 1 \le x \le 2 \end{cases}. \tag{7}$$

The Appendix contains some explicit expressions for the cdfs from a selection of parameters that will be considered in more detail in the course of the paper.

With the help of these results, it is possible to calculate (in the final step numerically) the true SCRs for the individual risks as well as for the aggregated risk. Note that for a risk X with density given in Eq. (5), we have $E(X) = (n+1)/(n+m+2)$, and, hence

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$$\mathrm{SCR}_X(\alpha) = \mathrm{VaR}_X(\alpha) - E(X) = F_X^{-1}(1-\alpha;\ n,m) - \frac{n+1}{n+m+2}. \tag{8}$$

Table 1 shows some selected results.

Table 1. Solvency capital requirements for *individual* risk distributions.

(n,m)	(0,0)	(1,0)	(2,0)	(3,0)	(0,1)	(0,2)	(0,3)	(1,2)	(1,3)	(1,4)	(2,1)	(3,1)	(4,1)
SCR_X (0.01)	0.4900	0.3283	0.2466	0.1974	0.5666	0.5345	0.4837	0.4900	0.3283	0.2466	0.1974	0.5666	0.5345
SCR_X (0.005)	0.4950	0.3308	0.2483	0.1987	0.5959	0.5790	0.5340	0.4950	0.3308	0.2483	0.1987	0.5959	0.5790

Table 2 contains the true SCR values for the aggregated risk $S = X + Y$, with independent Beta distributed risks X and Y, in comparison to the values $\mathrm{SCR}^{\sqrt{}}$ obtained via the square root formula (1).

It is evident that the square root formula in most cases significantly underestimates the true SCR, particularly in cases where the distribution of the aggregate risk is skewed to the *left* (lines 2–4); but this also holds true in some cases where the distribution of the aggregate risk is symmetric. Lines 5 and 6 show cases where the distributions of the aggregate risk are skewed to the right, yet the square root formula produces deviations in both directions! Interestingly, a major deviation also occurs if the individual risks, and, hence, also the aggregated risk are symmetrically distributed (line 1). However, there are also symmetric cases where the square root formula overestimates the true SCR, as can be seen from the last line in Table 2.

A closer analysis shows that for a special case of symmetry, namely for parameters of the form $(n_1,m_1,n_2,m_2) = (0,n,n,0)$ we have

$$F_S(x;\ 0,n,n,0) \sim \frac{n+1}{n+2} x^{n+2} \tag{9}$$

for small values of x which, by symmetry, leads to

$$\mathrm{SCR}_S(\alpha;\ n) \sim \mathrm{SCR}_{\mathrm{app}}(\alpha;\ n) = 1 - \left(\frac{n+2}{n+1}\alpha\right)^{1/(n+2)} \tag{10}$$

for small values of α. Table 3 shows the corresponding values for $\alpha = 0.005$.

On the other hand, an asymptotic expansion of $\mathrm{SCR}^{\sqrt{}}(\alpha; n)$ for this case shows that

$$\mathrm{SCR}^{\sqrt{}}(\alpha; n) = \sqrt{\left((1-\alpha)^{1/(n+1)} - \frac{n+1}{n+2}\right)^2 + \left(\frac{n+1}{n+2} - \alpha^{1/(n+1)}\right)^2}$$

$$\sim \mathrm{SCR}^{\sqrt{}}_{\mathrm{app}}(\alpha; n) = \frac{\sqrt{(n+1)^2+1}}{n+2} - \frac{n+1}{\sqrt{(n+1)^2+1}}\alpha^{1/(n+1)} \tag{11}$$

for small values of α. Table 4 shows the corresponding values for $\alpha = 0.005$.

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Table 2. Solvency capital requirements for *aggregate* risk distributions.

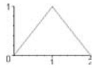
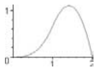
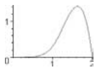
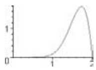
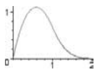
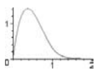
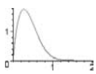
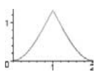
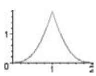
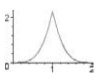
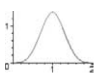
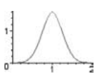
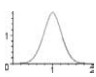
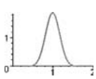
(n_1,m_1,n_2,m_2)	Density f_s	SCR_S (0.01)	$SCR^{\surd}$(0.01)	Error in%	SCR_S (0.005)	$SCR^{\surd}$(0.005)	Error in%
Line 1: (0,0,0,0)		0.8585	0.6929	−19.28	0.9000	0.7000	−22.21
Line 2: (1,0,1,0)		0.5942	0.4643	−21.85	0.6158	0.4678	−24.02
Line 3: (2,0,2,0)		0.4512	0.3488	−22.70	0.4658	0.3511	−24.61
Line 4: (3,0,3,0)		0.3633	0.2792	−23.12	0.3743	0.2810	−24.91
Line 5: (0,1,0,1)		0.8384	0.8013	−4.41	0.9171	0.8428	−8.10
Line 6: (0,2,0,2)		0.7352	0.7559	2.81	0.8187	0.8188	0.01
Line 7: (0,3,0,3)		0.6436	0.6841	6.30	0.7229	0.7553	4.47
Line 8: (0,1,1,0)		0.7479	0.6549	−12.44	0.8008	0.6816	−14.89
Line 9: (0,2,2,0)		0.6478	0.5887	−9.13	0.7056	0.6300	−10.71
Line 10: (0,3,3,0)		0.5656	0.5225	−7.61	0.6239	0.5698	−8.66
Line 11: (1,2,2,1)		0.6331	0.5822	−8.04	0.6851	0.6136	−10.44
Line 12: (1,3,3,1)		0.5758	0.5367	−6.79	0.6276	0.5729	−8.70
Line 13: (1,4,4,1)		0.5252	0.4933	−6.06	0.5760	0.5321	−7.62
...	...	...	...	...	...	...	...
Line 14: (4,8,8,4)		0.3643	0.3910	7.33	0.3816	0.4251	11.41

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Table 3. Corresponding values for $\alpha = 0.005$.

n	0	1	2	3
$SCR_S\ (\alpha;n)$	0.9000	0.8008	0.7056	0.6239
$SCR_{app}\ (\alpha;n)$	0.9000	0.8042	0.7142	0.6376

Table 4. Corresponding values for $\alpha = 0.005$.

n	0	1	2	3
$SCR^{\surd}(\alpha;n)$	0.7000	0.6816	0.6300	0.5698
$SCR^{\surd}_{app}(\alpha;n)$	0.7035	0.6821	0.6283	0.5666

Some further analysis shows that for large values of n, we obtain

$$\lim_{n\to\infty} \frac{SCR^{\surd}(\alpha;n)}{SCR_{app}(\alpha;n)} = L(\alpha) = -\frac{\sqrt{(1+\ln\alpha)^2 + (1+\ln(1-\alpha))^2}}{\ln\alpha}. \tag{12}$$

This indicates that for the considered kind of symmetry, $(n_1,m_1,n_2,m_2) = (0,n,n,0)$, the square root formula produces SCR values that are systematically too low compared with the true SCR values for the aggregate risk (Figure 1).

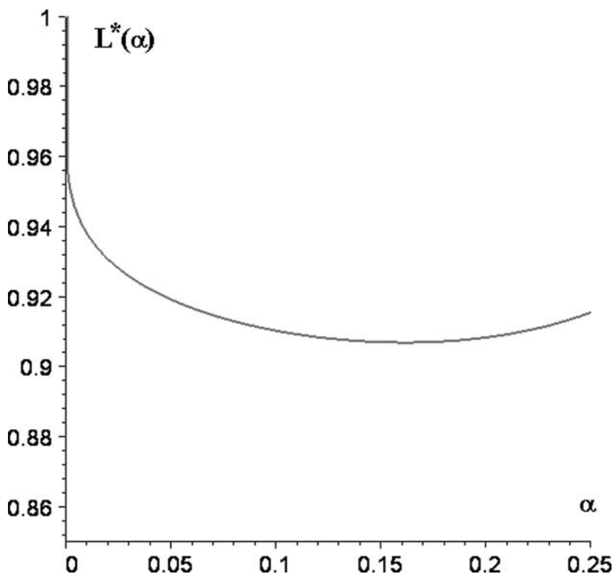

Figure 1.

3. Aggregated SCRs for uncorrelated risks

In this section, we investigate the behaviour of the square root formula for uncorrelated, but stochastically dependent risks. As a modelling tool, we use grid type copulas that were

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introduced in Straßburger & Pfeifer (2005) (see also Straßburger (2006)). Recall in brief that a copula C is a multivariate distribution function of a random vector that has continuous uniform margins. Its general importance is described in the following theorem.

Sklar's Theorem. *Let H denote a d-dimensional distribution function with margins $F_1,\ldots, F_d$. Then there exists a d-copula C such that for all $(x_1,\ldots,x_d)\in\mathbb{R}^d$,*

$$H(x_1,\ldots,x_d)=C(F_1(x_1),\ldots,F_d(x_d)). \tag{13}$$

If all the margins are continuous, then the copula is unique, and is determined uniquely on the ranges of the marginal distribution functions otherwise. Moreover, the converse of the above statement is also true. If we denote by $F_1^{-1},\ldots,F_d^{-1}$ the generalized inverses of the marginal distribution functions, then for every $(u_1,\ldots, u_d)$ in the unit d-cube,

$$C(u_1,\ldots,u_d)=H(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)). \tag{14}$$

Copulas can be estimated below and above by the so called Fréchet-Hoeffding bounds:

$$\begin{aligned}\mathcal{W}(u_1,\ldots,u_d)=\max(u_1+\cdots+u_d-d+1,0)&\leq C(u_1,\ldots,u_d)\\ &\leq\min(u_1,\ldots,u_d)=\mathcal{M}(u_1,\ldots,u_d).\end{aligned} \tag{15}$$

Note, however, that the lower Fréchet-Hoeffding bound is a copula only for $d=2$, while the upper bound is a copula for all $d\in\mathbb{N}$. In two dimensions, the pair $(U,1-U)$ has the lower Fréchet-Hoeffding bound as copula, while the d-dimensional random vector $\mathbf{U}=(U,\ldots,U)$ has the upper Fréchet-Hoeffding bound as copula; here U denotes a uniformly distributed random variable over the unit interval. For further detail on copulas, especially in connection with risk management, see McNeil *et al.* (2005).

A grid type copula is defined as:

Definition. Let d, $n\in\mathbb{N}$ and define intervals $I_{i_1,\ldots,i_d}(n):=\bigtimes_{j=1}^{d}\left(\frac{i_j-1}{n},\frac{i_j}{n}\right]$ for all possible choices $i_1,\ldots, i_d\in N_n:=\{1,\ldots, n\}$. If $a_{i_1,\ldots,i_d}(n)$ are non-negative real numbers with the property

$$\sum_{(i_1,\cdots,i_d)\in J(i_k)} a_{i_1,\cdots,i_d}(n)=\frac{1}{n} \tag{16}$$

for all $k\in\{1,\ldots,d\}$ and $i_k\in\{1,\ldots,n\}$, with $J(i_k):=\{(j_1,\ldots,j_n)\in N_n^d|j_k=i_k\}$, then the function $c_n:=n^d\sum_{(i_1,\ldots,i_d)\in N_n^d} a_{i_1,\ldots,i_d}(n)\,\mathbb{1}_{I_{i_1,\ldots,i_d}(n)}$ is the density of a d-dimensional copula, called *grid-type copula* with parameters $\{a_{i_1,\ldots,i_d}(n)|(i_1,\ldots,i_d)\in N_n^d\}$. Here, $\mathbb{1}_A$ denotes the indicator random variable of the event A, zas usual.

It is easy to see that in case of an absolutely continuous d-dimensional copula C, with continuous density

$$c(u_1,\ldots,u_d)=\frac{\partial^d}{\partial u_1\ldots\partial u_d}C(u_1,\ldots,u_d),\ (u_1,\ldots,u_d)\in(0,1)^d, \tag{17}$$

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c can be approximated arbitrarily close by a density of a grid-type copula. A this point, the classical *multivariate mean-value-theorem* of calculus tells us that we only have to choose

$$a_{i_1,\ldots,i_d}(n) := \int_{\frac{i_d-1}{n}}^{(i_d/n)} \cdots \int_{\frac{i_1-1}{n}}^{(i_1/n)} c(u_1,\ldots,u_d)du_1\ldots du_d, \quad i_1,\ldots,i_d \in N_n. \tag{18}$$

Another interpretation of grid type copulas is given by the observation that a random vector $\mathbf{U}=(U_1,\ldots,U_d)$ possesses a grid type copula type iff

$$P^{\mathbf{U}}(\bullet|\mathbf{U}\in I_{i_1,\ldots,i_d}(n)) = \mathcal{R}(I_{i_1,\ldots,i_d}(n)) \tag{19}$$

with $P(\mathbf{U}\in I_{i_1,\ldots,i_d}(n)) = a_{i_1,\ldots,i_d}(n)$, i.e. the conditional distribution of $\mathbf{U}$ given the hypercube $I_{i_1,\ldots,i_d}(n)$ is d-dimensional continuous uniform (denoted by $\mathcal{R}(\cdots)$).

A major advantage of grid type copulas is that they allow the explicit calculation of sums of *dependent* uniformly distributed random variables. This is essentially due to the following result.

Lemma. *Let $U_1,\ldots,U_d$ be independent standard uniformly distributed random variables and let f_d and F_d denote the density and cdf of $S_d := \Sigma_{i=1}^d U_i$, resp., for $d\in\mathbb{N}$. Then*

$$f_d(x) = \frac{1}{2(d-1)!}\sum_{k=0}^{d}(-1)^k\binom{d}{k}(x-k)^{d-1}\operatorname{sgn}(x-k)\mathbb{1}_{[0,d]}(x)$$
$$F_d(x) = \frac{1}{2d!}\sum_{k=0}^{d}(-1)^k\binom{d}{k}((-k)^d+(x-k)^d\operatorname{sgn}(x-k))\mathbb{1}_{[0,d]}(x)+\mathbb{1}_{(d,\infty]}(x) \tag{20}$$

for $x\in\mathbb{R}$. This follows the example from Uspensky (1937), Example 3, p. 277, who attributes this result already to Laplace.

Theorem. *Let $\mathbf{U}=(U_1,\ldots,U_d)$ be a random vector whose joint cdf is given by a grid-type copula with density $c_n := \sum_{(i_1,\ldots,i_d)\in N_n^d} a_{i_1,\ldots,i_d}(n)\mathbb{1}_{I_{i_1,\ldots,i_d}(n)}$. Then the density and cdf $\tilde{f}_d(n;\ \bullet)$ and $\tilde{F}_d(n;\ \bullet)$, resp., for the sum $S_d := \sum_{i=1}^d U_i$ is given by*

$$\begin{aligned}\tilde{f}_d(n;x) &= n\sum_{(i_1,\ldots,i_d)\in N_n^d} a_{i_1,\ldots,i_d}(n)\cdot f_d\left(nx+d-\sum_{j=1}^d i_j\right)\\ \tilde{F}_d(n;x) &= \sum_{(i_1,\ldots,i_d)\in N_n^d} a_{i_1,\ldots,i_d}(n)\cdot F_d\left(nx+d-\sum_{j=1}^d i_j\right)\end{aligned} \quad \text{for } x\in\mathbb{R}, \tag{21}$$

with f_d and F_d as defined in Eq. (20).

Example. Consider the weights $a_{ij}(n)$, $n=3$ for a copula density given in matrix form

$$A(3) = [a_{ij}(3)] = \begin{bmatrix} a & b & 1/3-a-b \\ c & 1-4a-2b-2c & -2/3+4a+2b+c \\ 1/3-a-c & -2/3+4a+b+2c & 2/3-3a-b-c \end{bmatrix} \tag{22}$$

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with suitable real numbers $a,b,c \in [0,1/3]$. It follows that the covariance of the corresponding random variables $(U_1,U_2)=(X,Y)$ is given by

$$E(XY)-E(X)E(Y)=\frac{1}{9}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}(3)(i-2)(j-2)=0,$$

i.e. the random variables (risks) X, Y are *uncorrelated* but in general *dependent* (unless $a=b=c=1/9$). If we denote $\gamma=(a,b,c)$ for short, the above theorem implies the following explicit representation of the cdf $\tilde{F}_2(3;\gamma;x)$ of the aggregated risk $S=X+Y$:

$$\tilde{F}_2(3;\gamma;x)=\begin{cases}0, & x\le 0\\ \frac{9a}{2}x^2, & 0\le x\le\frac{1}{3}\\ \frac{9}{2}(-a+[b+c])x^2+3(2a-[b+c])x+\frac{1}{2}(-2a+[b+c]), & \frac{1}{3}\le x\le\frac{2}{3}\\ \frac{5}{2}(3-18a-12[b+c])x^2+(-10+36a+27[b+c])x+\frac{1}{6}(20-66a-57[b+c]), & \frac{2}{3}\le x\le 1\\ \frac{9}{2}(-3+14a+6[b+c])x^2+(32-144a-63[b+c])x+\frac{1}{6}(-106+237a+213[b+c]), & 1\le x\le\frac{4}{3}\\ \frac{9}{2}(2-22a-4[b+c])x^2+(-28+156a-57[b+c])x+\frac{1}{6}(134-726a-267[b+c]), & \frac{4}{3}\le x\le\frac{5}{3}\\ \frac{3}{2}(-6+9a+3[b+c])x^2+3(4-6a-18[b+c])x+(-11+54a+18[b+c]), & \frac{5}{3}\le x\le 2\\ 1, & x\ge 2\end{cases} \quad (23)$$

Figure 2 shows visualizations of these cdfs, for various parameter choices.

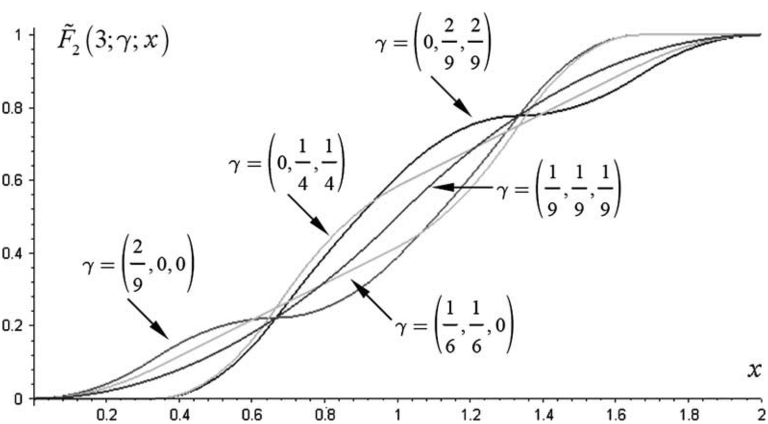

Figure 2.

From Eq. (23), we also obtain explicitly the corresponding quantile functions $Q_2(3;\gamma;\ \bullet\ )$ because only quadratic equations have to be solved for this purpose. The following formula shows the results for three selected parameter vectors γ in the range relevant for solvency purposes:

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$$Q_2(3;\gamma;1-\alpha)=\begin{cases}\begin{cases}2-\sqrt{\alpha}, & 0\le\alpha\le\frac{1}{9},\\ \frac{4}{3}+\frac{1}{3}\sqrt{2-9\alpha}, & \frac{1}{9}\le\alpha\le\frac{2}{9},\end{cases} & \gamma=\left(0,\frac{2}{9},\frac{2}{9}\right):\ \text{case [1]}\\ 2-\sqrt{2\alpha}, \quad 0\le\alpha\le\frac{2}{9}, & \gamma=\left(\frac{1}{9},\frac{1}{9},\frac{1}{9}\right):\ \text{case [2]}\\ \frac{5}{3}-\frac{1}{2}\sqrt{2\alpha}, \quad 0\le\alpha\le\frac{2}{9}, & \gamma=\left(\frac{2}{9},0,0\right):\ \text{case [3]}\end{cases} \tag{24}$$

Note that case [1] ('upper positive dependence') and case [3] ('upper negative dependence') correspond in a sense to the extreme cases under the above setup (see Figure 2). Case [2] corresponds to the independent case (cf. the first line in Table 2). Using Eq. (24), we can explicitly calculate the correct SCR values for the aggregate risk; in Table 5 these are compared with the former results of the square root formula (note that due to the zero covariance of X and Y, no correction term is necessary).

Table 5.

	SCR_S (α)			
α	Case [1]	Case [2]	Case [3]	$SCR^{\sqrt{}}(\alpha)$
0.01	0.9000	0.8585	0.5960	0.6929
Error in%	−23.01	−19.28	16.25	
0.005	0.9293	0.9000	0.6167	0.7000
Error in%	−24.67	−22.21	13.50	

It is perhaps surprising to see again a huge amount of instability in the square root formula here, from severe underestimation of the true SCR (as we have seen for left-skewed aggregated – independent – risks before), up to a significant overestimation of the true SCR. Note that the original risks as well as the aggregate risk have a symmetric distribution in all three cases.

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4. Further problems with the aggregation formula

In this section, we return to the setup of Section 2, but this time we allow for dependencies for the risks X and Y based on the upper and lower Fréchet-Hoeffding copulas as extreme cases of stochastic dependence. For simplicity, we concentrate on the symmetry case $(n_1,m_1,n_2,m_2)=(0,n,n,0)$ again. According to the comment after relation (15) and Sklar's Theorem, we can represent the risks X and Y as functions of just one uniformly distributed random variable U via

$$\begin{aligned} X &= F_X^{-1}(U;0,n) = U^{1/(n+1)} \\ Y &= F_X^{-1}(1-U;n,0) = 1-(1-U)^{1/(n+1)} \end{aligned} \tag{25}$$

for the lower Fréchet-Hoeffding copula, case [*l*] (this follows readily from Eq. (5); see also Appendix, Table 10) and

$$X = F_X^{-1}(U; 0, n) = U^{1/(n+1)}$$
$$Y = F_X^{-1}(U; n, 0) = 1 - U^{1/(n+1)} \tag{26}$$

for the upper Fréchet-Hoeffding copula, case [u]. Thus, the aggregated risk S has the representation

$$S = \begin{cases} 1 + U^{1/(n+1)} - (1-U)^{1/(n+1)} & \text{for case } [l] \\ 1 & \text{for case } [u], \end{cases} \tag{27}$$

which by monotonicity arguments, implies that the corresponding quantile function Q_S^* is similarly given by

$$Q_S^*(u; 0, n, n, 0) = 1 + u^{1/(n+1)} - (1-u)^{1/(n+1)} \text{ for } 0 \le u \le 1, \quad \text{for case } [l]. \tag{28}$$

As a simple consequence, the exact SCR for the aggregate risk for case [l] can be written as follows:

$$\mathrm{SCR}_S^*(\alpha; n) = (1-\alpha)^{1/(n+1)} - \alpha^{1/(n+1)} \tag{29}$$

while the adjusted SCR from the square root formula (1) is given by

$$\mathrm{SCR}^{\surd}(\alpha; n) = \sqrt{\left((1-\alpha)^{1/(n+1)} - \frac{n+1}{n+2}\right)^2 + \left(\frac{n+1}{n+2} - \alpha^{1/(n+1)}\right)^2 + 2\rho_n \left((1-\alpha)^{1/(n+1)} - \frac{n+1}{n+2}\right) \cdot \left(\frac{n+1}{n+2} - \alpha^{1/(n+1)}\right)} \tag{30}$$

for $0 \le \alpha \le 1/2$, where ρ_n denotes the correlation between $U^{1/(n+1)}$ and $1-(1-U)^{1/(n+1)}$. This, again, can be exactly calculated via the following intermediate formula, with $m = n+1$,

$$\int_0^1 u^{1/m}(1-(1-u)^{1/m})\,du = \frac{m}{m+1} - \int_0^1 (u(1-u))^{1/m}\,du = \frac{m}{m+1} - \frac{4^{1/m}\sqrt{\pi}\,\Gamma\left(1+\frac{1}{m}\right)}{2\,\Gamma\left(\frac{3}{2}+\frac{1}{m}\right)} \tag{31}$$

which gives

$$\rho_n = (n+1)(n+3) - \frac{(n+2)^2(n+3)}{n+1} \cdot \frac{4^{1/(n+1)}\sqrt{\pi}\,\Gamma\left(1+\frac{1}{n+1}\right)}{2\,\Gamma\left(\frac{3}{2}+\frac{1}{n+1}\right)}. \tag{32}$$

Note that $\lim_{n\to\infty} \rho_n = (\pi^2/6) - 1 = 0.6449\ldots$ Table 6 shows some numerical results.

Table 6. Numerical results.

	$\mathrm{SCR}_S^*(\alpha; n)$				$\mathrm{SCR}^{\surd}(\alpha; n)$			
α	$n=0$	$n=1$	$n=2$	$n=3$	$n=0$	$n=1$	$n=2$	$n=3$
0.01	0.9800	0.8949	0.7812	0.6812	0.9800	0.8806	0.7584	0.6561
Error in%					0.00	−1.60	−2.91	−3.68
0.005	0.9900	0.9267	0.8273	0.7328	0.9900	0.9120	0.8038	0.7068
Error in%					0.00	−1.58	−2.83	−3.54

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The asymptotic error for $n\to\infty$ is -6.19% for $\alpha=0.01$ and -5.57% for $\alpha=0.005$. This again indicates that the square root formula systematically underestimates the required SCR for this symmetry case, even with the proper correction term for correlation. Figure 3 shows the asymptotic ratio $\mathrm{L}^*(\alpha)=\lim_{n\to\infty}((\mathrm{SCR}^{\surd}(\alpha;n))/(\mathrm{SCR}_S^*(\alpha;n)))$ for $0\le\alpha\le.25$.

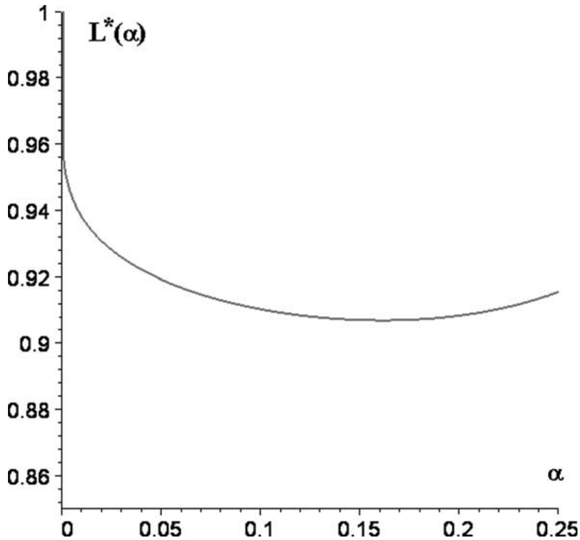

Figure 3.

The explicit form of this limit function is given by

$$\mathrm{L}^*(\alpha)=\frac{\sqrt{9(A(\alpha)+B(\alpha))^2+(36-3\pi^2)(A(\alpha)+B(\alpha))-3\pi^2A(\alpha)B(\alpha)+(36-3\pi^2)}}{3(A(\alpha)-B(\alpha))} \quad (33)$$

for $0\le\alpha\le1/2$, with $A(\alpha)=\ln(1-\alpha)$, $B(\alpha)=\ln\alpha$. Note that $\lim_{\alpha\downarrow0}\mathrm{L}^*(\alpha)=1$.

For case [u], it is easy to see that because S is a constant, the true SCR is zero, while now

$$\mathrm{SCR}^{\surd}(\alpha;n)=\sqrt{\left((1-\alpha)^{1/(n+1)}-\frac{n+1}{n+2}\right)^2+\left(\frac{n+1}{n+2}-\alpha^{1/(n+1)}\right)^2-2\rho_n\left((1-\alpha)^{1/(n+1)}-\frac{n+1}{n+2}\right)\cdot\left(\frac{n+1}{n+2}-\alpha^{1/(n+1)}\right)} \quad (34)$$

for $0\le\alpha\le1/2$ which is strictly positive for all n, with limit zero for $n\to\infty$ (Table 7). (Note that the correlation ρ_n from Eq. (32) changes its sign here.)

Table 7.

	$\mathrm{SCR}^{\surd}(\alpha;n)$							
α	$n=0$	$n=1$	$n=2$	$n=3$	$n=10$	$n=20$	$n=50$	$n=100$
0.01	0	0.2869	0.3434	0.3399	0.2074	0.1249	0.0563	0.0293
0.005	0	0.3119	0.3842	0.3870	0.2460	0.1501	0.0682	0.0356

The correlation adjusted square root formula hence significantly overestimates the true SCR, except for the trivial case $n=0$.

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5. Discussion

The foregoing analysis clearly shows that necessary calibrations of the standard SCR aggregation formula based on skewness and/or correlation alone cannot be sufficient for general purposes. For the class of risk distributions considered above, the square root formula tends to underestimate the true aggregate SCR considerably, for both kinds of skewness, although in some cases also the converse is true. Table 2 shows examples where the square root formula overestimates the true SCR even in cases of skewness to the right! This seems to be a general drawback of the standard deviation oriented SCR aggregation formula outside the world of *normal* or, more generally, *elliptically contoured* risk distributions. In our opinion, the general implementation of such a rule in a European standard formula should be performed only after a very thorough market-wide investigation of the type and shape of risk distributions that occur in practice. Otherwise, there is a danger that companies that use more sophisticated internal models are 'punished' by higher SCRs in comparison with those companies that only use a standard approach.

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From a mathematical point of view, the only reasonable 'all-purpose' calibration seems to be the application of the maximum possible value 1 for the correlations in the square root formula, which is equivalent to the additivity rule for aggregate SCRs, i.e.

$$\mathrm{SCR} = \sqrt{\sum_{i=1}^{n} \mathrm{SCR}_i^2 + 2\sum_{i<j} \mathrm{SCR}_i\ \mathrm{SCR}_j} = \sqrt{\left(\sum_{i=1}^{n} \mathrm{SCR}_i\right)^2} = \sum_{i=1}^{n} \mathrm{SCR}_i. \tag{35}$$

This would at least be generally consistent with the use of coherent (in particular, sub-additive) risk measures R for the calculation of the individual SCRs as

$$\mathrm{SCR}_i = R(X_i) - E(X_i) \tag{36}$$

where X_i denotes the risk pertaining to lob i because of the inequality

$$\mathrm{SCR}_{\text{total}} = R\left(\sum_{i=1}^{n} X_i\right) - E\left(\sum_{i=1}^{n} X_i\right) \le \sum_{i=1}^{n} R(X_i) - \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \mathrm{SCR}_i. \tag{37}$$

Formula (35) would hence produce a value that is generally sufficiently large to maintain the prescribed 99.5% confidence level. Although VaR is not coherent in all cases (see e.g. the discussion in McNeil *et al.* (2005), Chapter 6; or Straßburger (2006), Chapter 7), there are certainly more situations in which formula (35) provides a sufficiently large SCR on the VaR basis compared with formula (1). This holds at least true for all of the examples considered in this paper. For instance, the modified Table 2 reads (with $\mathrm{SCR}^{+}(\alpha)$ denoting the SCR according to rule (35)) as Table 8.

As is clearly seen, the overestimation of the true SCR is moderate for left skewed risk distributions (where formula (1) produces a severe underestimation), but certainly unacceptably high for right skewed distributions. Similarly, the modified Table 5 reads as Table 9.

Table 8.

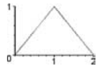
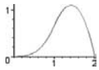
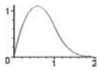
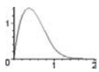
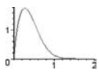
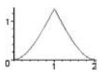
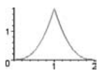
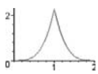
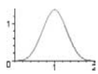
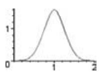
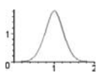
(n_1,m_1,n_2,m_2)	Density f_s	SCR_S (0.01)	SCR^+ (0.01)	Error in %	SCR_S (0.005)	SCR^+ (0.005)	Error in %
Line 1: (0,0,0,0)		0.8585	0.9800	14.15	0.9000	0.9900	10.00
Line 2: (1,0,1,0)		0.5942	0.6566	10.50	0.6158	0.6616	7.44
Line 3: (2,0,2,0)		0.4512	0.4932	9.31	0.4658	0.4966	6.61
Line 4: (3,0,3,0)		0.3633	0.3948	8.67	0.3743	0.3974	6.17
Line 5: (0,1,0,1)		0.8384	1.1332	35.16	0.9171	1.1918	29.95
Line 6: (0,2,0,2)		0.7352	1.0690	45.40	0.8187	1.1580	41.44
Line 7: (0,3,0,3)		0.6436	0.9674	50.31	0.7229	1.0680	47.74
Line 8: (0,1,1,0)		0.7479	0.8949	19.66	0.8008	0.9267	15.72
Line 9: (0,2,2,0)		0.6478	0.7811	20.58	0.7056	0.8273	17.25
Line 10: (0,3,3,0)		0.5656	0.6811	20.42	0.6239	0.7327	17.44
Line 11: (1,2,2,1)		0.6331	0.8171	29.06	0.6851	0.8596	25.47
Line 12: (1,3,3,1)		0.5758	0.7451	29.40	0.6276	0.7919	26.18
Line 13: (1,4,4,1)		0.5252	0.6788	29.25	0.5760	0.7271	26.23

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Table 9.

	$SCR_S(\alpha)$			
α	Case [1]	Case [2]	Case [3]	$SCR^+(\alpha)$
0.01	0.9000	0.8585	0.5960	0.9800
Error in%	8.16	12.40	39.18	
0.005	0.9293	0.9000	0.6167	0.9900
Error in%	6.13	9.09	37.71	

The same effect as before is visible here: the overestimation error decreases for 'upper positively' dependent risks (case [1]), while the converse is true for 'upper negatively' dependent risks (case [3]).

Finally, it should be noted that comparable results to those in Sections 2–5 hold true under the (throughout coherent) risk measure TVaR (see e.g. Straßburger (2006), Chapter 7).

A pragmatic way out of the problems outlined so far does not seem to be easy; a solution might be to allow the classical formula (1) only for certain classes of risk distributions (or lobs) where such severe misspecifications typically do nor occur, while formula (35) should be applied in all other cases.

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Appendix

Table 10 shows the expanded cdfs for the *individual* risks with density given by Eq. (5), for some selected parameters, in the range $0 \leq x \leq 1$.

Table 11 shows the cdfs for the *aggregated* risk for some selected parameter choices. Note that the corresponding densities $f_s(x;n_1,m_1,n_2,m_2)$ can easily be obtained from this by differentiation.

Table 10.

(n,m)	$F_X(x;n,m) = \int_0^x f_X(u)\,du$
(0,0)	x
(1,0)	x^2
(2,0)	x^3
(3,0)	x^4
(0,1)	$-x^2+2x$
(0,2)	x^3-3x^2+3x
(0,3)	$-x^4+4x^3-6x^2+4x$
(1,2)	$3x^4-8x^3+6x^2$
(1,3)	$-4x^5+15x^4-20x^3+10x^2$
(1,4)	$5x^6-24x^5+45x^4-40x^3+15x^2$
(2,1)	$-3x^4+4x^3$
(3,1)	$-4x^5+5x^4$
(4,1)	$-5x^6+6x^5$

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Table 11.

	$F_S\,(x;n_1,m_1,n_2,m_2)$	
(n_1,m_1,n_2,m_2)	$0\le x\le 1$	$1\le x\le 2$
(1,0,1,0)	$\frac{1}{6}x^4$	$-\frac{1}{6}x^4+2x^2-\frac{8}{3}x+1$
(2,0,2,0)	$\frac{1}{20}x^6$	$-\frac{1}{20}x^6+2x^3-\frac{9}{2}x^2+\frac{18}{5}x-1$
(3,0,3,0)	$\frac{1}{70}x^8$	$-\frac{1}{70}x^8+2x^4-\frac{32}{5}x^3+8x^2-\frac{32}{7}x+1$
(0,1,0,1)	$\frac{1}{6}x^4-\frac{4}{3}x^3+2x^2$	$-\frac{1}{6}x^4+\frac{4}{3}x^3-4x^2+\frac{16}{3}x-\frac{5}{3}$
(0,2,0,2)	$\frac{1}{20}x^6-\frac{3}{5}x^5+3x^4-6x^3+\frac{9}{2}x^2$	$-\frac{1}{20}x^6+\frac{3}{5}x^5-3x^4+8x^3-12x^2+\frac{48}{5}x-\frac{11}{5}$
(0,3,0,3)	$\frac{1}{70}x^8-\frac{8}{35}x^7+\frac{8}{5}x^6-\frac{32}{5}x^5+14x^4-16x^3+8x^2$	$-\frac{1}{70}x^8+\frac{8}{35}x^7-\frac{8}{5}x^6+\frac{32}{5}x^5-16x^4+\frac{128}{5}x^3-\frac{128}{5}x^2+\frac{512}{35}x-\frac{93}{35}$
(0,0,0,0)	$\frac{1}{2}x^2$	$-\frac{1}{2}x^2+2x-1$
(0,1,1,0)	$-\frac{1}{6}x^4+\frac{2}{3}x^3$	$\frac{1}{6}x^4-\frac{2}{3}x^3+\frac{8}{3}x-\frac{5}{3}$
(0,2,2,0)	$\frac{1}{20}x^6-\frac{3}{10}x^5+\frac{3}{4}x^4$	$-\frac{1}{20}x^6+\frac{3}{10}x^5-\frac{3}{4}x^4+2x^3-6x^2+\frac{48}{5}x-\frac{23}{5}$
(0,3,3,0)	$-\frac{1}{70}x^8+\frac{4}{35}x^7-\frac{2}{5}x^6+\frac{4}{5}x^5$	$\frac{1}{70}x^8-\frac{4}{35}x^7+\frac{2}{5}x^6-\frac{4}{5}x^5+\frac{32}{5}x^3-\frac{96}{5}x^2+\frac{832}{35}x-\frac{349}{35}$
(1,2,2,1)	$-\frac{9}{70}x^8+\frac{36}{35}x^7-\frac{14}{5}x^6+\frac{12}{5}x^5$	$\frac{9}{70}x^8-\frac{36}{35}x^7+\frac{14}{5}x^6-\frac{12}{5}x^5-\frac{32}{5}x^3+\frac{96}{5}x^2-\frac{576}{35}x+\frac{163}{35}$
(1,3,3,1)	$\frac{4}{63}x^{10}-\frac{40}{63}x^9+\frac{5}{2}x^8-\frac{100}{21}x^7+\frac{10}{3}x^6$	$-\frac{4}{63}x^{10}+\frac{40}{63}x^9-\frac{5}{2}x^8+\frac{100}{21}x^7-\frac{10}{3}x^6-8x^5+40x^4-\frac{640}{7}x^3+\frac{800}{7}x^2-\frac{640}{9}x+\frac{1087}{63}$
(1,4,4,1)	$-\frac{25}{924}x^{12}+\frac{25}{77}x^{11}-\frac{23}{14}x^{10}+\frac{95}{21}x^9-\frac{195}{28}x^8+\frac{30}{7}x^7$	$\frac{25}{924}x^{12}-\frac{25}{77}x^{11}+\frac{23}{14}x^{10}-\frac{95}{21}x^9+\frac{195}{28}x^8-\frac{30}{7}x^7-\cdots$
		$-\frac{216}{7}x^5+\frac{1080}{7}x^4-\frac{6880}{21}x^3+\frac{2560}{7}x^2-\frac{16000}{77}x+\frac{10919}{231}$

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