Mortality modelling with Lévy processes.

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Abstract

This paper addresses the modelling of human mortality by the aid of doubly stochastic processes with an intensity driven by a positive Lévy process. We focus on intensities having a mean reverting stochastic component. Furthermore, driving Lévy processes are pure jump processes belonging to the class of α-stable subordinators. In this setting, expressions of survival probabilities are inferred, the pricing is discussed and numerical applications to actuarial valuations are proposed.

Keywords : Stochastic mortality, longevity risk, Lévy processes.

1 Introduction.

Other the last decades, significant improvements in the duration of life have been observed in most countries, see e.g. the comparative study of MacDonald et al. (1998). As a consequence, a bad anticipation of this evolution threads life insurers: traditional valuation methods relying on deterministic mortality models (for a survey, see the international comparative study of mortality tables for pension fund retirees, GC 2005) may lead to underestimate prices and reserves related to contract providing long-term living benefits. See e.g. the paper of Olivieri (2001), which investigates the effects of uncertainty coming from projections.

One can cope with the uncertainty over mortality trends by modelling the mortality as a stochastic process. The recent literature on this topic is prolific and widely inspired from credit risk theory. In many papers, the mortality hazard rate depends on an affine process, see for instance Biffis (2005 a), Biffis et al. (2005 b), Biffis and Denuit (2006), Dahl (2004), Dahl and Møller (2006), Luciano and Vigna (2005), Schrager (2006). The main interesting feature of affine models is their analytical and computational tractability (see Duffie 2001 for an introduction).

Most recently, Cariboni and Schoutens (2006) have investigated the use of Lévy processes to price defaultable bonds. They assume that the intensity of default follows an Ornstein-Uhlenbeck (OU) dynamic, driven by positive Lévy processes, namely subordinators. Such models were initially developed by Barndorff-Nielsen and Shepard (2001a, 2002b) and used to price options on stocks with stochastic volatility (Nicolato and Vernados 2003). Our work aims to model human mortality in a similar way: as in Biffis (2005 a), we consider that the mortality hazard rate is the sum of one deterministic component and of one mean reverting stochastic process, driven by tempered α-stable subordinators (for an introduction see Cont and Tankov 2004). In this setting, survival probabilities are inferred from the knowledge of the Lévy process characteristic function.

We also presents two pricing approaches of life insurance claims. The first method is directly inspired from the practice of financial markets and assumes that the pricing is done under an

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equivalent measure. However, as the insurance market is incomplete, this pricing measure depends on the insurer's preferences. The second approaches is the indifference pricing and is based on the specification of insurer's utility function.

The outline of the paper is as follows. First, we briefly review the features of doubly stochastic processes and how to use them to model mortality. Section 3 presents the dynamic of mortality rates. In section 4, we first develop some general results when mortality is driven by a α stable subordinators and next study two particular cases, namely Gamma processes and Inverse Gaussian processes. Section 5 addresses the issue of pricing life insurance contracts. Section 6 briefly depicts the recent mortality trends of the Belgian population and defines some demographic indicators. Section 7 is finally devoted to numerical applications. We present examples of mortality projections and discuss briefly the pricing of life annuities.

2 Stochastic mortality.

As in paper of Dahl (2004), Dahl & Møller (2006), we consider an insurance portfolio consisting of n insured lives of the same age x . All processes mentioned hereafter are defined on a probability space $(\Omega, \mathcal{F}, P)$. The policyholder's remaining lifetimes are identically distributed random variables, denoted $T_1,\ T_2, \ldots, T_n$. For $i = 1, \ldots, n_x$, $\left(\mathcal{H}_t^i\right)_{t\geq 0}$ is the smallest $\sigma-$algebra with respect to which T_i is a stopping time. We note by μ_t^x the intensity of $T_{i=1\ldots n}$, at time t, for an individual of age x at $t = 0$. This intensity, also called the mortality rate is a non negative predictable process, defined on a filtration $\{\mathcal{G}_t\ :\ t \geq 0\}$, and such that $\int_0^t \mu_s^x.ds < \infty$ almost surely. Moreover, we assume that

$$P(T_i > t \,|\, \mathcal{G}_{T^*}) \quad = \quad e^{-\int_0^t \mu_s^x ds} \qquad 0 \leq t \leq T^*$$

This entails that conditionally to $\mathcal{G}_\infty$, the remaining lifetimes are independent inhomogeneous Poisson processes. The information available up to time t is contained in the filtration $\left(\mathcal{F}_t\right)_t$:

$$\mathcal{F}_t \quad = \quad \mathcal{G}_t \vee \mathcal{H}_t^1 \vee \ldots \vee \mathcal{H}_t^{n_x}$$

The subfiltration $\left(\mathcal{G}_t\right)_t$ provides hence enough information about the evolution of the intensity of mortality but is insufficient to determine the actual occurrence of death at time t. The number of deaths observed till time t is a counting process noted N_t and is a sum of indicator variables:

$$N_t = \sum_{i=1}^{n_x} I(T_i \leq t)$$

N_t is a $\mathcal{F}_t$-markov process, whose stochastic intensity is informally given by:

$$\mathbb{E}\left(dN_t | \mathcal{F}_t\right) \quad = \quad (n_x - N_{t-}).\mu_t^x.dt \tag{2.1}$$

We refer the interest reader to Brémaud (2001) chapter 2 and to Duffie (2001), appendix I, for further details on doubly stochastic processes. In this setting, the expected number of survivors at time $s > t$ is equal to:

$$\begin{aligned}
\mathbb{E}\left((n_x - N_s) | \mathcal{F}_t\right) \quad &= \quad \mathbb{E}\left(\sum_{i=1}^{n_x} I(T_i > s) | \mathcal{F}_t\right) \\
&= \quad \sum_{T_i > t} P\left(I(T_i > s) | \mathcal{F}_t\right) \\
&= \quad (n_x - N_t).\underbrace{\mathbb{E}\left(\exp\left(-\int_t^s \mu_u^x.du\right) | \mathcal{G}_t\right)}_{{}_{s-t}p_{x+t}}
\end{aligned} \tag{2.2}$$

where ${}_{s-t}p_{x+t}$ is the probability that an individual of age $x + t$, survives till age $x + s$. The next paragraph details the dynamic of μ_t^x.

3 Mean reverting intensity.

As in examples of affine mortality models developed in the work of Biffis (2005 a), we have assumed that the mortality rate is the sum of one deterministic component $\mu_x(t)$ and of one random variable Y_t:

$$\mu_t^x = \mu_x(t) + Y_t \tag{3.1}$$

$\mu_x(t)$ may be seen as a best estimate assumption or an available demographic basis whereas the variable Y_t represents random departure from the deterministic mortality table. The probability that an individual of age $x+t$ survives till age $x+s$ can be split as follows:

$$_{s-t}p_{x+t} = \underbrace{\exp\left(-\int_t^s \mu_x(u).du\right)}_{_{s-t}\bar{p}_{x+t}} . \underbrace{\mathbb{E}\left(\exp\left(-\int_t^s Y_u.du\right)|\mathcal{G}_t\right)}_{ADJ(t,s,Y_t)}$$

where

- $_{s-t}\bar{p}_{x+t}$ is the survival probability related to deterministic mortality rates.
- $ADJ(t,s,Y_t)$ is an adjustment factor, taking into account the evolution of future mortality.

The dynamic of Y_t is assumed to be mean reverting:

$$dY_t = a.\left(b - Y_t\right).dt + \sigma.dZ_t \tag{3.2}$$

where a , b are constant and Z_t is a Lévy process with an initial value $Z_0 = 0$. A Lévy process has independent, stationary increments and is continuous in probability (for an introduction see Cont and Tankov 2004 or Applebaum 2004). The characteristic function of Z_t denoted $\phi_t(u)$ is infinitely divisible: for every positive integer n, $\phi_t(u)$ is the nth power of a characteristic function. A Lévy process Z_t may be decomposed in three components (Lévy Ito decomposition): one deterministic drift $\gamma.t$, one Brownian motion of variance A and one jump process of intensity $\nu(z)$, the Lévy measure of Z_t. The triplet $(\gamma,\ A,\ \nu(z))$ fully determines the characteristic function of Z_t:

$$\begin{aligned}
\phi_t(u) &= \mathbb{E}\left(\exp\left(i.u.Z_t\right)\right) \\
&= \exp\left(t.\left(i.\gamma.u - \frac{1}{2}.u.A.u + \int_{\mathbb{R}} \left(e^{i.u.z} - 1\right)\nu(dz)\right)\right)
\end{aligned}$$

The random measure associated to Z_t is denoted $J_Z(dt,dz)$ and is such that:

$$Z_t = \int_0^t \int_{-\infty}^{+\infty} z.J_Z(dt,dz)$$

Furthermore, the compensator of Z_t is $C_t = t.\int_{-\infty}^{+\infty} z.\nu(dz)$. As the mortality hazard rate μ_t^x has to be a non negative $\mathcal{G}_t$- predictable process, one limits the study to Lévy processes having almost surely increasing trajectories, namely subordinators. By definition, the Lévy decomposition of such processes doesn't have any Brownian component ($A = 0$). Furthermore, one works with subordinators without any deterministic drift , $\gamma = 0$. The process Y_s , for all $s > t$, may easily be rewritten as follows:

$$Y_s = \left(1 - e^{-a.(s-t)}\right).b + e^{-a.(s-t)}.Y_t + \sigma.\int_t^s e^{-a.(s-\theta)}.dZ_\theta \tag{3.3}$$

To proof this, we apply the Ito's lemma to $\eta_s = e^{a.s}.\left(b - Y_s\right)$:

$$\begin{aligned}
d\eta_s &= a.e^{a.s}.\left(b - Y_s\right).ds - e^{a.s}.dY_s \\
&= -\sigma.e^{a.s}.dZ_s
\end{aligned}$$

Therefore, η_s is such that:

$$\eta_s = \eta_t - \sigma. \int_t^s e^{a.\theta}.dZ_\theta$$

and using the relation $Y_s = b - e^{-a.s}.\eta_s$ leads to the desired result (3.3). The integral $\Lambda_{t,s} = \int_t^s Y_u du$, which is involved in the calculation of $ADJ(t,s,Y_t)$ is equal to:

$$\int_t^s Y_u du = b.(s-t) + \frac{1}{a}.(Y_t - b).\left(1 - e^{-a.(s-t)}\right) + \frac{\sigma}{a}.\int_t^s \left(1 - e^{-a.(s-\theta)}\right).dZ_\theta \tag{3.4}$$

Characteristic functions of Y_s and $\Lambda_{t,s}$ will be useful in the sequel of this work to valuate the expectation of Y_t and survival probabilities:

$$\mathbb{E}\left(\exp\left(i.u.Y_s\right)|\mathcal{F}_t\right) = \exp\left(i.u.\left(\left(1 - e^{-a.(s-t)}\right).b + e^{-a.(s-t)}.Y_t\right)\right) .\mathbb{E}\left(\exp\left(i.u.\sigma.\int_t^s e^{-a.(s-\theta)}.dZ_\theta\right)|\mathcal{F}_t\right) \tag{3.5}$$

$$\mathbb{E}\left(\exp\left(i.u.\Lambda_{t,s}\right)|\mathcal{F}_t\right) = \exp\left(i.u.\left(b.(s-t) + (Y_t - b).\left(1 - e^{-a.(s-t)}\right).\frac{1}{a}\right)\right) .\mathbb{E}\left(\exp\left(i.u.\frac{\sigma}{a}.\int_t^s \left(1 - e^{-a.(s-\theta)}\right).dZ_\theta\right)|\mathcal{F}_t\right) \tag{3.6}$$

The adjustment factor $ADJ(t,s,Y_t)$ is equal to the characteristic function of $\Lambda_{t,s}$ valued at point $u = i$. The expectation in former equations can be calculated by the proposition of Eberlein and Raible (1999).

Proposition 3.1. *Let Z_t be a subordinator with the cumulant transform*

$$k(\theta) = \log \mathbb{E}(\exp(\theta.Z_1))$$

and let f:$\mathbb{R}_+ \to \mathbb{C}$ be a complex valued left continuous function such that $|Re(f)| \leq M$ then

$$\mathbb{E}\left(\exp\left(\int_0^t f(\theta).dZ_{\beta.\theta}\right)\right) = \exp\left(\int_0^t \beta.k(f(\theta)).d\theta\right) \tag{3.7}$$

Amongst positive Lévy processes, the class of tempered α-stable subordinators presents many interesting features such diversified distributions of Z_t or analytical tractability. For this reason, the rest of the paper focuses on this category of processes.

4 Tempered α-stable subordinators.

4.1 General case.

A process Z_t is said to be a α-stable subordinator, with $\alpha \in [0,1[$ if for all $a > 0$, the following equality holds in distribution:

$$\left(\frac{Z_{a.t}}{a^{\frac{1}{\alpha}}}\right)_{t\geq 0} =_d Z_t$$

Again, we refer to Applebaum (2004) for a detailed presentation of this class of subordinators. The Lévy measure of α-stable subordinators is:

$$\nu(z) = \frac{c}{z^{\alpha+1}}.I_{z>0} \qquad \alpha \in [0,1[$$

As explained in Cont and Tankov (2004), tempered stable processes are obtained by multiplying the Lévy measure of a stable process with a decreasing exponential:

$$\nu(z) = \frac{c.e^{-\lambda.z}}{z^{\alpha+1}}.I_{z>0} \qquad \alpha \in [0,1[$$

where λ is a non negative constant. This exponential softening amortizes the size of large jumps whereas it keeps the stable-like behavior of small jumps. In this setting, it is easily proved that the cumulant transform $k(\theta) = \log \mathbb{E}(\exp(\theta.Z_1))$ is worth:

$$\begin{aligned}
k(\theta) &= \int_0^{+\infty} \left(e^{\theta.z} - 1\right).\nu(dz) \\
&= c.\Gamma(-\alpha).\{(\lambda-\theta)^\alpha - \lambda^\alpha\} \qquad if\ \alpha \neq 0 \\
&= -c.\log\left(1 - \frac{\theta}{\lambda}\right) \qquad if\ \alpha = 0
\end{aligned} \tag{4.1}$$

If we apply the proposition (3.1) to the characteristic function (3.5) of Y_t, and differentiate it, we obtain the first moment of Y_t :

$$\begin{aligned}
\mathbb{E}\left(Y_s|\mathcal{F}_t\right) &= \left(1 - e^{-a.(s-t)}\right).b + e^{-a.(s-t)}.Y_t \\
&\quad -c.\Gamma(-\alpha).\alpha.\lambda^{\alpha-1}.\frac{\sigma}{a}.\left(1 - e^{-a.(s-t)}\right) \qquad if\ \alpha \neq 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left(Y_s|\mathcal{F}_t\right) &= \left(1 - e^{-a.(s-t)}\right).b + e^{-a.(s-t)}.Y_t \\
&\quad +c.\frac{1}{\lambda}.\frac{\sigma}{a}.\left(1 - e^{-a.(s-t)}\right) \qquad if\ \alpha = 0
\end{aligned}$$

Differentiating twice the characteristic function leads to the second moment of Y_t and its variance:

$$\mathbb{V}\left(Y_s|\mathcal{F}_t\right) = c.\Gamma(-\alpha).\alpha.(\alpha-1).\lambda^{\alpha-2}.\frac{\sigma^2}{2.a}.\left(1 - e^{-2.a.(s-t)}\right) \qquad if\ \alpha \neq 0$$

$$\mathbb{V}\left(Y_s|\mathcal{F}_t\right) = \frac{c}{\lambda^2}.\frac{\sigma^2}{2.a}.\left(1 - e^{-2.a.(s-t)}\right) \qquad if\ \alpha = 0$$

By taking limits for $s \to +\infty$ of expectations and variances, one infers the long term behavior of Y_t:

$$\begin{aligned}
\mathbb{E}\left(Y_\infty\right) &= b - c.\Gamma(-\alpha).\alpha.\lambda^{\alpha-1}.\frac{\sigma}{a} \qquad if\ \alpha \neq 0 \\
&= b + c.\frac{1}{\lambda}.\frac{\sigma}{a} \qquad if\ \alpha = 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}\left(Y_\infty\right) &= c.\Gamma(-\alpha).\alpha.(\alpha-1).\lambda^{\alpha-2}.\frac{\sigma^2}{2.a} \qquad if\ \alpha \neq 0 \\
&= \frac{c}{\lambda^2}.\frac{\sigma^2}{2.a} \qquad if\ \alpha = 0
\end{aligned}$$

The expectation and variance of the long term departure Y_∞ from the floor $\mu_x(t)$ can be used to set the parameters in order to have reasonable mortality declines over the remaining lifetime of the cohort.

By applying the proposition (3.1) to the characteristic function (3.6) of $\Lambda_{t,s}$, one infers the following expression for the adjustment factor $ADJ(t,s,Y_t)$:

$$\begin{aligned}
ADJ(t,s,Y_t) &= \mathbb{E}\left(\exp\left(-\Lambda_{t,s}\right)|\mathcal{F}_t\right) \\
&= \exp\left(-\left(b.(s-t) + (Y_t - b).\left(1 - e^{-a.(s-t)}\right).\frac{1}{a}\right)\right). \\
&\quad \exp\left(\int_t^s k\left(-\frac{\sigma}{a}.\left(1 - e^{-a.(s-\theta)}\right)\right).d\theta\right)
\end{aligned} \tag{4.2}$$

where $k(.)$ is the cumulant transform (4.1). Excepted in two particular cases developed in next subsections, it is not possible to calculate explicitly the integral of $k(.)$ in the expression of the adjustment factor. However, it can be easily numerically computed.

4.2 Gamma process.

When $\alpha = 0$, the subordinator is called "Gamma process" because its increments are Gamma random variables. The probability density of Z_t is in this case:

$$p_{Z_t}(z) = \frac{\lambda^{c.t}}{\Gamma(c.t)}.z^{c.t-1}.e^{-\lambda.z} \qquad \forall z > 0$$

The expectation and variance of Z_t are in this case:

$$\begin{aligned} \mathbb{E}(Z_t) &= \frac{c.t}{\lambda} \\ \mathbb{V}(Z_t) &= \frac{c.t}{\lambda^2} \end{aligned}$$

As the distribution of Z_t is known, gamma processes are easily simulated. The adjustment factor is rewritten as:

$$\begin{aligned} ADJ(t,s,Y_t) &= \exp\left(-\left(b.(s-t)+(Y_t-b).\left(1-e^{-a.(s-t)}\right).\frac{1}{a}\right)\right). \\ & \exp\left(-c.\int_t^s \log\left(1+\frac{\sigma}{a.\lambda}.\left(1-e^{-a.(s-\theta)}\right)\right).d\theta\right) \end{aligned}$$

The integral of the cumulant transform is here equal to:

$$\begin{aligned} \int_t^s \log\left(1+\frac{\sigma}{a.\lambda}.\left(1-e^{-a.(s-\theta)}\right)\right).d\theta = \Bigg[\frac{1}{a}.dilog\left(\frac{\sigma}{\sigma+a\lambda}e^{-a.(s-\theta)}\right) \\ +\frac{1}{a}.\left(\log\left(1+\frac{\sigma}{a\lambda}\left(1-e^{-a.(s-\theta)}\right)\right)\log\left(\frac{\sigma}{\sigma+a\lambda}e^{-a.(s-\theta)}\right)\right)\Bigg]_{\theta=t}^{\theta=s} \end{aligned}$$

Note that $dilog(x)$ is the dilogarithm function and is defined as:

$$dilog(x) = \int_1^x \frac{\log(u)}{1-u}.du$$

4.3 Inverse Gaussian process.

When α =1/2, the subordinator is called "Inverse Gaussian process" because its increments are Inverse Gaussian random variables. The probability density of Z_t is in this case:

$$p_{Z_t}(z) = \frac{c.t}{z^{3/2}}.e^{2.c.t.\sqrt{\pi.\lambda}}.e^{-\lambda.z-\pi.c^2.t^2.\frac{1}{z}} \qquad \forall z > 0$$

Let $\delta(t)$ and $\eta(t)$ be two functions of time, such that $c.t = \sqrt{\frac{\delta(t)}{2\pi}}$ and $\lambda = \frac{\delta(t)}{2.^2}$. The density of Z_t may then be rewritten as follows:

$$p_{Z_t}(z) = \sqrt{\frac{\delta(t)}{2.\pi}}\frac{1}{z^{3/2}}.\exp\left(-\frac{\delta(t).\left(z-\eta(t)\right)^2}{2.\eta(t)^2}.\frac{1}{z}\right) \qquad \forall z > 0$$

and corresponds to the law of an Inverse Gaussian random variable. As $\delta(t)$ tends to infinity, the inverse Gaussian distribution becomes more like a Gaussian distribution. The expectation and variance of Z_t are:

$$\begin{aligned} \mathbb{E}(Z_t) &= \eta(t) \\ \mathbb{V}(Z_t) &= \frac{\eta(t)^3}{\delta(t)} \end{aligned}$$

The adjustment factor is rewritten as:

$$\begin{aligned} ADJ(t,s,Y_t) &= \exp\left(-\left(b.(s-t)+(Y_t-b).\left(1-e^{-a.(s-t)}\right).\frac{1}{a}\right)\right). \\ &\quad \exp\left(-2.c.\sqrt{\pi}\int_t^s\left(\lambda+\frac{\sigma}{a}.\left(1-e^{-a.(s-\theta)}\right)\right)^{1/2}.d\theta\right). \\ &\quad \exp\left(+2.c.\sqrt{\pi}.\sqrt{\lambda}.(s-t)\right) \end{aligned}$$

Where

$$\int_t^s\left(\lambda+\frac{\sigma}{a}.\left(1-e^{-a.(s-\theta)}\right)\right)^{1/2}.d\theta =$$

$$\frac{2}{a}.\left[\left(\lambda+\frac{\sigma}{a}\left(1-e^{-a.(s-\theta)}\right)\right)^{1/2}-\left(\lambda+\frac{\sigma}{a}\right)^{1/2}.atanh\left(\frac{\left(\lambda+\frac{\sigma}{a}\left(1-e^{-a.(s-\theta)}\right)\right)^{1/2}}{\left(\lambda+\frac{\sigma}{a}\right)^{1/2}}\right)\right]_{\theta=t}^{\theta=s}$$

5 Market price of longevity risk.

Up to now, all the calculations have been made with respect to the historical measure P. In particular, survival probabilities appear to be mathematical expectations under this measure without any market price of risk. As the longevity risk is not diversifiable, it seems reasonable to assume that the market adds some longevity loadings to price insurance products. In the next two subsections, we will focus on two pricing methods. The first one consists in changing of measure from P to Q, a pricing measure. Whereas the second approach is based on the equivalence pricing.

5.1 Change of measure.

In order to price insurance payoffs, we postulate that we are given an equivalent martingale measure Q , under which the market value of insurance contracts is equal to the expectation of their payoffs, discounted at risk free rate. In theory, the mortality risk is fully diversifiable. Q incorporates then only preferences regarding longevity risks. In the setting of Lévy processes, we can define a class of probabilities equivalent to P by theorems 33.1 and 33.2 of Sato (1999) which state that for tempered stable processes, equivalent measures are defined by the following Radon-Nykodyn derivative:

$$\xi_t=\mathbb{E}\left(\frac{dQ}{dP}|\mathcal{G}_t\right) = \exp\left((\lambda-\lambda').Z_t-t.\int_0^{+\infty}\left(e^{(\lambda-\lambda').z}-1\right).\nu(dz)\right)$$

The process ξ_t is a martingale under P with respect to the filtration $(\mathcal{G}_t)_t$ and $\mathbb{E}\left(\frac{dQ}{dP}\right)=1$. Under Q, Z_t has the following Lévy measure:

$$\nu'(z)=\frac{c.e^{-\lambda'.z}}{z^{\alpha+1}}.I_{z>0} \qquad \alpha\in[0,1[$$

Note that it is not possible to modify parameters c and α by a change of measure (for details, see Cont and Tankov 2004, page 309). As mentioned in the paper of Biffis et al. (2005 b) p4, by construction, we have that every $\mathcal{G}_t$-martingale is a $\mathcal{F}_t$-martingale given that the conditional probability of death happening before t, $P(T_i\leq t|\mathcal{G}_t)$ only depends on the evolution of mortality risk factors up to time t, and not upon their whole path. Under this assumption ξ_t is also a martingale under P with respect to the filtration $(\mathcal{F}_t)_t$ and defines well a change of measure from P to Q, on $(\mathcal{F}_t)_t$. If conditional expectations and probabilities under Q are denoted by $\mathbb{E}^Q(.|\mathcal{G}_t)$ and $P^Q(.|\mathcal{G}_t)$, we have that:

$$P^Q\left(I(T_i>t)|\mathcal{F}_t\right) = I(T_i>t).\ \underbrace{\mathbb{E}^Q\left(\exp\left(-\int_t^s\mu_u^x.du\right)|\mathcal{G}_t\right)}_{{}_{s-t}p^Q_{x+t}}$$

And ${}_{s-t}p^{Q}_{x+t}$ may be split as follows:

$$ {}_{s-t}p^{Q}_{x+t} = \underbrace{\exp\left(-\int_t^s \mu_x(u).du\right)}_{{}_{s-t}\bar{p}_{x+t}} . \underbrace{\mathbb{E}^Q\left(\exp\left(-\int_t^s Y_u.du\right)|\mathcal{G}_t\right)}_{ADJ^Q(t,s,Y_t)} $$

Where $ADJ^Q(t,s,Y_t)$ is obtained by substituting λ' to λ in the expressions of $ADJ(t,s,Y_t)$. Let r be the constant risk free rate. The price at time t, of one pure endowment policy ($n_x = 1$), delivering a capital K at time s if the insured is alive at age $x+t$, is then equal to:

$$ \bar{E}^{Q}_{x+t,s-t}(K) \quad = \quad K.e^{-r.(s-t)}.{}_{s-t}p^{Q}_{x+t} \tag{5.1} $$

5.2 Indifference pricing.

This method is particularly well adapted for pricing purposes in incomplete markets and is based on the insurer's preferences. More precisely, this approach compares the following possibilities: either the insurer can choose to accept the risk, receive some premium and invest in the financial market, or the insurer can decide not to insure the risk and simply invest his wealth in the market. The price is here defined as the premium at which the insurer is indifferent between these two options. Let $U(W_s)$ be the insurer's utility, drawn from his wealth, W_s, at time s. we will focus on the class of exponential utility functions:

$$ U(W_s) \quad = \quad -\frac{1}{\rho}.\exp\left(-\rho.W_s\right) $$

where ρ is the parameter of risk aversion. As it appears in the literature, working with this category of utility presents the advantage that the indifference prices are independent from the insurer's wealth, because the absolute risk aversion $-\frac{U''(w)}{U'(w)}$ is constant (equal to ρ). We consider that the insurer is initially endowed with wealth W_t at time t, and can invest this amount in a cash account, earning a constant risk free rate, r :

$$ dW_t \quad = \quad r.W_t.dt $$

If the insurer doesn't accept any insurance risk, his expected utility , at the end of his time horizon s is given by:

$$ \begin{aligned} V_1(t,W_t) \quad &= \quad \mathbb{E}\left(U(W_s)\,|\,\mathcal{F}_t\right) \\ &= \quad -\frac{1}{\rho}.\exp\left(-\rho.W_t.e^{r.(s-t)}\right) \end{aligned} $$

Assume now that the insurer accepts to cover a pool of n_x individuals. The insurer will deliver a global payoff F_s at time s, in exchange of a total premium P. This payoff may be a function of $(Y_t)_t$ and of the number of deaths $(N_t)_t$. If we denote $W'_t = W_t + P$, his expected terminal utility is then equal to:

$$ \begin{aligned} V_2(t,W'_t,Y_t,N_t) \quad &= \quad \mathbb{E}\left(U(W_s - F_s)\,|\,\mathcal{F}_t\right) \\ &= \quad -\frac{1}{\rho}.\exp\left(-\rho.W'_t.e^{r.(s-t)}\right).\mathbb{E}\left(\exp\left(\rho.F_s\right)\,|\,\mathcal{F}_t\right) \end{aligned} \tag{5.2} $$

When the expression (5.2) is too difficult to calculate, we can rely on dynamic programming to build a PDE whose solution is V_2. Indeed, Applying the Itô's lemma for semimartingales with W'_t, Y_t and N_t as state variables, leads to the following equation:

$$\mathbb{E}\left(V_2(t+\Delta t, W'_{t+\Delta t}, Y_{t+\Delta t}, N_{t+\Delta t}) - V_2(t, W'_t, Y_t, N_t)|\mathcal{F}_t\right) =$$

$$\mathbb{E}\left(\int_t^{t+\Delta t} \frac{\partial V_2}{\partial u} + r.W'_u.\frac{\partial V_2}{\partial W'_u} + a.\left(b - Y_{u-}\right).\frac{\partial V_2}{\partial Y_u}.du\,|\mathcal{F}_t\right)$$

$$+\mathbb{E}\left(\int_t^{t+\Delta t}\int_0^{+\infty} V_2(u, W'_u, Y_{u-} + \sigma.z, N_{u-}) - V_2(u, W'_u, Y_{u-}, N_{u-}).J_Z(du, dz).du\,|\mathcal{F}_t\right)$$

$$+\mathbb{E}\left(\int_t^{t+\Delta t} \left(V_2(u, W'_u, Y_{u-}, N_u) - V_2(u, W'_u, Y_{u-}, N_{u-})\right).dN_u|\mathcal{F}_t\right) \quad (5.3)$$

By taking the limit of this last equation when Δt tends to zero and according to eq. (2.1), we get that V_2 is solution of SDE:

$$\begin{aligned}
0 \quad = \quad & \frac{\partial V_2}{\partial t} + r.W'_t.\frac{\partial V_2}{\partial W'_t} + a.\left(b - Y_{t-}\right).\frac{\partial V_2}{\partial Y_t} + \\
& \int_0^{+\infty} V_2(t, W'_t, Y_{t-} + \sigma.z, N_{t-}) - V_2(t, W'_t, Y_{t-}, N_{t-}).\nu(dz) + \\
& \left(V_2(t, W'_t, Y_{t-}, N_{t-} + 1) - V_2(t, W'_t, Y_{t-}, N_{t-})\right).(n_x - N_{t-}).\mu_t^x
\end{aligned} \quad (5.4)$$

Once that V_2 is determined, the indifference premium P is such that $V_1(t, W_t) = V_2(t, W_t + P)$.

If the pool counts only one members ($n_x = 1$), the payoff, F_s , of one pure endowment policy delivering a capital K at time s if the insured survives till age $x + s$, is worth:

$$F_s \quad = \quad (1 - N_s).K$$

and the insurer's expected utility is equal to:

$$\begin{aligned}
V_2(t, W_t + P, Y_t, 0) \quad = \quad & -\frac{1}{\rho}.\exp\left(-\rho.e^{r.(s-t)}.\left(W_t + P\right)\right) \\
& .\left(1 - {}_{s-t}p_{x+t}.\left(1 - \exp\left(\rho.K\right)\right)\right)
\end{aligned} \quad (5.5)$$

The proof of this result is given in appendix A. The indifference price of one endowment is then given by:

$$P = \bar{E}^{eq}_{x+t,s-t}(K) \quad = \quad \frac{1}{\rho}.e^{-r.(s-t)}.\ln\left(1 - {}_{s-t}p_{x+t}.\left(1 - \exp\left(\rho.K\right)\right)\right) \quad (5.6)$$

and is well independent of the insurer's wealth. This last formula will be used to price annuities by the principle of equivalent utilities, in the section devoted to numerical applications. Note that developments done in this section may be easily extended to address ALM issues. As in Young & Zariphopoulou (2002), we can indeed consider a market made up of cash and stocks. The indifference price and the related optimal asset allocation can then be found by solving the Hamilton Jacobi Bellman equation, which is similar to the SDE (5.4).

6 Demographic trends and indicators.

Figures (6.1) and (6.2) emphasizes the recent mortality trends observed in Belgium (for the total population, women and men). Every survival function and curve of deaths is built upon 3 years of observations and smoothed. As underlined by Pitacco and Olivieri (2006), two distinguished tendencies affect the evolution of mortality:

- Rectangularization: one observes an increasing concentration of deaths around the mode (at old ages) of the curve of deaths. This entails that the shape of survival functions evolves towards a rectangle. The interested reader should refer to Wilmoth and Horiuchi (1999) for a analysis of the rectangularization process.
- Expansion: The mode of the curve of deaths moves towards older age. this implies a movement of the survival function towards the maximum age reachable by a human being.

Figure 6.1: Evolution of survival functions.

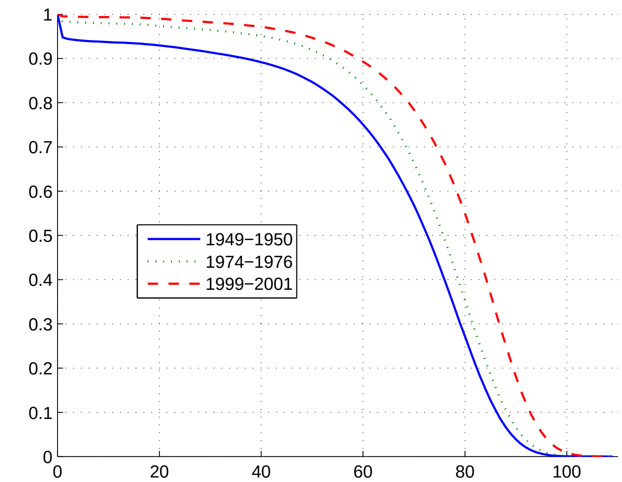
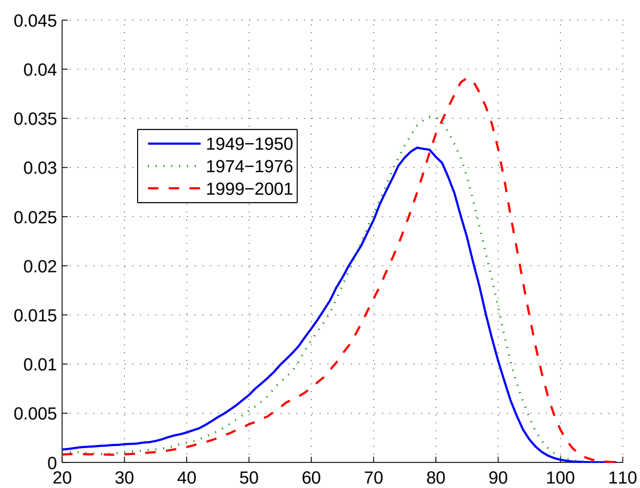


Figure 6.2: Curves of deaths.



A measure of the degree of rectangularization is given by the index H sometimes referred to as the entropy of the survival function. H measures the degree of concavity in the survival function and is defined by

$$H_x = -\frac{\int_0^{Tmax} \log\left({}_tp_x\right) . {}_tp_x . dt}{\int_0^{Tmax} {}_tp_x . dt}$$

where $Tmax$ is the maximum reachable age by a human being. Remark that $H = 0$ means that the survival function is a perfect rectangle: as a consequence, the lower its value for the cohort considered, the greater the joint effect of rectangularization and expansion. Another measure considered is the expectation of life at age x, denoted $\dot{e}_x$:

$$\dot{e}_x = \int_0^{Tmax} {}_s p_x ds$$

Table (6.1) presents average life expectations and entropies at birth and at retirement for a Belgian citizen. In fifty years, the life expectation at birth has grown of 11.74 years whereas the entropy has fallen of 0.0861 .

Table 6.1: Demographic indicators.

	1949-1951	1974-1976	1999-2001
$\dot{e}_0$	66.74	72.56	78.48
H_0	0.2246	0.1636	0.1385
$\dot{e}_{65}$	13.66	14.71	18.48
H_{65}	0.4887	0.4713	0.3882

In the next section, we will shows that our mortality models can reproduce most of the observed mortality tendencies, such the rectangularization and the expansion.

7 Numerical applications.

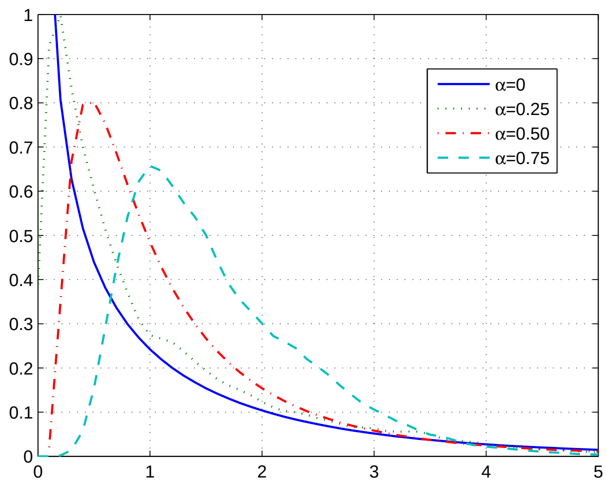
In this section, we present some examples of mortality projections. The deterministic mortality rates, $\mu_x(t)$ are set to observed mortality rates of the Belgian, male population for the period 1999 to 2001. x is set to 65 years (which is the legal retirement age in Belgium). Parameters $(a, b, c, \lambda, \alpha, \sigma)$ and Y_0 associated to the process Y_t are presented in table (7.1) .

Table 7.1: Parameters.

	a	b	c	λ	σ	Y_0
$\alpha = 0$, men	0.5	-0.035	1/2	1/2	0.01	0
$\alpha = 0.25$, men	0.5	-0.035	1/2	1/2	0.01	0
$\alpha = 0.50$, men	0.5	-0.035	1/2	1/2	0.01	0
$\alpha = 0.75$, men	0.5	-0.035	1/2	1/2	0.01	0

Figure (7.1) compares the probability densities of $Z_{t=1}$, for the chosen α values. Contrary to Gamma ($\alpha = 0$) and Inverse Gaussian ($\alpha = 0.5$) cases, there are no analytic expressions of Z_t densities for $\alpha = 0.25$ and $\alpha = 0.75$. However, we can compute them by simulations using the series representation of Lévy processes. The algorithm used is reminded in appendix. One clearly observes that the higher is α, the higher is the scattering of the Z_t density. and the lesser is the asymmetry.

Figure 7.1: Densities of $Z_{t=1}$.



Before any other numerical results, we provide an interpretation to the model underlying the evolution of mortality rates, and in particular to equations (3.1) and (3.2). If σ is set to zero, the mortality correction, $Y_t^{\sigma=0}$ is deterministic and obeys the ODE:

$$dY_t^{\sigma=0} = a.(b - Y_t^{\sigma=0}).dt$$

which admits the following solution:

$$Y_t^{\sigma=0} \quad = \quad e^{-a.t}.Y_0 + b.\left(1 - e^{-a.t}\right)$$

If a , b, Y_0 are respectively positive, negative and null, $Y_t^{\sigma=0}$ corresponds to a decrease of mortality rates and tends to b when $t \to +\infty$. Best estimate mortality rates are in this case:

$$\mu_t^{x,\sigma=0} = \mu_x(t) + Y_t^{\sigma=0}$$

When σ is positive, mortality rates are the sum of previous best estimate mortality rates and of a positive stochastic integral:

$$\mu_t^x \quad = \quad \mu_t^{x,\sigma=0} + \sigma. \int_0^t e^{-a.(t-\theta)}.dZ_\theta \tag{7.1}$$

It means that best estimate mortality rates are affected by an infinity of positive small jumps, which may be seen as small random worsening of life conditions. E.g. those mortality rate increases can result from bad food habits, epidemics, economic crisis, etc. The amplitude of those small jumps depends on the choice of parameters (α, c , λ, σ).

Figure 7.2: Evolution of survival functions, Inv. Gaussian model.

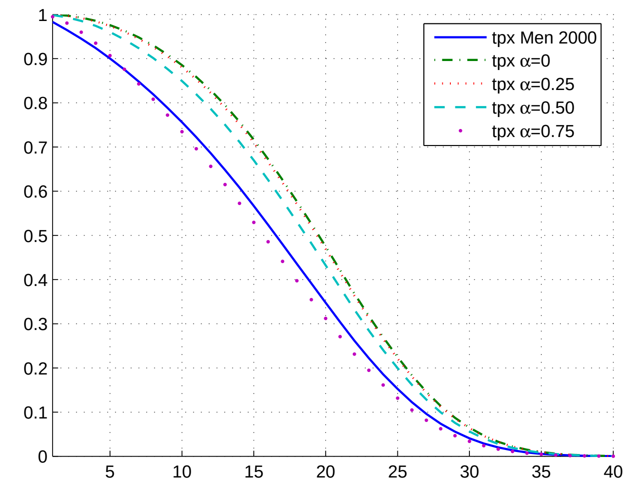
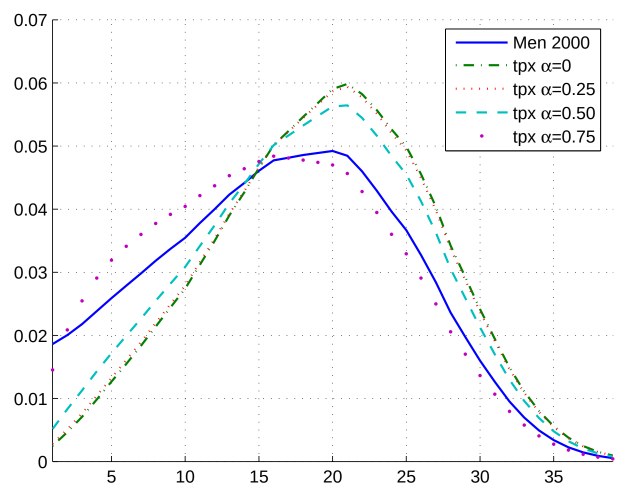


Figure 7.3: Curves of deaths, Inv. Gaussian model..



Figures (7.2) and (7.3) show projected survival functions and curves of deaths. Corresponding entropies and life expectancies are detailed in table (7.2). Small values of α (0 or 0.25) entail a small variance and amplitude of jumps affecting best estimate mortality rates. Those projections anticipate an improve of the human longevity: expected lifetimes increase about 3 years whereas entropies fall of 0.11 . Above $\alpha = 0.75$, the foreseen survival probabilities are inferior to the current ones: the life expectation slightly falls of 0.55 years and entropy increase of 0.0079 .

Table 7.2: Demographic indicators.

	$\dot{e}_{65}$	H_{65}	Y_∞
Men y. 2000	15.81	0.4581	-
$\alpha = 0$	18.76	0.3423	-0.0156
$\alpha = 0.25$	18.64	0.3464	-0.0144
$\alpha = 0.50$	17.87	0.3729	-0.0099
$\alpha = 0.75$	15.26	0.4660	0.0081

7.1 Annuity pricing under a change of measure.

This subsection briefly analyzes the pricing under a change of measure, of a life annuity purchased at age x and guaranteeing a continuous income of one unit till the insured's death. If T is the random time of death, the annuity price under Q , denoted $\bar{a}_x^Q$, is given by :

$$\begin{aligned}
\bar{a}_x^Q &= \int_0^{Tmax} e^{-r.s}.\mathbb{E}^Q\left(I(T \geq s)\,|\,\mathcal{F}_0\right).ds \\
&= \int_0^{Tmax} \bar{E}_{x,s}^Q(1).ds \\
&= \int_0^{Tmax} e^{-r.s}.{}_s\bar{p}_x.ADJ^Q(0,s,Y_0).ds
\end{aligned}$$

where r , the risk free rate, is worth 3.25%. x is set to 65 years and all the other parameters are identical to those presented in table 7.1. Table 7.3 compares the annuity prices for different changes of measure and distributions of subordinator:

Table 7.3: annuity prices by λ'.

	$\bar{a}_x^Q$, $\alpha = 0$	$\bar{a}_x^Q$, $\alpha = 0.25$	$\bar{a}_x^Q$, $\alpha = 0.50$	$\bar{a}_x^Q$, $\alpha = 0.75$
Men y. 2000	11.0090	11.0090	11.0090	11.0090
$\lambda' = \lambda - 0.2$	11.7537	11.9997	11.7959	10.4855
$\lambda' = \lambda - 0.1$	12.4727	12.5198	12.1679	10.7226
$\lambda' = \lambda$	12.9464	12.8768	12.4357	10.9024
$\lambda' = \lambda + 0.1$	13.2817	13.1392	12.6411	11.0464
$\lambda' = \lambda + 0.2$	13.5315	13.3416	12.8053	11.1659

The higher is λ', the smaller is the amplitude of large jumps of mortality rates. It entails that choosing a $\lambda' > \lambda$ (when $\sigma > 0$) increases the life expectancy and the annuity price. So as to emphasize the dependence of prices on the choice of model, we define $ADJ^Q(0,s,Y_0,\omega)$ as an occurrence of the random variable $\exp\left(-\int_0^s Y_u.du\right)$ under Q and introduce $\bar{a}_x^Q(\omega)$:

$$\bar{a}_x^Q(\omega) = \int_0^{Tmax} e^{-r.s}.{}_s\bar{p}_x.ADJ^Q(0,s,Y_0,\omega).ds$$

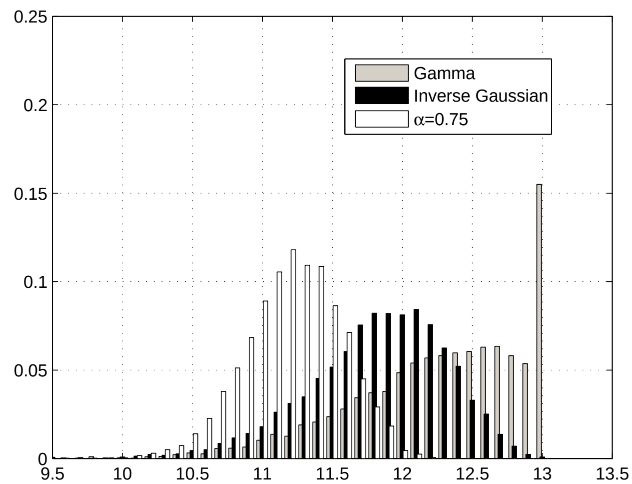
such that $\bar{a}_x^Q = \mathbb{E}^Q\left(\bar{a}_x^Q(\omega)|\mathcal{G}_0\right)$. $\bar{a}_x^Q(\omega)$ is the annuity price for a given evolution of mortality rates. Table (7.4) contains the annuity prices $\bar{a}_{65}^Q$, standard deviations and percentiles of $\bar{a}_{65}^Q(\omega)$ when $\lambda' = \lambda$. Those statistics are obtained by Monte-Carlo simulations (6000 scenarios). The 95% percentile is the price that the insurer should ask to cover an adverse deviation of longevity in 95% percent of cases.

Table 7.4: statistics over annuity prices.

	$\bar{a}_x^Q$	std $\bar{a}_x^Q(\omega)$	5% pctile $\bar{a}_x^Q(\omega)$	95% pctile $\bar{a}_x^Q(\omega)$
Men y. 2000	11.0090	-	-	-
$\alpha = 0$	12.3004	0.6541	11.1095	13.2320
$\alpha = 0.25$	12.2394	0.5121	11.3415	13.0142
$\alpha = 0.50$	11.8218	0.5009	10.9153	12.5447
$\alpha = 0.75$	11.2184	0.3432	10.6329	11.7659

Figure (7.4) presents the empirical distribution of $\bar{a}_x^Q(\omega)$, for $\alpha = 0$, 0.50. The higher is α, the smaller is the scattering of $\bar{a}_x^Q(\omega)$ and the smaller is their average.

Figure 7.4: Annuity pricing.



7.2 Indifference pricing of annuities.

In this subsection, we present some numerical results about the indifference pricing of an annuity. Adopting the same notations as previously, the annuity indifference price is defined by:

$$\begin{aligned}
\bar{a}_x^{eq} &= \int_0^{Tmax} \bar{E}_{x,s}^{eq}(1).ds \\
&= \int_0^{Tmax} \frac{1}{\rho}.e^{-r.s} \ln\left(1 -_s p_x.\left(1 - e^{\rho}\right)\right).ds
\end{aligned}$$

The risk free rate, r , and the insured's age, x, are respectively worth 3.25% an 65 years. All the other parameters are identical to those presented in table 7.1.

Table 7.5: annuity prices by ρ.

	$\bar{a}_x^{eq}$, $\alpha = 0$	$\bar{a}_x^{eq}$, $\alpha = 0.25$	$\bar{a}_x^{eq}$, $\alpha = 0.50$	$\bar{a}_x^{eq}$, $\alpha = 0.75$
Men y. 2000	11.0090	11.0090	11.0090	11.0090
$\rho = 1.50$	13.5538	13.4889	13.0755	11.6060
$\rho = 1.75$	13.8436	13.7810	13.3815	11.9470
$\rho = 2.00$	14.1239	14.0635	13.6777	12.2794
$\rho = 2.25$	14.3945	14.3363	13.9638	12.6025
$\rho = 2.50$	14.6554	14.5994	14.2397	12.9157

The smaller is the risk aversion ρ, the smaller is the annuity price. The choice of subordinator influences the price in a similar way to what we have observed for pricing under a change of measure: the higher is α, the smaller is the price.

8 Conclusions.

This paper shows that human mortality can be modeled by doubly stochastic processes with a mean reverting intensity. The stochastic component of the intensity belongs to the family of α-stable subordinators. This category of Lévy processes contains a wide variety of asymmetric distributions, are analytical tractable and numerically simulable.

In a first step, we have proposed a general expression of survival probabilities, whatsoever the choice of subordinator and that is easily computable. We have next studied two particular cases, namely the Gamma and Inverse Gaussian processes, which lead to analytical formulas for survival probabilities and for probability densities of driving Lévy processes. We also have addressed the issue of pricing and investigated two methods: the change of measure and the indifference pricing.

After a brief review of recent mortality trends, examples of mortality projections are proposed to illustrate their abilities in capturing the dynamic of the mortality evolution. Projected mortality rates are the sum of best estimate mortality rates and of an infinity of positive small jumps, which may be seen as small random worsening of life conditions. Finally, we have presented prices of life annuities obtained either by a change of measure or either by the indifference approach.

Appendix A.

We prove here that the insurer's expected utility V_2, eq. (5.5) is well solution of the SDE (5.4). Remember that:

$$\begin{aligned} V_2(t, W_t^{'}, Y_t, 0) &= -\frac{1}{\rho}.\exp\left(-\rho.e^{r.(s-t)}.W_t^{'}\right) \\ &\quad .\left(1 - {}_{s-t}\, p_{x+t}.\left(1-\exp\left(\rho.K\right)\right)\right) \end{aligned}$$

and that the survival probability is given by:

$$\begin{aligned} {}_{s-t}p_{x+t} &= {}_{s-t}\bar{p}_{x+t}.\exp\left(-\left(b.(s-t) + (Y_t - b).\left(1 - e^{-a.(s-t)}\right).\frac{1}{a}\right)\right). \\ &\quad \exp\left(\int_t^s k\left(-\frac{\sigma}{a}.\left(1-e^{-a.(s-\theta)}\right)\right).d\theta\right) \end{aligned} \tag{8.1}$$

Partial derivatives of ${}_{s-t}p_{x+t}$ with respect to t and Y_t are:

$$\frac{\partial_{s-t}p_{x+t}}{\partial t} = {}_{s-t}p_{x+t}.\left(\mu_x(t) + b + (Y_t - b).e^{-a.(s-t)} - k\left(-\frac{\sigma}{a}.\left(1-e^{-a.(s-t)}\right)\right)\right)$$

$$\frac{\partial_{s-t}p_{x+t}}{\partial Y_t} = -{}_{s-t}p_{x+t}.\left(1-e^{-a.(s-t)}\right).\frac{1}{a}$$

and they are useful to compute partial derivatives of V_2 with respect to t, $W_t^{'}$ and Y_t:

$$\begin{aligned} \frac{\partial V_2}{\partial t} &= V_2.\rho.e^{r.(s-t)}.W_t^{'}.r \\ &\quad + \frac{1}{\rho}.\exp\left(-\rho.e^{r.(s-t)}.W_t^{'}\right)\left(1-\exp\left(\rho.K\right)\right).\frac{\partial_{s-t}p_{x+t}}{\partial t} \end{aligned}$$

$$\frac{\partial V_2}{\partial W_t'} = -\rho . e^{r.(s-t)} . V_2$$

$$\frac{\partial V_2}{\partial Y_t} = \frac{1}{\rho} . \exp\left(-\rho . e^{r.(s-t)} . W_t^{'}\right) . \left(1 - \exp\left(\rho . K\right)\right) . \frac{\partial_{s-t} p_{x+t}}{\partial Y_t}$$

By definition of the cumulant transform eq. (4.1), we have that:

$$\begin{aligned} &\int_0^{+\infty} V_2(t, W_t', Y_{t-} + \sigma . z, 0) - V_2(t, W_t', Y_{t-}, 0) . \nu(dz) = \\ &\qquad \frac{1}{\rho} . \exp\left(-\rho . e^{r.(s-t)} . W_t^{'}\right) ._{s-t} p_{x+t} . \left(1 - \exp\left(\rho . K\right)\right) . \\ &\qquad \underbrace{\int_0^{+\infty} \left(\exp\left(-\frac{\sigma . z}{a} . \left(1 - e^{-a.(s-t)}\right)\right) - 1\right) . \nu(dz)}_{k\left(-\frac{\sigma}{a} . \left(1 - e^{-a.(s-t)}\right)\right)} \end{aligned}$$

Finally, the term in the SDE (5.4) related to the number of deaths, is rewritten as follows:

$$\begin{aligned} &\left(V_2(u, W_t', Y_{t-}, 1) - V_2(u, W_t', Y_{t-}, 0)\right) . \mu_t^x = \\ &\qquad -_{s-t} p_{x+t} . \frac{1}{\rho} . \exp\left(-\rho . e^{r.(s-t)} . W_t^{'}\right) \left(1 - \exp\left(\rho . K\right)\right) . \left(\mu_x(t) + Y_t\right) \end{aligned}$$

It suffices then to develop the SDE eq. (5.4) to prove that V_2 cancels it.

Appendix B.

We reproduce here the algorithm that allows us to simulate subordinators on [0,1] by series representation (for details see Cont and Tankov 2004, chapter 6). Let $U(z) = \int_z^{\infty} \nu(dz)$ be the tail integral of the Lévy measure of a subordinator. The inverse function of this integral is denoted $U^{-1}(z)$. Fix a number τ depending on the required precision.

Initialize k:=0

REPEAT WHILE $\sum_{i=1}^{k} T_i < \tau$
Set k = k+1
Simulate T_k : standard exponential.
Simulate V_k : uniform on $[0, 1]$
END WHILE

The occurrence of Z_t is : $Z_t = \sum_{i=1}^{k} 1_{V_i \leq t} U^{-1}\left(\Gamma_i\right)$ where $\Gamma_i = \sum_{j=1}^{i} T_j$

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