Multi-factor adjustment

Multi-factor Merton-type portfolio models of credit risk have become very popular among risk management practitioners. Practical implementations of these models mostly rely on Monte Carlo simulations, while analytical methods have been limited to the one-factor case. Here, Michael Pykhtin presents an analytical method for calculating portfolio value-at-risk and expected shortfall in the multi-factor Merton framework. This method is essentially an extension of the granularity adjustment technique to a new dimension

Most of the portfolio models of credit risk used in the banking industry are based on the conditional independence framework.[1] In these models, defaults of individual borrowers depend on a set of common systematic risk factors describing the state of the economy. Merton-type models, such as PortfolioManager and CreditMetrics, have become very popular. However, implementation of these models requires time-consuming Monte Carlo simulations, which significantly limits their attractiveness.

For one-factor Merton-type models, several analytical techniques have been developed. One such technique is the limiting loss distribution discovered by Vasicek (1991) for homogeneous portfolios and extended by Gordy (2003) and Vasicek (2002) to non-homogeneous portfolios. This approximation replaces the original loss distribution with the loss distribution for an infinitely fine-grained portfolio, whose value-at-risk and expected shortfall (ES) can be calculated analytically. However, the difference between the VARs (or ESs) of the original and the limiting loss distributions can be significant if the portfolio is not large enough. To estimate this difference, the granularity adjustment technique was introduced by Gordy (2003). Wilde (2001) and Martin & Wilde (2002) have derived a general closed-form expression for the granularity adjustment for portfolio VAR. More specific expressions for a one-factor default-mode Merton-type model (known as the Vasicek model) have been derived by Wilde (2001) and Pykhtin & Dev (2002). Emmer & Tasche (2003) have developed an analytical formulation for calculating VAR contributions from individual exposures. Gordy (2004) has derived a granularity adjustment for ES.

For multi-factor Merton-type models, no purely analytical methods for estimating portfolio VAR or ES have been reported. Although Gordy (2003) has shown that the limiting loss distribution is still applicable to large enough portfolios, calculation of the portfolio VAR still requires Monte Carlo simulation of the systematic risk factors. Moreover, it is not clear how large the portfolio needs to be to ensure applicability of the limiting loss distribution. In this article, we present an analytical method for calculating portfolio VAR and ES in the multi-factor Merton framework. This method is essentially an extension of the granularity adjustment technique to a new dimension.[2]

Model

Let us first set up a multi-factor default-mode Merton model. We consider a portfolio of loans to M distinct borrowers. To avoid cumbersome notations, we assume that each borrower has exactly one loan with principal A_i. We also define the weight of a loan in the portfolio as the ratio of its principal to the total principal of the portfolio, $w_i = A_i/\Sigma_{j=1}^{M} A_j$. Borrower i will default within a chosen time horizon (typically, one year) with probability p_i. Default happens when a continuous variable X_i describing the financial well-being of borrower i at the horizon falls below a threshold. We assume that variables $\{X_i\}$ (which may be interpreted as the standardised asset returns) have standard normal distribution. The default threshold for borrower i is given by $N^{-1}(p_i)$, where $N^{-1}(\cdot)$ is the inverse of the cumulative normal distribution function.

We assume that asset returns depend linearly on N normally distributed systematic risk factors with a full-rank correlation matrix. These systematic factors represent industry, geography, global economy or any other relevant indexes that may affect borrowers' defaults in a systematic way. Borrower i's standardised asset return is driven by a certain borrower-specific combination of these systematic factors Y_i (known as a composite factor):

$$X_i = r_i Y_i + \sqrt{1 - r_i^2}\,\xi_i \tag{1}$$

where ξ_i is the standardised normally distributed idiosyncratic shock. Factor loading r_i measures borrower i's sensitivity to the systematic risk.

Since it is more convenient to work with independent factors, we assume that N original correlated systematic factors are decomposed into N independent standard normal systematic factors Z_k (k = 1, ... , N). The relation between $\{Z_k\}$ and the composite factor is given by:

$$Y_i = \sum_{k=1}^{N} \alpha_{ik} Z_k \tag{2}$$

where coefficients α_{ik} must satisfy the relation $\Sigma_{k=1}^{N} \alpha_{ik}^2 = 1$ to ensure that Y_i has unit variance. Asset correlation between distinct borrowers i and j is given by $\rho_{ij} = r_i r_j \Sigma_{k=1}^{N} \alpha_{ik}\alpha_{jk}$.

If borrower i defaults, the amount of loss is determined by its loss-given default (LGD) stochastic variable Q_i with mean μ_i and standard deviation σ_i. We assume that these LGD variables are independent between themselves as well as from all the other variables in the model. We do not make any specific assumptions about the probability distribution of Q_i.

Finally, portfolio loss rate L can be written as the weighted average of individual loss rates L_i:

$$L = \sum_{i=1}^{M} w_i L_i = \sum_{i=1}^{M} w_i 1_{\{X_i \le N^{-1}(p_i)\}} Q_i \tag{3}$$

where $1_{\{\cdot\}}$ is the indicator function. Equation (3) describes the distribution of the portfolio losses at the horizon and thus completes the model.

A traditional approach to estimating quantiles of the portfolio loss distribution in the multi-factor framework is Monte Carlo simulation. If the portfolio is large enough to be considered fine-grained, most of the idiosyncratic risk in the portfolio is diversified away and portfolio losses are driven primarily by the systematic factors. In this case, equation (3) can be replaced by the limiting loss distribution of an infinitely fine-grained portfolio. The limiting loss is given by the expected loss conditional on the systematic risk factors, as can be shown by applying the law of large numbers conditionally on the factors (see Gordy, 2003, for details):

[1] *See Bluhm, Overbeck & Wagner (2002) for an excellent review of portfolio credit risk models*

[2] *Another interesting extension of the granularity adjustment technique is given in Canabarro, Picoult & Wilde (2003), where the authors use it for analysing derivatives counterparty credit risk*

$$L^{\infty} = E\left[L\left|\{Z_k\}\right.\right] = \sum_{i=1}^{M} w_i \mu_i N\left[\frac{N^{-1}(p_i) - r_i \sum_{k=1}^{N} \alpha_{ik} Z_k}{\sqrt{1-r_i^2}}\right] \quad (4)$$

Although equation (4) is much simpler than equation (3), it still requires Monte Carlo simulation of the systematic factors $\{Z_k\}$ when the number of factors is greater than one. Moreover, it is not clear how large the portfolio needs to be for equation (4) to become accurate.

In what follows, we design an analytical method for calculating tail quantiles and tail expectations of the portfolio loss L given by equation (3). The method is based on derivatives of VAR introduced by Gourieroux, Laurent & Scaillet (2000) and perfected by Martin & Wilde (2002).

Derivatives of VAR

We are interested in calculating the quantile at a confidence level q of the portfolio loss L. We will denote this quantile as $t_q(L)$. Let us assume that we have constructed a random variable $\bar{L}$ such that its quantile at level q, $t_q(\bar{L})$, can be calculated analytically and is close enough to $t_q(L)$. We will think of the portfolio loss as the variable $\bar{L}$ plus perturbation U defined as $U = L - \bar{L}$. To describe the scale of the perturbation, let us also introduce a perturbed variable $L_\varepsilon = \bar{L} + \varepsilon U$. Martin & Wilde (2002) have shown that, for high enough confidence levels q, $t_q(L_\varepsilon)$ can be calculated via the expansion in powers of ε around $t_q(\bar{L})$. By keeping terms up to quadratic and setting $\varepsilon = 1$ in this Taylor series, we can hope to calculate the portfolio loss quantile as:

$$t_q(L) = t_q(\bar{L}) + \left.\frac{dt_q(L_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2}\left.\frac{d^2 t_q(L_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} \quad (5)$$

Gourieroux, Laurent & Scaillet (2000) have derived the first two derivatives of VAR. The first derivative is given by the expectation of the perturbation conditional on $\bar{L} = t_q(\bar{L})$:

$$\left.\frac{dt_q(L_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = E\left[U\left|\bar{L} = t_q(\bar{L})\right.\right] \quad (6)$$

while the second derivative is:

$$\left.\frac{d^2 t_q(L_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = -\frac{1}{f_{\bar{L}}(l)}\frac{d}{dl}\left(f_{\bar{L}}(l)\,\mathrm{var}\left[U\left|\bar{L} = l\right.\right]\right)\bigg|_{l=t_q(\bar{L})} \quad (7)$$

where $f_{\bar{L}}(\cdot)$ is the probability density function for $\bar{L}$ and $\mathrm{var}[U|\bar{L} = l]$ is the variance of U conditional on $\bar{L} = l$. The problem is now reduced to finding appropriate $\bar{L}$.

Comparable one-factor model

We define $\bar{L}$ via the limiting loss distribution for the same portfolio as defined above, but in the one-factor Merton framework:

$$\bar{L} = l(\bar{Y}) = \sum_{i=1}^{M} w_i \mu_i \hat{p}_i(\bar{Y}) \quad (8)$$

where $\bar{Y}$ is the single systematic risk factor having the standard normal distribution, and $\hat{p}_i(y)$ is the probability of default of borrower i conditional on $\bar{Y} = y$, which is given by:

$$\hat{p}_i(y) = N\left[\frac{N^{-1}(p_i) - a_i y}{\sqrt{1-a_i^2}}\right] \quad (9)$$

where a_i is the effective factor loading for borrower i. Since $\bar{L}$ is a deterministic monotonically decreasing function of $\bar{Y}$, the quantile of $\bar{L}$ at level q can be calculated analytically simply as the function value at $Y = N^{-1}(1 - q)$:

$$t_q(\bar{L}) = l\left(N^{-1}(1-q)\right) \quad (10)$$

and therefore can be used as the zeroth-order approximation to $t_q(L)$.

Let us note that the derivatives of VAR in equations (6) and (7) are given by expressions conditional on $\bar{L} = t_q(\bar{L})$. Since $\bar{L}$ is a deterministic monotonically decreasing function of $\bar{Y}$, this conditioning is equivalent to conditioning on $\bar{Y} = N^{-1}(1 - q)$. The first and second derivatives of VAR can now be stated as:

$$\left.\frac{dt_q(L_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = E\left[U\left|\bar{Y} = N^{-1}(1-q)\right.\right] \quad (11)$$

and:

$$\left.\frac{d^2 t_q(L_\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = -\frac{1}{n(y)}\frac{d}{dy}\left(n(y)\frac{v(y)}{l'(y)}\right)\bigg|_{y=N^{-1}(1-q)} \quad (12)$$

respectively, where $v(\cdot)$ is the conditional variance of U defined as $v(y) = \mathrm{var}[U|\bar{Y} = y]$, $l'(\cdot)$ is the first derivative of $l(\cdot)$ and $n(\cdot)$ is the standard normal density (it appears here as the probability density of $\bar{Y}$).

To relate random variable $\bar{L}$ to the portfolio loss L, we need to relate the effective systematic factor $\bar{Y}$ to the original systematic factors $\{Z_k\}$. We assume a linear relation given by:

$$\bar{Y} = \sum_{k=1}^{N} b_k Z_k \quad (13)$$

where the coefficients must satisfy $\Sigma_{k=1}^{N} b_k^2 = 1$ to preserve unit variance of $\bar{Y}$. Now, we need to specify the set of M effective factor loadings $\{a_i\}$ and N coefficients $\{b_k\}$ to complete the specification of $\bar{L}$.

Our first step in determining $\{a_i\}$ and $\{b_k\}$ will be the requirement that $\bar{L}$ equals the expected loss conditional on $\bar{Y}$ (that is, $\bar{L} = E[L|\bar{Y}]$) for any portfolio composition. Apart from being very appealing intuitively, this requirement guarantees that the first-order term in the Taylor series, given by equation (11), vanishes for any confidence level q. To calculate $E[L|\bar{Y}]$, let us represent the composite risk factor for borrower i as:

$$Y_i = \bar{\rho}_i \bar{Y} + \sqrt{1-\bar{\rho}_i^2}\,\eta_i \quad (14)$$

where η_i is a standard normal variable independent of $\bar{Y}$ (but, in contrast to the true one-factor case, variables $\{\eta_i\}$ are inter-dependent) and $\bar{\rho}_i$ is the correlation between Y_i and $\bar{Y}$ given by:

$$\bar{\rho}_i \equiv \mathrm{cor}(Y_i, \bar{Y}) = \sum_{k=1}^{N} \alpha_{ik} b_k \quad (15)$$

Using these notations, we can rewrite the asset return given by equation (1) as:

$$X_i = r_i \bar{\rho}_i \bar{Y} + \sqrt{1-(r_i \bar{\rho}_i)^2}\,\zeta_i \quad (16)$$

where ζ_i is a standard normal variable independent of $\bar{Y}$. Therefore, the conditional expectation of L is:

$$E\left[L\left|\bar{Y}\right.\right] = \sum_{i=1}^{M} w_i \mu_i N\left[\frac{N^{-1}(p_i) - r_i \bar{\rho}_i \bar{Y}}{\sqrt{1-(r_i \bar{\rho}_i)^2}}\right] \quad (17)$$

By comparing equations (17) and (8), we see that $\bar{L}$ equals $E[L|\bar{Y}]$ for any portfolio composition if and only if the effective factor loadings are defined as:

$$a_i = r_i \bar{\rho}_i = r_i \sum_{k=1}^{N} \alpha_{ik} b_k \quad (18)$$

From now on, we assume that the effective factor loadings $\{a_i\}$ are given by equation (18) and that the correction to $t_q(\bar{L})$ is given by the second derivative of VAR (equation (12)).

While equation (18) is critical to the presented method, the choice of the coefficients $\{b_k\}$ is not. The choice of $\{b_k\}$ specifies the zeroth-order term $t_q(\bar{L})$ in the Taylor series of equation (5). Therefore, the method will work with many alternative specifications of $\{b_k\}$ that yield $t_q(\bar{L})$ close enough to the unknown target function value $t_q(L)$. Ideally, we would want to find a set $\{b_k\}$ that minimises the difference between the two quantiles. However, finding such a set is not an easy task, and an alternative, easy to calculate specification of $\{b_k\}$ is desirable.

Intuitively, one would expect the optimal single effective risk factor $\bar{Y}$ to have as much correlation as possible with the composite risk factors $\{Y_i\}$. We can express this intuition mathematically by requiring that the set of coefficients $\{b_k\}$ solve the following maximisation problem:

$$\max_{\{b_k\}}\left(\sum_{i=1}^{M} c_i \operatorname{cor}\left(\bar{Y}, Y_i\right)\right) \text{ such that } \sum_{k=1}^{N} b_k^2 = 1 \tag{19}$$

Taking into account equation (15), we can find the solution to this maximisation problem given by:

$$b_k = \sum_{i=1}^{M} (c_i / \lambda)\alpha_{ik} \tag{20}$$

where positive constant λ is the Lagrange multiplier chosen so that $\{b_k\}$ satisfy the constraint.

Unfortunately, it is not clear how to choose the coefficients $\{c_i\}$. However, some intuition about their possible form can be developed by minimisation of the conditional variance $v(y)$ (more precisely, its systematic part given by equation (28) below). Under an additional assumption that all r_i are small, this minimisation problem has a closed-form solution given by equation (20) with $c_i = w_i\mu_i n[N^{-1}(p_i)]$.[3] Even though the assumption of small r_i is often unrealistic and the performance of this solution is sub-optimal, it may serve as a starting point in a search of optimal $\{c_i\}$. After trying several different specifications, we have found that the set given by:

$$c_i = w_i\mu_i N\left[\frac{N^{-1}(p_i) + r_i N^{-1}(q)}{\sqrt{1-r_i^2}}\right] \tag{21}$$

is one of the best-performing choices. We used it in all examples discussed below.

Multi-factor adjustment

The remaining task is to derive an explicit algebraic form for equation (12). First, by taking the derivative with respect to y in equation (12), we can write the correction to $t_q(\bar{L})$ due to perturbation U as[4]:

$$\Delta t_q \equiv t_q(L) - t_q(\bar{L}) = -\frac{1}{2l'(y)}\left[v'(y) - v(y)\left(\frac{l''(y)}{l'(y)} + y\right)\right]\Bigg|_{y=N^{-1}(1-q)} \tag{22}$$

The first and second derivatives of function $l(y)$ required in equation (22) are obtained by differentiation of equation (8):

$$l'(y) = \sum_{i=1}^{M} w_i\mu_i \hat{p}_i'(y) \tag{23}$$

and:

$$l''(y) = \sum_{i=1}^{M} w_i\mu_i \hat{p}_i''(y) \tag{24}$$

where $\hat{p}_i'(y)$ and $\hat{p}_i''(y)$ are the first and second derivatives of the conditional probability of default. The latter is given by equation (9), whose differentiation yields:

$$\hat{p}_i'(y) = -\frac{a_i}{\sqrt{1-a_i^2}} n\left[\frac{N^{-1}(p_i) - a_i y}{\sqrt{1-a_i^2}}\right]$$

and:

$$\hat{p}_i''(y) = -\frac{a_i^2}{1-a_i^2}\frac{N^{-1}(p_i) - a_i y}{\sqrt{1-a_i^2}} n\left[\frac{N^{-1}(p_i) - a_i y}{\sqrt{1-a_i^2}}\right]$$

Since $\bar{L}$ is a deterministic function of $\bar{Y}$, the conditional variance of U is the same as the conditional variance of L, that is, $v(y) = \operatorname{var}(L|\bar{Y} = y)$. If, conditional on $\bar{Y}$, individual loss contributions were independent, equation (22) would be equivalent to Wilde's granularity adjustment. However, even though the second term in the asset return in equation (16) is independent of $\bar{Y}$, it gives rise to a non-zero conditional asset correlation between two distinct borrowers i and j. This becomes clear if we rewrite equation (16) as:

$$X_i = a_i\bar{Y} + \sum_{k=1}^{N}(r_i\alpha_{ik} - a_i b_k)Z_k + \sqrt{1-r_i^2}\xi_i \tag{25}$$

With $\{a_i\}$ defined according to equation (18), the second term (given by the sum over k) is independent of $\bar{Y}$. However, this term is responsible for the conditional asset correlation, which can be obtained directly from equation (25) and taking into account equation (18) along with the constraint $\Sigma_{k=1}^{N} b_k^2 = 1$:

$$\rho_{ij}^{Y} = \frac{r_i r_j \sum_{k=1}^{N}\alpha_{ik}\alpha_{jk} - a_i a_j}{\sqrt{\left(1-a_i^2\right)\left(1-a_j^2\right)}} \tag{26}$$

Although ρ_{ij}^{Y} has the meaning of the conditional asset correlation only for distinct borrowers i and j, we extend equation (26) to include the case $j = i$.

Nevertheless, conditional on $\{Z_k\}$, the asset returns are independent, and we may decompose the conditional variance as the sum of systematic and idiosyncratic parts:

$$\operatorname{var}\left[L\middle|\bar{Y} = y\right] = \operatorname{var}\left[E\left(L\middle|\{Z_k\}\right)\middle|\bar{Y} = y\right] + E\left[\operatorname{var}\left(L\middle|\{Z_k\}\right)\middle|\bar{Y} = y\right] \tag{27}$$

The first term of the right-hand side of equation (27) is the conditional on $\bar{Y} = y$ variance of the limiting portfolio loss L^∞ given by equation (4). It quantifies the difference between the multi-factor and one-factor limiting loss distributions (we will denote this term as $v_\infty(y)$) and is given by:

$$v_\infty(y) = \sum_{i=1}^{M}\sum_{j=1}^{M} w_i w_j \mu_i \mu_j \left[N_2\left(N^{-1}\left[\hat{p}_i(y)\right], N^{-1}\left[\hat{p}_j(y)\right], \rho_{ij}^{Y}\right) - \hat{p}_i(y)\hat{p}_j(y)\right] \tag{28}$$

where $N_2(\cdot,\cdot,\cdot)$ is the bivariate normal cumulative distribution function.[5] Differentiating equation (28) with respect to y yields:

$$v_\infty'(y) = 2\sum_{i=1}^{M}\sum_{j=1}^{M} w_i w_j \mu_i \mu_j \hat{p}_i'(y) \left[N\left(\frac{N^{-1}\left[\hat{p}_j(y)\right] - \rho_{ij}^{Y} N^{-1}\left[\hat{p}_i(y)\right]}{\sqrt{1-\left(\rho_{ij}^{Y}\right)^2}}\right) - \hat{p}_j(y)\right] \tag{29}$$

The second term of the right-hand side of equation (27) describes the effect of the finite number of loans in the portfolio. This term, which we will denote as $v_{GA}(y)$, describes the granularity adjustment and vanishes in the limit $M \to \infty$.[6] It is given by:

$$v_{GA}(y) = \sum_{i=1}^{M} w_i^2 \left(\mu_i^2\left[\hat{p}_i(y) - N_2\left(N^{-1}\left[\hat{p}_i(y)\right], N^{-1}\left[\hat{p}_i(y)\right], \rho_{ii}^{Y}\right)\right] + \sigma_i^2 \hat{p}_i(y)\right) \tag{30}$$

while its derivative is:

[3] *The solution is obtained by using the tetrachoric expansion of the bivariate normal in equation (28) (see Vasicek, 1998, for details) and expanding the terms of the resulting expression in powers of r_i and r_j up to the second order*

[4] *The relation $n'(y) = -yn(y)$ has been used*

[5] *Algorithms for evaluation of this function are discussed in great detail in Vasicek (1998)*

[6] *Provided that $\Sigma_{i=1}^{M} w_i^2 \to 0$ while $\Sigma_{i=1}^{M} w_i = 1$ (see Vasicek, 2002, or Emmer & Tasche, 2003)*

$$v'_{GA}(y)=\sum_{i=1}^{M}w_i^2\hat{p}'_i(y)$$
$$\left(\mu_i^2\left[1-2N\left(\frac{N^{-1}\left[\hat{p}_i(y)\right]-\rho_{ii}^Y N^{-1}\left[\hat{p}_i(y)\right]}{\sqrt{1-\left(\rho_{ii}^Y\right)^2}}\right)\right]+\sigma_i^2\right) \quad (31)$$

Since equation (22) is linear in the conditional variance $v(y) = v_\infty(y) + v_{GA}(y)$ and its first derivative, the quantile correction (we will call it multi-factor adjustment) is also the sum of the 'systematic' and 'granularity-adjustment' terms: $\Delta t_q = \Delta t_q^\infty + \Delta t_q^{GA}$. Each term in the multi-factor adjustment is obtained by substituting the corresponding conditional variance and its first derivative into equation (22). In the limit $M \to \infty$, Δt_q^{GA} vanishes, so we can interpret $t_q(\bar{L}) + \Delta t_q^\infty$ as the quantile of L^∞.

Expected shortfall

The approximation developed above allows for calculation of the portfolio VAR in the multi-factor Merton framework. While VAR is still used as a measure of risk by most financial institutions, it is known to have certain shortcomings (see Szegö, 2002, for a discussion). As an alternative to VAR, Acerbi & Tasche (2002) have proposed ES. Ignoring discontinuities of the portfolio loss rate distribution at its quantile $t_q(L)$, ES at a confidence level q for L is defined as the expected loss above the q-quantile:

$$ES_q(L)=E\left[L\middle|L\geq t_q(L)\right] \quad (32)$$

This definition is free from the shortcomings of VAR and is gaining popularity among practitioners. In this section, we extend our multi-factor adjustment method to ES. This extension is done similarly to the derivation of the one-factor granularity adjustment for ES in Gordy (2004).

As has been shown by Acerbi & Tasche (2002), we can rewrite equation (32) as:

$$ES_q(L)=\frac{1}{1-q}\int_q^1 ds\, t_s(L) \quad (33)$$

Since we know how to calculate $t_s(L)$ for any confidence level s, we can just integrate equation (33) numerically to arrive at the ES. However, we can do better than this. If we substitute the quantile of L in the form $t_s(L) = t_s(\bar{L}) + \Delta t_s(L)$ into equation (33), we immediately obtain the expected shortfall in the form:

$$ES_q(L)=ES_q(\bar{L})+\frac{1}{1-q}\int_q^1 ds\Delta t_s(L) \quad (34)$$

where the first term is the ES for our comparable one-factor portfolio and the second term is the ES multi-factor adjustment, which we will denote as $\Delta ES_q(L)$. If we assume that effective one-factor loadings $\{a_i\}$ are the same for all confidence levels above q, we can find both terms in closed form. One might argue that our definition of $\{b_k\}$ in equation (20) involves the coefficients $\{c_i\}$ (equation (21)) dependent on the confidence level, which makes the factor loadings $\{a_i\}$ depend on s. However, this problem can easily be avoided by redefining $\{b_k\}$ to be the same for all s above q. In the examples below, we used $\{b_k\}$ defined according to equation (20) with the confidence level q for all s above q.

To find $ES_q(\bar{L})$, we will use the ES definition given by equation (32). If we recall that $\bar{L}$ is the monotonic deterministic function $l(\bar{Y})$, we can write it as:

$$ES_q(\bar{L})=E\left[l(\bar{Y})\middle|\bar{Y}\leq N^{-1}(1-q)\right]=\frac{1}{1-q}\int_{-\infty}^{N^{-1}(1-q)} dy\, n(y)\, l(y) \quad (35)$$

Substituting $l(\bar{Y})$ from equation (8) into equation (35) and evaluating the integral[7] yields:

$$ES_q(\bar{L})=\frac{1}{1-q}\sum_{i=1}^{M}w_i\mu_i N_2\left[N^{-1}(p_i),N^{-1}(1-q),a_i\right] \quad (36)$$

To find $\Delta ES_q(L)$, we will recall that $\Delta t_s(L)$ equals one half of the second derivative of VAR. Using the second derivative of VAR in the form of equation (12), the second term in equation (34) can be written as:

$$\Delta ES_q(L)=-\frac{1}{2(1-q)}\int_q^1 ds\frac{1}{n(y)}\frac{d}{dy}\left(n(y)\frac{v(y)}{l'(y)}\right)\Bigg|_{y=N^{-1}(1-s)} \quad (37)$$

Changing the integrating variable from s to $y = N^{-1}(1 - s)$ yields:

$$\Delta ES_q(L)=-\frac{1}{2(1-q)}n\left[N^{-1}(1-q)\right]\frac{v\left[N^{-1}(1-q)\right]}{l'\left[N^{-1}(1-q)\right]} \quad (38)$$

As with the multi-factor adjustment to VAR, the adjustment to ES given by equation (38) is linear in the conditional loss variance. Therefore, it can also be represented as the sum of the systematic and idiosyncratic parts: $\Delta ES_q(L) = \Delta ES_q^\infty(L) + \Delta ES_q^{GA}(L)$. As in the case of VAR, $ES_q(\bar{L}) + \Delta ES_q^\infty(L)$ can be interpreted as the ES of L^∞.

Two-factor examples

We will use a two-factor set-up as a starting point for our performance testing of the multi-factor adjustment approximation. We assume that the loans in the portfolio are grouped into two buckets: A and B. Bucket u (index u can take values A or B) contains M_u identical loans characterised by a single probability of default p_u, expected LGD μ_u, standard deviation of LGD σ_u, composite factor Y_u and composite factor loading r_u. The composite factors are correlated with correlation ρ. In these notations, the asset correlation inside bucket u is r_u^2, while the asset correlation between the buckets is $\rho r_A r_B$. We also introduce bucket weights ω_u defined as the ratio of the net principal of all loans in bucket u to the net principal of all loans in the portfolio. Individual loan weights are related to bucket weights as $\omega_u = w_i M_u$.

□ **Homogeneous case, $M = \infty$.** Let us look first at the performance of the systematic part of the multi-factor adjustment. Figure 1(a) compares $t_{99.9\%}(\bar{L}) + \Delta t_{99.9\%}^\infty$ (dashed blue curves) with the exact 99.9% quantile of L^∞[8] (solid red curves) for the homogeneous case – when loans in both buckets have identical characteristics. In this example, we assume $p_A = p_B = 0.5\%$, $\mu_A = \mu_B = 40\%$, $\sigma_A = \sigma_B = 20\%$ and $r_A = r_B = 0.5$. The quantile is plotted as a function of the correlation ρ between the composite risk factors at three different bucket weights ω_A. The method performs very well except for the case of equal bucket weights ($\omega_A = \omega_B = 0.5$) at low ρ. For all choices of bucket weights, performance of the method improves with ρ. At any given ρ, performance of the method improves as one moves away from the $\omega_A = \omega_B$ case. This behaviour is natural because any of the limits $\rho = 1$, $\omega_A = 0$ and $\omega_A = 1$ corresponds to the one-factor case where the approximation becomes exact. As one moves away from one of the exact limits, the error of the approximation is expected to increase. The performance of the approximation is the worst when one is as far from the limits as possible – the case of equal bucket weights and low ρ.

□ **Non-homogeneous case, $M = \infty$.** Figure 1(b) compares the performance of the systematic part of the multi-factor adjustment with the exact solution for a non-homogeneous case. Bucket A is now characterised by the PD $p_A = 0.1\%$ and the composite factor loading $r_A = 0.5$, while bucket B has $p_B = 2.0\%$ and $r_B = 0.2$. The LGD parameters are left at the same values as before. This choice of parameters (assuming one-year horizon) is reasonable if we interpret bucket A as the corporate sub-portfolio (lower PD and higher asset correlation) and bucket B as the consumer sub-portfolio (higher PD and lower asset correlation). From figure 1(b), one can

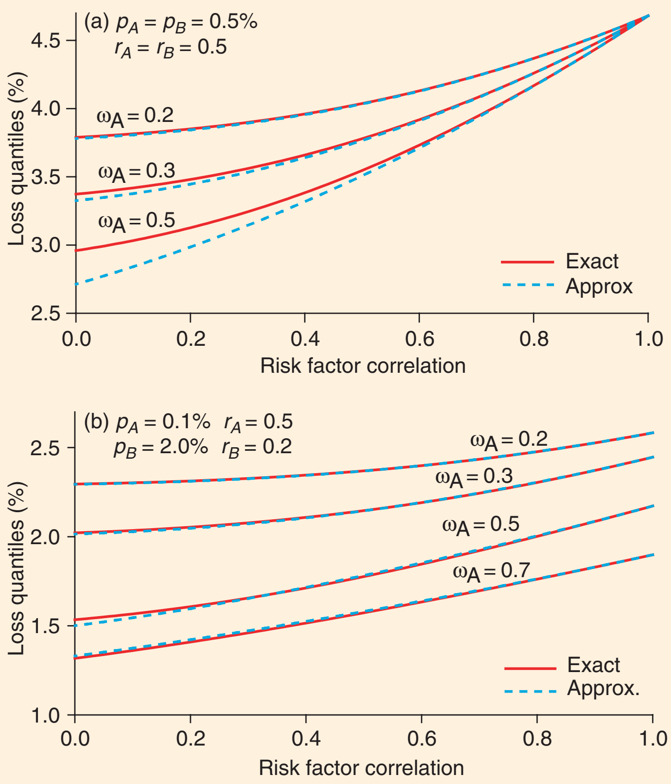
[7] *Evaluating the integral amounts to establishing the validity of the relation:*

$$\int_{-\infty}^{z}dy\, n(y)N\left(\left[x-ay\right]/\sqrt{1-a^2}\right)=N_2(x,z,a)$$

which can be verified by differentiating $N_2(x, z, a)$ with respect to x

[8] *By "exact 99.9% quantile of L^∞" we mean a quantile value calculated numerically, but without simulations*

1. VAR in two-factor set-up, infinite M

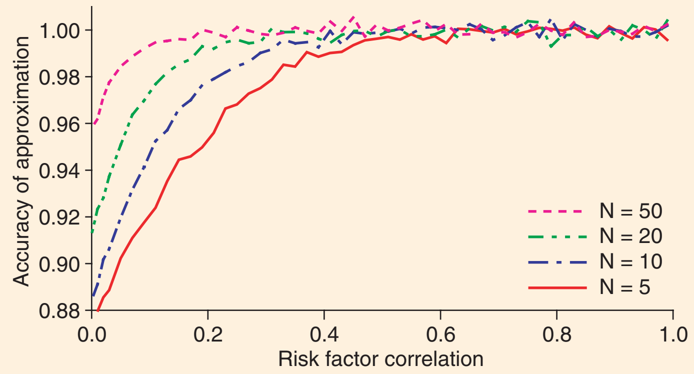


A. VAR in two-factor set-up, finite M

w_A	M	M_A	M_B	Portfolio loss quantiles Approximation	Monte Carlo
0.7	∞	∞	∞	1.58%	1.57%
	1,000	200	800	1.76%	1.76%
		500	500	1.68%	1.67%
		800	200	1.70%	1.69%
	200	40	160	2.49%	2.49%
		100	100	2.07%	2.04%
		160	40	2.18%	2.24%
0.3	∞	∞	∞	2.15%	2.15%
	1,000	200	800	2.30%	2.30%
		500	500	2.38%	2.37%
		800	200	2.71%	2.68%
	200	40	160	2.93%	2.87%
		100	100	3.30%	3.20%
		160	40	4.97%	4.48%

Note: 99.9% quantiles of the loss distribution in the two-factor two-bucket non-homogeneous set-up with ρ = 0.5 at varying number of loans in each bucket

2. Accuracy of approximation for VAR in multi-factor homogeneous set-up, infinite M



see that performance of the systematic part of the multi-factor adjustment is excellent for all choices of the bucket weights and the risk factor correlation. This example illustrates a general observation that the method performs much better in non-homogeneous cases than it does in homogeneous ones.

☐ **Non-homogeneous case, finite *M*.** Since it is impossible to calculate a quantile of the loss distribution exactly, we use Monte Carlo simulation as a benchmark for comparison. Table A compares the 99.9% quantile calculated with our method with the one obtained via a Monte Carlo simulation at varying bucket population. The comparison is made for the cases $w_A = 0.3$ and $w_A = 0.7$ assuming the risk factor correlation $\rho = 0.5$. As with Wilde's one-factor granularity adjustment, performance of the granularity adjustment part of our method generally improves as the number of loans in the portfolio increases. However, this improvement is not uniform across all bucket weights and population choices.

Multi-factor examples

Now we assume that there are more than two systematic risk factors in the model. We will consider a multi-factor set-up, which is a simplified version of the KMV/CreditMetrics systematic factor structure. Let us assume that there are $N - 1$ industry-specific (that is, independent) systematic factors $\{Z_k\}_{k=1}^{N-1}$ and one global systematic factor Z_N. Composite systematic factors have the form:

$$Y_i = \alpha_i Z_N + \sqrt{1-\alpha_i^2}\, Z_{k(i)} \quad (39)$$

where $k(i)$ denotes the industry that borrower i belongs to. For illustrational simplicity, we assume that all the loans in the same industry are grouped into a homogeneous bucket.[9] Thus, all M_u loans in bucket u are characterised by the same PD p_u, expected LGD μ_u, standard deviation of LGD σ_u, composite systematic risk factor Y_u and composite factor loading r_u. The weight of the global factor is assumed to be the same for all composite factors: $\alpha_i = \alpha$. The correlation between any pair of the composite systematic factors is $\rho = \alpha^2$, while the asset correlation inside bucket u is r_u^2 and the asset correlation between buckets u and v is $\rho r_u r_v$. As before, bucket weights ω_u are defined as the ratio of the net principal of all loans in bucket u to the net principal of all loans in the portfolio.

☐ **Homogeneous case, $M = \infty$.** We assume that the buckets are identical and are populated by a very large number of identical loans. In figure 2, we show the accuracy of the approximation as a function of ρ for several values of N at fixed $p_u = 0.5\%$, $\mu_u = 40\%$, $\sigma_u = 20\%$ and $r_u = 0.5$.[10] The accuracy is defined as the ratio of $t_{99.9\%}(\bar{L}) + \Delta t^{\infty}_{99.9\%}$ to the 99.9% quantile of L^{∞} obtained via Monte Carlo simulation. Apart from the noise coming from the simulation, the accuracy quickly improves as ρ increases. This behaviour is universal because in the limit of $\rho = 1$ the model is reduced to the one-factor framework. A more intriguing observation from figure 2 is that, at any given ρ, the approximation based on a one-factor model works better as the number of factors increases. This happens because, in the homogeneous case with composite risk factor correlation ρ, the limit $N - 1 = M$ is equivalent to the one-factor set-up with the factor loading $r_u\sqrt{\rho}$. When we increase the number of the systematic risk factors, we move towards this one-factor limit and the quality of the approximation is bound to improve.

☐ **Non-homogeneous case, VAR and ES.** Now we compare 99.9% quantiles and ESs of the portfolio loss calculated using the multi-factor adjustment approximation with the ones obtained from a Monte Carlo simulation

[9] *This assumption is not critical to the approximation performance*
[10] *These are the parameters we used in the two-factor homogeneous example*

B. Model parameters for 11-factor non-homogeneous set-up

u	1	2	3	4	5	6	7	8	9	10
p_u	0.1%	0.2%	0.2%	0.5%	0.5%	1.0%	1.0%	2.0%	2.0%	5.0%
μ_u	50%	30%	50%	30%	50%	30%	50%	30%	50%	30%
σ_u	20%	10%	20%	10%	20%	10%	20%	10%	20%	10%
r_u	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.2
ω_u	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
M_u^I	50	100	50	100	50	100	50	100	50	100
M_u^{II}	10	20	10	20	10	20	10	20	10	20
M_u^{III}	10	20	50	50	100	100	200	200	500	1,000

Note: model parameters for the 11-factor 10-bucket non-homogeneous set-up. The expected loss rate is 0.451% for all portfolios

C. VAR and ES in 11-factor non-homogeneous set-up

ρ	Limiting loss		Portfolio I		Portfolio II		Portfolio III	
	Approx	MC	Approx	MC	Approx	MC	Approx	MC
				VAR				
0.50	2.15%	2.15%	2.33%	2.34%	3.06%	3.09%	2.32%	2.36%
0.40	1.91%	1.91%	2.11%	2.12%	2.91%	2.92%	2.09%	2.13%
0.30	1.68%	1.68%	1.90%	1.90%	2.80%	2.78%	1.87%	1.92%
0.20	1.45%	1.47%	1.71%	1.71%	2.75%	2.65%	1.66%	1.73%
0.10	1.23%	1.26%	1.55%	1.54%	2.82%	2.54%	1.46%	1.55%
				Expected shortfall				
0.50	2.56%	2.57%	2.76%	2.77%	3.55%	3.60%	2.77%	2.83%
0.40	2.24%	2.23%	2.46%	2.48%	3.33%	3.35%	2.46%	2.52%
0.30	1.94%	1.96%	2.18%	2.19%	3.15%	3.16%	2.16%	2.25%
0.20	1.64%	1.67%	1.93%	1.94%	3.06%	2.99%	1.88%	2.02%
0.10	1.36%	1.43%	1.71%	1.72%	3.09%	2.85%	1.62%	1.82%

Note: 99.9% quantiles and expected shortfalls of the loss distribution for 11-factor 10-bucket non-homogeneous portfolios at varying ρ

for the case of 10 industries at several values of the composite risk factor correlation ρ. The parameters of the buckets are shown in table B. All buckets have equal weights $\omega_u = 0.1$, so the net exposure is the same for each bucket. The comparison for three portfolios (denoted as I, II and III), which only differ by the number of loans in the buckets, is given in table C. The number of loans in each bucket u for these portfolios (denoted as M_u^I, M_u^{II} and M_u^{III}) is shown in the last three rows of table B. First, let us compare the calculated quantiles and ESs for the asymptotic loss (L^∞ is the same for all three portfolios). The performance of the method is excellent even for very low levels of ρ. Similar to the two-factor set-up, the performance of the approximation for L^∞ in non-homogeneous cases is typically much better than it is in homogeneous cases (the appropriate homogeneous case for VAR is shown by the blue dash-dotted curve in figure 2). Comparing the calculated quantiles and ESs of L for portfolio I, we see that the method performs as impressively as it does for the asymptotic loss at all levels of ρ. This is because the largest exposure in the portfolio is rather small – only 0.2% of the portfolio exposure. In portfolio II, we have decreased the number of loans in each of the buckets uniformly by a factor of five, which brought the largest exposure to 1% of the portfolio exposure. The method's performance is still very good at high to medium values of risk factor correlation, but is rather disappointing at low ρ. Portfolio III has the same largest exposure as portfolio II, but much higher dispersion of the exposure sizes than either portfolio I or portfolio II. Although the resulting loss quantile is very close to the one for portfolio I, the approximation does not perform as well as it does for portfolio I because of the higher largest exposure.

Conclusion

Analytical methods for credit risk of loan portfolios have been mostly limited to one-factor models. In this article, we have presented a technique for calculating VAR and ES in the multi-factor Merton framework analytically. Application of this technique allows one to avoid slowly converging time-consuming Monte Carlo simulations and, at the same time, keep all the benefits of a multi-factor model.

The technique is based on finding a comparable one-factor portfolio whose loss distribution has properties similar to the ones of the original multi-factor loss distribution. VAR (or ES) for the original portfolio is calculated as the sum of VAR (or ES) for the limiting loss distribution of the comparable portfolio and the multi-factor adjustment. Calculation of the multi-factor adjustment is based on analytical expressions for the derivatives of VAR and is closely related to the granularity adjustment method.

The performance of the multi-factor adjustment approximation is excellent throughout a wide range of model parameters, as we have illustrated by several examples. Generally, the accuracy of the approximation improves as the number of the systematic factors and/or the correlation between the factors increase. Additionally, the accuracy of the multi-factor granularity adjustment improves as the size of the largest exposure in the portfolio becomes a smaller fraction of the entire portfolio exposure. ■

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