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A Technical Note on the Smith-Wilson Method

1. Introduction

The Smith-Wilson technique is a macroeconomic approach: a spot (i.e. zero coupon) rate curve is fitted to observed prices of financial instruments, with the macroeconomic ultimate long term forward rate as input parameter.[1]

The output from the Smith-Wilson calculation is the discount factor *P(t), t>0*. *P(t)* is the market price at valuing time for a zero coupon bond paying 1 at some future date *t* (the maturity).

Depending on whether we need the spot rates as continuously compounded rates $\tilde{R}_t$ or as rates R_t with annual compounding, the following relation between the discount factor and the spot rate can be used to assess the spot rates: $P(t) = \exp(-t \cdot \tilde{R}_t)$ for continuously compounded rates, and $P(t) = (1 + R_t)^{-t}$ for annual compounding.

The relation between the two rates is $\tilde{R}_t = \ln(1 + R_t)$.

The aim is to assess the price function *P(t)* for <u>all maturities</u> *t, t > 0.* From the relations referred to above it can be seen that therewith the whole risk-free term structure at valuing date is defined.

In its most general form the input data for the Smith-Wilson method can consist of different financial instruments that relate to interest rates. We will first present the formulae in the case where the inputs are zero coupon bond prices. The formulae in this simple case are quite easy to understand and straightforward to implement. Then we will present the formulae for the general case, where a large set of arbitrary financial instruments can be the input.

All financial instruments specified through

- their market price at valuation date,
- the cash payment dates up to maturity, and
- the size of the cash flows at these dates,

can be input instruments for the Smith-Wilson method.

In the last part of this note we will look at the input for fitting to zero coupon bond rates, to coupon bond rates and to par swap rates.

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[1] The mathematical background and a further discussion of the method can be found in the original paper by Andrew Smith and Tim Wilson, see Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Woodrow. (Remark: We will refer later on to an actualised version of the paper.)

We will proceed as follows: After some general remarks on extrapolation techniques in section 2 we list the advantages and disadvantages of the Smith-Wilson technique in section 3, give the formulae in section 4, apply these formulae to different input instruments in section 5 and illustrate the method through two worked examples for par swap rates in section 6.

2. Some general remarks

Most extrapolation methods start from the price function, and assume that the price function is known for a fixed number of *N* maturities. In order to get the price function for all maturities, more assumptions are needed.

The most common procedure is to impose – in a first step – a functional form with *K* parameters on the price function, on the spot rate curve or on the forward rate curve[2]. These functional forms could be polynomials, splines, exponential functions, or a combination of these or different other functions[3].

In some of the methods, in a second step, the *K* parameters are estimated by minimizing the sum of the squares of the differences between estimated data and market data at each point in time where market data is given. In other methods *K* equations are set up from which the *K* parameters are calculated.

The equations are – as a rule – set up in a manner that guarantees that *P* has (most of) the features desired for a price function. The desired features are:

- *P* is a positive function,
- strictly decreasing,
- with value 1 at time $t=0$,
- passing through all given data points,
- to a certain degree smooth, and
- with values converging to 0 for large t.

In some of the methods the term structure is estimated by using one approach for all maturities, in others different methods are used depending on whether spot rates are assessed in the liquid part or in the extrapolated part of the term structure. The most prominent examples of the first procedure are the Svensson method and the Nelson-Siegel method[4], where the same parametric form is used throughout the whole term structure. BarrieHibbert on the other hand apply splines for the liquid part and Nelson-Siegel for the extrapolated part of the term structure.

In the Smith-Wilson method the pricing function $P(t)$, for all $t>0$, is set up as the sum of a term $e^{-UFR \cdot t}$ for the asymptotical long term behavior of the discount factor and a linear combination of *N* kernel functions[5] $K_i(t),\ i=1,2,\ldots,N$ (the number *N* of kernel functions being equal to the number of input instruments).

The kernel functions are appropriately defined functions of the input market data and two input parameters: the ultimate forward rate (*UFR*) and a parameter (*alpha*) that determines how fast the estimated forward rates converge to *UFR*.

[2] Svensson imposes a parametric form with 6, Nelson-Siegel one with 4 parameters.
[3] BarrieHibbert use cubic splines for the liquid part and Nelson-Siegel for the extrapolated part.
[4] The method used by the ECB and many other central banks, when assessing the published zero coupon rates.
[5] The idea behind the choice of the kernel functions can be found in Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Woodrow.

If *N* input instruments are given, we know *N* market prices and can thus set up *N* linear equations. In most of the cases the resulting system of linear equations (SLE) can be solved automatically, i.e. without interfering from the outside. By plugging the solution of the SLE (the solution assessed for the maturities of the *N* input instruments) into the Smith-Wilson pricing function at any given time *t* we receive the discount function for maturity *t*. With the discount function, the spot rate curve is known.

3. Advantages and disadvantages of the Smith-Wilson (S-W) method

Compared to the other extrapolation methods, the main advantages can be summed up as follows:

- S-W is a method in the open domain. Both the formulae and a computing tool can be published on CEIOPS homepage. Thus, the method is wholly transparent and fully accessible to all companies, at all times.
- S-W is very flexible concerning the input, and at the same time it is very easy to implement. The risk-free term structure can be assessed from a choice of bonds (with or without coupons) or from swap rates, all by using one simple[6] excel spreadsheet.
- S-W can be used as a widely mechanized approach. In most cases the assessment of the extrapolated rates will consist in automatically applying the formulas to the input data, but in some situations – where the input data is biased, or where the linear equations that have to be solved are linearly dependent or nearly linearly dependent[7] – judgment may still be needed.
- S-W provides a perfect fit of the estimated term structure to the liquid market data. In many other methods the term structure is assessed as a smoothed curve that is only reasonably close to the market data[8]. A trade-off is often made

[6] Especially, VBA code will not be needed, as in many companies the opening of macro code from sources outside the company is considered a breach of security and will not be allowed.

[7] The system of linear equations that have to be solved can become linearly dependent or nearly linearly dependent for certain input data. This will require that the user of the method has to decide to remove some of the data from the input in order to compute a valid solution. The function W(t,u) can be interpreted as the covariance function of an Integrated Ornstein Uhlenbeck yield curve model. From this follows that linear dependency can only occur in cases where two or more of the input instruments have the same maturity; these are cases in which also the other extrapolation methods will have a problem. For details see Frankland, Smith, Wilkins, Varnell, Holtham, Biffis, Eshun, Dullaway – "Modelling Extreme Market Values – A Report of the Benchmarking Stochastic Models Working Party" (2008). The paper can be downloaded at:
http://www.actuaries.org.uk/__data/assets/pdf_file/0007/140110/sm20081103.pdf

[8] The Svensson and Nelson-Siegel method can be used as macroeconomic methods if the parameter defining the flat component of the curve is taken as UFR. The other 5 (3) parameters will be determined through a least square optimisation. For a market with a large set of market data the estimated term structure will not fit the market data exactly. Another example is given by the method CRO-Forum used to assess the risk-free rates from par swap rates when they proposed the input for QIS5 for CEIOPS. They use a "regression spline with smoothing constraints" method, the "Barrie&Hibbert standard yield curve fitting methodology". They clarify on page 8 of their note "QIS5 Technical Specification Risk –free interest rates" the following: "This method produces rates that are very close to but not exactly equal to market rates. The average absolute error is generally less than 1 basis point." It is not very clear whether this means that the error is assessed by first netting out positive and negative deviations for each currency, and so taking the average of the absolute value of these netted errors over all currencies. Should this be implied by what CRO-Forum writes, the fit of the term structures to the market data could be much worse than the 1 bps suggest.

between the goodness of fit and the smoothness of the term structure. In S-W all relevant data from the liquid market is taken as input, no smoothing is performed.

- S-W is based on solving a linear system of equations analytically. This is an advantage compared to methods that are based on e.g. minimizing sums of least square deviations, as these are susceptible to "catastrophic" jumps when the least-squares fit jumps from one set of parameters to another set of quite different values[9]. This problem is due to the non-linearity in the least squares formula which can give rise to more than one local minimum.
- S-W can be applied directly to the raw data from financial markets. No bootstrapping or other methods are needed to transform market par swap rates into zero coupon bond rates, as is the case for example in the linear extrapolation method, where the input has to be first converted into zero coupon bond rates.
- S-W is a uniform approach, both interpolation between the liquid market data points and extrapolation beyond the last data point are performed. For many other methods interpolation and extrapolation are done separately, often based on different principles and mostly using different kinds of functions for assessing the different parts of the curve. This can lead to inconsistencies between the interpolated and extrapolated part of the same curve and also to inconsistencies over time for each part of the curve. (If e.g. due to higher liquidity at the long end, the entry point for the extrapolation changes significantly from one period to the next, the rates for maturities between these two points in time will be assessed with quite different methods from one period to the next.)
- In S-W the ultimate forward rate will be reached asymptotically[10]. How fast the extrapolated forward rates converge to the *UFR* will depend on how the rates in the liquid part of the term structure behave and on an exogenous parameter *alpha*. For higher *alpha* the extrapolated forward rates converge faster to the *UFR*, i.e. the market data from the liquid part of the curve are of less impact for the extrapolated rates.

Some of the disadvantages of the Smith-Wilson approach:

- The parameter *alpha* has to be chosen outside the model. Thus, in general, expert judgment would be needed to assess this input parameter for each currency and each point in time separately. In order to have a harmonized approach over all currencies in Solvency II we will for all currencies use the Smith-Wilson approach with the parameter *alpha* starting at 0.1[11]. If this alpha is not appropriate for the currency it is applied to, we will increase it iteratively, until it is deemed – based on given criteria – to be appropriate. A lot more work needs to be done here to develop objective criteria for setting the alpha, in order to avoid that expert judgment is needed in all these cases.
- There is no constraint forcing the discount function *P(t)* to decrease. In the liquid part of the assessed term structure we could have cases where *P(t)* is a decreasing function on the given liquid market data points, but becomes a locally

[9] For a thorough discussion of these problems see Andrew J. G. Cairns – "Descriptive Bond-Yield and Forward-rate models for the British Government Securities' Market" (1997).
The paper can be found at http://www.ma.hw.ac.uk/~andrewc/papers/ajgc11.pdf

[10] Introducing the maturity T2 as the maturity where the UFR is reached literally can be avoided if the S-W outcome is no longer compared to the linear extrapolation outcome.

[11] Larger values of alpha give greater weight to the ultimate forward rate, while smaller values of alpha give more weight to the liquid market data. More work has to be done in order to see if a lower value of alpha than 0.1 could be more appropriate as starting value, as the resulting curves could be deemed to be more objective and market consistent.

increasing function on the interpolated curve. This can happen if a smooth curve is fitted between two neighboring market data points for which the *P(t)* values are quite near[12]. Many other methods would have the same problem here.

- Beyond the liquid part of the curve, *P(t) may become negative*. This situation can arise when the last forward rate in the liquid part of the curve is high compared to the sum of *UFR* and *alpha*. This is a disadvantage of S-W compared e.g. to parametric methods, as parametric methods often are based on formulas for the spot rate which per definition can not produce negative discount functions. If for certain sets of input market data *P(t)* will become negative, one has to take higher *alpha*s. This procedure will have to be based on expert judgment.

4. The Smith-Wilson technique

We will now explain how the term structure can be assessed by using the S-W technique.

Smith-Wilson technique for zero coupon bond prices as input

We start by assuming that in the liquid part of the term structure the price function is known for a fixed number of *N* maturities: u_1, u_2, u_3, up to u_N. This is the same as saying that the risk-free zero coupon rates for these *N* liquid maturities are given beforehand.

Depending on whether the market data spot rates are given as continuously compounded rates $\tilde{R}_{u_i}$ or as rates R_{u_i} with annual compounding, the input zero bond prices at maturities u_j can be expressed as:

$m_i = P(u_i) = \exp(-u_i \cdot \tilde{R}_{u_i})$ for continuously compounded rates, and

$m_i = P(u_i) = (1 + R_{u_i})^{-u_i}$ for annual compounding.

In this case, where zero coupon bond prices are the input, the task consists in assessing the price function, i.e. the spot rates for the remaining maturities. These can be both maturities in the liquid end of the term structure where risk-free zero coupon rates are missing (interpolation) and maturities beyond the last observable maturity (extrapolation).

The pricing function proposed by Smith and Wilson[13] reduces in this simple case to:

$$P(t) = e^{-UFR \cdot t} + \sum_{j=1}^{N} \zeta_j \cdot W(t, u_j), \quad t \geq 0 \tag{1}$$

[12] As an example *P(0)*=1, *P(1)*=0.95001, *P(2)*=0.95000, *P(3)*=0.9, and so on. When we fit a smooth curve through this points we will for large *alpha* get a curve between *P(1)* and *P(2)* that will bend down (i.e. *P(t)*<*P(2)*=0.95 for some 1<*t*<2) because of the enforced smooth continuation of the fit between *P(0)* and *P(1)*.

[13] Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Woodrow. Referred to in Michael Thomas, Eben Maré: "Long Term Forecasting and Hedging of the South African Yield Curve", Presentation at the 2007 Convention of the Actuarial Society of South Africa.

Andrew Smith: Pricing Beyond the Curve – derivatives and the Long Term (2001), presentation to be found at http://www.cfr.statslab.cam.ac.uk/events/content/20001/asmith2001.pdf

With the symmetric Wilson $W(t, u_j)$ functions defined as:

$$W(t,u_j) = e^{-UFR\cdot(t+u_j)} \cdot \left\{\alpha \cdot \min(t,u_j) - 0.5 \cdot e^{-\alpha\cdot\max(t,u_j)} \cdot (e^{\alpha\cdot\min(t,u_j)} - e^{-\alpha\cdot\min(t,u_j)})\right\} \quad (2)$$

The following notation holds:

- N, the number of zero coupon bonds with known price function
- m_i, $i=1, 2, \dots N$, the market prices of the zero coupon bonds
- u_i, $i=1, 2, \dots N$, the maturities of the zero coupon bonds with known prices
- t, the term to maturity in the price function
- *UFR*, the ultimate unconditional forward rate, continuously compounded
- α, mean reversion, a measure for the speed of convergence to the UFR
- ζ_i, $i=1, 2, \dots N$, parameters to fit the actual yield curve

The so called kernel functions $K_j(t)$ are defined as functions of the maturity t:

$$K_j(t) = W(t,u_j),\ \mathrm{t} > 0 \ \text{ and } \ j=1,2,3,\dots,N \quad (3)$$

They depend only on the input parameters and on data from the input zero coupon bonds. For each input bond a particular kernel function is computed from this definition. The intuition behind the model is to assess the function $P(t)$ as the linear combination of all the kernel functions. This is similar to the Nelson-Siegel method, where the forward rate function is assessed as the sum of a flat curve, a sloped curve and a humped curve, and the Svensson method, where a second humped curve is added to the three curves from Nelson-Siegel.

The unknown parameters needed to compute the linear combination of the kernel functions, ζ_j, $j=1, 2, 3 \dots N$, are given as solutions of the following linear system of equations:

$$\begin{aligned}
m_1 &= P(u_1) = e^{-UFR\cdot u_1} + \sum_{j=1}^{N} \zeta_j \cdot W(u_1,u_j) \\
m_2 &= P(u_2) = e^{-UFR\cdot u_2} + \sum_{j=1}^{N} \zeta_j \cdot W(u_2,u_j) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
m_N &= P(u_N) = e^{-UFR\cdot u_N} + \sum_{j=1}^{N} \zeta_j \cdot W(u_N,u_j)
\end{aligned} \quad (4)$$

In vector space notation this becomes:

$$\mathbf{m} = \mathbf{p} = \boldsymbol{\mu} + \mathbf{W}\boldsymbol{\zeta}\ , \quad (5)$$

with:

$$\mathbf{m} = (m_1, m_2, \dots\dots, m_N)^T,$$
$$\mathbf{p} = (P(u_1), P(u_2), \dots\dots P(u_N))^T,$$

$$\boldsymbol{\mu} = (e^{-UFR\cdot u_1}, e^{-UFR\cdot u_2}, \dots. e^{-UFR\cdot u_N})^T,$$

$$\zeta = (\zeta_1, \zeta_2, ...\zeta_N)^T,$$

where the superscript *T* denotes the transposed vector and

$\mathbf{W} = (W(u_i, u_j))_{i=1,2,...N;\, j=1,2,...N}$ is a *NxN*-matrix of certain Wilson functions.

It follows from (5) that the solution $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3, ..., \zeta_N)^T$ is the product of the inverted *NxN*-matrix **W** and the difference between the **p**-vector and the **μ**-vector (i.e. the difference between the market prices of the zero coupon bonds and the asymptotical term), that is:

$$\zeta = \mathbf{W}^{-1}(\mathbf{p} - \boldsymbol{\mu}) = \mathbf{W}^{-1}(\mathbf{m} - \boldsymbol{\mu}), \qquad (6)$$

We can now plug these parameters (i.e. $\zeta_1, \zeta_2, \zeta_3, ..., \zeta_N$) into the pricing function and get the value of the zero coupon bond price for all maturities *t* for which no zero coupon bonds prices are given to begin with:

$$P(t) = e^{-UFR \cdot t} + \sum_{j=1}^{N} \zeta_j \cdot W(t, u_j), \quad t>0 \qquad (7)$$

From this value it is straightforward to calculate the spot rates by using the definition of the zero coupon bond price. The spot rates are calculated as $\tilde{R}_t = \frac{1}{t} \cdot \ln(\frac{1}{P(t)})$ for continuous compounded rates and $R_t = (\frac{1}{P(t)})^{\frac{1}{t}} - 1$ if annual compounding is used.

Smith-Wilson technique for a set of general inputs

We now assume that we have *N* interest related financial instruments as input from the liquid part of the term structure and that *J* is the number of different dates at which a cash payment has to be made on behalf of at least one of these instruments. The following input shall be given:

- The market prices m_i of the instruments *i* at valuation date, for $i=1,2,3,...,N$.
- All cash payment dates $u_1, u_2, u_3, ..., u_J$, for the instruments.
- The cash flows $c_{i,1}, c_{i,2}, c_{i,3}, ..., c_{iJ}$ that are due for instrument *i* at time $u_1, u_2, ...u_J$, for all *i*. (If no cash payment is due at time $t = u_j$ on instrument *i*, then $c_{i,j}$ is set to nil).

The general pricing function at valuing time proposed by Smith and Wilson[14] is:

$$P(t) = e^{-UFR \cdot t} + \sum_{i=1}^{N} \zeta_i \cdot (\sum_{j=1}^{J} c_{i,j} \cdot W(t, u_j)), \quad t \geq 0 \qquad (8)$$

[14] Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Woodrow. Referred to in Michael Thomas, Eben Maré: "Long Term Forecasting and Hedging of the South African Yield Curve", Presentation at the 2007 Convention of the Actuarial Society of South Africa.

with the symmetric Wilson-functions $W(t, u_j)$ defined as in (2) above and the same notation for t, *UFR*, *α* and ζ_i as was given for the zero coupon case.

The function defined by the inner parenthesis in (8) is called the kernel functions $K_i(t)$:

$$K_i(t) = \sum_{j=1}^{J} c_{i,j} \cdot W(t, u_j), \ \mathrm{t} > 0, \mathrm{i} = 1,2,3,...\mathrm{N} \tag{9}$$

For each input instrument a particular kernel function is computed. The intuition here is to assess the function *P(t)* as the linear combination of all the kernel functions.

In the simple case, where the zero coupon prices $P(u_i)$ for certain maturities are given as market price input m_i, i.e. where m_i equals $P(u_i)$ for $i=1,2,3,\dots,N$, the left side of the linear system of equations (LSE) in (1) is known and it was straightforward to compute the ζ_i from this LSE. In the general case we have the market prices m_i of the instruments, but the zero coupon prices $P(u_i)$ are not known.

We do know how to assess the market price of an instrument i if all cash payment dates $u_1, u_2, u_3, \dots, u_J$ for the instrument, the cash flows $c_{i,1}, c_{i,2}, c_{i,3}, \dots, c_{iJ}$ at times $u_1, u_2, \dots u_J$, and the discount factors $P(u_j)$, $j=1,2,3,\dots,J$, are known. Then we have to discount the cash flows $c_{i,j}$ to the valuation date (i.e. multiply $c_{i,j}$ with $P(u_j)$ and sum over all cash flow dates).

$$m_i = \sum_{j=1}^{J} c_{i,j} \cdot P(u_j), \ \mathrm{i} = 1,2,3,....\mathrm{N} \tag{10}$$

In the above relation, we know the market prices m_i and the cash flows $c_{i,j}$.

We set the definition of the price function for $P(u_j)$ (8) into relation (10) and get the LSE:

$$\begin{aligned}
m_1 &= \sum_{j=1}^{J} c_{1,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{1,j} \cdot (e^{-UFR \cdot u_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{J} c_{l,k} \cdot W(u_j, u_k)) \\
m_2 &= \sum_{j=1}^{J} c_{2,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{2,j} \cdot (e^{-UFR \cdot u_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{J} c_{l,k} \cdot W(u_j, u_k)) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
m_N &= \sum_{j=1}^{J} c_{N,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{N,j} \cdot (e^{-UFR \cdot u_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{J} c_{l,k} \cdot W(u_j, u_k))
\end{aligned} \tag{11}$$

We can rearrange the above expressions to get:

$$\begin{aligned}
\sum_{j=1}^{J} c_{1,j} \cdot P(u_j) &= \sum_{j=1}^{J} c_{1,j} \cdot e^{-UFR \cdot u_j} + \sum_{l=1}^{N} (\sum_{k=1}^{J} (\sum_{j=1}^{J} c_{1,j} \cdot W(u_j, u_k)) \cdot c_{l,k}) \cdot \zeta_l \\
\sum_{j=1}^{J} c_{2,j} \cdot P(u_j) &= \sum_{j=1}^{J} c_{2,j} \cdot e^{-UFR \cdot u_j} + \sum_{l=1}^{N} (\sum_{k=1}^{J} (\sum_{j=1}^{J} c_{2,j} \cdot W(u_j, u_k)) \cdot c_{l,k}) \cdot \zeta_l \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\sum_{j=1}^{J} c_{N,j} \cdot P(u_j) &= \sum_{j=1}^{J} c_{N,j} \cdot e^{-UFR \cdot u_j} + \sum_{l=1}^{N} (\sum_{k=1}^{J} (\sum_{j=1}^{J} c_{N,j} \cdot W(u_j, u_k)) \cdot c_{l,k}) \cdot \zeta_l
\end{aligned} \tag{12}$$

In vector space notation we write the left side of (11) as:

$$\mathbf{m} = \mathbf{Cp}, \tag{13}$$

and (12) as:

$$\mathbf{Cp} = \mathbf{C\mu} + (\mathbf{CWC^T})\boldsymbol{\zeta}, \tag{14}$$

with:

$$\mathbf{m} = (m_1, m_2, ..., m_N)^T,$$

$$\mathbf{p} = (P(u_1), P(u_2), ...P(u_J))^T,$$

$$\mathbf{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & \cdots & c_{1,j} & \cdots & c_{1,J} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & \cdots & c_{2,j} & \cdots & c_{2,J} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i,1} & c_{i,2} & c_{i,3} & \cdots & \cdots & c_{i,j} & \cdots & c_{i,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{N,1} & c_{N,2} & \cdots & \cdots & \cdots & c_{N,j} & \cdots & c_{N,J} \end{bmatrix}. \quad \text{the NxJ cash flow matrix}$$

$$\boldsymbol{\mu} = (e^{-UFR \cdot u_1}, e^{-UFR \cdot u_2}, ...e^{-UFR \cdot u_J})^T,$$

$$\boldsymbol{\zeta} = (\zeta_1, \zeta_2, ...\zeta_N)^{\mathrm{T}},$$

and

$$\mathbf{W} = \begin{bmatrix} w(u_1,u_1) & w(u_1,u_2) & \cdots & \cdots & w(u_1,u_i) & \cdots & \cdots & w(u_1,u_J) \\ w(u_2,u_1) & w(u_2,u_1) & \cdots & \cdots & w(u_2,u_i) & \cdots & \cdots & w(u_2,u_J) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w(u_i,u_1) & w(u_i,u_2) & \cdots & \cdots & w(u_i,u_i) & \cdots & \cdots & w(u_i,u_J) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w(u_J,u_1) & \cdots & \cdots & \cdots & w(u_J,u_i) & \cdots & \cdots & w(u_J,u_J) \end{bmatrix}$$

the *JxJ* matrix of certain Wilson functions.

Combining (13) and (14) leads to:

$$\mathbf{m} = \mathbf{C\mu} + (\mathbf{CWC^T})\boldsymbol{\zeta}\ , \tag{15}$$

and we see at once that the solution ζ_1, ζ_2, ζ_3, … ζ_N is calculated by inverting the *NxN*-matrix $\mathbf{CWC^T}$ and multiplying it with the difference of the market value vector and the vector assessed as product of matrix **C** with the **μ**-vector, the asymptotical term:

$$\boldsymbol{\zeta} = (\mathbf{CWC}^{\mathrm{T}})^{-1}(\mathbf{m} - \mathbf{C}\boldsymbol{\mu}) \tag{16}$$

Now we can plug these parameters ζ_1, ζ_2, ζ_3, … ζ_N for $t= 1,2,3,…$ into the pricing function $P(t)$ and get the value of the discount function for all maturities, and thus the term structure for the spot rates.

Remark: When using swap rates to fit the risk-free term structure, an adjustment to allow for the credit risk in swaps has to be made. Assuming that the adjustment can be expressed as a delta credit risk spread of Δcr basis points of swaps above basic risk-free rates, it seems adequate to use one of two following adjustments:

1. Adjust the continuously compounded spot rates with Δcr basis points. This means that Δcr basis points are subtracted from the continuously compounded spot rates, which were assessed with the S-W technique from the unadjusted swaps. This is equivalent to multiplying the discount factors $P(t)$ (assessed from un-adjusted swaps), with an adjustment factor $e^{(\Delta cr/10000)\cdot t}$.
2. Adjust the input data with Δcr basis points. This means that Δcr basis points are subtracted from the raw par swap rates. These adjusted rates are then used to assess the spot rates with the S-W technique.

5. Fitting the spot rate term structure to bond prices and swap rates

With the Smith-Wilson technique the term structure can be fitted to all the different financial instruments that may be eligible as basis for assessing the risk-free interest rate curve.

Each set of instruments that is taken as input is defined by

- the vector of the market prices (of N instruments) at valuation date,
- the vector of the cash payment dates (J different dates) up to the last maturity, and
- the NxJ-matrix of the cash flows on the instruments in these dates.

We will now look at this input when the spot rate curve is fitted to zero coupon bond rates, to coupon bond rates and to par swap rates. We will furthermore give some simple computed examples for par swap rates as input.

Instruments	Market prices	Cash payment dates	Cash flow matrix
Zero coupon bonds	• Market prices of the *N* input instruments, given as the per cent amount of the notional amount • The market prices of the zero coupon input bonds translate at once into spot rates for input maturities	• The cash payment dates are the maturity dates of the *N* zero coupon input bonds (i.e. $J=N$)	• An NxN matrix with entries: - $c_{i,j} =1$ for $i=j$, and - $c_{i,j} =0$ else. • Remark: The **C** matrix reduces to the unity matrix. It can easily be seen that all the complex formulae given in (8,9,11,16) reduce to the simpler ones in (1,3,4,6).
Coupon bonds	• Market prices of the *N* coupon input bonds, given as the per cent amount of the notional amount of the bond.	• The cash payment dates are, in addition to the maturity dates of the input bonds all coupon dates. • Order these *J* cash payment dates in increasing order, i.e. $u_1 < u_2 < \ldots < u_J$ • Order the bonds depending on their time to maturity, such that if the time of maturity of the i^{th} bond is denominated with $u_{t(i)}$, the following holds: $u_{t(1)} < u_{t(2)} \ldots < u_{t(N)} = u_J$	• An NxJ matrix with the following entries (all i): - $c_{i,j} = r_c(i)/s,\ j<t(i)$ - $c_{i,t(i)} = 1+r_c(i)/s,$ - $c_{i,j} = 0,\ j>t(i),$ where $r_c(i)$ is the coupon rate of i^{th} bond, and s is the settlement frequency. • Remark: We propose to take the simple approach, and to not allow for day count details

Instruments	Market prices	Cash payment dates	Cash flow matrix
Par swap rates	• The market prices of the *N* par swap input instruments are taken as unit (i.e. 1). • To receive the swap rate, a floating rate has to be earned, that can be swapped against the fixed rate. To earn the variable rate a notional amount has to be invested. At maturity, the notional amount is de-invested.	• The cash payment dates are, in addition to the maturity dates of the swap agreements all swap rate payment dates. • Order these *J* cash payment dates in increasing order, i.e. $u_1 < u_2 < \ldots < u_J$ • Order the swaps depending on their time to maturity, such that if the time to maturity of the i^{th} swap is denominated with $u_{i,t(i)}$, the following holds: $u_{1,t(1)} < u_{2,t(2)} \ldots < u_{N,t(N)} = u_J$	• An *NxJ* matrix with the following entries (all i): - $c_{i,j} = r_c(i)/s,\ j<t(i)$ - $c_{i,t(i)} = 1+r_c(i)/s$, - $c_{i,j} = 0,\ j>t(i)$, where $r_c(i)$ is the swap rate of agreement i, and s is the settlement frequency. • Remark: We propose to take the simple approach, and to not allow for day count details.

6. Worked examples

When fitting the spot rate term structure to the input data from the following examples, we set the long term forward rate to 4.2% (for annual compounding; i.e. ln(1+0.042) = 0.04114 or 4.114% for continuous compounding), and the alpha parameter to 0.1.

Example 1

Market data input for example 1

Par swap rates	**Market prices**	**Cash payment dates**	**Transposed cash flow matrix C^T (i.e. cash flows of instrument i in column i)**
Maturities: 1,2,3,5 years s=1 coupon rates $r_c(1)$ 1% $r_c(2)$ 2% $r_c(3)$ 2.6% $r_c(5)$ 3.4%	i m_i 1 1 2 1 3 1 4 1	• u_1 =1, u_2=2, u_3=3, u_4=4, u_5=5 • $u_{1,t(1)}$ =1, $u_{2,t(2)}$ =2, $u_{3,t(3)}$ =3, $u_{4,t(4)}$ =5	$\begin{array}{lllll} & 1 & 2 & 3 & 4 \\ u_1 & 1.01 & 0.2 & 0.026 & 0.034 \\ u_2 & 0 & 1.02 & 0.026 & 0.034 \\ u_3 & 0 & 0 & 1.026 & 0.034 \\ u_4 & 0 & 0 & 0 & 0.034 \\ u_5 & 0 & 0 & 0 & 1.034 \end{array}$

The steps in the S-W technique

The 5x5 matrix of Wilson functions is computed straightforward from formula (2):

$$\mathbf{W} = \begin{bmatrix} 0.009 & 0.016 & 0.022 & 0.027 & 0.031 \\ 0.016 & 0.030 & 0.041 & 0.051 & 0.058 \\ 0.022 & 0.041 & 0.058 & 0.072 & 0.083 \\ 0.027 & 0.051 & 0.072 & 0.090 & 0.104 \\ 0.031 & 0.058 & 0.083 & 0.104 & 0.122 \end{bmatrix}$$

If multiplied with **C** from the right and **C^T** from the left, the resulting 4x4 matrix is:

$$\mathbf{CWC^T} = \begin{bmatrix} 0.009 & 0.017 & 0.023 & 0.035 \\ 0.017 & 0.032 & 0.045 & 0.067 \\ 0.023 & 0.045 & 0.065 & 0.097 \\ 0.035 & 0.067 & 0.097 & 0.150 \end{bmatrix}$$

The inverse of this matrix $\mathbf{CWC^T}$ is computed as:

$$
\left(\mathbf{CWC^T}\right)^{-1} = \begin{bmatrix} 10658.6 & -10190.4 & 3653.0 & -268.0 \\ -10190.4 & 14337.6 & -7987.5 & 1116.2 \\ 3653.0 & -7987.5 & 6252.3 & -1323.2 \\ -268.0 & 1116.2 & -1323.2 & 426.6 \end{bmatrix}
$$

We first multiply the cash flows in **C** with the vector **μ** of the asymptotic terms, and then subtract this vector from the vector of the market values:

$$
\mathbf{m - C\mu} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.969 \\ 0.959 \\ 0.956 \\ 0.965 \end{bmatrix} = \begin{bmatrix} 0.031 \\ 0.041 \\ 0.044 \\ 0.035 \end{bmatrix}
$$

Multiply $\mathbf{(CWC^T)^{-1}}$ with **m-μ**. The resulting vector represents the solution of the LSE that was set up in (15):

$$
\zeta = \begin{bmatrix} 57.79 \\ -33.5 \\ 111.40 \\ -5.47 \end{bmatrix}
$$

To assess the discount function *P(t)* in arbitrary *t, t>0*, the Wilson functions *W(t,uj)*, *j=1,2,3…J* have to be assessed and multiplied with **C**, as defined in (8). We want to compute the discount factor for *t=4*, and calculate therefore $\mathbf{w^T}$ = $(W(4,u_j))_{j=1,2,3,4,5}$

$$
\mathbf{w}^T = [0.27 \quad 0.051 \quad 0.072 \quad 0.090 \quad 0.104]
$$

This vector multiplied with $\boldsymbol{C^T}$ gives the values of the kernel functions in *t=4*, i.e.:

$$
(\mathrm{K_i}(4))_{\mathrm{i}=1,2,3,4} = \mathbf{w^T C^T} = [0.027 \quad 0.052 \quad 0.076 \quad 0.116]
$$

From the linear combination of these kernel functions we get:

$$
(\mathbf{w^T C^T})\zeta = 0.037
$$

and adding the asymptotical factor $e^{-0.04114\times 4} = 0.8483$, the discount function at maturity 4 years has the value *P(4)*= 0.848+0.037=0.885. This gives a spot rate (with annual compounding) of 3.10%.

We can table the Wilson functions for all maturities (years, month, days) for which risk-free spot rates will be needed, perform the above calculation for each maturity, and thus assess the risk-free interest rate term structure.

Example 2

Market data input for example 2

Par swap rates	**Market prices**	**Cash payment dates**	**Transposed cash flow matrix $\mathbf{C^T}$ (i.e. cash flows of instrument i in column i)**
Maturities: 1,2,3,5 years s=1 coupon rates $r_c(1)$ 1% $r_c(2)$ 2% $r_c(3)$ 2.6% $r_c(5)$ 3.4%	i m_i 1 1 2 1 3 1 4 1	• u_1=0.25, u_2=0.5, u_3=0.75, u_4=1, u_5=1.25, ……. u_{19}=4.75, u_{20}=5, • $u_{1,t(1)}$ =1, $u_{2,t(2)}$ =2, $u_{3,t(3)}$ =3, $u_{4,t(4)}$ =5	(see matrix below)

	1	2	3	4
u_1	0.0025	0.005	0.0065	0.0085
u_2	0.0025	0.005	0.0065	0.0085
u_3	0.0025	0.005	0.0065	0.0085
u_4	1.0025	0.005	0.0065	0.0085
u_5	0	0.005	0.0065	0.0085
u_7	0	0.005	0.0065	0.0085
u_8	0	1.005	0.0065	0.0085
u_9	0	0	0.0065	0.0085
..	0	0	…	…
u_{19}	0	0	0	0.0085
u_{20}	0	0	0	1.0085

The steps in the S-W technique

The 20x20 matrix of Wilson functions is computed straightforward from formula (2):

$$\mathbf{W} = \left[\begin{matrix}
0.001 & 0.001 & 0.002 & 0.002 & 0.003 & 0.003 & 0.004 & 0.004 & 0.005 & 0.005 & 0.005 & 0.006 & 0.006 & 0.006 & 0.007 & 0.007 & 0.007 & 0.007 & 0.008 & 0.008 \\
0.001 & 0.002 & 0.003 & 0.004 & 0.005 & 0.006 & 0.007 & 0.008 & 0.009 & 0.010 & 0.011 & 0.011 & 0.012 & 0.013 & 0.013 & 0.014 & 0.014 & 0.015 & 0.015 & 0.016 \\
0.002 & 0.003 & 0.005 & 0.007 & 0.008 & 0.009 & 0.011 & 0.012 & 0.013 & 0.014 & 0.016 & 0.017 & 0.018 & 0.019 & 0.019 & 0.020 & 0.021 & 0.022 & 0.023 & 0.023 \\
0.002 & 0.004 & 0.007 & 0.009 & 0.011 & 0.012 & 0.014 & 0.016 & 0.018 & 0.019 & 0.020 & 0.022 & 0.023 & 0.024 & 0.026 & 0.027 & 0.028 & 0.029 & 0.030 & 0.031 \\
0.003 & 0.005 & 0.008 & 0.011 & 0.013 & 0.015 & 0.017 & 0.020 & 0.022 & 0.023 & 0.025 & 0.027 & 0.029 & 0.030 & 0.032 & 0.033 & 0.034 & 0.036 & 0.037 & 0.038 \\
0.003 & 0.006 & 0.009 & 0.012 & 0.015 & 0.018 & 0.021 & 0.023 & 0.026 & 0.028 & 0.030 & 0.032 & 0.034 & 0.036 & 0.037 & 0.039 & 0.041 & 0.042 & 0.044 & 0.045 \\
0.004 & 0.007 & 0.011 & 0.014 & 0.017 & 0.021 & 0.024 & 0.027 & 0.029 & 0.032 & 0.034 & 0.037 & 0.039 & 0.041 & 0.043 & 0.045 & 0.047 & 0.049 & 0.050 & 0.052 \\
0.004 & 0.008 & 0.012 & 0.016 & 0.020 & 0.023 & 0.027 & 0.030 & 0.033 & 0.036 & 0.039 & 0.041 & 0.044 & 0.046 & 0.049 & 0.051 & 0.053 & 0.055 & 0.057 & 0.058 \\
0.005 & 0.009 & 0.013 & 0.018 & 0.022 & 0.026 & 0.029 & 0.033 & 0.036 & 0.040 & 0.043 & 0.046 & 0.049 & 0.051 & 0.054 & 0.056 & 0.059 & 0.061 & 0.063 & 0.065 \\
0.005 & 0.010 & 0.014 & 0.019 & 0.023 & 0.028 & 0.032 & 0.036 & 0.040 & 0.043 & 0.047 & 0.050 & 0.053 & 0.056 & 0.059 & 0.062 & 0.064 & 0.067 & 0.069 & 0.071 \\
0.005 & 0.011 & 0.016 & 0.020 & 0.025 & 0.030 & 0.034 & 0.039 & 0.043 & 0.047 & 0.051 & 0.054 & 0.058 & 0.061 & 0.064 & 0.067 & 0.070 & 0.072 & 0.075 & 0.077 \\
0.006 & 0.011 & 0.017 & 0.022 & 0.027 & 0.032 & 0.037 & 0.041 & 0.046 & 0.050 & 0.054 & 0.058 & 0.062 & 0.065 & 0.069 & 0.072 & 0.075 & 0.078 & 0.080 & 0.083 \\
0.006 & 0.012 & 0.018 & 0.023 & 0.029 & 0.034 & 0.039 & 0.044 & 0.049 & 0.053 & 0.058 & 0.062 & 0.066 & 0.070 & 0.073 & 0.077 & 0.080 & 0.083 & 0.086 & 0.089 \\
0.006 & 0.013 & 0.019 & 0.024 & 0.030 & 0.036 & 0.041 & 0.046 & 0.051 & 0.056 & 0.061 & 0.065 & 0.070 & 0.074 & 0.078 & 0.081 & 0.085 & 0.088 & 0.091 & 0.094 \\
0.007 & 0.013 & 0.019 & 0.026 & 0.032 & 0.037 & 0.043 & 0.049 & 0.054 & 0.059 & 0.064 & 0.069 & 0.073 & 0.078 & 0.082 & 0.086 & 0.089 & 0.093 & 0.096 & 0.099 \\
0.007 & 0.014 & 0.020 & 0.027 & 0.033 & 0.039 & 0.045 & 0.051 & 0.056 & 0.062 & 0.067 & 0.072 & 0.077 & 0.081 & 0.086 & 0.090 & 0.094 & 0.097 & 0.101 & 0.104 \\
0.007 & 0.014 & 0.021 & 0.028 & 0.034 & 0.041 & 0.047 & 0.053 & 0.059 & 0.064 & 0.070 & 0.075 & 0.080 & 0.085 & 0.089 & 0.094 & 0.098 & 0.102 & 0.105 & 0.109 \\
0.007 & 0.015 & 0.022 & 0.029 & 0.036 & 0.042 & 0.049 & 0.055 & 0.061 & 0.067 & 0.072 & 0.078 & 0.083 & 0.088 & 0.093 & 0.097 & 0.102 & 0.106 & 0.110 & 0.113 \\
0.008 & 0.015 & 0.023 & 0.030 & 0.037 & 0.044 & 0.050 & 0.057 & 0.063 & 0.069 & 0.075 & 0.080 & 0.086 & 0.091 & 0.096 & 0.101 & 0.105 & 0.110 & 0.114 & 0.118 \\
0.008 & 0.016 & 0.023 & 0.031 & 0.038 & 0.045 & 0.052 & 0.058 & 0.065 & 0.071 & 0.077 & 0.083 & 0.089 & 0.094 & 0.099 & 0.104 & 0.109 & 0.113 & 0.118 & 0.122
\end{matrix}\right]$$

If multiplied with **C** from the right and $\mathbf{C^T}$ from the left, the resulting 4x4 matrix is:

$$\mathbf{CWC^T} = \begin{bmatrix} 0.009 & 0.016 & 0.023 & 0.034 \\ 0.016 & 0.031 & 0.044 & 0.066 \\ 0.023 & 0.044 & 0.063 & 0.095 \\ 0.034 & 0.066 & 0.095 & 0.147 \end{bmatrix}$$

The inverse of this matrix $\mathbf{CWC^T}$ is computed as:

$$\left(\mathbf{CWC^T}\right)^{-1} = \begin{bmatrix} 10765.8 & -10328.0 & 3708.9 & -272.7 \\ -10328.0 & 14579.0 & -8136.8 & 1139.6 \\ 3708.9 & -8136.8 & 6381.5 & -1353.7 \\ -272.7 & 1139.6 & -1353.7 & 437.4 \end{bmatrix}$$

We first multiply the cash flows in **C** with the vector **μ** of the asymptotic terms, and then subtract the vector from the vector of the market values:

$$\mathbf{m - C\mu} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \mathbf{C} \cdot \begin{bmatrix} 0.9898 \\ 0.9796 \\ 0.9696 \\ 0.9597 \\ 0.9499 \\ 0.9402 \\ 0.9305 \\ 0.9210 \\ 0.9116 \\ 0.9023 \\ 0.8930 \\ 0.8839 \\ 0.8748 \\ 0.8659 \\ 0.8570 \\ 0.8483 \\ 0.8396 \\ 0.8310 \\ 0.8225 \\ 0.8141 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.969 \\ 0.959 \\ 0.957 \\ 0.967 \end{bmatrix} = \begin{bmatrix} 0.031 \\ 0.041 \\ 0.043 \\ 0.033 \end{bmatrix}$$

Multiply $\mathbf{(CWC^T)^{-1}}$ with **m-μ**. The resulting vector represents the solution of the LSE that was set up in (14):

$$\zeta = \begin{bmatrix} 58.6 \\ -34.1 \\ 11.8 \\ -5.7 \end{bmatrix}$$

To assess the discount function *P(t)* in arbitrary *t*, *t*>0, the Wilson functions $W(t,u_j)$, *j=1,2,3…20* have to be assessed and multiplied with **C**, as defined in (7). We want to compute the discount factor for *t*=4. We calculate $\mathbf{w^T}=(W(4,u_j))_{j=1,2,3,..20}$, multiply it with $\mathbf{C^T}$ and get the values of the kernel functions for *t*=4, i.e.:

$$(\mathrm{K}_i(4))_{i=1,2,3,4} = \mathbf{w}^\mathbf{T}\mathbf{C}^\mathbf{T} = [0{,}027 \quad 0{,}052 \quad 0{,}075 \quad 0{,}115]$$

From the linear combination of these kernel functions we get

$$(\mathbf{w}^\mathbf{T}\mathbf{C}^\mathbf{T})\boldsymbol{\zeta} = 0.0353$$

and adding the asymptotical factor $e^{-0.04114\times 4} = 0.8483$, the discount function at maturity 4 years has the value *P(4)* = 0.0353 +0.8483=0.8836. This gives a spot rate (with annual compounding) of 3.141%.