MODERN TONTINES

A Viable Alternative for Retirement Plans?

Pascal Winter[1]
Actuary IA - Qualified
www.winter-aas.com
pascal.winter@winter-aas.com

Frédéric Planchet[2]
Actuary IA - Certified
www.ressources-actuarielles.net
frederic@planchet.net

Copyright ©2020 by the authors
Working Paper – September 2020 – Version 1.8

In the context of global aging population, improved longevity and low interest rates, the question of pension plan under-funding and adequate elderly financial planning is gaining awareness worldwide, both among experts and in popular media. Additional emergence of societal changes - Peer to Peer business model and Financial Disintermediation – might have contributed to the resurgence of "Tontine" in various papers and the proposal of further models such as Tontine Pensions (Forman & Sabin, Survivor Funds, 2016), ITA - Individual Tontine Accounts (Fullmer & Sabin, 2019), Pooled-survival fund (Newfield, 2014), Pooled Annuity Funds (Donnelly, Actuarial fairness and solidarity in pooled annuity funds, 2015), and Modern Tontines (Weinert & Grundl, 2016) to name a few.

In this paper, we revisit the mechanism proposed by (Fullmer & Sabin, 2019) - which allows the pooling of Modern Tontines through a self-insured community. This "Tontine" generalization retains the flexibility of an individual design: open contribution for a heterogeneous population, individualized asset allocation and predesigned annuitization plan. The actuarial fairness is achieved by allocating the deceased proceedings to survivors using a specific individual pool share which is a function of the prospective expected payouts for the period considered.

After a brief introduction, this article provides a formalization of the mathematical framework with prospective analysis, characterizes the inherent bias, generalizes the mechanism to joint lives, and analyses simulated outcomes based on various assumptions. In particular, a reverse moral hazard limit is exposed and discussed (the "Term Dilemma"). Some solutions are then proposed to overcome scheme shortcomings and some requirements for a practical implementation are discussed.

---

[1] VP Finance & Actuarial
BNP Cardif TCB Life – Taipei - TAIWAN

[2] Partner Actuary at Prim'Act
Professor at ISFA

Contents

1 INTRODUCTION

Tontines have a controversial past and are infamous in popular culture[3]. In a form of tontine, the last survivor pockets all the gains making it comparable to a morbid lottery among subscribers, fueling abuses and collective imagination. However, it is probably more embezzlement, bankruptcy and abusive clauses that lead to their demise in the early 20th century. A more thorough review of Tontine history can be found in (McKeever, 2008) or (Milevsky, King William's tontine: Why the retirement annuity of the future should resemble its past, 2015). Additionally, Tontine were almost exclusively structured as fundraising tools while it seems that the interest to re-use them as retirement is a mostly recent endeavor.

The regulatory framework for Tontine is limited – though not forbidden as commonly believed. In France, the "Code des Assurances" stipulates special conditions to form Tontines. Le Conservateur is a typical example of such company, founded in 1844 and still present today in a niche and trending market. Notably, "FairTontine" is an example of an InsurTech which is planning to develop such funds linked to a crypto currency.

In parallel, adequate elderly financial planning is a concern worldwide. Low interest rates and longevity have put retirement schemes and pensions plans under pressure. The PEPP, "Pan-European Pension Plan" regulation issued in 2019 in Europe is a typical example of the political concern longevity can raise in our aging societies.

Pooling longevity risk among insured is common theme in pension research. (Piggott, Valdez, & Detzel, 2005) proposed the Grouped Self Annuitization or Pooled Annuity fund. (Goldsticker, 2007) discussed the possibility to use Mutual Funds to distribute Annuity like benefits. (Stamos, 2008) further analyzed the Pooled Annuity Funds. (Rotemberg, 2009) described a Continuously Liquidating Tontine (or Mutual Inheritance Fund) as a replacement for immediate annuities.

(Sabin, Fair Tontine Annuity, 2010) and (Sabin, A fast bipartite algorithm for fair tontines, 2011) studies in details the mechanism of an open-ended tontine fund with an allocation mechanism based on all member demographics. (Qiao & Sherris, 2013) further developed the GSP – Grouped Self Pooled funds, while (Donnelly, Actuarial fairness and solidarity in pooled annuity funds, 2015) studied the Actuarial Fairness and Solidarity in Pooled Annuity Funds. (Milevsky & Salisbury, Optimal retirement income tontines, 2015) and (Milevsky & Salisbury, Equitable Retirement Income Tontines: Mixing Cohorts Without Discriminating, 2016) proposed further optimization to income Tontines.

---

[3] "The Wrong Box" from Stevenson & Osbourne (1889) later adapted as a film in 1966

(Forman & Sabin, Tontine Pensions, 2015) (Forman & Sabin, Survivor Funds, 2016) (Forman & Sabin, Tontine Pensions Could Solve the Chronic Underfunding of State and Local Pension Plans, 2018) have been also active in the study of Tontine Pensions and Survivors Funds, while (Weinert & Grundl, 2016) provided an overview of such schemes named "Modern Tontines".

2 Modern Tontines Approach

In this section, we revisit the ITA – Individual Tontine Account - concept as defined by (Fullmer & Sabin, 2019). The reason for selecting this formulation is that it is one of the most practical oriented with an acceptable bias of "actuarial fairness" for a large population.

The Modern Tontine is a generalization of a classic Tontine – with an heterogenous population, an open subscription mechanism and flexible outgoes scheme (selected at issue). Once subscribed, there is no withdrawal and proceedings are to be paid upon survival following the schedule selected at onboarding. As a standard annuity or tontine, there are no benefits paid upon death, and the proceedings of deceased members are to be allocated among the survival peers.

2.1 Term, Outflows and Contribution Schedule

2.1.1 Contribution scheme

Since the Modern Tontine can be subscribed at any start of period[4], contribution scheme can be flexible: single, regular, and flexible payments. However, new money is subscribed at current conditions.

A particularity for regular contributions can be elaborated: as discussed below, a selection factor is proposed to be applied on the first 5-10 years for each new payment done (namely, it will reduce the Tontine Share amount to avoid the Term Dilemma effect and reverse moral hazard). We believe that this selection could lifted for regular premium contributions, provided there is no payment lapse. This should create a fidelity advantage for the members who commit and maintain fixed contribution during the accumulation period.

[4] For fairness concern, new members or subscriptions could be added in the fund just after a period end – after the Tontine Gains are allocated. However, one could envision some actuarial interpolation scheme that could allow a prioritization of the Tontine share to allow subscription at any time

2.1.2 Outflow scheme

Similarly, the outflow scheme is fully customizable: lump sum, life annuity, temporary annuity or a hybrid of these – with various weights. The mathematics below will consider a "flow intensity" to apprehend this flexibility.

To be noted is the possibility to design the "flow intensity" with weighs inversely proportional to the exponential probability of death[5] to normalize the gains over the course of the Tontine.

2.2 Reversion or Joint Survival Features

Common features in retirement schemes are the reversion benefit or joint survival life annuity.

A reversionary pension provides a reduced pension payout for the second life in case the first life death precedes the second, while the benefits are unchanged if the second life death happens before the first.

A joint survival annuity pays a given scheme while both lives are alive (the "joint life" status), and switches to another scheme when one life deceases – generally with lower payouts (common commercial proposed ratios range from 50% to 75%).

A generalization of the mathematical framework to encompass such options is proposed below.

2.3 Fund Investment and Allocation

2.3.1 Flexible Allocation

The fund allocation itself is also customizable – where a typical unit-linked mechanism could allow full flexibility for the members to manage and plan their allocation as per their preferences. As shown experimentally by (Fullmer & Sabin, 2019) the fund return volatility of individual member has only a second order impact on the individual performances of the Tontine – provided that the fund size is large enough.

2.3.2 Other Advanced features

Like 401k funds or standard Asset Management services, the investment platform can provide additional services such as a wide fund selection, predefined strategies such as lifestyle re-allocation, automatic arbitrage, portfolio replication, robot-advisor to name a few.

[5] Or the expected Tontine Gain to be more precise

2.4 Mechanism & Actuarial Fairness

2.4.1 Mortality redeemed amount

In a similar fashion to a standard Tontine, the Account Value of deceased members are re-allocated to survivors. In theory, actuarial fairness is maximized using a continuous time frame where proceedings are immediately allocated. In practice, this is hardly realistic

2.4.2 Intuition

The cornerstone of the model is the allocation key to assign mortality gains to the survival population at each time step. This allocation key is based on the mortality probabilities of each members for the assessed period, weighed by the projected account value. This value is referred below as the "Expected Survival Gain" for the whole member horizon or the "Tontine Share" for the specific period where the allocation is made.

Mathematically, this "Tontine Share" can be derived by ensuring that the "Expected Gain" is null. For a given member with a death probability q and an Account value AV, it can be expressed as follow:

$$E[Gain] = 0$$

$$Loss_{Death} + Gain_{Surv} = 0$$

$$-q.AV + TontShare(1 - q) = 0$$

$$TontShare = \frac{q}{(1-q)} AV$$

The resulting formula is familiar: the $\frac{q}{(1-q)}$ factor is the one found in the stepwise change in a recursive annuity reserve calculation (net of discount factor impact).

It is also notable that this amount is independent from the other members. In theory, to be fully exact, the Tontine returns depends on the whole pool demographics. However, as shown experimentally below, the bias induced can be negligible provided the fund is large and the Tontine share are sufficiently homogeneous.

2.4.3 Actuarial fairness

- ⇨ Age / Gender and other characteristics are supposedly embedded in the mortality table assumptions. The table selection challenges are apprehended in the Discussion section.
- ⇨ Different horizon and payment terms are embedded in the prospective view, the Tontine Share defined at a time step level and the allocation of mortality gains only up to the maximum common period of the considered cash flows

- ⇨ The variation of Account Value among members is also reflected in the Tontine Share calculation. It is to be noted that large outliers will impact volatility of the Tontine returns and thus reduce mutualization
- ⇨ The Pool can welcome new entrants at the beginning of every recalculation period
- ⇨ The personalized asset allocation results in various account value evolution. The frequent recalculation of Tontine Share allows to reflect the impact of volatile results on the Tontine Share at each period beginning

2.5 WITH BEQUEST ALTERNATIVE

From a commercial perspective, offering only non-redeemable option without any benefit in case death is a key limiting factor. Though not explored in this article, it is to be noted that it is technically possible to bundle the Modern Tontine with a standard investment platform – inside which the members retain the flexibility of top-ups and withdrawals along with the balance returned to beneficiary in case of death. This fund could use the same fund management infrastructure –but, from an actuarial fairness perspective, this fund cannot benefit from the Modern Tontine additional longevity returns.

2.6 VALUE PROPOSITION – SUMMARY

2.6.1 Pros and Cons from Consumer and Insurer / Administrator perspective

	Advantages	Limits & Attention Points
Pool Member	Additional Gain thanks to Tontine Returns Lower charges – no risk premium Flexibility (payments, scheme, and investment) Transparency of mechanism “P2P” without the need for a carrier	No Benefits upon death & no redemption possible Volatility of returns (Longevity, Idiosyncratic Mortality, Market risk) Complexity of mechanism to be exposed
Insurer / Administrator	No underfunding risk (Longevity, Market risk) Synergies with Asset management activity	Regulatory framework Term dilemma & Adverse selection Mortality table choice & selection factors Survival checks

2.6.2 Features Comparison – Modern Tontine vs Standard Fund

`	**Standard Fund**	**Modern Tontine**
Fund Selection	Unrestricted	Unrestricted
Entry Age	Unrestricted	40 ~ 80*
Annuity Age	*NA*	40 ~ 100*
Change Scheduled Payments	Yes	**No**
Change Fund Selection	Yes	Yes
Contribution Scheme Modification	Yes	Yes**
Partial or Full Surrender	Yes	**No**
Capital on Death	Yes	**No**
Additional Survival Returns	**No**	Yes

*Entry Age limits and maximum contribution will have to be set according the bias and volatility discussion below

**Payment Lapses could trigger the loss of the selection factor waiving

3 MATHEMATICAL FRAMEWORK

3.1 FRAMEWORK

We denote:

- Time[6] $t \in [\![0, T]\!]$ with T being the term, index k.
- Members $n \in [\![0, N]\!]$, index i
- Survival probability of a member of age x at age $t + x$: ${}_tp_x$
- Account Value for member i at time t :
  - "beginning of period" $AVbop_t^i$ (includes contribution)
  - "middle of period" $AVmop_t^i$ (includes financial return, before decease redeem and survival allocation)
  - "end of period" $AVeop_t^i$ (after death redeem and survival allocation)

Calculations are made using the following conventions:

- Annuity payments made at period end
- Contribution made at period start
- t starts at 0 (first period is 0)

3.2 REDEEM AMOUNT & ALLOCATION MECHANISM

Note: calculation below are at the step t_c. Members age are noted x at time t_c.

3.2.1 Redeem Amount

- **<u>Tontine redeem</u>** $TontRedeem_{t_c}$ – is forfeited account value (after financial return) AV_{mop}^i of all the deceased numbers during the period t_c:

$$TontRedeem_{t_c} = \sum_{i:Death} AV_{mop}^i$$

3.2.2 Redeem Amount Allocation to survived members

- **<u>The Tontine Share</u>** for member n is the Fair Expected Survival Gain for the time t_c based on the member account value (after financial return) AV_{mop}^n :

$$TontShare_{t_c}^n = \left(\frac{q_{x_n}}{1 - q_{x_n}}\right) AV_{mop}^n$$

[6] For convenience, a yearly unit has been arbitrarily chosen, but a monthly or quarterly step could be equally considered.

- **The Tontine Return:** $TontReturn_{t_c}^n$ is the total forfeited account value $TontRedeem_{t_c}$ allocated to survived members using $TontShare_{t_c}^n$ as an allocation key:

$$\forall n: Survived \qquad TontReturn_{t_c}^n = TontRedeem_{t_c} \frac{TontShare_{t_c}^n}{\sum_{i:Surv} TontShare_{t_c}^i}$$

In order to recoup with the notation defined in (Fullmer & Sabin, 2019), we can define the "Group Gain" as:

$$\mathrm{G}_{t_c} = \frac{TontRedeem_{t_c}}{\sum_{i:Surv} TontShare_{t_c}^i}$$

and ensure:

$$\forall n: Survived \qquad TontReturn_{t_c}^n = TontShare_{t_c}^n . \mathrm{G}_{t_c}$$

3.3 Actuarial Fairness and Bias Analysis

Calculations below are at the step t_c. Members age are noted x at time t_c.

3.3.1 Consistency Check

Thanks to the Tontine Share construction, it is easy to prove that the Expected Value of $TontRedeem_{t_c}$ is equal to the Expected Value of all the Tontine Shares for survivors:

$$E\left[TontRedeem_{t_c}\right] = E\left[\sum_{i:Surv} TontShare_{t_c}^i\right]$$

By noting:

$$E\left[TontRedeem_{t_c}\right] = E\left[\sum_{i:Death} AV_{mop}^i\right]$$

and developing the second term:

$$E\left[\sum_{i:Surv} TontShare_{t_c}^i\right] = \sum_i (1 - q_{x_i}).TontShare_{t_c}^i \quad =$$

$$= \sum_i \left[(1 - q_{x_i}).\left(\frac{q_{x_i}}{1 - q_{x_i}}\right) AV_{mop}^i\right] = \sum_i q_{x_i}.AV_{mop}^i \; = E\left[\sum_{i:Death} AV_{mop}^i\right]$$

We get the equality. This is however not enough to prove that the allocation model works.

3.3.2 Model actuarial fairness* - Intuition

Having the allocation model work is equivalent to show that the Tontine Returns expected value are in line to the Tontine Returns for each individual member. In practice, this is not the case, as shown and discussed by (Donnelly, Actuarial fairness and solidarity in pooled annuity funds, 2015) and (Sabin & Forman, 2016)

This bias exists since the total mortality proceeds (Tontine Returns) of a given period depend on the individual member status (alive or not), creating a bias in the group gain.

A simple way to grasp the intuition is to create a fictive pool with 2 profiles:

1) A single member with a large Account Value and an extremely high death probability
2) Many other members with relatively low Account Value and low death probabilities

	Profile 1	Profile 2
# Member	1	5,000
AV	500,000	1,000
qx	5.00%	0.20%
Tontine Share	26,316	2.004
Exp. Tont. Red.	25,000	2.0

In this case, the pool is very unlikely to have enough depth to cover the 1st profile Tontine Share in case of survival, as shown in below table:

Pool level	Profile 1	Profile 2	Pool
Total AV	500,000	5,000,000	5,500,000
Tontine Share	26,316	10,020	36,336
Exp. Tont. Red.	25,000	10,000	35,000

3.3.3 Model actuarial fairness - Demonstration

To characterize the fairness, one can express the individual Expected Tontine Returns as a function of the individual Tontine Share:

$$\forall n \in [\![0, N]\!] \qquad E\left[TontReturn\ {}_{t_c}^{n}\right] = TontShare_{t_c}^{n}.\frac{(1-q_n)}{1-q_n+\frac{q_n AV_{mop}^{n}}{\sum_{i\neq n} q_i AV_{mop}^{i}}}$$

which can also be expressed as:

$$\forall n \in [\![0, N]\!] \qquad E\left[TontReturn\ {}_{t_c}^{n}\right] = TontShare_{t_c}^{n}.\frac{\sum_{i\neq n} q_i AV_{mop}^{i}}{TontShare_{t_c}^{n}+\sum_{i\neq n} q_i AV_{mop}^{i}}$$

This result shows that the upper bound presented in (Sabin & Forman, 2016) is in fact reached and provides the exact value of the bias. To demonstrate above equalities, one can write

$$TontReturn\,_{t_c}^{n} = TontShare_{t_c}^{n}.\frac{\sum_{i:Death} AV_{mop}^{i}\ {}_{t_c}}{\sum_{i:Surv} TontShare_{t_c}^{i}}$$

$$= TontShare_{t_c}^{n}.\frac{\sum_{i} AV_{mop}^{i}\ {}_{t_c}\ 1_{\{T_i=t_c\}}}{\sum_{i} TontShare_{t_c}^{i}\ 1_{\{T_i>t_c\}}}$$

thus

$$TontReturn\,_{t_c}^{n} \times \sum_{i} TontShare_{t_c}^{i}\ 1_{\{T_i>t_c\}} = TontShare_{t_c}^{n}.\times \sum_{i} AV_{mop}^{i}\ 1_{\{T_i=t_c\}}$$

By isolating the terms of individual n, one finds

$$TontReturn\,_{t_c}^{n} \times \left(TontShare_{t_c}^{n}\ 1_{\{T_n>t_c\}} + \sum_{i\neq n} TontShare_{t_c}^{i}\ 1_{\{T_i>t_c\}} \right)$$
$$= TontShare_{t_c}^{n}.\times \left(AV_{mop}^{n}\ 1_{\{T_n=t_c\}} + \sum_{i\neq n} AV_{mop}^{i}\ 1_{\{T_i=t_c\}} \right)$$

By taking the mathematical expectation and using the independence among tontine members, one get

$$E\left[TontReturn\,_{t_c}^{n}\right] \times \left(TontShare_{t_c}^{n} + \sum_{i\neq n} TontShare_{t_c}^{i} \times (1-q_i) \right)$$
$$= TontShare_{t_c}^{n}.\times \sum_{i\neq n} AV_{mop}^{i}\, q_i$$

Because $TontShare_{t_c}^{i} = \frac{q_i}{(1-q_i)} AV_{mop}^{i}$, this equality leads to

$$E\left[TontReturn\,_{t_c}^{n}\right] = TontShare_{t_c}^{n}.\frac{\sum_{i\neq n} q_i AV_{mop}^{i}}{TontShare_{t_c}^{n} + \sum_{i\neq n} q_i AV_{mop}^{i}}$$

By further replacing Tontine Share expression:

$$E\left[TontReturn\,_{t_c}^{n}\right] = TontShare_{t_c}^{n}.\frac{\sum_{i\neq n} q_i AV_{mop}^{i}}{\frac{q_n}{(1-q_n)} AV_{mop}^{n} + \sum_{i\neq n} q_i AV_{mop}^{i}}$$

Which can also be written:

$$E\left[TontReturn\ {}_{t_c}^{n}\right] = TontShare_{t_c}^{n}.\frac{(1-q_n)}{1-q_n+\frac{q_n AV_{mop}^{n}}{\sum_{i\neq n} q_i AV_{mop}^{i}}}$$

3.3.4 Individual Bias Error

Re-using above notations, we can identify the error bias as[7]:

$$error_n = \frac{E\left[TontReturn\ {}_{t_c}^{n}\right]}{TontShare_{t_c}^{n}} - 1 = \frac{(1-q_n)}{1-q_n+\frac{q_n AV_{mop}^{n}}{\sum_{i\neq n} q_i AV_{mop}^{i}}} - 1$$

Which can also be expressed as:

$$error_n = -\frac{TontShare_{t_c}^{n}}{TontShare_{t_c}^{n} + \sum_{i\neq n} q_i AV_{mop}^{i}}$$

It is to be noted that the error bias tends towards 0 when N is large: $\lim_{N\to\infty} error_n = 0$

For convenience, one can further approximate the error as follow:

$$error_n \sim -\frac{TontShare_{t_c}^{n}}{\sum_{i:Surv} TontShare_{t_c}^{i}}$$

Which experimentally shows that the bias is highly linked to the “atomization” of the Tontine Share – or the heterogeneity of the $q.AV$.

3.4 The Term Dilemma and Possible Moral Hazard

3.4.1 Positive Selection factor and Moral Hazard

The moral hazard in Annuities is a common subject in actuarial field. (Valdez, Piggott, & Wang, 2006) showed that adverse selection can exist in Annuities and GSA (Group Self Annuitization) – although the effect is expected to be less severe in GSA than in traditional annuity. While we do not think it should be an issue for non-voluntary contribution (compulsory retirement funds, proceeds from a term life…) – we believe that there will be a positive selection effect for elective contribution.

3.4.2 The Term Dilemma - Description

The Term dilemma arises from the fact that it is possible to “breakdown” a given investment in 2 sub-terms while keeping the same Tontine Returns. For instance, instead of investing for

[7] This formulation is mathematically equivalent to the one proposed by (Sabin & Forman, 2016)

a lumpsum target of 10 year, one could elect to invest in a 5 years term first, then reinvest 5 year later to reach the term of 10 years.

Of course, in the second case, the member would have an option not to follow its investment after the first 5 years (in case of health issues for instance), while the first choice locks the member for 10 years.

This raises an issue in terms of fairness and makes the Modern Tontine workable only if everybody elects to invest on the shortest period available – which is against the essence of the scheme.

3.4.3 The Term Dilemma – Proof in Lump Sum case[8]

Let us consider 2 terms $t_1 > t_2$ and 2 members with the same demographics:

- 1st member elects a lumpsum of term t_1 and initial contribution AV
- 2nd member elects a lumpsum of term t_2 and initial contribution AV, and will then re-elect a lumpsum with of term t_1 with the proceedings

We assume that both will survive on the whole period and for simplification, that the fund member is infinite (otherwise their survival would marginally impact the global return).

Using above notations, the 1st member has a Survival Expected Payout of:

$$SEP_{t_1}^1 = AV.\frac{1}{{}_{t_1+1}p_x}$$

The 2nd member has its first Survival Expected Payout as follow:

$$SEP_{t_2}^2 = AV.\frac{1}{{}_{t_2+1}p_x}$$

The 2nd member second and final Expected Survival Gain at t_1 is as follow (he/she reinvests its SEP):

$$SEP_{t_1}^2 = SEP_{t_2}^2.\frac{1}{{}_{t_1+1}p_{x+t_2}} = AV.\frac{1}{{}_{t_2+1}p_x}.\frac{1}{{}_{t_1+1}p_{x+t_2}} = AV.\frac{1}{{}_{t_1+1}p_x} = SEP_{t_1}^1$$

Which shows that both members have the same Survival Expected Survival Gain.

[8] The demonstration can be generalized to all $\{FN_t\}$, with the optimal strategy for the "moral hazard" benefactor being to invest on one year terms and reinvesting the proceedings minus the payment expected for the "normal" member at each time step

3.5 PROSPECTIVE VIEW AT A GIVEN TIME

The mathematic framework for a single period is enough to proceed to the simulation. However, flexible outflow selection scheme requires some prospective indicators. Additionally, some of the indicators will be used to test the adequacy of the model.

Notes:

1. *Calculation below are at a member level. For simpler notations, the member index n is not present in this section.*
2. *All the calculation below are seen from a given t_c. For simpler notations, the initial t_c is not present in this section*
3. *In practice, this prospective view will be updated at each time step since it depends on actual Tontine Returns and actual financial returns*

3.5.1 Expected Payouts (at member level)

- **Nominal Flow (at future time t):** this is the outflow intensity selected by the member at subscription. It is used to norm the outflows. $\{FN_t\}$
- The **Nominal Survival Expected Payout**[9] $nSEP$ - this is the basic unit to allocate the Initial Amount across the nominal flow:

$$nSEP = \frac{AV}{\sum_t FN_t \cdot {}_{t+1}p_x}$$

- The **Survival Expected Payouts** $\{SEP_t\}$ – payment amount made in case of survival based on current account value - can be easily derived:

$$SEP_t = nSEP.FN_t$$

- The **Total Survival Expected Payout** SEP is the sum of expected payouts

$$SEP = \sum_t SEP_t = nSEP.\sum_t FN_t = AV.\frac{\sum_t FN_t}{\sum_t FN_t \cdot {}_{t+1}p_x}$$

- The **Payout Present Value** $\{PPV_t\}$ are the outflows weighed by current Account Value, intensity, and survival probability. At a single future time step:

$$PPV_t = SEP_t \cdot {}_{t+1}p_x = nSEP.FN_t \cdot {}_{t+1}p_x = \frac{AV}{\sum_k FN_k \cdot {}_{k+1}p_x}.FN_t \cdot {}_{t+1}p_x$$

It is notable that by construction $\sum_t PPV_t = AV$

[9] $nSEP$ can be interpreted as the Face Amount of an Annuity with outflow intensity $\{FN_t\}$ and Premium AV.

3.5.2 Expected Survival Gain and Tontine Share (at member level)

- The **Expected Survival Gain** is defined as the surplus of the Total Survival Expected Payout compared with current Account Value on the whole term of the member ITA:

$$ESG = SEP - AV = AV\left(\frac{\sum_t FN_t}{\sum_t FN_t \cdot {}_{t+1}p_x} - 1\right)$$

- The prospective **Tontine Share[10] at a future time t** $TontShare_t$ is the Expected Survival Gain at each time step seen at time 0. A convenient way to express it is the prospective Account Value at beginning of the period t weighed by the decrement probability during period t:

$$TontShare_t = \left(\frac{q_{x+t}}{1 - q_{x+t}}\right)\frac{\sum_{k \geq t} PPV_k}{{}_tp_x} = nSEP\left(\frac{q_{x+t}}{1 - q_{x+t}}\right)\frac{\sum_{k \geq t} FN_k \cdot {}_{k+1}p_x}{{}_tp_x}$$

- The **Tontine Share at time 0**, is the allocation key that is used in the model. It is also consistent with the definition given above, provided the AV includes the financial return earned during the period:

$$TontShare_o = \left(\frac{q_{x+t}}{1 - q_{x+t}}\right)AV$$

- For consistency, we can ensure that $\sum_t TontShare_t = ESG$:

$$\sum_t TontShare_t = nSEP\sum_t\left(\frac{q_{x+t}}{1 - q_{x+t}}\right)\frac{\sum_{k \geq t} FN_k \cdot {}_{k+1}p_x}{{}_tp_x}$$

$$= nSEP\sum_t\sum_{k \geq t}\left(\frac{q_{x+t}}{1 - q_{x+t}}\right)\frac{FN_k \cdot {}_{k+1}p_x}{{}_tp_x}$$

$$= nSEP\sum_t\sum_{k \geq t}\frac{FN_k \cdot {}_{k+1}p_x \cdot q_{x+t}}{p_{x+t} \cdot {}_tp_x} = nSEP\sum_t\sum_{k \geq t}\frac{FN_k \cdot {}_{k+1}p_x \cdot q_{x+t}}{{}_{t+1}p_x}$$

By re-indexing the sum:

$$= nSEP\sum_k\sum_{t \leq k}\frac{FN_k \cdot {}_{k+1}p_x \cdot q_{x+t}}{{}_{t+1}p_x} = nSEP\sum_k\left[FN_k \cdot \sum_{t \leq k}\frac{{}_{k+1}p_x \cdot q_{x+t}}{{}_{t+1}p_x}\right]$$

and noting that:

$$\sum_{t \leq k}\frac{{}_{k+1}p_x \cdot q_{x+t}}{{}_{t+1}p_x} = \sum_{t \leq k} {}_{k-t}p_{x+t+1} \cdot q_{x+t} = \sum_{t \leq k} {}_{k-t}p_{x+t+1} \cdot (1 - p_{x+t})$$

$$= \sum_{t \leq k}\left({}_{k-t}p_{x+t+1} - {}_{k-t}p_{x+t+1} \cdot p_{x+t}\right) = \sum_{t \leq k}\left({}_{k-t}p_{x+t+1} - {}_{k-t+1}p_{x+t}\right) = 1 - {}_{k+1}p_x$$

---

[10] By construction, $TontShare_t$ can be interpreted as the stepwise change of mathematical reserve for an Annuity with outflow intensity $\{FN_t\}$ and Single Premium AV.

We obtain:

$$\sum_t TontShare_t = nSEP \sum_k [FN_k . (1 - {}_{k+1}p_x)]$$

$$= nSEP \sum_t (FN_t - FN_t . {}_{t+1}p_x) = nSEP \sum_t \left(FN_t - \frac{PPV_t}{nSEP}\right)$$

$$= nSEP \sum_t FN_t - AV = SEP - AV = ESG$$

3.5.3 Generalization with an expected return yield

In further model projection, it will be interesting to compare the SEP and $TontShare$ flows expected at issue date versus their actual counterparts. To do so effectively, an expected yield needs to be incorporated.

By noting v_t the discount factor at time t (derived from the expected return r_k):

$$\forall t \; v_t = \prod_{k \leq t} (1 + r_k)^{-1}$$

We get:

- The **<u>Survival Expected Payouts with expected returns</u>** $\{SEP'_t\}$ – payment amount made in case of survival based on current account value - become:

$$SEP'_t = \frac{nSEP . FN_t}{v_t}$$

- The **<u>Total Survival Expected Payout with expected returns</u>** SEP' is the sum of expected payouts

$$SEP' = \sum_t SEP'_t = nSEP . \sum_t \frac{FN_t}{v_t} = AV . \frac{\sum_t \frac{FN_t}{v_t}}{\sum_t FN_t . {}_{t+1}p_x}$$

- The **<u>Payout Present Value</u>** $\{PPV_t\}$ is unchanged by construction
- The prospective **<u>Tontine Share at a future time t with expected returns</u>** $TontShare'_t$ becomes:

$$TontShare'_t = \left(\frac{q_{x+t}}{1 - q_{x+t}}\right) \frac{\sum_{k \geq t} PPV_k}{{}_tp_x . v_t} = nSEP \left(\frac{q_{x+t}}{1 - q_{x+t}}\right) \frac{\sum_{k \geq t} FN_k . {}_{k+1}p_x}{{}_tp_x . v_t}$$

<u>Note:</u> the equality $\sum_t TontShare_t = ESG$ does not hold since the expected survival gain includes Financial returns as well.

3.6 JOINT LIFE AND REVERSIONARY FEATURES

3.6.1 Joint Life Generalization via synthetic schemes

A rather straightforward method to generalize the Tontine Share calculation to joint life is to create synthetic schemes that replicates the selected option payouts, as described by (Promislow, 2011). The idea is to express all possible joint life options as a sum of 3 basic components which are contingent to events that are easily manipulated from a probabilistic perspective: the 1st life survival, the 2nd life survival and a pure joint-life survival status (ie both are alive).

For instance, let's denote the nominal flows $\{FN_t^1\}$ that are paid when the 1st life is alive, $\{FN_t^2\}$ when the second life is alive and $\{FN_t^{pj}\}$ when both the first and second life are alive.

The Reversion option can be expressed as two individual life payout scheme and a negative pure joint life payout scheme:

- ⇨ $\{FN_t^1\}$ the initial outflow scheme, which applies to the joint life status and that will be maintained for the 1st life in case the 2nd life death precedes
- ⇨ $\{FN_t^2\}$ the reversion flows for the 2nd life in case the 1st life death precedes
- ⇨ $\{FN_t^{pj}\} = -\{FN_t^2\}$, the synthetic pure joint outflows

The Joint Survival Life option can similarly be expressed as two individual life payout scheme and a negative pure joint life payout scheme. By noting $\{FN_t^{joint}\}$ the amount to be paid when both lives are alive, one can write:

- ⇨ $\{FN_t^1\}$ the outflow for the 1st life in case the 2nd life death precedes
- ⇨ $\{FN_t^2\}$ the outflow for the 2nd life in case the 1st life death precedes
- ⇨ $\{FN_t^{pj}\} = \{FN_t^{joint}\} - \{FN_t^1\} - \{FN_t^2\}$, the synthetic pure joint outflows

3.6.2 Account Value allocation

Once the target joint life payout scheme has been broken down into the 3 basic components $\{FN_t^1\}$, $\{FN_t^2\}$, $\{FN_t^{joint}\}$ the mathematics is common to all joint life options.
By re-using previous notations, and noting ${}_tp_x$, ${}_tp_y$ and ${}_tp_{xy}$ the actuarial probabilities for 1st life, 2nd life and pure joint life status, one can get the **Nominal Payout Present Value**[11] $nPPV$ for each of the component:

$$nPPV^1 = \sum_t FN_t^1 \cdot {}_{t+1}p_x$$

[11] This equivalent to the nominal present value of an annuity with a schedule $\{FN_t\}$

$$nPPV^2 = \sum_t FN_t^2 \cdot {}_{t+1}p_y$$

$$nPPV^{pj} = \sum_t FN_t^{pj} \cdot {}_{t+1}p_{xy}$$

The current account value to be used in the Tontine Share allocation can then be allocated among the schemes. For instance:

$$AV^{pj} = AV. \frac{nPPV^{pj}}{nPPV^1 + nPPV^2 + nPPV^{pj}}$$

3.6.3 Tontine Share

The Tontine share calculation is then equivalent to the one done in the single life case – performed on each of the individual components using the appropriate decrements.

$$TontShare_t = \left(\frac{q_x}{1-q_x}\right) AV_{mop}^1 + \left(\frac{q_y}{1-q_y}\right) AV_{mop}^2 + \left(\frac{q_{xy}}{1-q_{xy}}\right) AV_{mop}^{pj}$$

4 Modelisation: Conventions & Hypothesis

4.1 Convention and hypothesis

4.1.1 Global conventions

- Annual step[12]
- Payment at the end of the period
- Contribution made at the beginning of the period
- Tontine Share are calculated after FIN Return and before survival
- 3 funds are projected: Low, Mid and High volatility (with Low, Mid and High returns)

4.1.2 Mortality

In terms of mortality, we will use the latest Taiwan TSO 2011. This is purely an arbitrary choice for illustration purpose – the choice of mortality will be further discussed below.
The selection factor used is arbitrary too and fixed to 40% in first year increasing to 90% with an annual step of 5% (member presence).

[12] For convenience, a yearly unit has been arbitrarily chosen, but a monthly or quarterly step could be equally considered.

4.2 ALGORITHM

The algorithm used can be summarized as follow:

Start period:

- ⇨ `Initialise (IF policies, px…)`
- ⇨ `Add new members and new contributions`

"Mid" period:

- ⇨ `Add Financial Return for the step`
- ⇨ `Calculate Tontine Share and Expected SEP`

End Period:

- ⇨ `Calculate Tontine Redeem amount`
- ⇨ `Remove deceased members from the pool`
- ⇨ `Allocate Tontine Redeem based on survivors Tontine Share`
- ⇨ `Prepare Next iteration (add new Insured, initialize bop variables)`

4.3 SCENARIOS GENERATION

1000 scenarios, including both mortality and fund scenarios.

4.3.1 Random Number Generator

The Mersenne Twister pseudo random number generator algorithm is being used.

4.3.2 Fund Scenarios

For the matter of generating stochastic returns, a standard Black & Scholes framework with intercorrelated Brownian motions has been selected.

Returns and Volatility:

	Return	**Volatility**
Low Vol	2%	5%
Mid Vol	4%	10%
High Vol	8%	20%

Correlations are as follow:

Correlation	**Low Vol**	**Mid Vol**	**High Vol**
Low Vol	1	0.2	0.1
Mid Vol		1	0.4
High Vol			1

4.3.3 Mortality Scenarios

The mortality scenarios are derived from a random uniform distribution (between 0 and 1) applied to the survival function at member enter date.

4.4 METRICS USED

To assess the model, following indicators will be extracted from the simulation tool.

- **Mortality A/E ratio:** this is the standard A/E ratio for mortality, expressed by count or amount
- **Tontine Share A/E ratio:** is the Tontine Gain compared to the Expected Tontine Share supposed to be accumulated. The Expected Tontine Share can either be the expected Share calculated at subscription (based on prospective indicators and average financial return) or the one calculated at beginning of each allocation step (path dependent).
- **SEP A/E ratio:** same as previous but for the Survival Expected Payouts vs Actual Payouts. The Expected Amount can also be estimated at subscription or at the star of each period (path dependent).

5 MODELISATION - GLOBAL

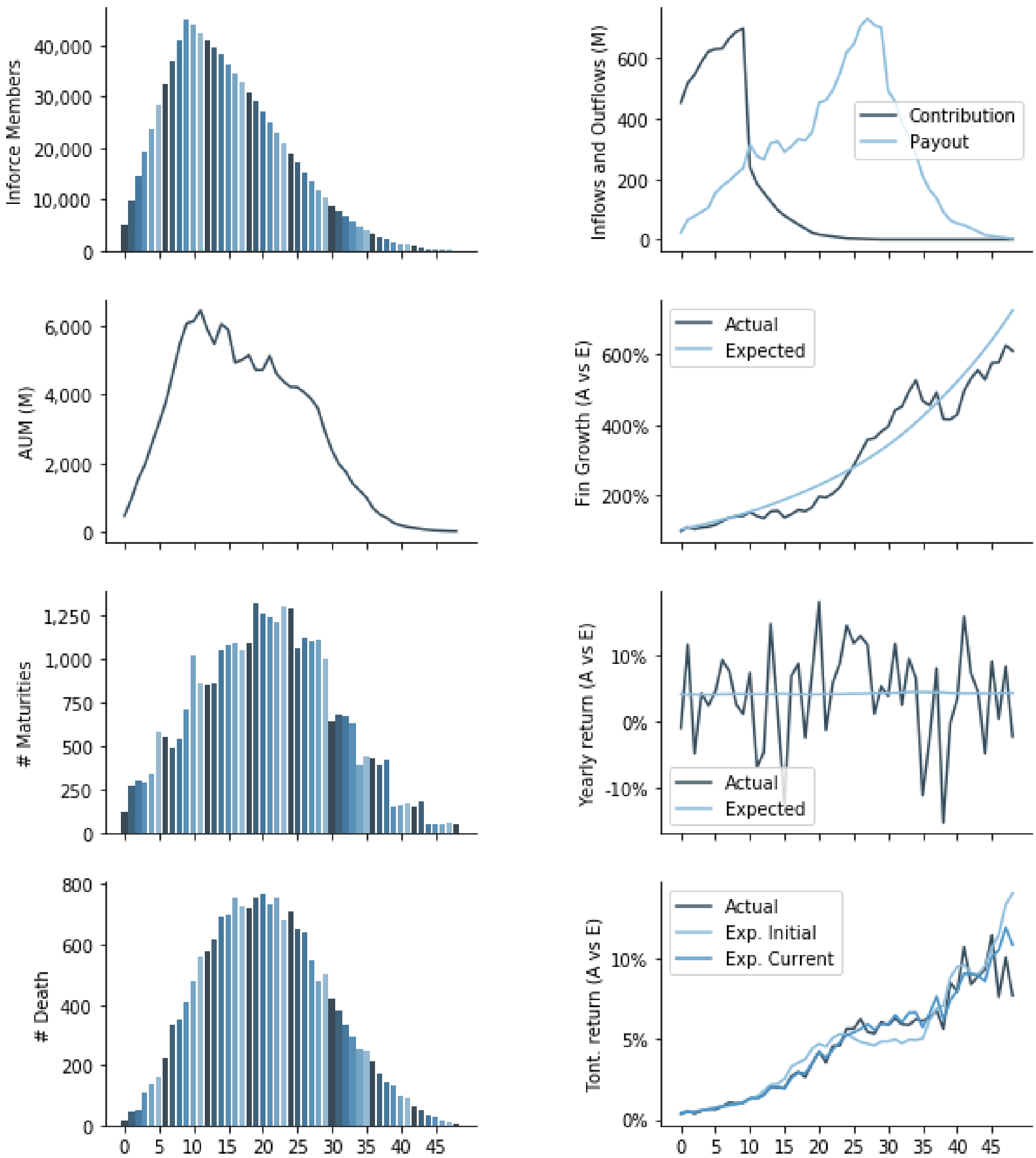
5.1 MODEL POINTS

5.1.1 Distribution

- ⇨ 5,000 insured per year for 10 years – then run-off
- ⇨ 40 to 70 years old entry age, Male & Female equal proportion
- ⇨ Distributed contribution: Single Pay, 5, 10, 15 and 20 year pay
- ⇨ Annuitization starts at 65 up to 100
- ⇨ Asset Allocation: Random among the 3 funds (by default, we assume rebalancing of asset at each step with the target allocation)

5.2 Single Simulation Result

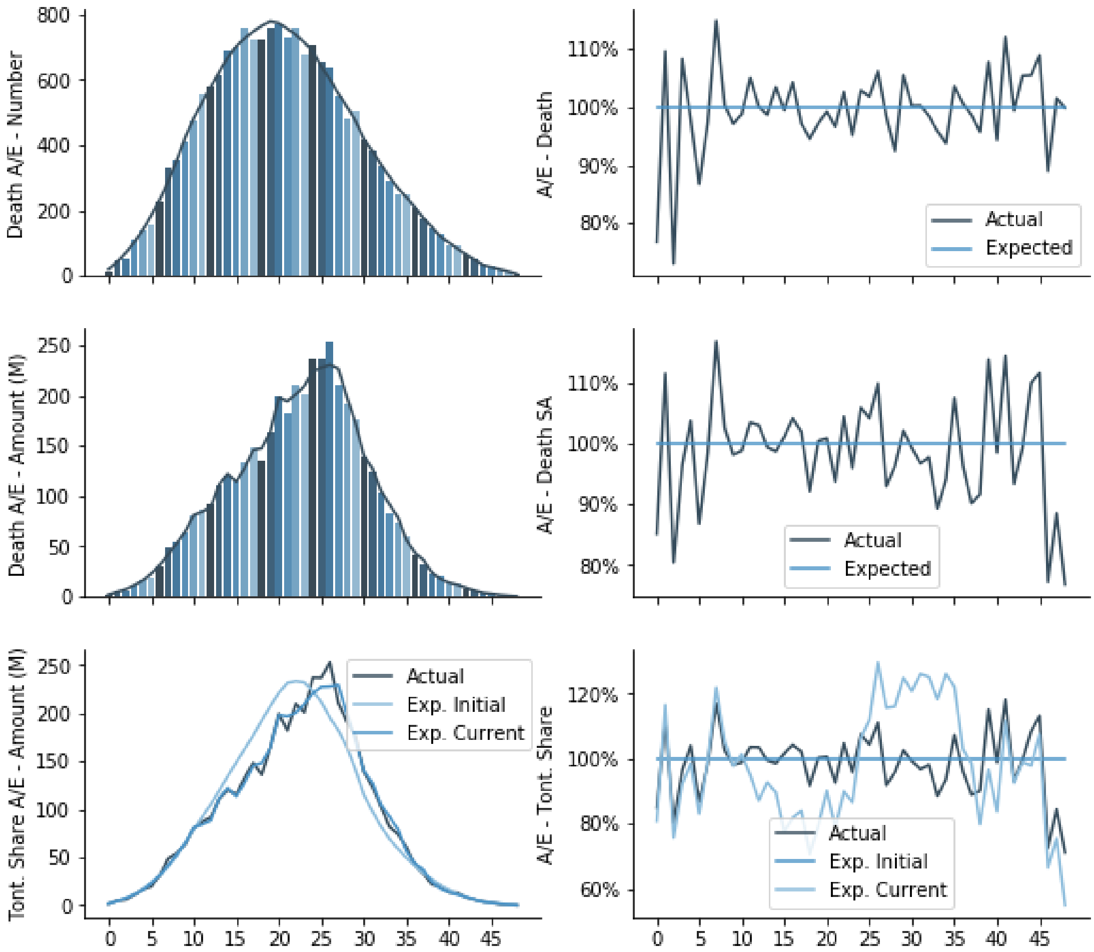
5.2.1 Fund Overview over the years



The above graphs show a single simulation of the fund over the year. The left column allows the apprehend the peak of population (reached around 10 years with little less than 50,000) and the extinction of the fund in 50 years. Maturities are staged depending on annuity length selected by members. Finally, death count is slightly skewed on the right compared with in force members count, logically increased by aging population.

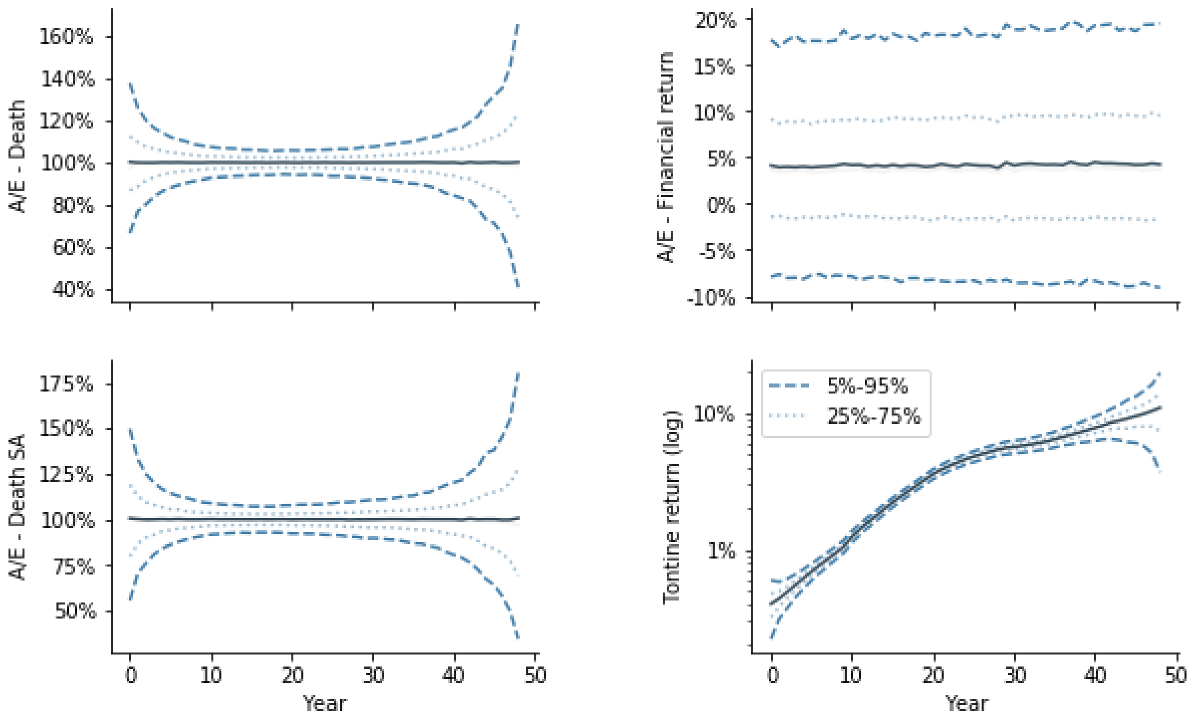
Right graphs illustrate contribution and payout, and the investment and Tontine returns compared with expected. The Tontine Returns increases with age – which is expected, but so is its deviation from expected from benchmark. Interestingly, we can see the impact of financial return on Tontine Returns with the “initial benchmark”.

5.2.2 Actual vs Expected Tontine Returns



Right side graphs show the correlation of A/E ratio on mortality with A/E ratios for Tontine Share. On the last graph, the “Current benchmark” stays fairly closes to 1 in middle of the Tontine and deviates increasingly and start and end, due to population size and idiosyncratic bias. Initial Benchmark shows the additional impact of Financial return on A/E for Tontine share. When recouping with previous graph, we can observe the similar movements with Financial return (lower, higher, lower).

5.1 Whole Simulation Distribution Analysis



As observed on a single simulation, the Tontine return follows a similar deviation at start and end of the fund, due to idiosyncratic mortality risk linked to small fund size. This shows the the importance of having the largest pool possible to neutralize this volatile effect.

6 Modelisation – Focused Tests

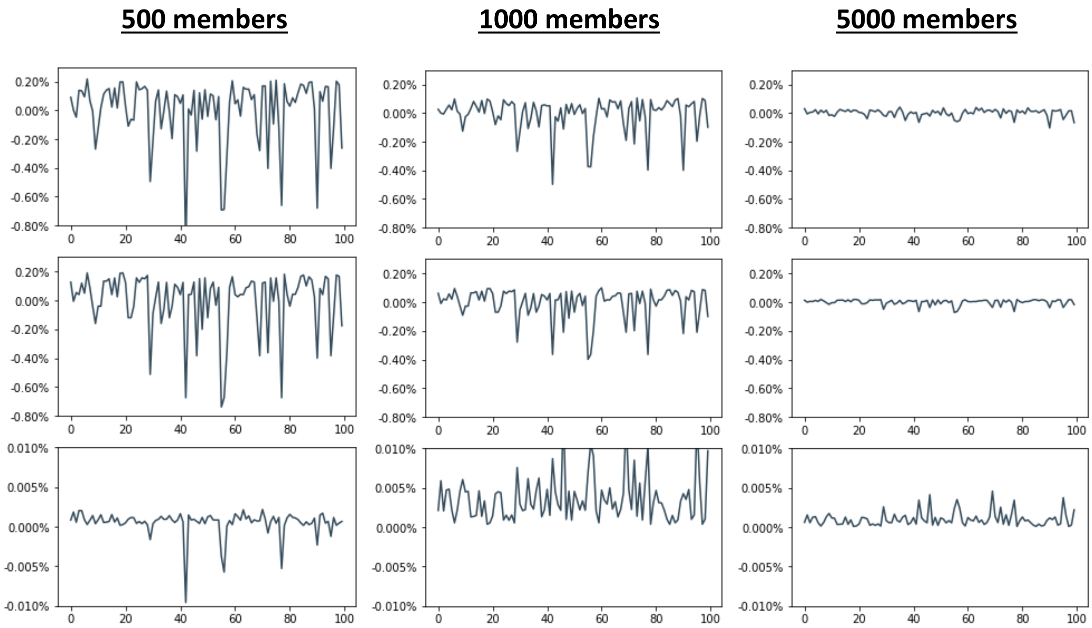
6.1 Actuarial Fairness Bias

6.1.1 Simulation

To isolate the bias, here are the projection hypothesis used:

- ⇨ **500, 1000 and 5,000** insured
- ⇨ 40 to 70 years old entry age, Male & Female equal proportion
- ⇨ Distributed contribution: **Single Pay only**
- ⇨ Only **one year projected**, **Lump Sum**
- ⇨ Asset Allocation: **financial return forced to 0**

6.1.2 Results



1st Line: Observed Bias, 2nd Line: Proxy Bias estimation, 3rd Line: The Bias reported

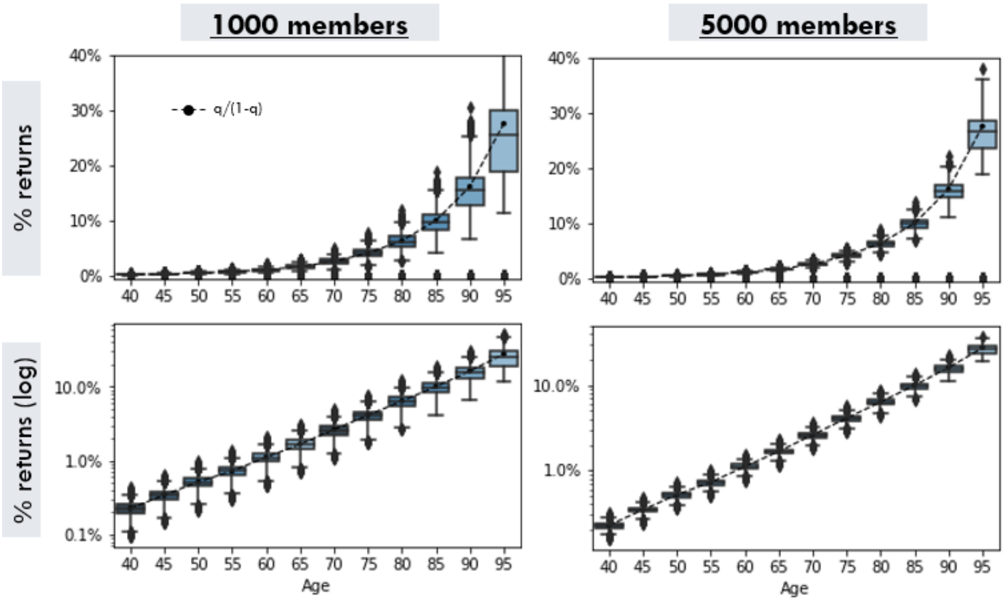
The impact of fund size of on bias is evident, with high deviation for 500 members mostly tampered when above 5000 members. To be noted the 2nd line of graph which shows the fitting of the bias proxy in this case.

6.2 SENSITIVITY ANALYSIS MORTALITY & LONGEVITY DEVIATIONS

6.2.1 Simulation

To isolate the bias, here are the projection hypothesis used:

- ⇨ **1000 and 5,000** insured
- ⇨ 40 to 70 years old entry age, **Male only**
- ⇨ Distributed contribution: **Single Pay only**
- ⇨ Only **one year projected**, **Lump Sum**
- ⇨ Asset Allocation: **financial return forced to 0**
- ⇨ **Selection factors forced to 1**



Based on above graphs (1st line), Tontine Returns are consistent with the mortality factor – the average mortality scenario. Of course, the returns in case of survival are limited for people at 40 years while they are indeed remarkably high for high ages (95). In current low rate environment, we could conclude that they are material after 65. Given this observation, low ages are not meaningful in terms of return for these solutions (a minimum age of 40 with long horizon), while higher ages could benefit from an outflow schedule that neutralizes the exponential growth of the force of mortality.

7 Discussion

7.1 Technical

7.1.1 Allocation Bias

As shown, the model has an inherent bias, linked to the inter-dependency of individual Tontine returns with pool returns. In practice, we observed that this bias is proportional to the ratio of the member tontine share divided by total tontine share. Also, this bias was small compared to the overall returns and mortality idiosyncratic risk.

To manage this bias, the key is to limit the "atomization" of the Tontine Share – ie ensure that there is no member with an abnormally high share compared to the rest. The Tontine share depending both on the death probability and the account value – it seems logical to introduce limitation in terms of maximum contribution and minimum / maximum age. Along with enough member participation (5,000 seems enough from our simulation), the bias becomes negligible.

7.1.2 Idiosyncratic Mortality Risk

The idiosyncratic is a significant source of volatility for the Modern Tontine returns. The same measures described in the bias mitigation can be taken: ensure large pool size (5,000 members and over seems ideal), limit the entry age (40 to 80), and limit the contribution size.

7.1.3 Financial Risk

The scheme being designed on a "Unit-Linked" concept, the financial risk will be bear exclusively by the members. Various investment strategies – passive and active – could be proposed as a service to each member to mitigate this risk and match their preferences.

As a note, it should be reminded that a member would primarily bear the risk linked to its own asset allocation. The return of the other member of the pool will only impact the members' Tontine Returns – and thus the overall investment only as a second order factor.

7.1.4 Reverse Moral Hazard and Term Dilemma

Due to non-refundable nature of the Tontine in case of death, once could expect a natural self-selection process on elective schemes – qualified as a "Reverse Moral Hazard"

Additionally, as shown above, the Term dilemma is a significant drawback for elective plans – and should be carefully considered. A way to characterize it is to consider an "option" for the member to discontinue the Tontine pooling in case additional information about his/her health arises. The value of such "option" could be tentatively valued, assuming one could separate the "sudden" from the "foreseeable" causes of death at a given horizon. This separation of the mortality could allow to use different decrements depending on the terms selected. In practice, this equates using selection factors calibrated on the "predictiveness" of the causes of death.

Practically, some mitigators to manage reverse moral hazard and term dilemma could be:

- ⇨ **Propose Modern Tontines only for "compulsory" plans** where outflows are preset and contribution not elective (government retirement plan).
- ⇨ **Create sub Modern Tontines funds for each maturity**. Given sensitivity of this scheme to Mortality idiosyncratic risk – this solution seems sub-efficient.
- ⇨ **Introduce a selection factor for early years**: Like a Term Life with strict underwriting, one could imagine a selection factor to be applied on the mortality table selected for the first years of a member in the Modern Tontine. This will favor longer terms return-wise and should counterbalance the Term Dilemma benefits.
- ⇨ **Increase minimum maturity:** Along with the selection factor introduction – allowing a minimum term between the first investment and the outgoes would allow to level

the adverse selection risk. A minimum term of 5 ~ 10 years seems aligned with the purpose of a retirement plan.

7.1.5 The Step Length Selection

The step at which at which the Tontine mechanism is triggered is an important consideration practically. On a pure theoretical standpoint, the “instantaneous” allocation is the most accurate. For modelling purpose, an annual step has been used. Some constraints arise: existing members need to prove their survival and new entrants would expect to join the pool as soon as possible. Given the importance of Tontine Returns in the scheme, we tend to prioritize the survival checks - especially when some actuarial interpolation techniques could be applied to the new joiners. The driver here to select the step would probably be the technology used for the survival checks.

7.2 PRACTICAL / COMMERCIAL

7.2.1 Regulatory Framework

PEPP – Pan European Pension Plan – shows the attention politics and regulators give to adequate elderly financial planning. Further work is however expected to fit the Modern Tontines in an existing framework.

7.2.2 No Benefit Upon Death

Though not exclusive to Tontine – this is a limitation from consumer perspective. Providing a with bequest alternative (and thus no Tontine returns) could respond to this drawback. A reversion scheme could also be designed by generalizing the mathematics or using automatic transfer from the with-bequest to the tontine fund with appropriate time and weights.

7.2.3 Complexity of Mechanism to be exposed

Exposing the mechanism to consumer will be a limitation – especially given the possible volatility on returns and the “sharing” nature of the mortality proceeds. Illustration, transparency, and regular communication will be required.

7.2.4 Mortality Table

The Tontine share – cornerstone of the allocation model – is highly dependent on the mortality assumption retained. Choosing the mortality across different generation raises several questions: best estimate assessment, segmentation, and re-evaluation.

Best estimate assessment: As most of actuarial study, it should be appropriate with the target population and available experience, either internal and/or external.

Segmentation: Up to which level the segmentation of mortality assessment should be done is left open. The model shown used a standard Age / Gender segmentation as per the mortality table used for illustration. This question goes beyond the sole technical point and is ultimately an arbitrage between fairness, solidarity, and regulation.

Re-evaluation of assumptions: Once size and experience is large enough; it should be possible to develop "internal" experience benchmarks. The question on whether and how to impact existing and new joiners is left open.

7.2.5 Selection Factors

As discussed above – selection factors are expected to be a key mitigator for the "Term Dilemma" and "Reverse Moral Hazard" on elective schemes. From insurance lines in case of death, one can observe that the underwriting selection effect generally lasts around 5 years, and seldom last more than 10 years. Ideally, these factors could be further calibrated by entry age – especially if the expected mortality gap is wide from a member to another.

Several methods could be used to derive these factors, among them:

- ⇨ Calibration from experience on Annuity portfolios with similar features
- ⇨ Approximation from other lines underwriting effect
- ⇨ Mortality causes analyses and separation among "sudden" and "foreseeable"

7.2.6 Regular Survival Checks

As the history of tontine has shown, fraud is a possibility that cannot be excluded. Survival checks can be time consuming and would directly impact the operation of the pools, as discussed during the step selection. The technology used to realize this task will directly impact the administration efficiency and benefits for members.

8 Conclusion

"The Tontine is perhaps the most discredited financial instrument in history"[13].

Used primarily as a fund-raising vehicle, their history is indeed tainted with scandals, bankruptcies, and a popular belief of "indecency" toward gamble on human life.

In the current context of pension fund underfunding epidemic, Modern Tontines, a generalization of the tontine scheme, could become a viable retirement instrument and fill part of the increasing need for adequate elderly financial planning. Without the need for a carrier, they could offer attractive returns in a highly flexible, "P2P" retirement plan. However, there are limitations since the financial, idiosyncratic mortality and longevity risks would be borne by the pool.

Technically, the method presented contain an inherent fairness bias – linked to the atomization of the Tontine Share. We have observed mathematically and experimentally that this bias could be negligible with appropriate limits sets in terms of fund size, demographics, and contribution size. Similarly, the idiosyncratic mortality risk is a direct function of the pool size and its homogeneousness. With the financial risk being a consequence of the member choice and preferences in terms of allocation and strategy, the pool is left with the longevity risk which is much more complex to mitigate.

Operationally, annuities can be subject to moral hazard and in extreme cases fraud. The "Term Dilemma" is a serious drawback of the model – which can however be mitigated by setting adequate minimum term limits and introducing some selection factors on the mortality to favor longer terms. The survival check will also be a key operational challenge and its implementation will dictate the robustness of the pool along with some of its characteristics. Aside from the Tontine mechanism, Modern Tontines as presented here are close to a classic asset management or unit-linked insurance activity – a well-known and developed activity. Finally, Modern Tontines implementation is tightly linked with regulatory framework.

[13] Attributed to Edward Chancellor - (McKeever, 2008)

9 BIBLIOGRAPHY

Chen, A., Hieber, P., & K., K. J. (2018). Tonuity: A novel individual-oriented retirement plan. *Astin Bulletin 49(1)*, 5-30.

Donnelly, C. (2014). Bringing cost transparency to the life annuity market. *Insurance: Mathematics and Economics 56*, 14-27.

Donnelly, C. (2015). Actuarial fairness and solidarity in pooled annuity funds. *ASTIN Bulletin 45 (1): 49-74*.

Forman, J. B., & Sabin, M. J. (2015). Tontine Pensions. *University of Pennsylvania Law Review 163 (3)*, 755–831.

Forman, J. B., & Sabin, M. J. (2016). Survivor Funds. *Pace Law Review 37 (1)*, 204 - 91.

Forman, J. B., & Sabin, M. J. (2018). Tontine Pensions Could Solve the Chronic Underfunding of State and Local Pension Plans. *SOA - Retirement 20/20 Papers*.

Fullmer, R. K., & Sabin, M. J. (2019). Individual Tontine Accounts. *Journal of Accounting and Finance, 19(8)*, Doi.org/10.33423/jaf.v19i8.2615.

Goldsticker, R. (2007). A Mutual Fund to Yield Annuity-Like Benefits. *Financial Analysts Journal 63 (1)*, 63-67.

McKeever, K. (2008). A Short History of Tontines. *Fordham Journal of Corporate and Financial Law 15 (2)*, Article 5.

Milevsky, M. A. (2015). *King William's tontine: Why the retirement annuity of the future should resemble its past.* Cambridge University Press.

Milevsky, M. A., & Salisbury, T. S. (2015). Optimal retirement income tontines. *Insurance: Mathematics and Economics 64*, 91-105.

Milevsky, M. A., & Salisbury, T. S. (2016). Equitable Retirement Income Tontines: Mixing Cohorts Without Discriminating. *ASTIN Bulletin 45 (1)*, 49-74.

Milevsky, M. A., Salisbury, T. S., Gonzalez, G., & Jankowski, H. (2019). Annuities Versus Tontines in the 21st Century: A Canadian Case Study. *SOA: Retirement Section Research Committee*.

Newfield, P. (2014). The Tontine: An Improvement on the Conventional Annuity? *The Journal of Retirement Winter 2014, 1 (3)*, 37- 48.

Piggott, J., Valdez, E. A., & Detzel, B. (2005). The Simple Analytics of a Pooled Annuity Fund. *Journal of Risk and Insurance 72 (3)*, 497 - 520.

Promislow, S. D. (2011). *Fundamentals of Actuarial Mathematics: Second Edition.* John Wiley & Sons, Ltd. ISBN: 978-0-470-68411-5.

Qiao, C., & Sherris, M. (2013). Managing Systematic Mortality Risk With Group Self-Pooling and Annuitization Schemes. *Journal of Risk and Insurance 80 (4)*, 949-74.

Rotemberg, J. J. (2009). Can a Continuously-Liquidating Tontine (or Mutual Inheritance Fund) Succeed Where Immediate Annuities Have Floundered? *Harvard Business School BGIE Unit Working Paper No. 09-121, Harvard Business School Finance Working Paper No. 09-121*.

Sabin, M. J. (2010). Fair Tontine Annuity. *Available at SSRN: https://ssrn.com/abstract=1579932 or http://dx.doi.org/10.2139/ssrn.1579932*.

Sabin, M. J. (2011). A fast bipartite algorithm for fair tontines. *Available at SSRN: https://ssrn.com/abstract=1848737 or http://dx.doi.org/10.2139/ssrn.1848737* .

Sabin, M. J., & Forman, J. B. (2016). *The Analytics of a Single-Period Tontine.* Available at SSRN: https://ssrn.com/abstract=2874160.

Stamos, M. Z. (2008). Optimal consumption and portfolio choice for pooled annuity funds. *Insurance: Mathematics and Economics 43 (1)*, 56-68.

Valdez, E. A., Piggott, J., & Wang, L. (2006). Demand and adverse selection in a pooled annuity fund. *Insurance: Mathematics and Economics 39 (2)*, 251-266.

Weinert, J.-H., & Grundl, H. (2016). The Modern Tontine: An Innovative Instrument for Longevity Risk Management in an Aging Society. *ICIR Working Paper Series No. 22/2016*.