Heat Kernel Framework for
Asset Pricing in Finite Time

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Abstract

A heat kernel approach is proposed for the development of a general, flexible, and mathematically tractable asset pricing framework in finite time. The pricing kernel, giving rise to the price system in an incomplete market, is modelled by weighted heat kernels that are driven by multivariate Markov processes and that provide enough degrees of freedom in order to calibrate to relevant data, e.g. to the term structure of bond prices. It is shown how, for a class of models, the prices of bonds, caplets, and swaptions can be computed in closed form. The dynamical equations for the price processes are derived, and explicit formulae are obtained for the short rate of interest, the risk premium, and for the stochastic volatility of prices. Several of the closed-form asset price models presented in this paper are driven by combinations of Markovian jump processes with different probability laws. Such models provide a rich basis for consistent applications in several sectors of a financial market including equity, fixed-income, commodities, and insurance. The flexible, multidimensional and multivariate structure, on which the asset price models are constructed, lends itself well to the transparent modelling of dependence across asset classes. As an illustration, the impact on prices by spiralling debt, a typical feature of a financial crisis, is modelled explicitly, and contagion effects are readily observed in the dynamics of asset returns.

1 Introduction

In this paper, we shall take the view that in a modern asset pricing framework (i) pricing models should be coherent across all asset classes traded in a financial market, (ii) securities pricing used in the front offices of financial firms should be compatible with asset risk management, and (iii) pricing formulae should be applicable in the banking industry and also in the insurance sector. Expressed in other words, these three requirements state that modern pricing models ought to be consistent under the real probability measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$, and at the same time they should retain a high degree of flexibility and mathematical ease while guaranteeing the coherence of the price system for all financial assets.

In what follows, we propose an asset pricing framework that can be applied, in principle, to all asset classes and that is mathematically tractable so that Monte Carlo techniques are never necessary for scenario simulations of asset price dynamics. The proposed approach includes partial automatic calibration to market data such as initial prices of assets. The price system of assets traded in a financial market shall be developed by modelling the pricing kernel (state-price density) first. Once the stochastic framework for the pricing kernel is built and the connection with bond prices is established, we go on to show how price processes for other asset classes can be derived in a natural way. We also consider how the situation, in which the debt of a sovereign country gets out of control, can be incorporated in the same pricing framework without introducing extra assumptions to include effects of credit risk.

The general setup of the asset pricing framework is developed in finite time, $t \in [0, U]$ for $U < \infty$. We model a financial market by a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\})$ where $\mathbb{P}$ denotes the real probability measure, and where $\{\mathscr{F}_t\}$ is the market filtration. We consider a (multi-dimensional) process $\{X_t\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$, and assume that the market filtration is generated by $\{X_t\}$. Furthermore, we assume that $\{X_t\}$ has the Markov property with respect to $\{\mathscr{F}_t\}$, its natural filtration. Then, we introduce the pricing kernel process $\{\pi_t\}$ to model the market agent's preferences and the dynamics of interest rates in the economy which $\{\pi_t\}$ is associated with. We write $\{S_t\}_{0 \leq t \leq T < U}$ for the price process of a dividend-paying asset, and let $\{D_t\}_{0 \leq t \leq T < U}$ denote the (continuous) dividend stream up until T. Then the price S_t at time t is given by

$$S_t = \frac{1}{\pi_t} \mathbb{E}^{\mathbb{P}} \left[\pi_T S_T + \int_t^T \pi_u D_u \mathrm{d}u \,\middle|\, \mathscr{F}_t \right]. \tag{1.1}$$

In order to calculate asset prices explicitly, the following ingredients need to be specified: (i) The Markov process $\{X_t\}$ that generates the market filtration, and thus the market information; (ii) the pricing kernel $\{\pi_t\}$, and thus the dynamics of the interest rates and the agent's preferences; (iii) the random variable S_T and the process $\{D_t\}$, thus the asset's terminal cash flow and the dividend stream, respectively. All ingredients are specified in such a way that the price process $\{S_t\}$ is adapted to the market filtration generated by $\{X_t\}$. For textbooks about the theory of pricing kernels, preferences, asset pricing, and interest rates modelling, one may consult, e.g., Back (2010), Björk (2009), Cochrane (2005), Duffie

(2001), and Brigo & Mercurio (2006).

In the next section, we introduce weighted heat kernels to define the class of pricing kernels, and thus give rise to the asset pricing framework proposed in this paper. Heat kernel models for the development of stochastic price systems have been proposed by Akahori et al. (2009) in an infinite-time setting, and more recently by Akahori & Macrina (2012) in a finite-time context. We further develop the weighted heat kernel pricing models introduced in the latter work, and derive formulae for the price processes of discount bonds and swaptions, and the associated interest rate process. The stochastic short rate of interest is by construction non-negative.

In Section 3, we construct a class of pricing kernels which leads to closed-form price processes for bonds, caplets, and swaptions—all in one. Explicit price models are then obtained by specifying the dynamics of the market information flow and the degrees of freedom in the formulation of a particular pricing kernel model. The dynamics of certain price processes have time-dependent lower and upper bounds, a feature we not necessarily view as being a shortcoming.

In Section 4, we derive the dynamical equation for the bond price processes introduced in the previous section for the case that the market filtration is generated by a diffusion process. The market price of risk process is also obtained endogenously, which, for this class of pricing models, incorporates a discernible part that can be identified as the incentive for accepting model risk.

In Section 5, we introduce multivariate Lévy random bridges and extend the pricing framework to an incomplete market. A formula for the computation of option prices by Fast-Fourier-Transform methods is given, and asset price models with higher-order rational structures are developed.

In Section 6, the proposed pricing kernel approach is applied to general asset pricing, and we show how asset price models constructed under the $\mathbb{P}$-measure translate into asset price models equipped with stochastic interest rate and stochastic volatility under the $\mathbb{Q}$-measure. The interaction between the bond price process and, e.g., the equity component of the discounted share price process is clearly identifiable. This property renders the herewith proposed asset pricing framework also appealing for the construction of hedging strategies against losses due to the exposure of a financial position to a specific market sector. Additional examples of explicit multivariate asset pricing models driven by jump processes are given.

In the last chapter, spiralling sovereign deficit is modelled and its impact on the price dynamics of sovereign bonds is shown. The flexibility of the considered heat kernel state-price density approach allows for the construction of explicit dependence models linking the price evolution of, e.g., bonds issued by several sovereign governments. Contagion effects arise endogenously, and the graphs in Section 7 illustrate the impact of dependent economies and markets on the price dynamics of assets.

The list of investigations connected with the asset pricing approach presented in this paper is by no means complete. The developed theory is applicable to the modelling of foreign exchange rates and the pricing of foreign exchange securities. Inflation-linked bonds and other indexed assets can be priced in a similar way. Then, there are commodity assets, including energy and agricultural products, which may involve insurance contracts to hedge

against substantial losses due to averse weather conditions. In the last section of this paper, effects on asset prices by perceived solvency risk are considered in the dynamics of their returns. Further work in this direction, may include the modelling of "costs of funding" arising from shifts in asset prices due to the deterioration of a creditor's economic situation. Even though full-fledged credit risk models have not yet been developed within the present asset pricing framework (and this is yet another project), the material in Section 7 may also be looked at from the perspective of modelling "credit valuation adjustments". The bounds, within which the dynamics of certain price processes are confined, could be exploited to model the levels of sustainability for the costs of funding, or applied by financial regulators to impose time-dependent capital requirements. Although treated in this paper, the explicit modelling of dependence structures for asset portfolios remains somewhat in the background. The weighted heat kernel approach however offers a versatile basis for the inclusion of manageable dependence models that could be useful for the risk analysis of portfolios. One could begin with mean-variance optimization whereby covariance matrices are explicitly modelled, and partial information about the underlying risk factors drives the optimal portfolio. Another investigation may concern models for volatility surfaces, which arise from the selection of particular pricing kernels and their application to specific asset classes. Such an investigation extends to the analysis of the derived option price models and their calibration to data relevant for the pricing of assets.

2   Pricing kernel models and the pricing of bonds, caplets and swaptions

We proceed to specify the form of the pricing kernel model $\{\pi_t\}$. We consider the following class:

$$\pi_t = f_0(t) + f_1(t) \int_0^{U-t} \mathbb{E}\left[F(t+u, X_{t+u}) | X_t\right] w(t,u)\, \mathrm{d}u, \tag{2.1}$$

where $\{X_t\}$ is an unspecified Markov process defined for $t \in [0, U]$ such that $t+u \leq U$, $f_0(t)$ and $f_1(t)$ are deterministic, non-negative, non-increasing functions, $F(t,x)$ is a positive measurable function, and $w(t,u)$ is a so-called weight function satisfying

$$w(t, u-s) \leq w(t-s, u) \tag{2.2}$$

for $s \leq t \wedge u$. We introduce the following short-hand notation:

$$\rho(t, u, X_t) := \mathbb{E}\left[F(t+u, X_{t+u}) | X_t\right]. \tag{2.3}$$

It can be proven that the considered pricing kernel processes are indeed supermartingales adapted to the filtration generated by $\{X_t\}$. We refer to Akahori & Macrina (2012) for a proof that can be applied also in the present context.

   As a special case of the general pricing formula (1.1), the price process of a discount bond with maturity T, is given by

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}\left[\pi_T | X_t\right], \tag{2.4}$$

4

where $0 \leq t \leq T < U$. We keep in mind that the market filtration is generated by the Markov process $\{X_t\}$, and thus it suffices to take the expectation conditional only on X_t. The conditional expectation of π_T can be computed explicitly to obtain

$$\mathbb{E}\left[\pi_T \,|\, X_t\right] = f_0(T) + f_1(T) \int_{T-t}^{U-t} \rho(t, u, X_t) w(T, u - T + t) \, \mathrm{d}u, \tag{2.5}$$

where the tower property is invoked and a variable substitution is applied. We define

$$Y_{tT} = \int_{T-t}^{U-t} \rho(t, u, X_t) w(T, u - T + t) \, \mathrm{d}u. \tag{2.6}$$

The bond price process can then be written in the compact form

$$P_{tT} = \frac{f_0(T) + f_1(T) Y_{tT}}{f_0(t) + f_1(t) Y_{tt}}, \tag{2.7}$$

and the initial term structure is given by

$$P_{0t} = \frac{f_0(t) + f_1(t) Y_{0t}}{f_0(0) + f_1(0) Y_{00}}. \tag{2.8}$$

We deduce that

$$f_0(t) = P_{0t}\left[1 + f_1(0) Y_{00}\right] - f_1(t) Y_{0t}, \tag{2.9}$$

where we may set $f_0(0) = 1$ with no loss of generality. By inserting (2.9) in (2.7), we obtain

$$P_{tT} = \frac{P_{0T} + y(T)\left(Y_{tT} - Y_{0T}\right)}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}, \tag{2.10}$$

where

$$y(t) = \frac{f_1(t)}{1 + f_1(0) Y_{00}}, \tag{2.11}$$

for $\leq t \leq T$. Similarly, the expression for the pricing kernel can be written in the form

$$\pi_t = \pi_0\left[P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)\right], \tag{2.12}$$

where $\pi_0 = 1 + f_1(0) Y_{00}$. Assuming that the bond price function is differentiable with respect to T, the expression for the instantaneous forward rate $\{r_{tT}\}$ is given by

$$\begin{aligned} r_{tT} &= -\partial_T \ln\left(P_{tT}\right), \\ &= -\frac{\partial_T P_{0T} + \left(Y_{tT} - Y_{0T}\right) \partial_T y(T) + y(T)\left(\partial_T Y_{tT} - \partial_T Y_{0T}\right)}{P_{0T} + y(T)\left(Y_{tT} - Y_{0T}\right)}. \end{aligned} \tag{2.13}$$

The process $\{r_t\}$ for the short rate of interest can then be deduced by setting $r_t = r_{tT}\big|_{T=t}$:

$$r_t = -\frac{\partial_t P_{0t} + \left(Y_{tt} - Y_{0t}\right)\left\{\partial_T y(T)\right\}_{T=t} + y(t)\left\{\left(\partial_T Y_{tT} - \partial_T Y_{0T}\right)\right\}_{T=t}}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}. \tag{2.14}$$

Once the bond price system is derived, one can calculate the price of fixed-income derivatives such as caplets and swaptions. We consider a t-maturity swaption contract with strike K, which is written on a collection of discount bonds P_{tT_i} with maturities $\{T_i\}_{1,\dots,n}$. An application of the pricing formula (1.1) shows that the price Swp_{0t} of the swaption at time zero is

$$Swp_{0t} = \frac{1}{\pi_0}\mathbb{E}\left[\pi_t\left(1 - P_{tT_n} - K\sum_{i=1}^{n}P_{tT_i}\right)^+\right], \tag{2.15}$$

where

$$P_{tT_i} = \frac{P_{0T_i} + y(T_i)\left(Y_{tT_i} - Y_{0T_i}\right)}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}. \tag{2.16}$$

Then, by use of (2.12), we obtain

$$
\begin{aligned}
Swp_{0t} \;=\; \mathbb{E}\Bigg[& \left(P_{0t} - P_{0T_n} - K\sum_{i}^{n}P_{0T_i} + y(t)\left(Y_{tt} - Y_{0t}\right)\right. \\
& \left. -y(T_n)\left(Y_{tT_n} - Y_{0T_n}\right) - K\sum_{i=1}^{n}y(T_i)\left(Y_{tT_i} - Y_{0T_i}\right)\right)^+\Bigg].
\end{aligned}
\tag{2.17}
$$

The price of caplets can be calculated in an analogous way. Further details for the calculation of caplets and swaptions follow in the next section.

We may wonder at this stage whether it might be possible to construct a class of discount bond price processes for which the associated prices of interest rate derivatives can be calculated in closed form. We shall present such bond price models in detail in the next section. Before, we prepare the ground by making the following observation.

Proposition 2.1. *Let $\{M_t\}_{0 \leq t < U}$ be an $\{\mathscr{F}_t\}$-adapted $\mathbb{P}$-martingale that induces a change-of-measure from $\mathbb{P}$ to an equivalent auxiliary probability measure $\mathbb{M}$. Analogous to (2.6), let $\{Y_{tT}^{\mathbb{M}}\}_{0 \leq t \leq T < U}$ be defined by*

$$Y_{tT}^{\mathbb{M}} = \int_{T-t}^{U-t} \rho^{\mathbb{M}}(t,u,X_t)\,w(T,u-T+t)\,\mathrm{d}u, \tag{2.18}$$

where

$$\rho^{\mathbb{M}}(t,u,X_t) = \mathbb{E}^{\mathbb{M}}\left[F(t+u,X_{t+u})\,|\,X_t\right]. \tag{2.19}$$

Then the process

$$\pi_t = \pi_0\left[P_{0t} + y(t)\left(Y_{tt}^{\mathbb{M}} - Y_{0t}^{\mathbb{M}}\right)\right]M_t \tag{2.20}$$

is a positive $(\{\mathscr{F}_t\},\mathbb{P})$-supermartingale where

$$P_{0t} = \frac{f_0(t) + f_1(t)\,Y_{0t}^{\mathbb{M}}}{f_0(0) + f_1(0)\,Y_{00}^{\mathbb{M}}}, \tag{2.21}$$

and

$$y(t) = \frac{f_1(t)}{1 + f_1(0)Y_{00}^{\mathbb{M}}}. \tag{2.22}$$

Proof. By construction $\{\pi_t/M_t\}$ is a positive $(\{\mathscr{F}_t\}, \mathbb{M})$-supermartingale. That is, for $0 \le t \le u < U$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\pi_u \,|\, X_t\right] &= \mathbb{E}^{\mathbb{P}}\left[\pi_0\left[P_{0u} + y(u)\left(Y_{uu}^{\mathbb{M}} - Y_{0u}^{\mathbb{M}}\right)\right] M_u \,|\, X_t\right], \\
&= \mathbb{E}^{\mathbb{M}}\left[\pi_0\left[P_{0u} + y(u)\left(Y_{uu}^{\mathbb{M}} - Y_{0u}^{\mathbb{M}}\right)\right] X_t\right] M_t, \\
&\le \pi_0\left[P_{0t} + y(t)\left(Y_{tt}^{\mathbb{M}} - Y_{0t}^{\mathbb{M}}\right)\right] M_t = \pi_t.
\end{aligned}
\tag{2.23}
$$

$\square$

Thus, if convenient, we can construct pricing kernel models by considering $\mathbb{M}$-propagators $\{\rho^{\mathbb{M}}(t,u,X_t)\}$ while using the form (2.20). In such a case, the form of the asset price formulae remain unchanged, and for the discount bond price process one has

$$
P_{tT} = \frac{P_{0T} + y(T)\left(Y_{tT}^{\mathbb{M}} - Y_{0T}^{\mathbb{M}}\right)}{P_{0t} + y(t)\left(Y_{tt}^{\mathbb{M}} - Y_{0t}^{\mathbb{M}}\right)}.
\tag{2.24}
$$

3 Closed-form and explicit price models

In this section, we construct a class of pricing kernels for which the price processes of underlying and derivative assets are obtained analytically. We explicitly calculate the price processes of bonds, caplets, and swaptions, and note that the derived examples lead to bounded price processes—a property we not necessarily view as a shortcoming.

Proposition 3.1. *Let $\{M_t\}_{0 \le t < U}$ define a measure $\mathbb{M}$ on $\mathscr{F}_t$ such that $\mathbb{M}$ is equivalent to $\mathbb{P}$ on $t \in [0, U)$. Let $\{A_t\}_{0 \le t < U}$ be an $(\{\mathscr{F}_t\}, \mathbb{M})$-martingale and let $b(t)$ be a deterministic and non-increasing function such that $\pi_t > 0$ for all $t \in [0, U)$. Then,*

$$
\pi_t = \pi_0\left[P_{0t} + b(t)A_t\right] M_t
\tag{3.1}
$$

is a pricing kernel, and the discount bond system P_{tT} takes the form

$$
P_{tT} = \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t}.
\tag{3.2}
$$

Proof. To verify that (3.1) is indeed a positive supermartingale, one follows the proof of Proposition 2.1. The discount bond price process (3.2) is derived by applying (2.4) and by observing that

$$
\begin{aligned}
P_{tT} &= \frac{\mathbb{E}^{\mathbb{P}}\left[\left[P_{0T} + b(T)A_T\right] M_T \,|\, X_t\right]}{\left[P_{0t} + b(t)A_t\right] M_t}, \\
&= \frac{P_{0T} + b(T)\mathbb{E}^{\mathbb{M}}\left[A_T \,|\, X_t\right]}{P_{0t} + b(t)A_t}.
\end{aligned}
\tag{3.3}
$$

Here we have used the Bayes formula to change the measure from $\mathbb{P}$ to $\mathbb{M}$. Since $\{A_t\}$ is by definition an $(\{\mathscr{F}_t\}, \mathbb{M})$-martingale, the result (3.2) holds. $\square$

Lemma 3.1. *The short rate of interest process $\{r_t\}$ associated with the pricing kernel models (3.1) is of the form*

$$r_t = -\frac{\partial_t P_{0t} + A_t \partial_t b(t)}{P_{0t} + b(t)A_t}. \tag{3.4}$$

Proof. We first derive the instantaneous forward rate $\{r_{tT}\}$ as in (2.13), and then we set $r_t = r_{tT}\big|_{T=t}$. $\qquad\square$

Lemma 3.2. *In the pricing system specified in Proposition (3.1), the prices Cp_{0t} and Swp_{0t} at time zero of t-maturity caplet and swaption contracts are as follows:*

Caplet:

$$Cp_{0t} = -\left(\partial_t P_{0t} + K P_{0t}\right) \int_{a^C} a\, p(a)\, \mathrm{d}a - \left[\partial_t b(t) + K b(t)\right] \int_{a^C} a\, p(a)\, \mathrm{d}a, \tag{3.5}$$

where $P[A_t \in \mathrm{d}a] = p(a)\mathrm{d}a$ and

$$a^C := \left\{ a : a < -\frac{\partial_t P_{0t} + K P_{0t}}{\partial_t b(t) + K b(t)} \right\}. \tag{3.6}$$

Swaption:

$$Swp_{0t} = \left(P_{0t} - P_{0T_n} - K \sum_{i=1}^{n} P_{0T_i} \right) \int_{a^S} p(a)\, \mathrm{d}a$$
$$+ \left[b(t) - b(T_n) - K \sum_{i=1}^{n} b(T_i) \right] \int_{a^S} a\, p(a)\, \mathrm{d}a, \tag{3.7}$$

where $P[A_t \in \mathrm{d}a] = p(a)\mathrm{d}a$ and

$$a^S := \left\{ a : a > \frac{K \sum_{i=1}^{n} P_{0T_i} - P_{0t} + P_{0T_n}}{b(t) - b(T_n) - K \sum_{i=1}^{n} b(T_i)} \right\}. \tag{3.8}$$

Explicit models. In order to obtain explicit pricing models, the following quantities need to be specified in the definition of the pricing kernel (2.1): (i) The finite-time Markov process $\{X_t\}$ that generates the market filtration and drives all prices, (ii) the positive function $F(t+u, x)$ that, to a great extent, characterises the type of pricing model, (iii) the weight function $w(t, u)$, and (iv) the deterministic functions $f_0(t)$ and $f_1(t)$. The particular class of models considered in this paper allows for explicit calibration of $f_0(t)$ to the initial term structure, and for a one-to-one correspondence between the degree of freedom $f_1(t)$ and option data (e.g., caplets and swaptions). Thus we only specify the Markov process $\{X_t\}$, $F(t+u, x)$, and $w(t, u)$. In the following two examples, we let the Markov process $\{X_t\}$ be given by the Brownian random bridge $\{L_{tU}\}$ defined by

$$L_{tU} = \sigma t X_U + \beta_{tU}, \tag{3.9}$$

where σ is a constant parameter, X_U is a random variable with *a priori* density $p(x)$, and $\{\beta_{tU}\}_{0 \le t \le U}$ is an independent standard Brownian bridge where U is fixed. The change-of-measure density martingale $\{M_t\}$ in (3.1) that is associated with such an information model, satisfies

$$\mathrm{d}M_t = -\frac{\sigma U}{U-t}\mathbb{E}^{\mathbb{P}}\left[X_U \,|\, L_{tU}\right] M_t \,\mathrm{d}W_t^{\mathbb{P}} \tag{3.10}$$

where $\{W_t^{\mathbb{P}}\}$ is defined by (4.6) in the next section.

Quadratic models: We choose $F(t+u, x) = x^2$, and set $w(t, u) = U - t - u$. For this class of models, we obtain

$$A_t = \frac{U}{(U-t)^2} L_{tU}^2 - \frac{t}{U-t} \quad \text{and} \quad b(t) = \frac{(U-t)^4 f_1(t)}{4U\left[1 + \frac{1}{12}f_1(0)U^3\right]}, \tag{3.11}$$

which determine explicitly the processes for the pricing kernel, the bond price, and the associated interest rate. Furthermore, we can now work out the explicit expressions for the caplet and swaption prices via (3.5) and (3.7), respectively. For the caplet price, we obtain

$$
\begin{aligned}
Cp_{0t} \;&=\; \left[\frac{t}{U-t}\left(\partial_t b(t) + K b(t)\right) - \left(\partial_t P_{0t} + K P_{0t}\right)\right] N(\kappa) \\
&+ \; \frac{t}{U-t}\left[\partial_t b(t) + K b(t)\right]\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\kappa^2\right),
\end{aligned} \tag{3.12}
$$

where $N(x)$ denotes the cumulative normal distribution function, and κ is defined by

$$\kappa := \sqrt{\frac{U-t}{t}\left(\frac{t}{U-t} - \frac{\partial_t P_{0t} + K P_{0t}}{\partial_t b(t) + K b(t)}\right)}. \tag{3.13}$$

Exponential quadratic models: In this case we choose a function $F(t+u, x)$ that depends explicitly on time, that is:

$$F(t+u, x) = \exp\left(\frac{x^2}{2(U-t-u)}\right), \tag{3.14}$$

and take the weight function to be $w(t, u) = (U - t - u)^{\eta - 1/2}$ where $\eta > 1/2$. Then, we have:

$$A_t = \sqrt{1 - \frac{t}{U}}\exp\left(\frac{L_{tU}^2}{2(U-t)}\right) - 1 \quad \text{and} \quad b(t) = \frac{(U-t)^{\eta-1/2}U^{1/2}f_1(t)}{1 + f_1(0)U^{\eta}}. \tag{3.15}$$

As for the quadratic models, the pricing kernel, the bond price, and the associated interest rate are all determined in closed form. For the price of a swaption contract, we obtain, for this class of models, the following:

$$
\begin{aligned}
Swp_{0t} \;&=\; \left(P_{0t} - P_{0T_n} - K\sum_{i=1}^{n} P_{0T_i}\right) N(-\nu) \\
&+ \; \left[b(t) - b(T_n) - K\sum_{i=1}^{n} b(T_i)\right]\left[N(\nu) - N\left(\nu\sqrt{1-t/U}\right)\right], \tag{3.16}
\end{aligned}
$$

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where

$$v := \sqrt{\frac{2U}{t} \ln \left[\left(1 - \frac{t}{U} \right)^{-1/2} \left(\frac{K \sum_{i=1}^{n} P_{0T_i} + P_{0T_n} - P_{0t}}{b(t) - b(T_n) - K \sum_{i=1}^{n} b(T_i)} + 1 \right) \right]}. \qquad (3.17)$$

Analogous applications of Lemma 3.2 lead to the explicit expressions for the price of a swaption utilising the quadratic models, and for the price of caplets in the case of the exponential quaratic models. Although written in a less unifying form—and in the case of the quadratic models, including less useful degrees of freedom—the quadratic and exponential quadratic models were first developed in Akahori & Macrina (2012). We also note that interest rate models with a quadratic or exponential quadratic structure have been studied in Jamashidian (1996) and McCloud (2009, 2012), too.

Boundedness of prices. Bond prices fluctuate by construction between zero and one, and the associated interest rate is non-negative. However, the bond price processes produced by the above models have tighter bounds, and the same holds for the interest rate and the yield of the bond. As a consequence, the strike price of call bond options have an upper bound for in-the-money options. One might think that having bounded bond prices and associated interest rates is a shortcoming. On the contrary, we think that such a feature may be advantageous, especially if the time-dependent bounds are wide enough for the interest rate to have sufficient freedom. The bounds may be put in relation with economic policies of which goal is to keep bond prices within a certain range. In turn, this may suggest to use the additional degree of freedom $f_1(t)$, cast inside the deterministic function $b(t)$, to include regulators' policies, for instance. Research regarding bounded asset prices and the relation to regulators' policies and markets shall be continued elsewhere. We shall keep the boundedness property inherent in certain rational asset pricing models in our mind for when we later turn to general asset pricing, Section 6, and to the impact on prices by an economy's spiralling deficit, Section 7.

4   Dynamical equations

We derive the stochastic differential equation of the bond price for the case that the martingale $\{A_t\}$ in Proposition 3.1 is a continuous diffusion process. The drift term of the stochastic differential equation reveals the market price of risk, which is obtained endogenously. It turns out that the market price of risk process is constituted by two distinct components. One part is associated with the stochasticity of a financial market due to noisy information about the market factors. The second component of the market price of risk can be identified as model risk that is directly related to the choice of the class of price models. Furthermore, the Brownian motion that drives the bond price process emerges also endogenously, and in fact it is the innovations process updating the price process as the quality of market information improves.

Proposition 4.1. *Let $\{W_t^{\mathbb{P}}\}$ be a standard $(\{\mathscr{F}_t\}, \mathbb{P})$-Brownian motion. Let the $(\{\mathscr{F}_t\}, \mathbb{M})$-martingale $\{A_t\}$ satisfy*

$$\mathrm{d}A_t = v_t \left(\mathrm{d}W_t^{\mathbb{P}} + \vartheta_t \mathrm{d}t \right), \tag{4.1}$$

where $\{v_t\}$ and $\{\vartheta_t\}$ are well-defined $\{\mathscr{F}_t\}$-adapted processes. Then the dynamical equation of the discount bond with price process (3.2) is given by

$$\frac{\mathrm{d}P_{tT}}{P_{tT}} = \left(r_t + \lambda_t \Omega_{tT} \right) \mathrm{d}t + \Omega_{tT} \mathrm{d}W_t^{\mathbb{P}}, \tag{4.2}$$

where

$$r_t = -\frac{\partial_t P_{0t} + A_t \partial_t b(t)}{P_{0t} + b(t) A_t}, \tag{4.3}$$

$$\lambda_t = \vartheta_t - v_t \frac{b(t)}{P_{0t} + b(t) A_t}, \tag{4.4}$$

$$\Omega_{tT} = v_t \left[\frac{b(T)}{P_{0T} + b(T) A_t} - \frac{b(t)}{P_{0t} + b(t) A_t} \right]. \tag{4.5}$$

Proof. We apply Ito's Lemma to (3.2) and use (4.1). $\qquad\square$

By specifying the driving Markov process and by selecting a particular type of process $\{A_t\}$ in Proposition (4.1), we also obtain the result in the following lemma.

Lemma 4.1. *Let the Markov process generating the market filtration $\{\mathscr{F}_t\}$ be the information process (3.9). Then the Brownian motion $\{W_t^{\mathbb{P}}\}$ satisfies*

$$\mathrm{d}W_t^{\mathbb{P}} = \mathrm{d}L_{tU} - \frac{1}{U-t} \left(\sigma U \, \mathbb{E}^{\mathbb{P}} \left[X_U \, | \, L_{tU} \right] - L_{tU} \right) \mathrm{d}t, \tag{4.6}$$

and the process $\{\theta_t\}$ in Proposition 4.1 is given by

$$\vartheta_t = \frac{\sigma U}{U-t} \mathbb{E}^{\mathbb{P}} \left[X_U \, | \, L_{tU} \right]. \tag{4.7}$$

The process $\{v_t\}$ in Proposition 4.1 satisfies

$$v_t = \frac{2U}{(U-t)^2} L_{tU} \tag{4.8}$$

in the case of the quadratic models (3.11), and it satisfies

$$v_t = \frac{\left(\frac{U-t}{U} \right)^{1/2}}{U-t} L_{tU} \exp \left[\frac{L_{tU}^2}{2(U-t)} \right] \tag{4.9}$$

in the case of the exponential quadratic models (3.15).

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The process $\{W_t^{\mathbb{M}}\}$, satisfying $\mathrm{d}W_t^{\mathbb{M}} = \mathrm{d}W_t^{\mathbb{P}} + \vartheta_t \mathrm{d}t$, is an $(\{\mathscr{F}_t\}, \mathbb{M})$-Brownian motion. This follows from the Girsanov theorem. We emphasize here that the risk premium process $\{\lambda_t\}$ is constituted by the process $\{\vartheta_t\}$ and $\{\nu_t\}$. We observe (a) that $\{\vartheta_t\}$ is determined by the filtration model, that is, by the choice of the generating Markov process $\{L_{tU}\}$, and (b) that $\{\nu_t\}$ depends on the selection of the heat kernel models, that is, on the choice for $F(t,x)$ and $w(t,u)$. We can view $\{\vartheta_t\}$ as the risk premium component associated with the uncertainty in the market modelled via the information flow process $\{L_{tU}\}$. The component $\{\nu_t\}$ may however be interpreted as the premium associated with model risk since it is closely related to the choice of the specific asset price model. By applying Ito's formula to (2.13), we obtain the following result:

Proposition 4.2. *The instantaneous forward rate $\{r_{tT}\}$ of the bond price process (3.2) satisfies the dynamical equation*

$$\frac{\mathrm{d}r_{tT}}{r_{tT}} = -\sigma_{tT}\Omega_{tT}\mathrm{d}t + \sigma_{tT}\left(\mathrm{d}W_t^{\mathbb{P}} + \lambda_t \mathrm{d}t\right), \tag{4.10}$$

where the instantaneous forward rate volatility $\{\sigma_{tT}\}$ is defined by

$$\sigma_{tT} = \nu_t \left(\frac{b(T)}{P_{0T} + b(T)A_t} - \frac{\partial_T b(T)}{\partial_T P_{0T} + A_t \partial_T b(T)} \right). \tag{4.11}$$

Applying Ito's formula to (2.14) gives the stochastic differential equation for the short rate of interest $\{r_t\}$ in the same set-up:

$$\frac{\mathrm{d}r_t}{r_t} = \mu_t \mathrm{d}t + \sigma_t \left(\mathrm{d}W_t^{\mathbb{P}} + \lambda_t \mathrm{d}t\right), \tag{4.12}$$

where $\{\lambda_t\}$ is the instantaneous market price of risk (4.4), $\sigma_t = \sigma_{tT}\big|_{T=t}$, and

$$\mu_t = \frac{\partial_t P_{0t} + A_t \partial_t b(t)}{P_{0t} + b(t)A_t} - \frac{\partial_{tt} P_{0t} + A_t \partial_{tt} b(t)}{\partial_t P_{0t} + A_t \partial_t b(t)}. \tag{4.13}$$

The processes driving the dynamics of the presented models have Gaussian laws, and the dynamics of the deduced instantaneous forward rates have the HJM-form, c.f. Heath et al. (1992), Filipović (2009). It can be shown that $\{W_t^{\mathbb{P}}\}_{0 \leq t < U}$ is an $(\{\mathscr{F}_t\}, \mathbb{P})$-Brownian motion, see, e.g., Brody et al. (2008).

5 Incomplete market models driven by LRBs

In this section, we extend the pricing framework to include multi-dimensional risk factors, and we generate asset pricing models in an incomplete market. We consider a class of finite-time Markov processes, the so-called "Lévy random bridges" (LRBs), as constructed in Hoyle et al. (2011). An LRB can be interpreted as a Lévy process that is bound to have a prescribed, albeit arbitrary, distribution at a fixed future time. The LRB and the generating Lévy process are linked by a an equivalent probability measure with respect to which the LRB has the law of the generating Lévy process. Before we slightly extend this result appearing in Hoyle et al. (2011), we first give the definition of a multivariate LRB:

Definition 5.1. *We say that $\{L_{tU}\}_{0 \leq t \leq U}$ is a multivariate LRB on $\mathbb{R}^m$ if the following are satisfied:*

1. *The random variable L_{UU} on $\mathbb{R}^m$ has multivariate marginal law $\nu(z)$, $z \in \mathbb{R}^m$.*

2. *There exist a multivariate Lévy process $\{L_t\}_{0 \leq t \leq U}$ on $\mathbb{R}^m$ such that L_t has multivariate density function $\rho_t(x)$ on $\mathbb{R}^m$ for all $t \in (0, U]$.*

3. *The marginal law $\nu(z)$ concentrates mass where $\rho_U(z)$ is positive and finite, that is $0 < \rho_U(z) < \infty$ for $\nu(z)$ almost every z.*

4. *For every $n \in \mathbb{N}_+$, every $0 < t_1 < \ldots < t_n < U$, every $(x_1, \ldots, x_n) \in \mathbb{R}^m \times \mathbb{R}^n$, and $\nu(z)$ almost every z, we have*

$$\mathbb{P}\left[L_{t_1 U} \leq x_1, \ldots, L_{t_n U} \leq x_n \mid L_{UU} = z\right] = \mathbb{P}\left[L_{t_1} \leq x_1, \ldots L_{t_n} \leq x_n \mid L_U = z\right]. \quad (5.1)$$

Proposition 5.1. *Let $\{L_{tU}\}_{0 \leq t \leq U}$ denote a multivariate LRB with marginal law $\mathbb{P}[L_{UU} \in \mathrm{d}z] = \nu(\mathrm{d}z)$. Let the multivariate Lévy process $\{L_t\}_{0 \leq t \leq U}$, which generates the LRB, have density $\rho_t(x)$ for all $t \in (0, U]$. Under the measure $\mathbb{L}$ defined by*

$$\ell_t^{-1} := \left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{L}}\right|_{\mathscr{F}_t} = \int_{\mathbb{R}} \frac{\rho_{U-t}(z - L_{tU})}{\rho_U(z)} \nu(\mathrm{d}z), \quad (5.2)$$

the LRB $\{L_{tU}\}$ has the law of the generating Lévy process for $t \in [0, U)$.

Proof. The verification of this proposition follows closely the results leading to Proposition 3.7 in Hoyle et al. (2011). □

The measure $\mathbb{L}$ is rather useful for several calculations. LRBs, which have joint marginal law at $t = U$ and which are generated by independent Lévy processes, are independent under $\mathbb{L}$:

Proposition 5.2. *Let $\{L_{tU}\}_{0 \leq t \leq U}$ be a multivariate LRB on $\mathbb{R}^m$ generated by a Lévy process on $\mathbb{R}^m$ of which multivariate density function $\rho_t(z)$ factorises, that is*

$$\rho_t(z) = \prod_{k=1}^m \rho_t^k(z_k). \quad (5.3)$$

Under $\mathbb{L}$, the components, $\{L_{tU}^{(i)}\}$ and $\{L_{tU}^{(j)}\}$ $i \neq j$, of the LRB on $\mathbb{R}^m$ are independent and each LRB component has the law of the respective component of the generating Lévy process $\{L_t\}$, for $t \in [0, U)$.

Proof. We show that the generating function of the multivariate LRB factorises under

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$\mathbb{L}$. We have:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\ell_t \exp\left(\sum_{k=1}^m \alpha_k L_{tU}^{(k)}\right)\right] &= \mathbb{E}^{\mathbb{L}}\left[\exp\left(\sum_{k=1}^m \alpha_k L_{tU}^{(k)}\right)\right], \\
&= \int_{\mathbb{R}^m} \exp\left(\sum_{k=1}^m \alpha_k y_k\right) \mathbb{L}\left[L_{tU}^{(1)} \in \mathrm{d}y_1, \dots, L_{tU}^{(k)} \in \mathrm{d}y_m\right], \\
&= \int_{\mathbb{R}^m} \exp\left(\sum_{k=1}^m \alpha_k y_k\right) \prod_{k=1}^m \mathbb{L}\left[L_{tU}^{(k)} \in \mathrm{d}y_k\right], \\
&= \prod_{k=1}^m \int_{\mathbb{R}} \exp\left(\alpha_k y_k\right) \mathbb{L}\left[L_{tU}^{(k)} \in \mathrm{d}y_k\right], \\
&= \prod_{k=1}^m \mathbb{E}^{\mathbb{L}}\left[\exp\left(\alpha_k L_{tU}^{(k)}\right)\right]. \tag{5.4}
\end{aligned}
$$

$\square$

Next we make use of the $\mathbb{L}$-independence property of the LRBs to propose multi-factor pricing kernel models in the situation where the driving Markov process is a multivariate LRB. We assume that the market filtration $\{\mathscr{F}_t\}$ is generated by a two-dimensional LRB, of which first component is a Brownian random bridge,

$$
L_{tU}^{(1)} = \sigma X_U^{(1)} t + \beta_{tU}, \tag{5.5}
$$

and of which second component is a gamma random bridge defined by

$$
L_{tU}^{(2)} = X_U^{(2)} \gamma_{tU}. \tag{5.6}
$$

The Brownian bridge $\{\beta_{tU}\}$ and the gamma bridge $\{\gamma_{tU}\}$ are assumed independent of each other and also independent of the random variables $X_U^{(1)}$ and $X_U^{(2)}$. However, the two X random variables are dependent and have a priori joint marginal law $\nu(z_1, z_2)$. This set-up is of the kind considered in Proposition 5.2. Next we recall the bond pricing formula (2.24) where, this time,

$$
\begin{aligned}
Y_{tT}^{\mathbb{L}} = \int_{T-t}^{U-t} \int_{T-t}^{U-t} & w(T, u_1 - T + t, u_2 - T + t) \\
&\times \mathbb{E}^{\mathbb{L}}\left[F\left(t+u_1, t+u_2, L_{t+u_1,U}^{(1)}, L_{t+u_2,U}^{(2)}\right) \Big| L_{tU}^{(1)}, L_{tU}^{(2)}\right] \mathrm{d}u_1 \mathrm{d}u_2. \tag{5.7}
\end{aligned}
$$

Since, for $t \in [0, U)$, the two LRB components have each the law of the corresponding underlying Lévy process component—that is respectively the Brownian motion and the gamma process—the conditional expectation simplifies considerably under $\mathbb{L}$. That is,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{L}} &\left[F\left(t+u_1, t+u_2, L_{t+u_1,U}^{(1)}, L_{t+u_2,U}^{(2)}\right) \Big| L_{tU}^{(1)} = x_1, L_{tU}^{(2)} = x_2\right] \\
&= \mathbb{E}^{\mathbb{L}}\left[F\left(t+u_1, t+u_2, \left(L_{t+u_1,U}^{(1)} - L_{tU}^{(1)}\right) + x_1, \left(L_{t+u_2,U}^{(2)} - L_{tU}^{(2)}\right) + x_2\right)\right]. \tag{5.8}
\end{aligned}
$$

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Since the LRB components are $\mathbb{L}$-independent, and since the $\mathbb{L}$-laws of the LRB components are known, the probability densities of the increments in the above equation are known. We thus have:

$$\mathbb{L}\left[L^{(1)}_{t+u_1,U} - L^{(1)}_{tU} \in \mathrm{d}y_1\right] = \frac{1}{\sqrt{2\pi u_1}} \exp\left(-\frac{y_1^2}{2u_1}\right)\mathrm{d}y_1, \tag{5.9}$$

$$\mathbb{L}\left[L^{(2)}_{t+u_2,U} - L^{(2)}_{tU} \in \mathrm{d}y_2\right] = \frac{\mathbb{1}\{y_2 > 0\}}{\Gamma[mu_2]} y_2^{mu_2-1} \exp(-y_2)\,\mathrm{d}y_2, \tag{5.10}$$

where $m > 0$ and $\Gamma[x]$ is the gamma function. In order to work out an explicit example, we need to specify $F(t, y_1, y_2)$ and the weight function $w(t, u_1, u_2)$. We choose the following example:

$$F\left(t + u_1, t + u_2, y_1 + x_1, y_2 + x_2\right) = \exp\left[a\left(y_1 + x_1\right) - c\left(y_2 + x_2\right)\right], \tag{5.11}$$

$$w(t, u_1, u_2) = \exp\left(-\tfrac{1}{2}a^2(t + u_1)\right)(c + 1)^{m(t+u_2)}, \tag{5.12}$$

where $a \in [-\infty, \infty)$, $c \geq 0$ are constants. Then we insert (5.9) and (5.10) together with (5.11) and (5.12) in (5.7) and calculate the integrals over u_1 and u_2. The result is:

$$Y^{\mathbb{L}}_{tT} = (U - T)^2(c + 1)^{mt} \exp\left(a\,L^{(1)}_{tU} - c\,L^{(2)}_{tU} - \tfrac{1}{2}a^2 t\right). \tag{5.13}$$

The two-factor pricing kernel, jointly driven by a Brownian random bridge and a gamma random bridge, is thus given by a formula similar to (2.20) where $\{\ell_t\}$ is the reciprocal of (5.2) while $Y^{\mathbb{L}}_{tt}$, $Y^{\mathbb{L}}_{0t}$, and $Y^{\mathbb{L}}_{00}$ are deduced from (5.13). For this class of models, the bond price process can be written as follows:

$$P_{tT} = \frac{P_{0T} + b(T)A^{\mathbb{L}}_t}{P_{0t} + b(t)A^{\mathbb{L}}_t}, \tag{5.14}$$

where

$$b(t) = \frac{(U - t)^2 f_1(t)}{1 + f_1(0)U^2}, \tag{5.15}$$

$$A^{\mathbb{L}}_t = (c + 1)^{mt} \exp\left(a\,L^{(1)}_{tU} - c\,L^{(2)}_{tU} - \tfrac{1}{2}a^2 t\right) - 1, \tag{5.16}$$

for $0 \leq t \leq T < U$. It is straightforward to show that $\{A^{\mathbb{L}}_t\}$ is an $(\{\mathscr{F}_t\}, \mathbb{L})$-martingale for $t \in [0, U)$. Then the process $\{\ell_t A^{\mathbb{L}}_t\}$ is an $(\{\mathscr{F}_t\}, \mathbb{P})$-martingale, and the bond price process (5.14) has a representation in terms of $(\{\mathscr{F}_t\}, \mathbb{P})$-martingales, that is

$$P_{tT} = \frac{P_{0T}\,\ell_t + b(T)A^{\mathbb{P}}_t}{P_{0t}\,\ell_t + b(t)A^{\mathbb{P}}_t}, \tag{5.17}$$

where $A^{\mathbb{P}}_t = \ell_t A^{\mathbb{L}}_t$. Such models might be regarded as belonging to the finite-time equivalence class of Flesaker & Hughston (1996) bond price models. We note that pricing kernel models over infinite time, as in Flesaker & Hughston (1996), must be potentials of class D,

see Rogers (1997), Meyer (1962). In finite time, pricing kernel processes merely need to be positive $(\{\mathscr{F}_t\}, \mathbb{P})$-supermartingales to ensure non-negative interest rates. Furthermore, the martingale processes underlying the price models arise endogenously from the pricing kernel structure (2.1). Formula (2.1) can be viewed as a "machine" that implicitly produces martingales for pricing formulae with a rational form. Bermin (2012) revisits the Flesaker-Hughston approach to bond pricing and shows how yield curves may be inverted for any short rate process consistent with bond price processes that have an exponentially-affine structure.

In Section 3, explicit pricing models are derived, and one may ask at this point what the connection is between those pricing models and the ones specified in (5.14). It turns out, as expected, that the link is a change of probability measure. Let us consider an $(\{\mathscr{F}_t\}, \mathbb{L})$-martingale $\{A_t^{\mathbb{L}}\}$ and an $(\{\mathscr{F}_t\}, \mathbb{M})$-martingale $\{A_t^{\mathbb{M}}\}$. Furthermore, we introduce an $(\{\mathscr{F}_t\}, \mathbb{L})$-density-martingale $\{\eta_t\}_{0 \le t < U}$ that changes the probability measure $\mathbb{L}$ to the equivalent measure $\mathbb{M}$. Then we set $A_t^{\mathbb{L}} = \eta_t A_t^{\mathbb{M}}$, and finally observe, for $0 \le s \le t < U$, that

$$\mathbb{E}^{\mathbb{L}}\left[A_t^{\mathbb{L}} \,\middle|\, \mathscr{F}_s \right] = \mathbb{E}^{\mathbb{L}}\left[\eta_t A_t^{\mathbb{M}} \,\middle|\, \mathscr{F}_s \right] = \eta_s \,\mathbb{E}^{\mathbb{M}}\left[A_t^{\mathbb{M}} \,\middle|\, \mathscr{F}_s \right] = \eta_s A_s^{\mathbb{M}} = A_s^{\mathbb{L}}. \tag{5.18}$$

This type of relation is also what connects (5.14) and (5.17).

A useful formula. The fact that an LRB has the law of its generating Lévy process under the "Lévy probability measure" $\mathbb{L}$ can be exploited to derive a rather useful formula involving the characteristic function of a Lévy process and its Fourier transform. This formula might be particularly useful for numerical implementations of option pricing formulae based on Fast Fourier Transform techniques, see, e.g., Carr & Madan (1999). We consider again the $\mathbb{L}$-conditional expectation

$$\mathbb{E}^{\mathbb{L}}\left[F(t+u, L_{t+u,U}) \,|\, L_{tU} \right], \tag{5.19}$$

where $\{L_{tU}\}$ is an LRB. We specify $F(t,x)$ by its Fourier transform $\widehat{F}(t,y)$, that is

$$F(t,x) = \int_{\mathbb{R}} \exp(-i\,xy)\widehat{F}(t,y)\,\mathrm{d}y, \tag{5.20}$$

where $\widehat{F}(t,y)$ is selected such that $F(t,x)$ is positive and integrable. Then we have,

$$\mathbb{E}^{\mathbb{L}}\left[F(t+u, L_{t+u,U}) \,|\, L_{tU} \right] = \mathbb{E}^{\mathbb{L}}\left[\int_{\mathbb{R}} \exp(-iy L_{t+u,U})\widehat{F}(t+u,y)\,\mathrm{d}y \,\middle|\, L_{tU} \right]. \tag{5.21}$$

Assuming that Fubini's Theorem is herewith satisfied, we swap the expectation with the integral, and obtain

$$\mathbb{E}^{\mathbb{L}}\left[F(t+u, L_{t+u,U}) \,|\, L_{tU} \right] = \int_{\mathbb{R}} \mathbb{E}^{\mathbb{L}}\left[\exp\left(-iy L_{t+u,U} \right) \,\middle|\, L_{tU} \right] \widehat{F}(t+u,y)\,\mathrm{d}y. \tag{5.22}$$

Since the LRB has the $\mathbb{L}$-law of the underlying Lévy process for $t \in [0, U)$, we can calculate the conditional expectation by recalling that the increments of a Lévy process are independent and stationary:

$$\mathbb{E}^{\mathbb{L}}\left[\exp\left(-iyL_{t+u,U}\right)\,\middle|\,L_{tU}\right] = \exp\left(-iyL_{tU}\right)\mathbb{E}^{\mathbb{L}}\left[\exp\left(-iyL_{uU}\right)\right]. \tag{5.23}$$

The expectation on the right-hand-side of the equation above is the generating function of a Lévy process. We denote the characteristic function of a Lévy process by $\Psi(y)$, and thus write

$$\mathbb{E}^{\mathbb{L}}\left[\exp\left(-iyL_{uU}\right)\right] = \exp\left[-u\,\Psi(y)\right], \tag{5.24}$$

for $u \in [0, U)$. This leads to

$$\mathbb{E}^{\mathbb{L}}\left[F(t+u, L_{t+u,U})\,|\,L_{tU}\right] = \int_{\mathbb{R}}\exp\left[-iyL_{tU} - u\,\Psi(u)\right]\widehat{F}(t+u, y)\,\mathrm{d}y, \tag{5.25}$$

and hence to the useful formula

$$\begin{aligned}
Y_{tT}^{\mathbb{L}} &= \int_{T-t}^{U-t} w(T, u-T+t)\mathbb{E}^{\mathbb{L}}\left[F(t+u, L_{t+u,U})\,|\,X_{tU}\right]\mathrm{d}u, \\
&= \int_{T-t}^{U-t} w(T, u-T+t)\int_{\mathbb{R}}\exp\left[-iyL_{tU} - u\,\Psi(u)\right]\widehat{F}(t+u, y)\,\mathrm{d}y\,\mathrm{d}u. \tag{5.26}
\end{aligned}$$

Expression (5.26) is valid also in the multi-factor case, in which the LRB $\{L_{tU}\}$ is a multidimensional vector. The elements of the LRB vector may be dependent through their terminal marginal laws as considered at the beginning of this section. The model (5.13) may be derived as a special case of the formula (5.26).

We conclude this section by producing multi-dimensional and multi-factor asset price processes, and note that for the rest of this section the process $\{Y_{tT}^{(i)}\}$ that follows is defined under the $\mathbb{L}$-measure. For $i = 1, 2, \ldots, n$, let

$$Y_{tT}^{(i)} = \int_{T-t}^{U-t}\mathbb{E}^{\mathbb{L}}\left[F_i(t+u, L_{t+u,U})\,|\,L_{tU}\right]w_i(T, u-T+t)\,\mathrm{d}u, \tag{5.27}$$

where $F_i(t, x)$ is a positive and integrable function, and $w_i(t, u)$ is a weight function. We emphasize that the Markov process $\{L_{tU}\}_{0 \le t}$ may be multi-dimensional. Then, the following is a multi-dimensional and multi-factor model for the bond price:

$$P_{tT} = \frac{P_{0T} + \sum_{i=1}^{n} y_i(T)\left(Y_{tT}^{(i)} - Y_{0T}^{(i)}\right)}{P_{0t} + \sum_{i=1}^{n} y_i(t)\left(Y_{tt}^{(i)} - Y_{0t}^{(i)}\right)}, \tag{5.28}$$

where

$$y_i(t) = \frac{f_i(t)}{1 + \sum_{i=1}^{n} Y_{00}^{(i)}}. \tag{5.29}$$

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These models can be extended further. Let us assume that the multivariate LRBs generating the market filtration $\{\mathscr{F}_t\}$ satisfy Proposition 5.2. Then, the product of an $(\{\mathscr{F}_t\}, \mathbb{L})$-supermartingale is again an $(\{\mathscr{F}_t\}, \mathbb{L})$-supermartingale. This leads us to the construction of higher-order asset pricing models, hereunder applied to the pricing of bonds. We generalise the bond price model (5.14): For $0 \leq t \leq T$ and $N \in \mathbb{N}$,

$$P_{tT} = \frac{P_{0T} + \sum_{i=1}^{N} \Lambda_{tT}^{(i)}}{P_{0t} + \sum_{i=1}^{N} \Lambda_{tt}^{(i)}}, \tag{5.30}$$

where

$$\Lambda_{tT}^{(i)} = \sum_{j_{n_i-m_i}=1}^{n_i-m_i} b_{j_{n_i-m_i}}^{(i)}(T) A_t^{(i, j_{n_i-m_i})} \sum_{j_{n_i-(m_i-1)}=j_{n_i-m_i}+1}^{n_i-(m_i-1)} b_{j_{n_i-(m_i-1)}}^{(i)}(T) A_t^{(i, j_{n_i-(m_i-1)})}$$

$$\cdots \sum_{j_{n_i-1}=j_{n_i-2}+1}^{n_i-1} b_{j_{n_i-1}}^{(i)}(T) A_t^{(i, j_{n_i-1})} \sum_{j_{n_i}=j_{n_i-1}+1}^{n} b_{j_{n_i}}^{(i)}(T) A_t^{(i, j_{n_i})} \tag{5.31}$$

for $n_i \geq m_i \in \mathbb{N}$. By setting $T = t$ in (5.31) one obtains $\{\Lambda_{tt}\}$. The deterministic functions $b_1^{(i)}(t), b_2^{(i)}(t), \dots, b_n^{(i)}(t)$ are such that the product $b_1^{(i)}(t) b_2^{(i)}(t) \cdots b_n^{(i)}(t)$ is non-negative and non-increasing. The processes $\{A_t^{(i,j)}\}_{0 \leq t < U}$ and $\{A_t^{(i,k)}\}_{0 \leq t < U}$ are $(\{\mathscr{F}_t\}, \mathbb{L})$-martingales, and these are $\mathbb{L}$-independent for $j \neq k$. For instance, for $N = 1$, $n_1 = 3$ and $m_1 = 2$, one obtains

$$P_{tT} = \frac{P_{0T} + b_{123}(T) A_t^{(1)} A_t^{(2)} A_t^{(3)}}{P_{0t} + b_{123}(t) A_t^{(1)} A_t^{(2)} A_t^{(3)}}, \tag{5.32}$$

where $b_{123}(t) = b_1(t) b_2(t) b_3(t)$ for $0 \leq t \leq T$. For $N = 1$, $n_1 = 3$ and $m_1 = 1$, we have

$$P_{tT} = \frac{P_{0T} + b_{12}(T) A_t^{(1)} A_t^{(2)} + b_{13}(T) A_t^{(1)} A_t^{(3)} + b_{23}(T) A_t^{(2)} A_t^{(3)}}{P_{0t} + b_{12}(t) A_t^{(1)} A_t^{(2)} + b_{13}(t) A_t^{(1)} A_t^{(3)} + b_{23}(t) A_t^{(2)} A_t^{(3)}}, \tag{5.33}$$

where $b_{ij}(t) = b_j(t) b_k(t)$ for $j \neq k$ and $0 \leq t \leq T$. In order to lighten the notation, the i-index is suppressed in (5.32) and (5.33) since we have only one type of higher-order term in the sum over i. For $N = 2$, $n_1 = 3$ and $m_1 = 2$, $n_2 = 3$ and $m_2 = 1$, a combination of third-order and second-order models is obtained:

$$P_{tT} =$$
$$\frac{P_{0T} + b_{123}^{(1)}(T) A_t^{(1,1)} A_t^{(1,2)} A_t^{(1,3)} + b_{12}^{(2)}(T) A_t^{(2,1)} A_t^{(2,2)} + b_{13}^{(2)}(T) A_t^{(2,1)} A_t^{(2,3)} + b_{23}^{(2)}(T) A_t^{(2,2)} A_t^{(2,3)}}{P_{0t} + b_{123}^{(1)}(t) A_t^{(1,1)} A_t^{(1,2)} A_t^{(1,3)} + b_{12}^{(2)}(t) A_t^{(2,1)} A_t^{(2,2)} + b_{13}^{(2)}(t) A_t^{(2,1)} A_t^{(2,3)} + b_{23}^{(2)}(t) A_t^{(2,2)} A_t^{(2,3)}}$$
$$\tag{5.34}$$

We note that the construction of higher-order pricing formulae is not limited to models driven by LRBs. The pricing kernel model (2.1) can be used to construct higher-order price models driven by other Markov processes. Higher-order models gain in importance when considering general asset pricing including dependences across several types of assets.

6 General asset pricing in finite time

The pricing formula (1.1) states that the no-arbitrage price process of an asset is an $(\{\mathscr{F}_t\}, \mathbb{P})$-martingale. There are several ways of constructing $(\{\mathscr{F}_t\}, \mathbb{P})$-martingales—we consider however a natural method within the framework developed thus far. For some fixed T, we denote by $\{m_{tT}\}_{0 \leq t \leq T}$ an $(\{\mathscr{F}_t\}, \mathbb{P})$-martingale, and write

$$\pi_t S_{tT} = m_{tT}. \tag{6.1}$$

Definition 6.1. *Let $\{Z_{tT}\}_{0 \leq t \leq T < U}$ be defined by*

$$Z_{tT} = \int_{T-t}^{U-t} \mathbb{E}^{\mathbb{P}} \left[G\left(t + u, X_{t+u}\right) | X_t \right] \psi(T, u - T + t) \mathrm{d}u, \tag{6.2}$$

where the deterministic function $G(t, x)$ is measurable, and $\psi(t, u)$ is a deterministic function with the property $\psi(t, u - s) = \psi(t - s, u)$ for $s \leq t \wedge u$.

Proposition 6.1. *Let $0 \leq t \leq T < U$, and let $g_0(T)$ and $g_1(T)$ be deterministic functions. Then, for each fixed T,*

$$m_{tT} = g_0(T) + g_1(T) Z_{tT} \tag{6.3}$$

is an $(\{\mathscr{F}_t\}, \mathbb{P})$-martingale.

  Proof. This proposition can be proven by following the steps in the proof of Proposition 2.2 in Akahori & Macrina (2012). We observe that $\psi(t, u - s) = \psi(t - s, u)$ implies $\psi(t, u) = \psi(t + u)$. $\qquad\square$

  We now have the necessary ingredients in order to propose the following class of asset price models.

Proposition 6.2. *Let $\{\pi_t\}$ and $\{m_{tT}\}$ be the processes (2.12) and (6.3), respectively. Then the asset price model (6.1) takes the form*

$$S_{tT} = \frac{S_{0T} + z(T)\left(Z_{tT} - Z_{0T}\right)}{P_{0t} + y(t)\left(Y_{tt} - Y_{0t}\right)}, \tag{6.4}$$

where $z(T) = g_1(T)/\pi_0$.

  Proof. The relation (6.4) follows from (6.1) by inserting (2.12) and (6.3). The degree of freedom $g_0(T)$ can be calibrated to the asset price S_{0T} at time 0 via

$$g_0(T) = S_{0T} \pi_0 - g_1(T) Z_{0T}, \tag{6.5}$$

where $\pi_0 = 1 + f_1(0) Y_{00}$. $\qquad\square$

  As at the beginning of Section 3, we next derive general asset price models for which explicit expressions for derivatives can be computed. In doing so, we implicitly apply the results in Proposition 2.1.

Lemma 6.1. *Let $\{M_t\}_{0 \leq t < U}$ be the $\{\mathcal{F}_t\}$-adapted density martingale inducing a change-of-measure from $\mathbb{P}$ to $\mathbb{M}$. Let $\{A_t^{(i)}\}_{0 \leq t < U}^{i=1,2}$ be $(\{\mathcal{F}_t\}, \mathbb{M})$-martingales, and consider a pricing kernel model of the form (3.1). Furthermore, let*

$$S_{TT} = \frac{S_{0T} + b_1(T)A_T^{(1)}}{P_{0T} + b_2(T)A_T^{(2)}}, \tag{6.6}$$

where $b_i(T)$, $i = 1, 2$, are deterministic functions, and $b_2(t)$ is non-negative and non-increasing. Then the associated asset price process $\{S_{tT}\}$ is given by

$$S_{tT} = \frac{S_{0T} + b_1(T)A_t^{(1)}}{P_{0t} + b_2(t)A_t^{(2)}}. \tag{6.7}$$

Proof. The expression (6.7) is obtained by computing the expectation in

$$S_{tT} = \frac{1}{\pi_t} \mathbb{E}^{\mathbb{P}}\left[\pi_T S_{TT} \mid \mathcal{F}_t\right] \tag{6.8}$$

where, by use of $\{M_t\}$, the measure $\mathbb{P}$ is changed to $\mathbb{M}$ in order to exploit the martingale property of $\{A_t^{(i)}\}_{0 \leq t < U}^{i=1,2}$ under $\mathbb{M}$. $\qquad \square$

Dynamical equation of the asset price process. We consider the price process given in (6.7) and assume that it is adapted to a market filtration generated by two Brownian random bridges $\{L_{tU}^{(i)}\}$, $i = 1, 2$, c.f. (3.9). We then follow the calculations in Section 4 to deduce the subsequent result:

Proposition 6.3. *Let $\{\mathcal{F}_t\}$ be jointly generated by $\{L_{tU}^{(i)}\}$, $i = 1, 2$. Let the price process of an asset be of the form (6.7) where $\{A_t^{(i)}\}_{0 \leq t < U}$ satisfies*

$$dA_t^{(i)} = v_t^{(i)}\left(dW_t^{\mathbb{P}\ (i)} + \vartheta_t^{(i)}dt\right) \tag{6.9}$$

for $i = 1, 2$. The process $\{v_t^{(i)}\}$ is $\{\mathcal{F}_t\}$-adapted, and

$$\vartheta_t^{(i)} = \frac{\sigma_i U}{U - t} \mathbb{E}^{\mathbb{P}}\left[X_U^{(i)} \mid \mathcal{F}_t\right], \quad dW_t^{\mathbb{P}\ (i)} = dL_{tU}^{(i)} - \frac{1}{U - t}\left(\sigma_i U \mathbb{E}^{\mathbb{P}}\left[X_U^{(i)} \mid \mathcal{F}_t\right] - L_{tU}^{(i)}\right)dt$$

where σ_i is constant. The dynamical equation of such a price process is given by

$$\frac{dS_{tT}}{S_{tT}} = \left(r_t + \lambda_t \Sigma_{tT}\right)dt + \Sigma_{tT}\, dW_t^{\mathbb{P}}, \tag{6.10}$$

where

$$r_t = -\frac{\dot{P}_{0t} + \dot{b}_2(t)A_t^{(2)}}{P_{0t} + b_2(t)A_t^{(2)}}, \quad \lambda_t = \begin{pmatrix} \vartheta_t^{(1)} - \rho_{ij} v_t^{(2)} \frac{b_2(t)}{P_{0t}+b_2(t)A_t^{(2)}} \\[2mm] \vartheta_t^{(2)} - v_t^{(2)} \frac{b_2(t)}{P_{0t}+b_2(t)A_t^{(2)}} \end{pmatrix}, \quad \Sigma_{tT} = \begin{pmatrix} \frac{b_1(T)v_t^{(1)}}{S_{0T}+b_1(T)A_t^{(1)}} \\[2mm] -\frac{b_2(t)v_t^{(2)}}{P_{0t}+b_2(t)A_t^{(2)}} \end{pmatrix}.$$

The process $W_t^{\mathbb{P}} = (W_t^{\mathbb{P}\ (1)}, W_t^{\mathbb{P}\ (2)})$ is a two-dimensional $(\{\mathcal{F}_t\}, \mathbb{P})$-Brownian motion where $dW_t^{\mathbb{P}\ (i)} dW_t^{\mathbb{P}\ (j)} = \rho_{ij}\, dt$ for $i \neq j$, and $dW_t^{\mathbb{P}\ (i)} dW_t^{\mathbb{P}\ (j)} = dt$ for $i = j$.

As in Lemma 4.1, $\{v_t^{(i)}\}$, $i = 1, 2$, are determined by the specific choice of the models at the base of the processes $\{A_t^{(i)}\}$, $i = 1, 2$, and thus are model-specific. Furthermore, the dynamics (6.10) can be represented in the form so as to obtain the risk-neutral dynamical equation of the asset price process $\{S_{tT}\}_{0 \leq t \leq T}$. We have:

$$\frac{\mathrm{d}S_{tT}}{S_{tT}} = r_t \mathrm{d}t + \Sigma_{tT}\, \mathrm{d}W_t^{\mathbb{Q}}, \tag{6.11}$$

where $\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t^{\mathbb{P}} + \lambda_t \mathrm{d}t$ is the risk-neutral Brownian motion defined in terms of the $\mathbb{P}$-Brownian motion $\{W_t^{\mathbb{P}}\}$ and the market price of risk process $\{\lambda_t\}$. The solution to the stochastic differential equation (6.11) has the familiar $\mathbb{Q}$-log-normal form

$$S_{tT} = S_{0T} \exp\left(\int_0^t \left(r_s - \tfrac{1}{2}\Sigma_{sT}^2 \right) \mathrm{d}s + \int_0^t \Sigma_{sT}\, \mathrm{d}W_s^{\mathbb{Q}} \right). \tag{6.12}$$

How one may obtain "finite-time Black-Scholes-type models" from (6.12) can be deduced by consulting Brody et al. (2008), Section 9. In addition, appropriate choices for the functions $F(x)$, $G(x)$, and the related weight functions will need to be made.

Example. Let the market filtration be generated by two Brownian random bridges $\{L_{tU}^{(i)}\}$, $i = 1, 2$. Let us consider the quadratic model (3.11) for the pricing kernel modelled by $\{A_t^{(2)}\}$ driven by $\{L_{tU}^{(2)}\}$, and the exponential quadratic model (3.15) for the equity component modelled by $\{A_t^{(1)}\}$ and driven by $\{L_{tU}^{(1)}\}$. In such a case, one obtains an asset price process of the form

$$S_{tT} = \frac{S_{0T} + \frac{(U-T)^{\eta-1/2}U^{1/2}g_1(T)}{4U\left[1+(1/12)f_1(0)U^3\right]}\left[\sqrt{1-t/U}\,\exp\left(\frac{L_{tU}^{(1)\,2}}{2(U-t)} \right) - 1 \right]}{P_{0t} + \frac{(U-t)^4 f_1(t)}{4U\left[1+(1/12)f_1(0)U^3\right]}\left[\frac{U}{(U-t)^2}\,L_{tU}^{(2)\,2} - \frac{t}{U-t} \right]}. \tag{6.13}$$

In the case that $\{L_{tU}\}$ is multidimensional, that is, $L_{tU} = \{L_{tU}^{(1)}, L_{tU}^{(2)}, \ldots, L_{tU}^{(n)}\}$, the asset with price process (6.13) is traded in an incomplete market.

Example. Let the market filtration be generated by four LRBs: a stable-$1/2$ random bridge $\{L_{tU}^{(1)}\}$, two gamma random bridges $\{L_{tU}^{(2)}\}$ and $\{L_{tU}^{(3)}\}$, and a Brownian random bridge $\{L_{tU}^{(4)}\}$. For details about $1/2$-stable random bridges, we refer to Hoyle et al. (2010). The four random bridges are assumed to be independent under the "Lévy probability measure" $\mathbb{L}$. Next we make use of Definition 6.1 and Proposition 6.2. It is convenient to take the expectations under an auxiliary probability measure as presented in Proposition 2.1. Given that in this example the market filtration is generated by four LRBs, we choose to compute the expectations under the "Lévy probability measure" $\mathbb{L}$, under which the LRBs have the law of the generating underlying Lévy process. For the function $G(t+u, X_{t+u})$ in Definition (6.1), we set

$$G\left(t+u_1, t+u_2, L_{t+u_1,U}^{(1)}, L_{t+u_2,U}^{(2)} \right) = \exp\left(-\kappa L_{t+u_1,U}^{(1)} + c L_{t+u_2,U}^{(2)} \right) \tag{6.14}$$

where $\kappa \geq 0$, $0 \leq c \leq 1$. The Laplace transform of the stable-1/2 subordinator $\{L_t\}$, which is a Lévy process, is

$$\mathbb{E}\left[\exp\left(-\kappa L_t\right)\right] = \exp\left(-\frac{\alpha\sqrt{\kappa}}{\sqrt{2}}t\right) \tag{6.15}$$

where $\alpha > 0$ is the activity parameter that features in the subordinator's density function

$$\rho_t(y) = \mathbb{1}\{y > 0\}\frac{\alpha t}{\sqrt{2\pi}\,y^{3/2}}\exp\left(-\frac{\alpha^2 t^2}{2y}\right). \tag{6.16}$$

We now calculate (6.2) with the specification (6.14) and by setting

$$\psi(t+u) = \widetilde{\psi}(t,u_1,u_2) = \exp\left(\frac{\alpha\sqrt{\kappa}}{\sqrt{2}}(t+u_1)\right)(1-c)^{m(t+u_2)} \tag{6.17}$$

where $m > 0$. The result is:

$$Z_{tT} = (U-T)^2\exp\left(-\kappa L_{tU}^{(1)} + \frac{\alpha\sqrt{\kappa}}{\sqrt{2}}t\right)(1-c)^{mt}\exp\left(cL_{tU}^{(2)}\right). \tag{6.18}$$

In this case, the price process $\{S_{tT}\}$ given in (6.4) can be written in the form (6.7). For the denominator, we choose a discount factor of the kind (5.14). We obtain the following price process for, e.g., equity:

$$S_{tT} = \frac{S_{0T} + b_1(T)A_t^{(1)}}{P_{0t} + b_2(t)A_t^{(2)}}, \tag{6.19}$$

where, for $\eta \geq 0$, $a \in [-\infty, \infty)$, and $q > 0$,

$$A_t^{(1)} = (1-c)^{mt}\exp\left(-\kappa L_{tU}^{(1)} + \frac{\alpha\sqrt{\kappa}}{\sqrt{2}}t + cL_{tU}^{(2)}\right) - 1, \quad b_1(T) = \frac{(U-T)^2 g_1(T)}{1 + f_1(0)U^2},$$

$$A_t^{(2)} = (\eta+1)^{qt}\exp\left(-\eta L_{tU}^{(3)} + aL_{tU}^{(4)} - \frac{1}{2}a^2 t\right) - 1, \quad b_2(t) = \frac{(U-t)^2 f_1(t)}{1 + f_1(0)U^2}. \tag{6.20}$$

Another example could be constructed by generating equity models driven by "VG random bridges" and quoted in units of the natural numeraire at the basis of the model (5.14). In the case that the economic factors modelled by the random variables $L_{UU}^{(i)}$ are dependent, one obtains simple dependence structures between the dynamics of the equity and the associated discount bond system that determines the discount rate in the financial market. The more advanced dependence models introduced in the next section can also be applied to model more complex interactions between different segments of a financial market.

7 Spiralling debt and its impact on global bond markets

We now address in more detail the pricing of sovereign bonds. Even though the majority of sovereign bonds pay coupons, we focus on discount bonds, for convenience. This simplification does not affect the view taken or the problem we intend to tackle here. The emphasis

is shifted on the value of a sovereign bond that should reflect the level of economic health of the issuing country. News regarding the bond market over the last few years has constantly reminded us that investors frequently balance the capability of a sovereign economy to grow vis-a-vis the amount of accumulated debt held at any one time. We choose this point of view, and wonder how to construct asset pricing models, which take into account at least some of this perspective.

We consider a simple model for the economic structure of a country. We assume that a central government has a source of income, for instance taxes and the revenues of state-owned companies. On the other hand it also has expenditures in order, for instance, to finance armed forces, public education, a public health system, and other welfare. While the difference between income and expenditures fluctuates over the course of time, we assume that it is unlikely, at least in a well-run and periodically well-assessed economy, that this difference spikes for the better or for the worse. If it were the case, then we might see an economy's growth rate move from 1% to 10% within a few months, or a drop in the growth rate by a similar amount in the same time span. It is more likely though that a central government has to step-in to cover huge unexpected losses due to, e.g., the unfolding of an international financial crisis, domestic or international wars, natural catastrophes, and other calamities hard to predict and with disastrous impact on the economic health of a country. So, in addition to the "structural" income and outcomes of an economy, we consider the accumulation of significant debt due to severe losses, which may very well make the level of financial stress of an economy "jump". We model the structural part of the various cash flows of an economy by a Brownian random bridge

$$L_{tU}^{(1)} = \sigma t X_U + \beta_{tU}, \tag{7.1}$$

where X_U may represent the economic wealth of a country at a future time U. We model the spiralling cumulative debt amassed by a sovereign country in the time interval $[0, U]$ by a gamma random bridge $\{L_{tU}^{(2)}\}$. The random total (extraordinary) debt amassed by time U is modelled by $L_{UU}^{(2)} = X_U^{(2)}$. Its distribution can be arbitrarily specified. We imagine that the "structural" or "non-crisis" balance $L_{UU}^{(1)}$ is dependent on the total debt (losses) $L_{UU}^{(2)}$ accumulated by time U. For instance, a sovereign government may decide at time U to make substantial cuts to the expenditures for public welfare if the amount of "extraordinary losses (debt)" will have spiralled by time U beyond what is perceived to be manageable. Therefore $L_{UU}^{(1)}$ and $L_{UU}^{(2)}$ are assumed to have a joint marginal distribution, and this means that we are in the same modelling environment as in Section 5.

The bond pricing model presented next is one of the simplest, though still rich enough to capture the desiderata within this discussion. One can of course choose to develop more sophisticated models. We choose a class of bond price models similar to (5.14), and follow the steps from (5.7) to (5.16) with one minor change in equation (5.11). We consider

$$F(t + u_1, t + u_2, y_1 + x_1, y_2 + x_2) = \exp\left(-a(y_1 + x_1) + c(y_2 + x_2)\right),$$
$$w(t, u_1, u_2) = \exp\left(-\frac{a^2}{2}(t + u_1)\right)(1 - c)^{m(t + u_2)}. \tag{7.2}$$

The reason for the change in the sign is that this way the losses will be recognised as downward jumps in the time series of the bond price. We emphasize that expectations are computed under the $\mathbb{L}$-measure, under which LRBs with joint terminal marginals are nevertheless independent and inherit the law of the generating Lévy processes. The bond price is then given by

$$P_{tT} = \frac{P_{0T} + b(T)A_t^{\mathbb{L}}}{P_{0t} + b(t)A_t^{\mathbb{L}}},$$ (7.3)

where, for $0 \leq t \leq T < U$, $a \geq 0$, $0 \leq c \leq 1$, $m > 0$, we have

$$b(t) = \frac{(U-t)^2 f_1(t)}{1 + f_1(0)U^2}, \quad A_t^{\mathbb{L}} = (1-c)^{mt} \exp\left(-a\,L_{tU}^{(1)} - \tfrac{1}{2}a^2 t + c\,L_{tU}^{(2)}\right) - 1.$$ (7.4)

Dependence in international markets. The effects of losses getting out of control are not confined to ones domestic economy. Especially in a global financial market, the deterioration of an economy's health exposes, for instance, foreign creditors holding debt of the distressed economy to higher credit risk. Bond markets are global "debt networks" linking several national economies with one another. The result of such network might be "contagion": an ailing economy may severely damage creditors which, through financial exposure, can be affected by spiralling losses experienced by the debtor. For instance, a foreign investor may see their investments in foreign bonds significantly devalued if the bond price declines due to unexpected losses, or out-of-control debt management. The foreign investor's loss may be commensurate with the percentage investment in an ailing economy compared with their total financial exposure and reserves gained through income. Here size matters, of course, and the discussions about the magnitude of the Greek versus the Italian debt impacting on the Eurozone or world economy come to ones mind. The next example aims at illustrating how contagion effects can be modelled in the present asset pricing framework. We introduce a linear combination of cumulative random bridge processes defined by

$$\widetilde{L}_{tU}^{(j)} = \sum_i^n w_i^{(j)} L_{tU}^{(i)},$$ (7.5)

where $\{L_{tU}^{(i)}\}$ are, e.g., gamma random bridges with joint terminal distribution and generated by independent gamma processes. The weight parameter $w_i^{(j)}$ measures the level of exposure to each cumulative process $\{L_{tU}^{(i)}\}$. Since the linear combination may not be the same for any creditor exposed to the pool $\{\widetilde{L}_{tU}^{(j)}\}$, further freedom is given through j-indexing the exposed entity (sovereign state, private company, etc.). For instance *Country A* may have a financial exposure of 15% to *Country X* and 8% to *Country Y*. The percentage exposures, which can be collected form various financial intelligence organizations, can be used to determine the weights for *Country A*. On the other hand, *Country B* may be exposed by 34% to *Country Y*'s economic performance and by 65% to *Country Z*'s. One sees that, even though *Country A* and *B* may not hold any of each others financial assets, they are linked to each other through the common exposure to *Counrty Y*'s economy, albeit to different levels. The bonds of *Country A* and *Country B* are expected to both show the impact

of their respective exposures to *Country Y*. The bond price processes of *Country A* then may have the form (7.3) where

$$b(t) = \frac{(U-t)^{n+1}f_1(t)}{1+f_1(0)U^{n+1}}, \quad A_t^{\mathbb{L}} = \prod_{i=1}^{n}\left(1-w_i^{(j)}\right)^{m_i t}\exp\left(-a\,L_{tU}^{(1)} - \tfrac{1}{2}a^2 t + \widetilde{L}_{tU}^{(j)}\right) - 1,$$

(7.6)

and $0 \le w_i^{(j)} \le 1$, $m_i > 0$. The level of financial exposure to a specific economy can be measured, at least in part, in terms of debt instruments held. Hereafter, we look at a simulation of contagion due to exposure to sovereign debt of two foreign countries. In particular, we suppose that Germany and France are exposed to the Spanish and Italian economic environment. The level of exposures $w_i^{(j)}$, governing in part the levels of dependence among the debt holders, are selected as follows:

	GER	FRA	ESP	ITA
GER	1	0	0	0
FRA	0	1	0	0
ESP	0.57	0.47	1	0
ITA	0.49	0.25	0	1

In this basic illustration, Germany has no exposure to the French economy, however it has exposure levels 0.57 and 0.49 to Spain and Italy, respectively. The numbers listed in the table are not normalised. Dependence among the four economies is also subject to the joint marginal distribution of the LRBs underlying the dynamics of the yield processes. The closer the time horizon U, the more the joint marginal distribution of the multivariate random variable L_{UU} will govern the dynamics of the dependent yield processes. The simulations that follow are based on bond price models of the form (7.3) where $b(t)$ and $\{A_t^{\mathbb{L}}\}$ are specified in (7.6).
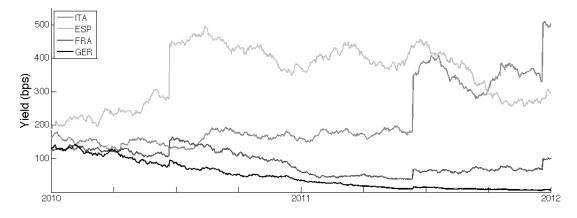

Figure 1: Simulation of the yield process of the two-year-maturity bonds issued by Germany, France, Italy, and Spain.

In Figure 1, we see the impact of spiralling debt on the yield process of sovereign bonds. The significant jumps in the yield processes of Spain and Italy (first and second

trajectories from above) are due to unexpected losses (e.g. bank bail-outs, natural disasters) over relatively short periods of time. These spikes then have repercussions on the yield processes of France and Germany to various degrees of severity. The level of repercussion on each exposed country is relative to the health of the own economy. In the simulation above, we see that France's yield process (third trajectory from above) needs more time than Germany's yield trajectory to recover from the shocks. The implication is that although Germany has a higher exposure level to Spain's and Italy's finances, it has a more robust domestic economy—with, e.g., higher growth rate—than France. Thus, Germany is in a position to better weather foreign economic shocks. Contagion effects, due to increased economic stress, are also observed in the behaviour of the spread process when comparing the performance of bonds issued by different sovereign states.
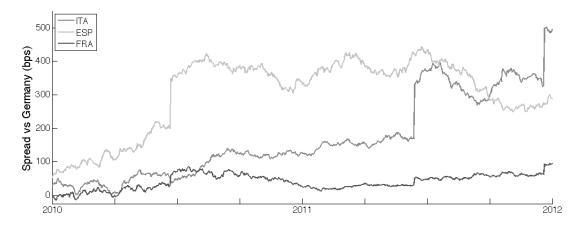

Figure 2: Simulation of the price spread process for the two-year-maturity bonds issued by France, Italy, and Spain compared with the two-year bond issued by Germany.

In Figure 2, we plot the difference between the yield processes of France, Spain, and Italy when compared with the yield process of the German sovereign bond. We observe the widening of the spread level for Spain and Italy, due to the two upward jumps in economic deficit. France keeps the evolution of the spread between the yield of its bond and the one by Germany in check even though it takes the hit from the exposures to the Spanish and Italian economies.

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