Constant Proportion Portfolio Insurance in presence of Jumps in Asset Prices

Rama Cont

IEOR Department

and

Center for Financial Engineering,

Columbia University, New York

E-mail: `Rama.Cont@columbia.edu`

Peter Tankov

Laboratoire de Probabilités et

Modèles Aléatoires

Université Paris VII

e-mail: `tankov@math.jussieu.fr`

(corresponding author)

Abstract

Constant proportion portfolio insurance (CPPI) allows an investor to limit downside risk while retaining some upside potential by maintaining an exposure to risky assets equal to a constant multiple of the *cushion*, the difference between the current portfolio value and the guaranteed amount. Whereas in diffusion models with continuous trading, this strategy has no downside risk, in real markets this risk is non-negligible and grows with the multiplier value. We study the behavior of CPPI strategies in models where the price of the underlying portfolio may experience downward jumps. Our framework leads to analytically tractable expressions for the probability of hitting the floor, the expected loss and the distribution of losses. This allows to measure the gap risk but also leads to a criterion for adjusting the multiplier based on the investor's risk aversion. Finally, we study the problem of hedging the downside risk of a CPPI strategy using options. The results are applied to a jump-diffusion model with parameters estimated from returns series of various assets and indices.

Key words: Portfolio insurance, CPPI, Lévy process, time-changed Lévy models, hedging, CPPI option, Value at Risk, expected loss.

MSC Classification (2000) : 60G51, 91B28, 91B30.

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1   Introduction

Portfolio insurance refers to portfolio management techniques designed to guarantee that the portfolio value at maturity or up to maturity will be greater or equal to a given lower bound (floor), typically fixed as a percentage of the initial investment [18]. These techniques allow the investor to limit downside risk while retaining some potential in case of an upside market move which is however reduced in comparison with the unprotected portfolio (see [14] for a comparison of cost of different portfolio insurance strategies).

Option based portfolio insurance [13, 18] combines a position in the risky asset with a put option on this asset. In many cases options on a given fund or portfolio may not be available in the market: an alternative approach is to use constant proportion portfolio insurance (CPPI), popularized by Black and Jones [5] and Perold [6, 19]. This strategy is based on the notion of *cushion*, defined as the difference between the fund value and the floor. An amount of wealth proportional to the cushion is invested into the risky asset —typically an index or a portfolio of stocks— and the remainder is used to buy riskless bonds. The exposure to the risky asset is thus gradually reduced when the markets move down and the portfolio value approaches the floor.

1.1   Constant Proportion Portfolio Insurance

The CPPI strategy is a self-financing strategy whose goal is to leverage the returns of a risky asset (typically a traded fund or index) through dynamic trading, while guaranteeing a fixed amount N of capital at maturity T. To achieve this, the portfolio manager shifts his/her position between the risky asset, whose price we denote by S_t and a *reserve* asset, typically a bond, whose price we denote by B_t. For simplification we will model the reserve asset as a zero-coupon bond with maturity T and nominal N. The exposure to the risky asset is a function of the *cushion* C_t, defined as

$$C_t = V_t - B_t$$

At any date t,

- if $V_t > B_t$, the exposure to the risky asset (wealth invested into the risky asset) is given by $mC_t \equiv m(V_t - B_t)$, where $m > 1$ is a constant multiplier.

- if $V_t \leq B_t$, the entire portfolio is invested into the zero-coupon.

Assume first that the interest rate r is constant and that the underlying asset follows a Black-Scholes model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

Then it is easy to see from the definition of the strategy that the cushion also satisfies the Black-Scholes stochastic differential equation

$$\frac{dC_t}{C_t} = (m\mu + (1-m)r)dt + m\sigma dW_t,$$

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which is solved explicitly by

$$C_T = C_0 \exp\left(rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2 T}{2}\right).$$

and hence

$$V_T = N + (V_0 - Ne^{-rT})\exp\left(rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2 T}{2}\right) \quad (1)$$

This means that in the Black-Scholes model with continuous trading, the CPPI strategy is equivalent to taking a long position in a zero-coupon bond with nominal N to guarantee the capital at maturity and investing the remaining sum into a (fictitious) risky asset which has m times the excess return and m times the volatility of S and is perfectly correlated with S.

1.2 Price jumps and "gap risk"

Formula (1) shows that in the Black-Scholes model with continuous trading there is no risk of going below the floor, regardless of the multiplier value. On the other hand the expected return of a CPPI-insured portfolio is

$$E[V_T] = N + (V_0 - Ne^{-rT})\exp(rT + m(\mu - r)T).$$

We then arrive to the paradoxical conclusion that in the Black-Scholes model, whenever $\mu > r$, the expected return of a CPPI portfolio can be increased indefinitely and without risk, by taking a high enough multiplier.

Yet the possibility of going below the floor, known as "gap risk", is widely recognized by CPPI managers: there is a nonzero probability that, during a sudden downside move, the fund manager will not have time to readjust the portfolio, which then crashes through the floor. In this case the issuer has to refund the difference, at maturity, between the actual portfolio value and the guaranteed amount N. It is therefore important for the issuer of the CPPI note to quantify and manage this "gap risk".

Beyond the (widely documented) econometric issue of whether jumps occur or not in a given asset's price, a fundamental point is one of liquidity of the underlying: many CPPI strategies are written on funds which may be thinly traded, leading to jumps in the market price due to liquidity effects. Since the volatility of V_t is proportional to m, the risk of such loss increases with m, and in practice, the value of m should be fixed by relating it to an acceptance threshold for the probability of loss or some other risk measure.

1.3 Outline

We study the behavior of CPPI strategies in models where the price of the underlying portfolio may experience downward jumps. This allows to quantify in a meaningful manner the "gap risk" of the portfolio while maintaining the analytical tractability of continuous–time models. We establish a direct relation

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between the value of the multiplier m and the risk of the insured portfolio, which allows to choose the multiplier based on the risk tolerance of the investor, and provide a Fourier transform method for computing the distribution of losses and various risk measures (VaR, expected loss, probability of loss) over a given time horizon, first in an exponential Lévy model and then in a more general framework with stochastic volatility.

Once jumps are introduced in the model, a natural question is the hedging of the gap risk. We deal with these questions in section 5 and show that the embedded option component of a CPPI fund can be hedged using out of the money put options.

The article is structured as follows. Section 2 defines the model setup and describes the dynamics of the CPPI strategy. In section 3, we present analytical formulae for computing various risk measures –VaR, expected loss, probability of loss– for the CPPI strategy when the log price follows a Lévy process. Section 4 extends this analysis to include the effect of stochastic volatility, in the framework of time-changed Lévy processes. In section 5, we study the hedging of the gap risk for a CPPI strategy using vanilla options written on the underlying fund or index. We conclude by a discussion of our results and their relevance to applications.

1.4 Relation to previous research

CPPI strategies in presence of jumps in stock prices were considered by Prigent and Tahar [20] in a diffusion model with (finite intensity) jumps. While the approach of [20] is to consider variants of the CPPI strategy which incorporate extra guarantees, our approach is to quantify the gap risk of classical CPPI strategies. Also, our study considers a more general modeling framework for the dynamics of the underlying asset, allowing for infinite activity jumps and stochastic volatility.

A related literature considers the risk resulting from relaxing the continuous trading assumption: in this case the manager also faces gap risk, but for different reasons. This possibility was already considered in Black & Perold [6] and has been revisited in [4] using large deviations methods to estimate the possible losses between two consecutive trading dates. However the frequency of trading interventions during a downside market move is hard to predict, making this parameter difficult to determine. On the other hand, the gap risk due to price jumps, which are a market reality, is qualitatively different from that generated by discrete trading in a Black-Scholes model. In particular, attributing gap risk to discrete trading gives the wrong impression that this risk can be eliminated by more frequent rebalancing, while our analysis points to a non-negligible residual risk even in presence of continuous trading.

2 Model setup

We assume that the price processes for the risky asset S and for the zero-coupon B may be written as

$$\frac{dS_t}{S_{t-}} = dZ_t \quad \text{and} \quad \frac{dB_t}{B_{t-}} = dR_t,$$

where Z and R are possibly discontinuous driving processes, modeled as semi-martingales. As a simple example, one can take Z a Lévy process and $R_t = rT$ for some constant interest rate r. A less trivial example of (B_t) is provided by the Vasicek model.

Example 1. The Vasicek model is a one-factor interest rate model where the short rate r_t follows (under the risk-neutral probability) an Ornstein-Uhlenbeck process:

$$dr_t = (\alpha - \beta r_t)dt + \sigma dW_t.$$

The zero coupon is given by

$$B_t = B(t, T) = E[e^{-\int_t^T r_s ds}].$$

It follows that in the Vasicek model the zero-coupon satisfies the stochastic differential equation

$$\frac{dB_t}{B_t} = r_t dt - \sigma \frac{1 - e^{-\beta(T-t)}}{\beta} dW_t.$$

We make the following assumption:

- $\Delta Z_t > -1$ almost surely.

- The zero-coupon price process B is continuous.

While the first hypothesis guarantees the positivity of the risky asset price, the second one allows to focus on the impact of jumps in the underlying asset. This assumption implies in particular

$$B_t = B_0 \exp\left(R_t - \frac{1}{2}[R]_t\right) > 0 \quad \text{a.s.}$$

Let $\tau = \inf\{t : V_t \leq B_t\}$. Since the CPPI strategy is self-financing, up to time τ the portfolio value satisfies

$$dV_t = m(V_{t-} - B_t)\frac{dS_t}{S_{t-}} + \{V_{t-} - m(V_{t-} - B_t)\}\frac{dB_t}{B_t},$$

which can be rewritten as

$$\frac{dC_t}{C_{t-}} = m dZ_t + (1 - m)dR_t,$$

where $C_t = V_t - B_t$ denotes the cushion.

Change of numeraire Introducing the discounted cushion $C_t^* = \frac{C_t}{B_t}$ and applying Itô formula to this process, we find

$$\frac{dC_t^*}{C_{t-}^*} = m(dZ_t - d[Z, R]_t - dR_t + d[R]_t), \tag{2}$$

Define

$$L_t \equiv Z_t - [Z, R]_t - R_t + [R]_t. \tag{3}$$

Equation (2) can then be rewritten in a more compact form as

$$C_t^* = C_0^* \mathcal{E}(mL)_t,$$

where $\mathcal{E}$ denotes the stochastic (Doléans-Dade) exponential defined by

$$\frac{d\mathcal{E}(mL)_t}{\mathcal{E}(mL)_{t-}} = m dL_t$$

(see Appendix A). After time τ, according to our definition of the CPPI strategy, the process C^* remains constant. Therefore, the discounted cushion value for this strategy can be written explicitly as

$$C_t^* = C_0^* \mathcal{E}(mL)_{t \wedge \tau},$$

or alternatively

$$\frac{V_t}{B_t} = 1 + \left(\frac{V_0}{B_0} - 1\right) \mathcal{E}(mL)_{t \wedge \tau}. \tag{4}$$

Since the stochastic exponential can become negative, in presence of negative jumps of sufficient size in the stock price, the capital N at maturity is no longer guaranteed by this strategy.

3 Measuring gap risk for CPPI strategies

3.1 Probability of loss

A CPPI-insured portfolio incurs a loss (breaks through the floor) if, for some $t \in [0, T]$, $V_t \leq B_t$. The event $V_t \leq B_t$ is equivalent to $C_t^* \leq 0$ and since R is continuous and $\mathcal{E}(X)_t = \mathcal{E}(X)_{t-}(1 + \Delta X_t)$, $C_t^* \leq 0$ for some $t \in [0, T]$ if and only if $m\Delta L_t \leq -1$ for some $t \in [0, T]$. This leads us to the following result:

Proposition 1. *Let L be of the form $L = L^c + L^j$, where L^c is a continuous process and L^j is an independent Lévy process with Lévy measure ν. Then the probability of going below the floor is given by*

$$P[\exists t \in [0, T] : V_t \leq B_t] = 1 - \exp\left(-T \int_{-\infty}^{-1/m} \nu(dx)\right).$$

Proof. This result follows from the fact that the number of jumps of the Lévy process L^j in the interval $[0, T]$, whose sizes fall in $(-\infty, -1/m]$ is a Poisson random variable with intensity $T\nu((-\infty, -1/m])$. $\qquad\square$

Corollary 1. *Assume that S follows an exponential Lévy model of the form*

$$S_t = S_0 e^{N_t},$$

where N is a Lévy process with Lévy measure ν. Then the probability of going below the floor is given by

$$P[\exists t \in [0, T] : V_t \leq B_t] = 1 - \exp\left(-T \int_{-\infty}^{\log(1-1/m)} \nu(dx)\right). \qquad (5)$$

Proof. It follows from proposition 4 that there exists another Lévy process L satisfying

$$\frac{dS_t}{S_{t-}} = dL_t.$$

The Lévy measure of L is given by

$$\tilde{\nu}^L(A) = \int 1_A(e^x - 1)\nu(dx).$$

Applying proposition 1 concludes the proof. $\qquad\square$

3.2   Expected loss

We now study the distribution of loss of a CPPI-managed portfolio given that a loss occurs, with the aim of computing its expectation and other functionals (risk measures).

To obtain some explicit formulae, we assume that the process L appearing in the stochastic exponential in (4) is a Lévy process, and we denote its Lévy measure by ν. We can always write $L = L^1 + L^2$ where L^2 is a process with piecewise constant trajectories and jumps satisfying $\Delta L_t^2 \leq -1/m$ and L^1 is a process with jumps satisfying $\Delta L_t^1 > -1/m$. In other words, L^1 has Lévy measure $\nu(dx)1_{x>-1/m}$ and L^2 has Lévy measure $\nu(dx)1_{x\leq-1/m}$, no diffusion component and no drift. Denote by $\lambda^* := \nu((-\infty, -1/m])$ the jump intensity of L^2, by τ the time of the first jump of L^2 (it is an exponential random variable with intensity λ^*) and by $\tilde{L}^2 = \Delta L_\tau^2$ the size of the first jump of L^2. Let ϕ_t be the characteristic function of the Lévy process $\log \mathcal{E}(mL^1)_t$ and $\psi(u) = \frac{1}{t}\log \phi_t(u)$. Finally, we suppose without loss of generality that the discounted cushion satisfies $C_0^* = 1$.

First we compute the expectation of loss.

Proposition 2. *Assume*

$$\int_1^\infty x\nu(dx) < \infty.$$

Then the expectation of loss conditional on the fact that a loss occurs is

$$E[C_T^*|\tau \leq T] = \frac{\lambda^* + m \int_{-1}^{-1/m} x\nu(dx)}{(1 - e^{-\lambda^* T})(\psi(-i) - \lambda^*)}(e^{-\lambda^* T}\phi_T(-i) - 1).$$

and the unconditional expected loss satisfies

$$E[C_T^* 1_{\tau \leq T}] = \frac{\lambda^* + m \int_{-1}^{-1/m} x\nu(dx)}{(\psi(-i) - \lambda^*)}(e^{-\lambda^* T}\phi_T(-i) - 1). \tag{6}$$

Proof. The discounted cushion satisfies

$$C_T^* = \mathcal{E}(mL^1)_{\tau \wedge T}(1 + m\tilde{L}^2 1_{\tau \leq T}) = \mathcal{E}(mL^1)_T 1_{\tau > T} + \mathcal{E}(mL^1)_\tau(1 + m\tilde{L}^2)1_{\tau \leq T}. \tag{7}$$

Since L^1 and L^2 are Lévy processes, τ, $\tilde{L}^2$ and L^1 are independent. Since by [21, Theorem 25.17], and by definition of ϕ_t,

$$E[\mathcal{E}(mL^1)_t] = \phi_t(-i),$$

we have

$$E[C_T^*|\tau \leq T] = \frac{E[1 + m\tilde{L}^2]}{1 - e^{-\lambda^* T}} \int_0^T \lambda^* e^{-\lambda^* t} E[\mathcal{E}(mL^1)_t]dt$$

$$= (\lambda^* + m \int_{-1}^{-1/m} x\nu(dx))\frac{1}{1 - e^{-\lambda^* T}} \int_0^T e^{-\lambda^* t}\phi_t(-i)dt.$$

and the result follows. $\square$

Remark 1. Suppose that $\int_{\mathbb{R}} |x|\nu(dx) < \infty$ and let (σ^2, ν, γ) be the characteristic triplet of L with respect to zero truncation function (general Lévy measures may be treated along the same lines with a slightly heavier notation). Proposition 4 and the Lévy-Khintchine representation then give the characteristic exponent of $\log \mathcal{E}(mL)_t$:

$$\psi(u) = -\frac{m^2\sigma^2 u^2}{2} + iu\left(m\gamma - \frac{\sigma^2 m^2}{2}\right) + \int_{z > -1/m}(e^{iu\log(1+mz)} - 1)\nu(dz)$$

$$\psi(-i) = m\gamma + m \int_{z > -1/m} z\nu(dz). \tag{8}$$

From equation (7) it follows that the expected gain conditional on the fact that the floor is not broken satisfies

$$E[C_T^*|\tau > T] = E[\mathcal{E}(mL^1)_T] = \phi_T(-i) = \exp\left\{Tm\gamma + Tm \int_{z > -1/m} z\nu(dz)\right\}.$$

Therefore, similarly to the Black-Scholes case, conditional expected gain in an exponential Lévy model is increasing with the multiplier, provided the underlying Lévy process has a positive expected return.

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3.3 Loss distribution

For computing risk measures, we need the distribution function of the loss given that a loss occurred, that is, we want to compute, for $x < 0$, the quantity

$$P[C_T^* < x | \tau \leq T].$$

Our approach for computing this conditional distribution function is to express its characteristic function explicitly in terms of the characteristic exponents of the Lévy processes involved and recover the distribution function by numerical Fourier inversion. A similar strategy was used in [8, 17] for pricing European options. In the theorem below,

$$\tilde{\phi} := \frac{1}{\lambda^*} \int_{-\infty}^{-1/m} e^{iu \log(-1 - mx)} \nu(dx)$$

denotes the characteristic function of $\log(-1 - m\tilde{L}^2)$.

Theorem 1. *Choose a random variable X^* with characteristic function ϕ^*, such that $E[|X^*|] < \infty$ and $\frac{|\phi^*(u)|}{1+|u|} \in L^1$. If*

$$\frac{|\tilde{\phi}(u)|}{(1 + |u|)|\lambda^* - \psi(u)|} \in L^1 \tag{9}$$

$$\int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} |\log|1 + mx|| \nu(dx) < \infty \tag{10}$$

for sufficiently small ε, then for every $x < 0$,

$$P[C_T^* < x | \tau \leq T] = P[-e^{X^*} < x]$$
$$+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iu \log(-x)} \left(\frac{\lambda^* \tilde{\phi}(u)}{iu(\lambda^* - \psi(u))} \frac{1 - e^{-\lambda^* T + \psi(u)T}}{1 - e^{-\lambda^* T}} - \frac{\phi^*(u)}{iu} \right) du. \tag{11}$$

Remark 2. The random variable X^* is needed only for the purpose of Fourier inversion: the cumulative distribution function of the loss distribution is not integrable and its Fourier transform cannot be computed, but the difference of two distribution functions has a well-defined Fourier transform. In practice, X^* can always be taken equal to a standard normal random variable.

Proof. From equation (7) it follows that the characteristic function of $\log(-C_T^*)$ conditionally on the fact that a loss occurs, satisfies

$$E[e^{iu \log(-C_T^*)} | \tau \leq T] = \frac{1}{1 - e^{-\lambda^* T}} \int_0^T \lambda^* e^{-\lambda^* t} E\left[e^{iu \log(-\mathcal{E}(mL^1)_t (1 + m\tilde{L}^2))} \right] dt$$
$$= \frac{1}{1 - e^{-\lambda^* T}} \int_0^T \lambda^* e^{-\lambda^* t} e^{t\psi(u)} \tilde{\phi}(u) dt$$
$$= \frac{\tilde{\phi}(u)(1 - e^{-\lambda^* T + \psi(u)T})}{(\lambda^* - \psi(u))(1 - e^{-\lambda^* T})}.$$

The integral in (11) converges at $u \to \infty$ due to the theorem's conditions and the fact that

$$\left| \frac{1 - e^{-\lambda^* T + \psi(u)T}}{1 - e^{-\lambda^* T}} \right| < \frac{1 + e^{-\lambda^* T}}{1 - e^{-\lambda^* T}},$$

On the other hand, condition (10) is equivalent to

$$E[|\log(-1 - m\tilde{L}^2)|] < \infty$$
$$E[|\log \mathcal{E}(mL^1)_T|] < \infty,$$

and together with the assumption $E[|X^*|] < \infty$, this proves that $\phi(u) = 1 + O(u)$, $\tilde{\phi}(u) = 1 + O(u)$ and $\phi^*(u) = 1 + O(u)$ as $u \to 0$, and therefore the integrand in (11) is bounded and therefore integrable in the neighborhood of zero. The proof is completed by applying Lemma 1. $\square$

4 Stochastic volatility

Empirical evidence suggests that independence of increments is not a property observed in historical return time series: stylized facts such as volatility clustering show that the amplitude of returns is positively correlated over time. This and other deviations from the case of IID returns can be accounted for introducing a "stochastic volatility" model for the underlying asset. It is well known that the stochastic volatility process with continuous paths

$$\frac{dS_t}{S_t} = \sigma_t dW_t$$

has the same law as a time-changed Geometric Brownian motion

$$S_t = e^{-\frac{v_t}{2} + W_{v_t}} = \mathcal{E}(W)_{v_t}, \quad \text{where} \quad v_t = \int_0^t \sigma_s^2 ds,$$

where the time change is given by the integrated volatility process v_t, provided that volatility is independent from the Brownian motion W governing the stock price.

In the same spirit, Carr et al. [7] have proposed to construct "stochastic volatility" models with jumps by time-changing an exponential Lévy model for the discounted stock price:

$$S_t^* = \mathcal{E}(L)_{v_t}, \qquad v_t = \int_0^t \sigma_s^2 ds$$

where L is a Lévy process and σ_t is a positive process. The stochastic volatility thus appears as a random time change governing the intensity of jumps, and can be seen as reflecting an intrinsic market time scale ("business time"). The volatility process most commonly used in the literature (and by practitioners) is the process

$$d\sigma_t^2 = k(\theta - \sigma_t^2)dt + \delta\sigma_t dW. \tag{12}$$

introduced in [12] which has the merit of being positive, stationary and ana-
lytically tractable. Other specifications such as positive Lévy-driven Ornstein-
Uhlenbeck process [1] can also be used. The Brownian motion W driving the
volatility is assumed to be independent from the Lévy process L. For the spec-
ification (12) Laplace transform of integrated variance v is known in explicit
form [12]:

$$\mathcal{L}(\sigma, t, u) := E[e^{-uv_t} | \sigma_0 = \sigma] = \frac{\exp\left(\frac{k^2 \theta t}{\delta^2}\right)}{\left(\cosh \frac{\gamma t}{2} + \frac{k}{\gamma} \sinh \frac{\gamma t}{2}\right)^{\frac{2k\theta}{\delta^2}}} \exp\left(-\frac{2\sigma_0^2 u}{k + \gamma \coth \frac{\gamma t}{2}}\right)$$

with $\gamma := \sqrt{k^2 + 2\delta^2 u}$.

Since the formulas for loss probability, unconditional expected loss and un-
conditional loss distribution have an exponential dependence on time remaining
to maturity, these formulas can be generalized to the case of stochastic volatility
in a straightforward way, by first conditioning on the trajectory of the volatility
process. We only give the final results leaving the details of the computation to
the reader. In all computations below, we place ourselves at date t and suppose
that *no losses have occurred yet*.

Probability of loss   Recall the notation $\lambda^* = \int_{-\infty}^{-1/m} \nu(dx)$ for the intensity
of large negative jumps that break the floor. Then, in presence of stochastic
volatility, the probability of going below the floor is given by

$$P[\exists s \in [t, T] : V_s \leq B_s | \mathcal{F}_t] = 1 - \mathcal{L}(\sigma_t, T - t, \lambda^*)$$

The gap risk is thus very sensitive to the volatility of the underlying, especially
given that a CPPI fund is usually a long-term investment. For instance, if the
initial loss probability was 5%, and the volatility increases by a factor of 2, the
loss probability changes to about 19%, leading to a much bigger cost for the
bank in terms of regulatory capital.

If the underlying is sufficiently liquid, this extra cost can be offset by buying
an asset with a positive sensitivity to volatility such as a variance swap, however
a simpler and more popular approach is to control the volatility exposure by
adjusting the multiplier.

Suppose that the multiplier (m_t) is a continuous stochastic process adapted
to the filtration generated by the volatility process (σ_t). The discounted cushion
is given by

$$C_t^* = \mathcal{E}\left(\int_0^{\cdot} m_s dL_{v_s}\right)_{\tau \wedge T}, \quad \tau = \inf\{t \geq 0 : m_t \Delta L_{v_t} \leq -1\},$$

and the probability of loss, once again by conditioning on the trajectory of the
volatility process, is given by

$$P[\tau \leq T] = 1 - E\left[\exp\left(-\int_0^T dt \, \sigma_t^2 \int_{-\infty}^{1/m_t} \nu(dx)\right)\right].$$

This means that the random time τ is characterized by a hazard rate λ_t, interpreted as the probability of loss "per unit time", given by

$$\lambda_t = \sigma_t^2 \int_{-\infty}^{1/m_t} \nu(dx) \tag{13}$$

In the situation where the portfolio manager must monitor not only potential losses at maturity but also the short term Value at Risk during the life of the contract, one can hedge this volatility exposure by choosing the multiplier m_t keeping the hazard rate constant:

$$\sigma_t^2 \int_{-\infty}^{1/m_t} \nu(dx) = \sigma_0^2 \int_{-\infty}^{1/m_0} \nu(dx),$$

where m_0 is the initial multiplier value fixed according to the desired loss probability level. If the jump size distribution is α-stable, the above formula amounts to $m_t = m_0 \left(\frac{\sigma_t}{\sigma_0} \right)^{-2/\alpha}$.

Expected loss   When volatility is stochastic, the unconditional expected loss can be computed using

$$E[C_T^* 1_{\tau \le T} | \mathcal{F}_t] = \frac{\lambda^* + m \int_{-1}^{-1/m} x\nu(dx)}{\psi(-i) - \lambda^*} (\mathcal{L}(\sigma_t, T - t, \lambda^* - \psi(-i)) - 1). \tag{14}$$

Over a short period of time $[t, t + dt]$, the expected loss is equal to

$$\sigma_t^2 dt \left(\lambda^* + m \int_{-1}^{-1/m} x\nu(dx) \right) = \sigma_t^2 dt \int_{-1}^{-1/m} (1 + mx)\nu(dx).$$

Hence, to keep the short-term expected loss at the same level, the variable multiplier m_t should be chosen to have

$$\sigma_t^2 \int_{-1}^{-1/m_t} (1 + m_t x)\nu(dx) = \text{const.}$$

Loss distribution   The conditional characteristic function of the loss logarithm can be computed similarly to other risk indicators:

$$E[e^{iu \log(-C_T^*)} | \tau \le T, \mathcal{F}_t] = \frac{E[e^{iu \log(-C_T^*)} 1_{\tau \le T} | \mathcal{F}_t]}{P[\tau \le T | \mathcal{F}_t]} = \frac{\tilde{\phi}(u)(1 - \mathcal{L}(\sigma_t, T - t, \lambda^* - \psi(u)))}{(\lambda^* - \psi(u))(1 - \mathcal{L}(\sigma_t, T - t, \lambda^*))}.$$

The loss distribution can then be computed by numerical inversion of this characteristic function as in theorem 1.

5   Pricing and hedging gap risk

The client of a CPPI fund may be insured against the gap risk meaning that if the fund breaks the floor, a third party such as the bank controlling the fund, reimburses the loss. In this case, the share of a CPPI fund is a financial product with an optional component. It is then equivalent to the sum of a self-financing portfolio corresponding to the uninsured CPPI strategy and an option to receive the difference between the floor and the portfolio value, if the portfolio goes below the floor. This option will be referred to as the CPPI-embedded option. The (discounted) payoff of this option is equal to $-C_T^* 1_{\tau \leq T}$.

If the underlying is liquid (such as a market index) and the market quotes for European options on this underlying are available, the CPPI-embedded option can be marked to market (priced) and its risk can be hedged away. The price of this option is equal to the expectation of its discounted pay-off under the risk-neutral probability Q which must be calibrated to the market-quoted prices of European options as in [10] or [3] (this is because as we will see below, a near-perfect hedge with European puts can be constructed). The price is then given by (6) in the Lévy setting or (14) in the stochastic volatility setting, where both expectations must be evaluated under the risk-neutral probability. Note that the option price does not depend on the current underlying value and is not sensitive to small movements of the underlying. The CPPI-embedded option is a pure 'gap risk' and volatility product.

A hedging strategy for a CPPI-embedded option can be constructed using short maturity put options. We suppose that at each date, the market quotes put options with time to maturity $h := T/n$ for some integer $n > 0$. In order to construct a perfect hedge, we modify the CPPI strategy in question: instead of continuous rebalancing, we suppose that the rebalancing step is equal to h. Later, we show that when $h \to 0$, the terminal value of the discretely rebalanced portfolio hedged using short maturity put options converges to the terminal value of a continuously rebalanced CPPI without the gap risk.

In this section we adopt an exponential Lévy model for the discounted stock price:

$$S_t^* = \mathcal{E}(L)_t,$$

where L is a Lévy process under Q. We denote by C_k^n and $C_k^{*,n}$ the values of, respectively, the non-discounted and the discounted cushion at the rebalancing date k and make the simplifying hypothesis that the interest rate is constant and equal to r. We consider two different hedging strategies: in the first case, the cost of put options used for hedging is considered exogenous, and we compute separately the terminal value of the hedge portfolio and the cost of hedging. In the second case we consider a more realistic situation where the cost of hedging is subtracted from the cushion (in the form of a management fee) at each step.

Case 1   By definition of the CPPI strategy we get

$$C_{k+1}^{*,n} = mC_k^{*,n} \frac{S_{(k+1)h}^*}{S_{kh}^*} + (1-m)C_k^{*,n}, \tag{15}$$

where S^* is the discounted stock price. This expression is negative if $S^*_{(k+1)h} < \frac{m-1}{m} S^*_{kh}$. To hedge this risk, we therefore buy put options with strike $\frac{m-1}{m} e^{rh} S_{kh}$ expiring at date $(k+1)h$. One such option has pay-off $e^{rh} S_{kh} \left(\frac{m-1}{m} - \frac{S^*_{(k+1)h}}{S^*_{kh}} \right)^+$, and if we buy $\frac{mC^n_k}{S_{kh}}$ units, the discounted pay-off is equal to $C^{*,n}_k \left(m - 1 - m \frac{S^*_{(k+1)h}}{S^*_{kh}} \right)^+$, that is, to the negative part of equation (15). Therefore, the discounted cushion in presence of hedging satisfies

$$
C^{*,n}_{k+1} = C^{*,n}_k \left(1 + m \left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1 \right) \right)^+ ,
$$

and

$$
C^{*,n}_n = C^*_0 \prod_{k=0}^{n-1} \left(1 + m \left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1 \right) \right)^+ . \tag{16}
$$

The cost of hedging is given by the sum of prices of all put options necessary for hedging:

$$
\text{Cost}^n = \sum_{k=0}^{n-1} E^Q \left[C^{*,n}_k \left(-1 - m \left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1 \right) \right)^+ \right] = E^Q[C^{*,n}_n] - C^*_0.
$$

Case 2 We now suppose that the cost of hedging is taken from the cushion at each step. In this case, by arguments similar to the ones used above, one can show that the discounted cushion satisfies

$$
\tilde{C}^{*,n}_n = C^*_0 \prod_{k=0}^{n-1} \frac{\left(1 + m \left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1 \right) \right)^+}{E^Q \left[\left(1 + m \left(\frac{S^*_{(k+1)h}}{S^*_{kh}} - 1 \right) \right)^+ | \mathcal{F}_{kh} \right]}.
$$

In this case, the portfolio is again self-financing and it is not surprising that the discounted cushion is a martingale.

The following proposition provides convergence results as the number of rebalancing dates goes to infinity.

Proposition 3. *Suppose that the Lévy measure ν of L has no atom at the point $-1/m$. Then*

    i. The discounted cushion in case 1 converges to the value obtained with a continuous-time CPPI but without the gap risk:

$$
\lim_{n \to \infty} C^{*,n}_n = C^*_T 1_{\tau > T} \quad a.s.
$$

Suppose, in addition that $\int_1^\infty x^m \nu(dx) < \infty$. Then

ii. The cost of hedging in case 1 converges to the expected loss under the risk-neutral probability:

$$\lim_{n\to\infty} \mathrm{Cost}^n \to -E^Q[C_T^* 1_{\tau \le T}].$$

iii. The discounted cushion in case 2 converges to the value obtained with a continuous-time CPPI but without the gap risk, divided by its risk-neutral expectation:

$$\lim_{n\to\infty} \tilde{C}_n^{*,n} = \frac{C_T^* 1_{\tau > T}}{E^Q[C_T^* 1_{\tau > T}]} \quad a.s.$$

Proof.

Part (i) Let X be a the Lévy process satisfying $e^X = \mathcal{E}(L)$. If $\tau \le T$ then starting from some n, $C_n^{*,n} \equiv 0$. Suppose that $\tau > T$. Then, starting from some n, all terms in the product (16) are positive and therefore we can write

$$\log C_n^{*,n} = \log C_0^* + \sum_{k=0}^{n-1} \log \left(1 + m(e^{X_{(k+1)h} - X_{kh}} - 1)\right).$$

Fix $\varepsilon > 0$ and let A and B be two disjoint sets of jumps of X such that $A \cup B$ exhausts the jumps of X on $[0, T]$, A is finite and $\sum_{s \in B} \Delta X_s^2 < \varepsilon$. We will use the notation $\mathcal{A} = \{k < n : A \cap (kh, (k+1)h] \ne \emptyset\}$ and $\tilde{\mathcal{A}} = \{k < n : k \notin \mathcal{A}\}$. Then, using the Taylor expansion up to second order for the function $f(x) = \log(1 + m(e^x - 1))$,

$$\begin{aligned}
\log C_n^{*,n} = \log C_0^* &+ \sum_{k \in \mathcal{A}} \log \left(1 + m(e^{X_{(k+1)h} - X_{kh}} - 1)\right) \\
&+ \sum_{k < n} \{m(X_{(k+1)h} - X_{kh}) + \frac{1}{2}(m - m^2)(X_{(k+1)h} - X_{kh})^2\} \\
&- \sum_{k \in \mathcal{A}} \{m(X_{(k+1)h} - X_{kh}) + \frac{1}{2}(m - m^2)(X_{(k+1)h} - X_{kh})^2\} \\
&+ \sum_{k \in \tilde{\mathcal{A}}} R(X_{(k+1)h} - X_{kh}),
\end{aligned}$$

where R stands for the residual term in the Taylor formula. Starting from some n, $\sup_{k \in \tilde{\mathcal{A}}} |X_{(k+1)h} - X_{kh}| \le 2\varepsilon$, hence $\sum_{k \in \tilde{\mathcal{A}}} |R(X_{(k+1)h} - X_{kh})| \le r(\varepsilon)[X]_T$ with $r(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$. The first sum in above converges to

$$\sum_{t \in A} \log(1 + m(e^{\Delta X_t} - 1))$$

as $n \to \infty$, the second sum converges to

$$mX_T + \frac{1}{2}(m - m^2)[X]_T,$$

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and the third sum gives

$$\sum_{t \in A} (m \Delta X_t + \frac{1}{2}(m - m^2)\Delta X_t^2).$$

Assembling all this together and making ε tend to 0, we conclude that on the set $\tau > T$,

$$\log C_n^{*,n} \to \log C_0^* + \sum_{t \leq T: \Delta X_t \neq 0} \{\log(1+m(e^{\Delta X_t}-1))-m\Delta X_t\} + mX_T + \frac{1}{2}(m-m^2)[X]_T^c.$$

almost surely as $n \to \infty$. Exponentiating both sides and going back to the process L using proposition 4, we finally find

$$\lim_{n \to \infty} C_n^{*,n} = C_T^* 1_{\tau > T} \quad \text{a.s.}$$

Part (ii) This follows from part (i) and the dominated convergence theorem using the bound

$$C_n^{*,n} = C_0^* \prod_{k=1}^{n-1} \left(1 + m(e^{X_{(k+1)h} - X_{kh}} - 1)\right) \leq C_0^* e^{mX_T}$$

which is integrable under the hypothesis $\int_1^\infty x^m \nu(dx) < \infty$.

Part (iii) This is a direct consequence of parts (i) and (ii). $\square$

6 Examples

Let us now illustrate the results of section 3 in the case of a jump diffusion model with a double-exponential jump size distribution [16] with parameters estimated from daily returns of various assets.

6.1 Model setup and parameter estimation

To estimate the parameters have used daily time series of log returns from December 1st 1996 to December 1st 2006 for

1. General Motors Corporation (GM)

2. Microsoft Corporation (MSFT)

3. Shanghai Composite index

making a total of 2500 data points for each series (expect for Shanghai Composite for which we have 9 years). We model these data sets using an exponential

Lévy model [16] where the driving Lévy process has a non-zero Gaussian component and a Lévy density of the form

$$\nu(x) = \frac{\lambda(1-p)}{\eta_+}e^{-x/\eta_+}1_{x>0} + \frac{\lambda p}{\eta_-}e^{-|x|/\eta_-}1_{x<0}. \tag{17}$$

Here, λ is the total intensity of positive and negative jumps, p is the probability that a given jump is negative and η_- and η_+ are characteristic lengths of respectively negative and positive jumps.

To estimate the model parameters from market data, we use the empirical characteristic function as suggested in [22, 23]: we find the parameter vector $\theta = (b, \sigma, \lambda, p, \eta_+, \eta_-)$ by minimizing

$$\int_{-K}^{K} |\psi_\theta(u) - \hat{\psi}(u)|^2 w(u) du.$$

where

$$\hat{\psi}(u) = \frac{1}{t}\log\frac{1}{N}\sum_{k=1}^{N}e^{iuX_i}$$

is the empirical characteristic exponent,

$$\psi_\theta(u) = -\frac{\sigma^2 u^2}{2} + i\gamma u + \frac{\lambda p}{1+iu\eta_-} + \frac{\lambda(1-p)}{1-iu\eta_+} - \lambda$$

is the characteristic exponent of the Kou model and w is the weight function.

Ideally, the weights should reflect the relative precision of $\hat{\psi}(u)$ as an estimate of $\psi_\theta(u)$, and one should choose w to be equal to the inverse of variance of $\hat{\psi}(u)$:

$$w(u) = \frac{1}{E[(\hat{\psi}(u) - E[\hat{\psi}(u)])\overline{(\hat{\psi}(u) - E[\hat{\psi}(u)])}]}$$
$$\approx \frac{t^2\phi_{\theta^*}(u)\overline{\phi_{\theta^*}(u)}}{E[(\hat{\phi}(u) - \phi_{\theta^*}(u))\overline{(\hat{\phi}(u) - \phi_{\theta^*}(u))}]},$$

where θ^* is the true parameter. However, this expression depends on the unknown parameter vector θ and cannot be computed. Since the return distribution is relatively close to Gaussian, the characteristic function in the weight w may be approximated with the Gaussian one

$$w(u) \approx \frac{e^{-\sigma_*^2 u^2}}{1 - e^{-\sigma_*^2 u^2}}$$

with $\sigma_*^2 = \text{Var}\, X$. The cutoff parameter K was fixed to 60 based on tests with simulated data (the estimated parameter values are not very sensitive to this parameter for $K > 50$).

The estimated parameter values are shown in table 1. Figure 1 shows that Kou model fits the smoothed returns density quite well, in particular, the exponential tail decay seems to be a realistic assumption.

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Series	μ	σ	λ	p	η_+	η_-
MSFT	-0.473	0.245	99.9	0.230	0.0153	0.0256
GM	-0.566	0.258	104	0.277	0.0154	0.0204
Shanghai composite	0.101	0.161	39.1	0.462	0.0167	0.0175

Table 1: Kou model parameters estimated from MSFT, GM and Shanghai Composite time series.
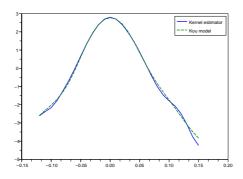

Figure 1: Logarithm of the density for MSFT time series. Solid line: kernel density estimator. Dashed line: Kou model with parameters estimated via empirical characteristic function.
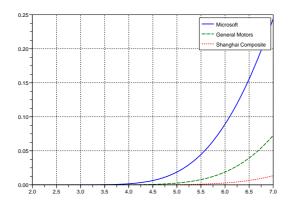
Figure 2: Probability of loss as a function of the multiplier.

6.2 Loss probability

Assuming the (discounted) risky asset price process follows the Kou model, equation (5) for loss probability yields

$$P[\exists t \in [0,T] : V_t \leq B_t] = 1 - \exp\left(-Tp\lambda \left(1 - 1/m\right)^{1/\eta_-}\right).$$

Figure 2 shows the dependence of the loss probability on the multiplier for a CPPI portfolios containing MSFT and GM stocks or Shanghai Composite index as risky assets. Although Microsoft is slightly riskier, the loss probabilities for the two stocks are quite similar. A 5% loss probability over 5 years corresponds to a multiplier value of about 5.5 for Microsoft and 6 for General Motors.

6.3 Expected loss

Once again, suppose that the discounted risky asset

$$S_t^* = \mathcal{E}(L)_t$$

follows the Kou model

$$S_t^* = S_0^* e^{N_t},$$

where N is a Lévy process with volatility σ, drift b and Lévy density ν of the form (17). For convenience of notation we write $\lambda_\pm = 1/\eta_\pm$; $c_- = p\lambda$; $c_+ = (1-p)\lambda$. We assume $\lambda_+ > 1$. Proposition 4 implies that L is a Lévy process with volatility σ, drift $b + \frac{\sigma^2}{2}$ and Lévy density

$$\nu_L(x) = \lambda_+ c_+ (1+x)^{-1-\lambda_+} 1_{x>0} + \lambda_- c_- (1+x)^{-1+\lambda_-} 1_{-1<x<0}.$$

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Splitting this Lévy density in two parts, we find on one hand

$$\lambda^* = c_-(1 - 1/m)^{\lambda_-},$$

$$1 + \frac{m}{\lambda^*} \int_{-1}^{-1/m} x\nu_L(dx) = -\frac{m-1}{\lambda_- + 1},$$

and on the other hand, from (8), the characteristic exponent of the Lévy process $\log \mathcal{E}(mL^1)_t$ at the point $-i$ is

$$\psi(-i) = m(b + \sigma^2/2) + \frac{c_+ m}{\lambda_+ - 1} - \frac{c_- m}{\lambda_- + 1}$$
$$- \frac{c_- \lambda_- m (1 - 1/m)^{\lambda_- + 1}}{\lambda_- + 1} + c_- m (1 - 1/m)^{\lambda_-}.$$

Finally, assembling the two factors we get the expected loss:

$$E[C_T^* | \tau \leq T] = -\frac{(m-1)(1 - e^{-\lambda^* T + \psi(-i)T})\lambda^*}{(\lambda_- + 1)(1 - e^{-\lambda^* T})(\lambda^* - \psi(-i))}.$$

Figure 3 shows the dependence on the multiplier of the unconditional expected loss and (expected loss conditional on a loss having occurred for a CPPI portfolio for various examples of risky assets and a time horizon of $T = 3$ years. While the loss probability for an Shanghai Composite-based fund is much smaller than that of the GM-based one, if a loss does occur, the CPPI fund based on Shanghai Composite will experience a much larger loss. This happens because the Shanghai Composite index has a larger expected return (about 14% per year compared to around zero for GM). Before the loss-making jump, the Shanghai Composite-based fund will have a better performance, leading to a larger proportion of risky asset in the portfolio and therefore a larger loss after a large negative jump. Of course, not only the expected loss but also the expected gain of the Shanghai Composite-based portfolio will be much greater than that of GM-based one.

6.4 Loss distribution

In the Kou model, the integrand in (11) must be evaluated numerically in general. But for illustration purposes we consider here a particular case of Kou model with $p = 1$ (only negative jumps) and $\eta_- = 1$. This amounts to saying that

$$L_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

where N is a Poisson process with intensity λ and $\{Y_i\}$ are independent uniforms on $[-1, 0]$. That is, we suppose that during a crash the jump size distribution is uniform on $[-1, 0]$. It is easy to check that the condition (10) is satisfied. This model can describe, for example, a defaultable asset with random recovery
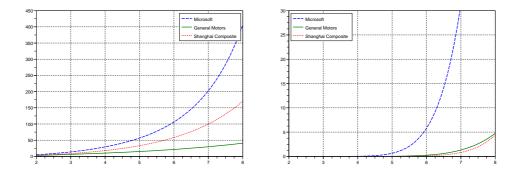
Figure 3: Expected loss over $T = 3$ years as a function of the multiplier, for nominal $N = 1000\$$ and $r = 0.04$. Left: expected loss conditional on a loss having occurred. Right: unconditional expected loss.

on the interval $[0, 1]$ in case of default. In this case, $\lambda^* = \lambda(1 - 1/m)$ and $\tilde{L}^2 \sim U([-1, -1/m])$. An easy computation then shows that

$$\tilde{\phi}(u) = E[e^{iu\tilde{L}^2}] = \frac{(m-1)^{iu}}{1 + iu},$$

hence, condition (9) is also satisfied. On the other hand,

$$\log \mathcal{E}(mL^1)_t = m(\mu - r)t + m\sigma W_t + \sum_{i=1}^{N_t} \log(1 + m\Delta L_t^1),$$

and therefore

$$\psi(u) = iu \left\{ m(\mu - r) - \frac{m^2\sigma^2}{2} \right\} - \frac{u^2 m^2 \sigma^2}{2} - \frac{\lambda}{m} \frac{iu}{1 + iu}.$$

We see that the expression under the integral in (11) is explicit and only the final integration must be done numerically.

Figure 4 shows the (unconditional) distribution of losses in this example, with data $\mu = 0.1$, $\sigma = 0.2$, $r = 0.03$, $\lambda = 1/3$. The initial capital was $N = 1000\$$, the time horizon $T = 2$ years and the multiplier $m = 2$. The 5% probability corresponds approximately to a loss size of 62\$ (this is the 5% VaR).

7 Discussion

We have argued that the study of CPPI strategies in continuous-time diffusion models fails to account for the possibility of a loss: to account for this "gap" risk jumps must be introduced in the dynamics of the underlying asset. We
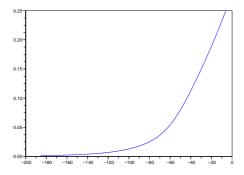
Figure 4: Probability of loss of a given size as a function of loss size (distribution function of losses).

have proposed a simple framework for studying this "gap risk" caused by downward jumps in the value of the underlying portfolio. Our framework leads to analytically tractable expressions for the probability of hitting the floor, the expected loss and the distribution of losses. This allows to measure the gap risk but also leads to a criterion for adjusting the multiplier based on the investor's risk aversion.

We have illustrated these computations in the case of a jump-diffusion model, with parameters estimated on daily stock and index returns. While these data reveal a relatively low level of jump risk, as revealed by past price history, we stress that they are not necessarily the ones to use from a risk management perspective: for a CPPI strategy, the choice of jump parameters can be used to design a stress test of the strategy and the values of jump parameters can be determined with this interpretation in mind. Examples of "crash notes" on CPPI funds have been recently observed in the over-the-counter market. The values of spreads paid for such notes (typically 100 bps above LIBOR for many CPPI funds) can be used as an indicator of "risk–neutral" jump parameters which can be used for pricing the gap risk. The results above apply equally well in both cases.

Finally, CPPI strategies are increasingly applied to credit portfolios. Credit CPPI products are based on CPPI-type strategies applied to a portfolio of defaultable bonds or credit default swaps. In this case the underlying portfolio naturally experiences downward jumps at each default event so the above framework can be useful for analyzing the risk of such products.

Appendix A

In this appendix, we recall two results from stochastic calculus. Details and proofs may be found in [9, Chapter 9] or [15].

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Let X be a semimartingale. Then the stochastic differential equation

$$\frac{dY_t}{Y_{t-}} = dX_t, \quad Y_0 = 1,$$

has a unique strong solution called stochastic or Doléans-Dade exponential of X, denoted by $\mathcal{E}(X)_t$ and written explicitly as

$$\mathcal{E}(X)_t = X_0 e^{X_t - \frac{1}{2}[X]_t^c} \prod_{s \leq t, \Delta X_s \neq 0} (1 + \Delta X_s) e^{-\Delta X_s}.$$

Let us now recall a result linking the ordinary and stochastic exponentials of a Lévy process:

Proposition 4. *1. Let $(X)_{t \geq 0}$ be a real valued Lévy process with Lévy triplet (σ^2, ν, γ) and $Z = \mathcal{E}(X)$ its stochastic exponential. If $Z > 0$ a.s. then there exists another Lévy process $(L_t)_{t \geq 0}$ such that $Z_t = e^{L_t}$ where*

$$L_t = \ln Z_t = X_t - \frac{\sigma^2 t}{2} + \sum_{0 \leq s \leq t} \left\{ \ln(1 + \Delta X_s) - \Delta X_s \right\}.$$

Its Lévy triplet $(\sigma_L^2, \nu_L, \gamma_L)$ is given by:

$$\sigma_L = \sigma,$$

$$\nu_L(A) = \nu(\{x : \ln(1 + x) \in A\}) = \int 1_A(\ln(1 + x))\nu(dx),$$

$$\gamma_L = \gamma - \frac{\sigma^2}{2} + \int \nu(dx) \left\{ \ln(1 + x) 1_{[-1,1]}(\ln(1 + x)) - x 1_{[-1,1]}(x) \right\}.$$

2. Let $(L)_{t \geq 0}$ be a real valued Lévy process with Lévy triplet $(\sigma_L^2, \nu_L, \gamma_L)$ and $S_t = \exp L_t$ its exponential. Then there exists a Lévy process $(X)_{t \geq 0}$ such that S_t is the stochastic exponential of X: $S = \mathcal{E}(X)$ where

$$X_t = L_t + \frac{\sigma^2 t}{2} + \sum_{0 \leq s \leq t} \left\{ 1 + \Delta L_s - e^{\Delta L_s} \right\}.$$

The Lévy triplet (σ^2, ν, γ) of X is given by:

$$\sigma = \sigma_L,$$

$$\nu(A) = \nu_L(\{x : e^x - 1 \in A\}) = \int 1_A(e^x - 1)\nu_L(dx),$$

$$\gamma = \gamma_L + \frac{\sigma_L^2}{2} + \int \nu_L(dx) \left\{ (e^x - 1) 1_{[-1,1]}(e^x - 1) - x 1_{[-1,1]}(x) \right\}.$$

Appendix B

Lemma 1. *Let X_1 and X_2 be random variables with $E[|X_i|] < \infty$, $i = 1, 2$, and denote, by F_i the distribution function of X_i, e.g. $F_i(x) = P[X_i \leq x]$, and by ϕ_i the characteristic function of X_i. Then*

$$\int_{\mathbb{R}} e^{iux}(F_1(x) - F_2(x))dx = \frac{\phi_2(u) - \phi_1(u)}{iu}, \quad u \neq 0. \tag{18}$$

In addition, if

$$\int_{\mathbb{R}} \frac{|\phi_i(u)|}{1 + |u|}du < \infty, \quad i = 1, 2$$

then

$$F_1(x) - F_2(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{\phi_2(u) - \phi_1(u)}{iu}du, \quad all\ x.$$

Proof. First part. Denoting by p_i the laws of X_i, $i = 1, 2$, we have

$$\int_{\mathbb{R}} e^{iux}(F_1(x) - F_2(x))dx$$
$$= \int_{\mathbb{R}} dx e^{iux} \int_{\mathbb{R}} dz \{1_{x<0}1_{z\leq x} - 1_{x\geq 0}1_{z>x}\}(p_1(dz) - p_2(dz)). \tag{19}$$

From the integrability of X_i, it follows that

$$\int_{-\infty}^{0} F_i(x)dx < \infty \quad \text{and} \quad \int_{0}^{\infty} (1 - F_i(x))dx < \infty, \quad i = 1, 2.$$

Therefore, we can use the Fubini theorem to interchange the integrals in (19), which produces

$$\int_{\mathbb{R}} e^{iux}(F_1(x) - F_2(x))dx$$
$$= \int_{\mathbb{R}} dz(p_1(dz) - p_2(dz))\left\{1_{z\leq 0}\frac{1 - e^{iuz}}{iu} - 1_{z\geq 0}\frac{e^{iuz} - 1}{iu}\right\} = \frac{\phi_2(u) - \phi_1(u)}{iu}$$

for every $u \neq 0$.

Second part. Without loss of generality, suppose that F_2 is continuous, because otherwise we could introduce a continuous CDF F_3 and decompose $F_1 - F_2 = F_1 - F_3 + (F_3 - F_2)$. Multiplying both sides of (18) by

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{u^2}{2\sigma^2}}e^{-iut}$$

and integrating with respect to u, we get

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{u^2}{2\sigma^2}}e^{-iut}\frac{\phi_2(u) - \phi_1(u)}{iu}du = \frac{\sigma}{\sqrt{2\pi}} \int e^{-\frac{\sigma^2}{2}(x-t)^2}(F_1(x) - F_2(x))dx.$$

When $\sigma \to \infty$, the right-hand side converges to $F_1(t) - F_2(t)$ in all continuity points of F_1. By the hypothesis $E[|X_i|] < \infty$, $i = 1, 2$, $\phi_i(u) = 1 + O(u)$, $i = 1, 2$, $\frac{\phi_2(u) - \phi_1(u)}{iu}$ is integrable near zero. Since it is also integrable at infinity by the lemma's condition, the left-hand side converges to

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iut} \frac{\phi_2(u) - \phi_1(u)}{iu} du$$

for every t, as $\sigma \to \infty$, and the proof is completed. $\qquad\square$

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