Non-affine stochastic volatility jump diffusion models

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Abstract

This paper proposes an alternative option pricing model in which the stock prices follow a diffusion process with non-affine stochastic volatility and random jumps.

Approximative European option pricing formulae are derived by transforming a non-linear PDE in an approximate linear PDE which is explicitly solved by using Fourier transformations. We check that these approximative prices are close to the Monte Carlo estimates and compare them with the prices in an affine stochastic volatility jump diffusion model.

Model parameters are estimated by using the method of simulated moments. We evaluate the impact of the different submodels on option prices and on implied volatility.

Keywords: jump-diffusion, non-affine stochastic volatility, method of simulated moments, implied volatility curves.

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1 Introduction

In the last decade, there has been evidence that stochastic volatility processes with jumps in the returns are important to model index return and volatility, but some recent studies on option pricing prove that most models still are incapable to capture some empirical characteristics observed in data. In particular the mean squared errors by the parameter estimation remain high and the derived implied volatility graphs are not satisfactory.

Some authors use the square root process to represent the dynamics of the instantaneous variance (e.g. Heston (1993)) but this square root model is generally rejected by many authors. The cause is principally due to the insufficient kurtosis generated by the model (see, e.g., Benzoni (1998) and Pan (2002)). They find that the parameter estimates are inconsistent with a time series analysis of the implied volatility series. Many recent papers (e.g. Bakshi et al. (1997, 2000), Bates (1996) and Duffie et al. (2000)) indicate that the square root model is incapable to explain the high moments of the spot volatility. Jones (2003) proves that the moments of stock return depend mainly on the variance of the volatility.

The purpose of this paper is to introduce a more general form of the stochastic volatility model which includes the affine class of random intensity models studied by Bates (1996) and Bakshi et al. (1997, 2000). Our class of models generalizes the square root model by allowing the instantaneous variance of the volatility to be proportional to any power of the variance, and then to be a non-affine process. We will call this a non-affine stochastic volatility jump diffusion model (NA-SVJD). The main goal of this paper is to check whether the NA-SVJD model might contribute to a better fit of the data by its supplementary parameter of the power of the variance, and this by analogy of the paper of Chan et al. (1992) in interest rate modelling where the generalisation of the square root in the Cox-Ingersoll-Ross (1985) process has been studied.

We first turn to the estimation of its parameters, a problem which is quite challenging. Several studies have pointed out that implicit volatility from option prices should theoretically summarize a rich information regarding expected future volatility. Therefore, we will use both the underlying asset prices and the prices of options on them in order to estimate the parameters of the model. Indeed, we propose to use a method of simulated moments (MSM) for estimating all parameters of the NA-SVJD model except the current return variance, and this by using both spot prices of the underlying assets and option prices on them. The method of simulated moments (MSM) approach has been developed by McFadden (1989), Pakes and Pollard (1989) and has been used in the financial literature by e.g. Duffie and Singleton (1993) and Bakshi, Cao and Chen (2000). The MSM technique is quite easy to use for complicated estimation problems in econometrics and it only asks a reasonable computation time. The daily initial return variance is then estimated by looking at the Mean Squared Error (MSE) between the approximative option prices and the observed market prices.

In the NA-SVJD model, however, explicit European call option prices can only be obtained in an approximative way. Indeed, the partial differential equation (PDE) approach introduced by Heston (1993), Bates (1996) and Bakshi et al. (1997) leads to a non-linear PDE which we will linearize in order to be able to solve it in an explicit way by using Fourier transformations. We compare the approximative European call option prices with the corresponding Monte Carlo estimates and the results in an affine stochastic volatility model.

This paper is composed as follows. In Section 2, we present the non-affine stochastic volatility jump diffusion model (NA-SVJD) that we study in this paper. Section 3 develops an approximative option pricing formula in this model. Section 4 describes the MSM estimation technique and the volatility filtering technique and reports the estimation results of the general model and various submodels. Section 5 compares the performance in option pricing among the different models, and this by studying the implied volatility graphs. Section 6 uses the previous estimated parameters to test the precision of the approximative European call option prices by Monte Carlo simulations.

2 Non-affine Stochastic Volatility Jump Diffusion Model

Throughout this paper, we consider a complete probability space $(\Omega, \mathcal{F}, P)$ with an information filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ satisfying the usual conditions. We denote by $S = \{S_t, 0 \leq t \leq T\}$ a stock price process (the risky asset price), where T is a finite time horizon.

We assume that under a risk-neutral probability measure Q, the asset price follows a jump diffusion with the instantaneous conditional variance

V following a non-affine stochastic process:

$$\begin{aligned} dS(t)/S(t) &= (r - \lambda\mu_J)dt + \sqrt{V(t)}dZ_s(t) + J(t)dq(t) & (1) \\ dV(t) &= k(\theta - V(t))dt + \sigma_v V^{\gamma/2}(t)dZ_v(t) & (2) \end{aligned}$$

with

$$cov(dZ_s(t), dZ_v(t)) = \rho dt \quad \text{and} \quad Q(dq(t) = 1) = \lambda dt \tag{3}$$

and

$$\ln(1 + J(t)) \sim N(\ln(1 + \mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2) \tag{4}$$

where r is the riskless interest rate which is assumed to be constant, λ is the annual frequency of the jumps and J is the percentage jump size (conditional on a jump occurring), identically and independently distributed over time, with unconditional mean μ_J. q is a Poisson jump counter with intensity λ and the parameters k, θ and σ_v are respectively the speed of adjustment, the long-run mean, and the variation coefficient of the diffusion volatility $V(t)$.

q and J are assumed to be independent and moreover, they are supposed to be independent with the correlated standard Brownian motions Z_s and Z_v.

The non-affine stochastic volatility model generalizes the square root model by allowing the instantaneous standard deviation of volatility to be proportional to any power of the volatility.

The above processes for asset prices contain models used by many authors, e.g., Merton (1993), Bates (1996), Bakshi et al. (1997), Duffie et al. (2000). The non-affine stochastic volatility model itself is also called the "Constant Elasticity of Variance" (CEV) model and has been investigated in many papers in interest rate modelling, and has been used by Chacko and Viceira (2003) and Jones (2003) to model stochastic variance processes.

3 European call option price

In this section, we want to price a European call option by using the PDE approach which is by now quite standard in literature (see e.g. Heston (1993), Bates (1996) and Bakshi et al. (1997)). We let C_t denote the price at time t of a European-style call option on S_t with strike price K and expiration time $T = t + \tau$. Using the fact that the terminal payoff of an European call option on the underlying stock S with strike price K is $\max(S_T - K, 0)$ and assuming that the short-term interest rate r is constant over the lifetime of the option, the price of the European call at time t equals

$$\begin{aligned} C(S(t), V(t), t) &= e^{-r(T-t)} E_t^Q (S_T - K)^+ \qquad (5) \\ &= e^{-r(T-t)} \Big(\int_K^\infty S_T P_t(S_T) dS_T - K \int_K^\infty P_t(S_T) dS_T \Big) \\ &= S(t)\pi_1 - K e^{-r(T-t)} \pi_2 \end{aligned}$$

where E_t^Q (resp. $P_t(.)$) is the conditional expectation operator with respect to the risk neutral probability measure Q of Section 2, (resp. is the conditional density function of S_T) conditional to the information $\mathcal{F}_t$.

$\pi_2 = Q(S_T \geqslant K)$ is one minus the risk-neutral distribution function and $\pi_1 = \int_K^\infty \frac{S_T}{E_t(S_T)} P_t(S_T) dS_T$ is also a probability. In fact, we compute some approximations of π_1 and π_2. Indeed, we have employed the partial differential equation (PDE) approach and we obtained non-linear PDE's for the characteristic functions f_1 and f_2 corresponding to π_1 and π_2. Therefore, we utilized a Taylor approximation to obtain linear PDE's, like Chacko and Viceira (2003) have done in another setting, and by solving these linearized PDE's, we obtain approximative characteristic functions f_1 and f_2 corresponding to π_1 and π_2. If $\theta^\gamma(1-\gamma) + \gamma\theta^{\gamma-1}V_t > 0$ and $\left|\rho(\theta^{\frac{\gamma+1}{2}}(\frac{1-\gamma}{2}) + \frac{\gamma+1}{2}\theta^{\frac{\gamma-1}{2}}V_t)\right| \leq \sqrt{V_t}\sqrt{\theta^\gamma(1-\gamma) + \gamma\theta^{\gamma-1}V_t}$, the approximative characteristic functions f_1 and f_2 correspond in fact to the linear SDE system with a time-dependent correlation between the Brownian motions:

$$\frac{dS_t}{S_t} = (r - \lambda\mu_J)dt + \sqrt{V_t} dZ_s(t) + J_t dq_t \qquad (6)$$

$$dV_t = k(\theta - V_t)dt + \sigma_v \sqrt{\theta^\gamma(1-\gamma) + \gamma\theta^{\gamma-1}V_t} dZ_v(t) \qquad (7)$$

with

$$\ln(1 + J(t)) \sim N(\ln(1 + \mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2) \qquad (8)$$

and

$$Q(dq(t) = 1) = \lambda dt \quad \text{and} \quad cov(dZ_s(t), dZ_v(t)) = \widetilde{\rho}_t dt \tag{9}$$

with

$$\widetilde{\rho}_t = \frac{\rho(\theta^{\frac{\gamma+1}{2}}(\frac{1-\gamma}{2}) + \frac{\gamma+1}{2}\theta^{\frac{\gamma-1}{2}}V_t)}{\sqrt{V_t}\sqrt{\theta^{\gamma}(1-\gamma) + \gamma\theta^{\gamma-1}V_t}}. \tag{10}$$

The explicit expressions and the details of the derivation of the approximative characteristic functions f_1 and f_2 are given in Appendix A. In this Appendix, the link with the linear SDE system (6)-(10) is explained as well.

The approximative probabilities π_1 and π_2 can be calculated by finding the inverse Fourier transforms of the approximative characteristic functions and are given by

$$\pi_1(S, K, T, r, t) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}(\frac{e^{-i\phi \ln K} f_1(S, K, T, r, t, \phi)}{i\phi}) d\phi \tag{11}$$

and

$$\pi_2(S, K, T, r, t) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}(\frac{e^{-i\phi \ln K} f_2(S, K, T, r, t, \phi)}{i\phi}) d\phi \tag{12}$$

where Re[.] denotes the real component of a complex number.

The infinite integrals involved by the inverse Fourier transforms can be evaluated by some numerical integration method like Simpson's Rule.

In Section 6, we will test the approximative pricing formula with Monte Carlo results and with results for an affine stochastic volatility model.

First, we will estimate the parameters of the NA-SVJD model and discuss the empirical results obtained.

4 Empirical results

4.1 Data description

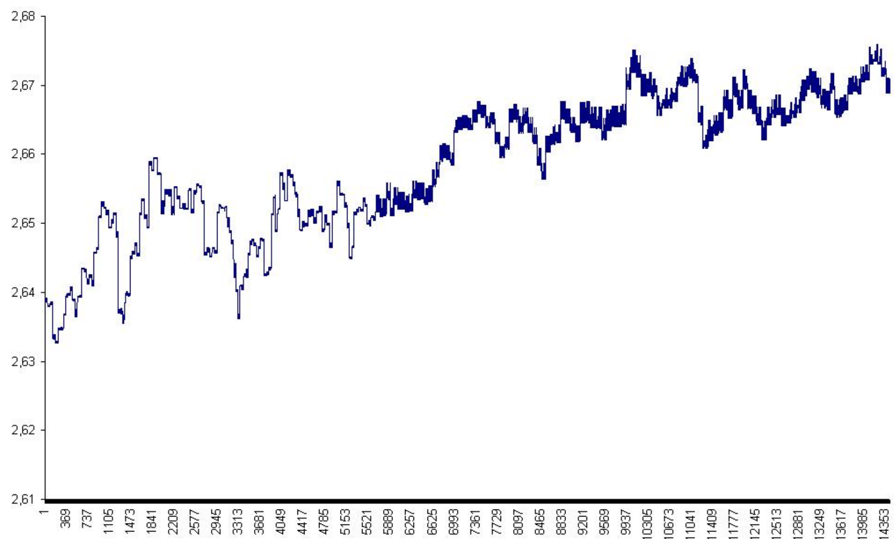

Figure 1: Time series of the S&P 500 stock index

The empirical analysis of this paper is based on a joint time series $\{S_n, C_n\}$ of the S&P 500 spot and option prices. The sample consists of daily CBOE closing prices, and have previously been used by Aït-Sahalia, Wang and Yared (2001) and Duffie, Pan and Singleton (2000) among others. The time series data covers the period from 1/4/1993 to 12/31/1993 providing 14431 observations. A time series plot of the S&P 500 stock index and a plot of the Black & Scholes implied volatilities can be found in Figure 1 and Figure 2 respectively. Summary statistics of the data are reported in Table 1.

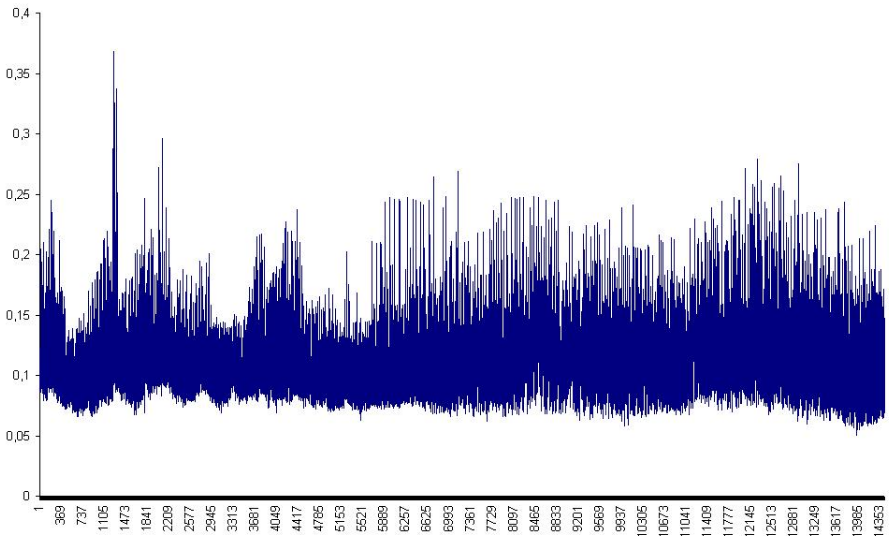

Figure 2: Black & Scholes implied volatilities

$Statistics$	$Mean$	$Std\ dev$	$Skewnes$	$Kurtosis$	Max	Min
$S\&P500$	6.1210	0.0227	$-0,4821$	-0.7007	6.1617	6.0619

Table 1: Summary statistics of stock returns

From Table 1 we see that the distribution of the S&P 500 stock returns is slightly negatively skewed (-0.4821) and have negative excess Kurtosis. When we look at the frequency distribution of the S&P 500 daily log return (Figure 3), we see that this distribution is highly peaked and heavily tailed in comparison with a normal distribution.

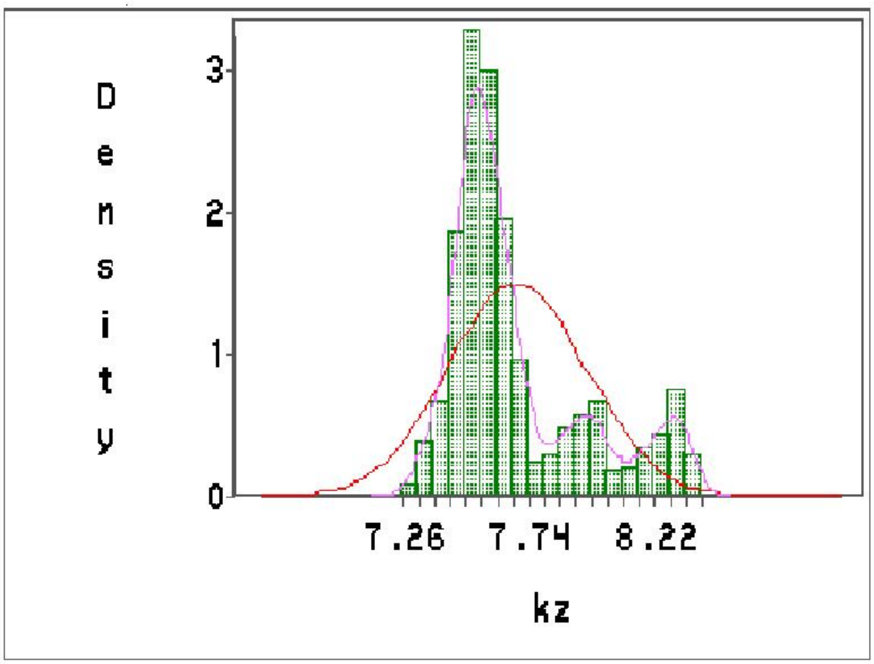

Figure 3: Frequency distribution of the S&P 500 daily log return compared with a Normal distribution

4.2 Estimation procedure

In this section, we focus on the estimation of the parametric model specified in Section 2 using the joint time series data $\{S_n,\, C_n\}$ of spot and option prices. The conditional likelihood function of the state vector $S(t)$ is not known for general non-affine models. Therefore we propose to use the method of simulated moments (MSM) proposed by Duffie and Singleton (1993) and Bakshi, Cao and Chen (2000).

Among many potential applications, the MSM technique is well suited to the estimation of systems of stochastic differential equations.

Following Bakshi et al. (2000), we first divide the option data into several categories based on moneyness and maturity. By the time to maturity, an option contract can be classified as:

(i) short-term ($T \leq 45$ days)

(ii) medium-term ($45 \leq T \leq 90$ days)

(iii) long-term ($T \geq 90$ days).

Let $F_t = e^{r(T-t)} S_t$ be the forward value at time t of the stock to be delivered at time T. Then a call contract is defined as being at-the-money (ATM) at time t if $0.95 \leq F_t/K \leq 1.05$; out-of-the-money (OTM) if $F_t/K \leq 0.95$; or in-the-money (ITM) if $F_t/K \geq 1.05$. The sample was further partitioned into three categories in terms of moneyness: short term (ST), medium term (MT) and long term (LT). Therefore, the proposed moneyness and maturity classification produces 9 categories. Table 2 below, shows the number of observations that falls within each category.

	ST	MT	LT
$OTM\ F/K \leq 0.95$	2101	2503	882
$ATM\ F/K \in (0.95, 1.05)$	3950	3134	408
$ITM\ F/K \geq 1.05$	349	681	415

Table 2: Number of observations within each category

For each category $j = 1, ..., 9$, we define the ratio of the observed option price on the strike price $\frac{C_{j,t}^{OBS}}{K_{j,t}}$ at day t, and we let g_t denote the 9-dimensional vector with $\frac{C_{j,t}^{OBS}}{K_{j,t}}$ in the j^{th} position defined as:

$$g_t = \begin{pmatrix} \frac{C_{1,t}^{OBS}}{K_{1,t}} \\ . \\ . \\ \frac{C_{9,t}^{OBS}}{K_{9,t}} \end{pmatrix}$$

where $t = 1, ..., N$, and where $K_{j,t}$ is the strike price of the observed call option in category j on day t, and N the number of days available in the dataset. We assume that the initial spot price and initial volatility to be equal to the averages of respectively the initial stock prices and of the market implied volatilities in the data base. We further denote the vector of the parameters to be estimated by Θ.

For an initial fixed value of the parameter vector Θ, and an arbitrarily large M, we simulate M series spot volatility and spot prices and calculate the call price in the j^{th} moneyness-maturity category by Monte Carlo simulations (see e.g. formula (15) below). Let $h_t(\Theta)$ denote the 9-dimensional vector with the average of the M simulated call prices divided by the strike price $\frac{E(C^{sim}_{j,t}(\Theta))}{K_{j,t}}$ in the j^{th} category:

$$h_t(\Theta) = \begin{pmatrix} \frac{E(C^{sim}_{1,t}(\Theta))}{K_{1,t}} \\ . \\ . \\ \frac{E(C^{sim}_{9,t}(\Theta))}{K_{9,t}} \end{pmatrix}$$

From g_t and h_t we build the corresponding means $G = \frac{1}{N}\sum_{t=0}^{N} g_t$ and $H(\Theta) = \frac{1}{N}\sum_{t=0}^{N} h_t(\Theta)$. We then search for the value of Θ for which G and $H(\Theta)$ are as close as possible in the sense that we minimize

$$\theta_{TH} = \underset{\Theta}{\operatorname{argmin}}(G - H(\Theta))'\Omega(G - H(\Theta)) \tag{13}$$

where Ω is an arbitrary weighting matrix which is symmetric and positive definite. If θ_{TH} is not sufficiently small enough, we repeat the procedure by using the obtained Θ as the initial value and we repeat this until θ_{TH} becomes small enough. The finally obtained value of Θ is the MSM estimator of the parameter vector.

Table 3 reports in the case of $r = 3.19\%$ and zero dividend yield, the estimations of the parameters obtained by the MSM and the standard deviations for each estimation. We also include the spot volatility and the ratio of the mean squared error (MSE) for each model to that of the Stochastic Volatility (SV) model.

The estimated volatility for each model can be backed out from option prices and parameter estimates of this model via non-linear least square methods by solving:

$$\min_{V(t)} \sum_{j=1}^{N} \left| C_j(S_t, K, T) - \hat{C}_j(S_t, K, T, \Theta) \right|^2 \tag{14}$$

where $C_j(S_t, K, T)$ is the actual price of an option with exercise price K and expiring at time T, and where $\hat{C}_j(S_t, K, T, \Theta)$ is the price of the option predicted by the model, given the parameters estimation Θ. N equals the number of options in the dataset on November 2, 1993, i.e. $N = 87$.

From the literature, it is clear that the parameter estimations depend a lot on the kind of dataset which has been used. Chacko and Viceira (2003) find for example higher values of γ and σ_v but they consider weekly and monthly stock prices. The estimates of the SV and SVJ models from Duffie et al. (2000) can neither be compared with ours since all parameters are estimated with only the data of November 2, 1993. Our estimated jump parameters in the SVJ model and the correlation parameters are consistent with the findings of Eraker et al. (2003).

	SV	$NA-SV$	SVJ	$NA-SVJD$
k	2.6557 (0.018)	2.0284 (0.017)	1.5120 (0.021)	1.9678 (0.018)
θ	0.0508 (0.048)	0.1188 (0.053)	0.1122 (0.047)	0.1118 (0.068)
σ_v	0.7518 (0.040)	0.6950 (0.042)	0.6666 (0.041)	0.5969 (0.048)
ρ	-0.3644 (0.014)	-0.4521 (0.022)	-0.3930 (0.027)	-0.4116 (0.021)
γ	1.0	1.5328 (0.044)	1.0	1.4406 (0.069)
λ			0.0667 (0.012)	0.0537 (0.012)
μ_J			-3.3785 (0.0517)	-3.1544 (0.0501)
σ_J			2.1681 (0.061)	1.8269 (0.062)
MSE %	2.5317 100%	1.7477 69.03	0.8863 35.01	0.8387 33.17
$\sqrt{V_0}$	9.06%	11.89%	12.19.%	14.91%

Table 3 : MSM parameter estimations of the four nested models

To compare the different submodels, we observe the ratios of the mean squared error (MSE) for each model to that of the Stochastic Volatility (SV) model. It turns out that the NA-SVJD superperforms the other subcases.

Indeed, the NA-SV model reduces the SV squared pricing error by more than 30 percent. The stochastic volatility with jumps (SVJ) model is a more powerful model, cutting SV errors by more than 60 percent and thus explaining an important part of the volatility smile. The NA-SVJD combines NA-SV and jump models and reduces the SV errors, but not in a significant way.

We find that the initial volatility $\sqrt{V_t}$ increases when comparing the different submodels. This fact is also observed in Eraker et al. (2003) when studying affine stochastic volatility jump models.

5 Implied volatility graph

In this research, another main diagnostic of relative model misspecification is to compare the implied volatility model pattern of each model across both moneyness and maturity. Also for this exercise, we use the subsample data on November, 2 1993, in which there are six different maturities.

Figure 4 and 5 compare the different nested submodels among each others. Figure 4 shows the implied volatility curves for short term maturity (17-days), who have the same form with the minimum around the moneyness equal to one.

In Figure 5, which focuses on a long term maturity (318 day), the implied volatility exhibits a moneyness-related U-shaped smile under the SV, NA-SV and NA-SVJD models, but under the SVJ model, the U-shaped form is difficult to be recognized. The SVJ's implied volatilities are always persistently higher (by about 1 percent on average).

Figure 6 shows the volatility smiles for different values of the parameter γ (gamma) in the power of volatility in the NA-SV model, namely for γ equal to resp. 0.5, 1.4406 and 2.0. We observe that the implied volatility curve is increasing in γ.

Figure 7 shows the volatility smiles for different values of the correlation coefficient ρ (rho), who have the same form and are much at the same level. For in-the-money-options, an increase of the correlation implies a small increase of the implied volatility, where as for out-of-the-money options it leads to a decrease.

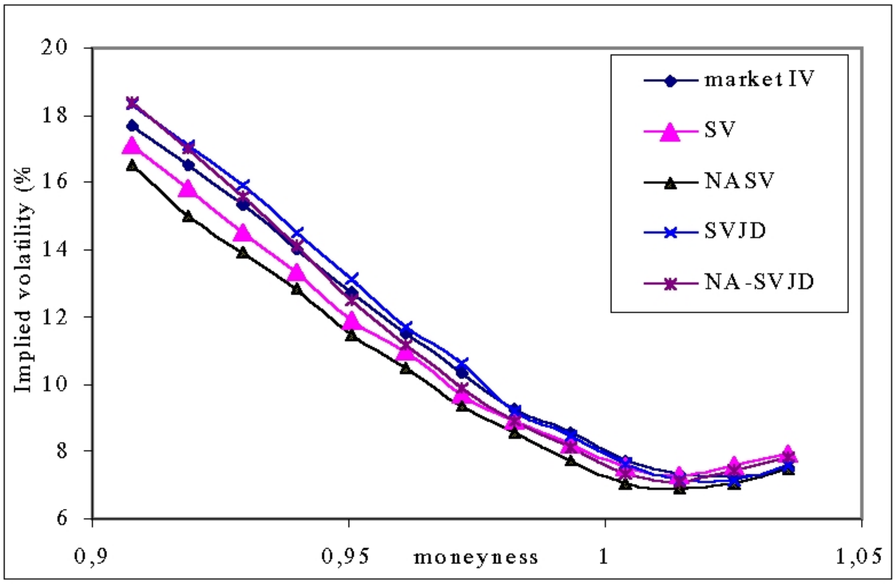

Figure 4 : Implied volatility curves for 17 days to maturity and for the different models

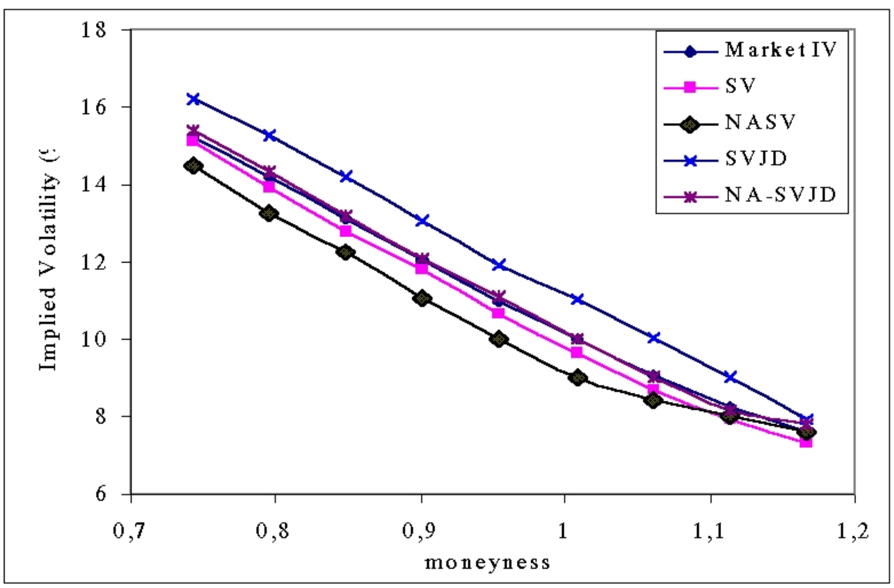

Figure 5 : Implied volatility curves for 318 days to maturity and for the different models

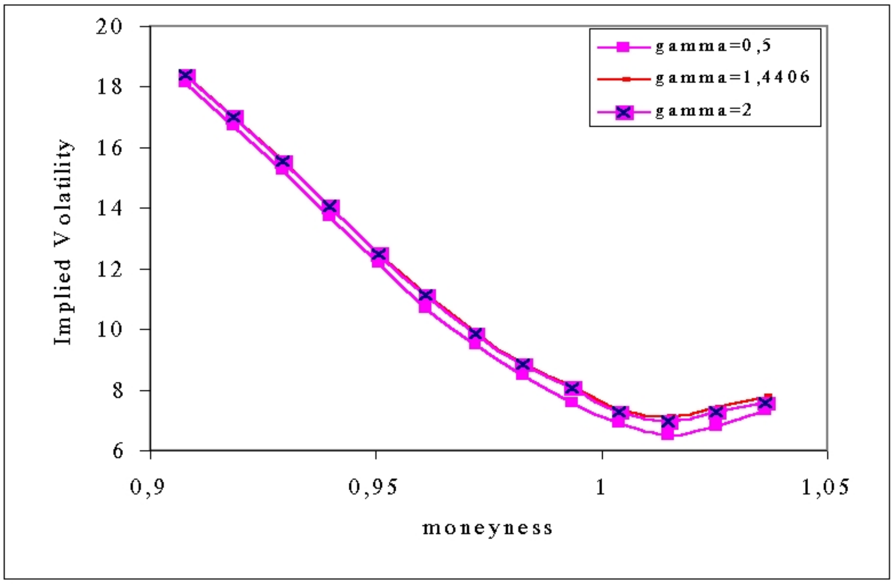

Figure 6 : smile curves for different values of gamma and maturity 17 days

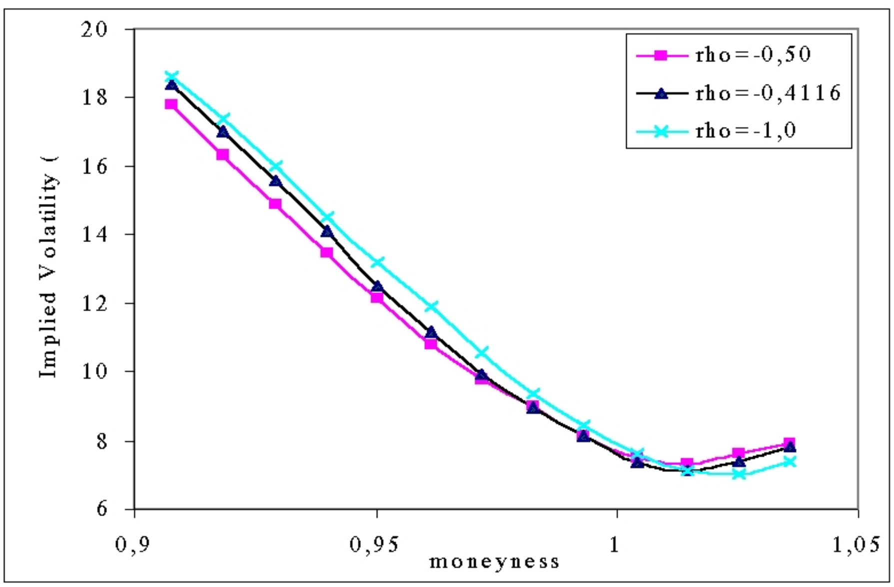

Figure 7 : : smile curves for different values of correlation and maturity 17 days

6 Explicit approximations versus Monte Carlo estimates.

In this section, we present and discuss the results of a comparison of the explicit approximative European call option prices versus the corresponding Monte Carlo estimates:

$$C_t = e^{-r(T-t)} \frac{1}{N} \sum_{l=1}^{N} (\max(0, S_l - K)) \tag{15}$$

where S_l is the $l - th$ realization of the state variable at time T and N is the number of the simulations. In order to test the approximative but closed formula for a European call option (5), we use the parameters summarized in Table 2 and perform N =1,000,000 iterations to obtain Monte Carlo estimates (15). For different maturities and different types of moneyness, the obtained values and corresponding CPU-times can be found in Table 4. The standard error of the Monte Carlo estimates are given between parentheses.

T days	F/K	Monte Carlo	CPU ($*10^4$)	NA-SVJD	CPU	SVJD	CPU
90	1.1467	77.4107 (0.0423)	3.7363	77.9350	0.1167	77.3546	0.1167
	1.0297	41.2622 (0.0271)	3.5842	41.6118	0.1333	41.2380	0.1480
	0.8957	19.9262 (0.0090)	1.9358	20.0981	0.1833	19.8885	0.1821
180	1.1467	97.6754 (0.0515)	3.6781	97.0183	0.1000	97.5584	0.1167
	1.0297	66.0112 (0.0363)	2.7396	66.4425	0.2167	66.0425	0.1333
	0.8957	32.7123 (0.1457)	2.2357	35.3652	0.2251	34.0695	0.2011
252	1.1467	113.9127 (0.0621)	3.3112	113.1437	0.1333	113.5184	0.1333
	1.0297	85.0112 (0.0260)	2.7501	85.9635	0.1811	85.0902	0.1491
	0.8957	52.0735 (0.1347)	2.3017	54.4086	0.2307	53.6663	0.2136

Table 4: Comparison of MC estimates, approximating NA-SVJD prices and SVJD prices.

When comparing the CPU-times necessary to calculate the Monte Carlo estimates and the other prices, the difference is striking: in order to obtain a Monte Carlo estimate with a reasonable standard error, a CPU-time is necessary which is 100,000 larger than the CPU-times for the other prices!

From Table 4, one observes that the approximating European call option prices in the NA-SVJD model are fairly precise in the ITM and ATM cases, whereas for the OTM cases a relative error of about 5% is possible. However, in all cases the prices obtained in the affine SVJD model are closer to the Monte-Carlo values than the approximating prices of the NA-SVJD model. Therefore, we conclude from Table 4 that for this dataset of Aït-Sahalia, Wang and Yared (2001), it might be better to use the affine SVJD model for option pricing than using the approximating prices obtained in the NA-SVJD model.

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Appendix A

The price of a European option $C(S(t), V(t), t)$ depends on the current value of the underlying asset, the current volatility and the time. When differentiating $C(S(t), V(t), t)$ with respect to $L(t) = \ln(S(t))$ and the residual time $\tau = T - t$, the value of the European call option $C(L(t), V(t), t)$ must satisfy the following partial differential equation (PDE):

$$\frac{1}{2}VC_{LL} + [r - \lambda\mu_J - \frac{1}{2}V]C_L + \rho\sigma_v V^{\frac{\gamma+1}{2}} C_{LV} + \frac{1}{2}\sigma_v^2 V^\gamma C_{VV} + k(\theta - V)C_V$$

$$-C_\tau - rC + \lambda E_t(C(L + \ln(1 + J), V, t) - C(L, V, t)) = 0 \qquad \text{(A: 1)}$$

The option pricing formula given by (5) has a structure like the Black & Scholes formula. Bakshi, Cao and Chen (1997) prove that the probabilities π_1 and π_2 must also satisfy the some corresponding PDE's. Consequently we obtain two new PDE's which are respectively given by:

$$\frac{1}{2}V\pi_{1LL} + [r - \lambda\mu_J + \frac{1}{2}V]\pi_{1L} + \rho\sigma_v V^{\frac{\gamma+1}{2}}\pi_{1LV} + \frac{1}{2}\sigma_v^2 V^\gamma \pi_{1VV} + [k(\theta - V) + \rho\sigma_v V^{\frac{\gamma+1}{2}}]\pi_{1V}$$

$$-\pi_{1\tau} - \lambda\mu_J\pi_1 + \lambda E_t((1 + \ln(1 + J))\pi_1(L + \ln(1 + J), V, t) - \pi_1(L, V, t)) = 0 \qquad \text{(A: 2)}$$

subject to the boundary condition at the expiration time T :

$$\pi_1(L, V, T) = 1_{L(T)\geq \ln K}$$

and

$$\frac{1}{2}V\pi_{2LL} + [r - \lambda\mu_J - \frac{1}{2}V]\pi_{2L} + \rho\sigma_v V^{\frac{\gamma+1}{2}}\pi_{2LV} + \frac{1}{2}\sigma_v^2 V^\gamma \pi_{2VV} + k(\theta - V)\pi_{2V}$$

$$-\pi_{2\tau} + \lambda E_t(\pi_2(L + \ln(1+J), V, t) - \pi_2(L, V, t)) = 0 \qquad \text{(A: 3)}$$

subject to the boundary condition at the expiration time T :

$$\pi_2(L, V, T) = 1_{L(T) \geq \ln K}$$

Equation (A: 2) and (A: 3) are non-linear PDE's. In order to have linear PDE's, we use the expansion of Chacko and Viceira (1999) of the power of V to get:

$$V^{\frac{\gamma+1}{2}} \approx \theta^{\frac{\gamma+1}{2}} \frac{1-\gamma}{2} + \frac{\gamma+1}{2} \theta^{\frac{\gamma-1}{2}} V \qquad \text{(A: 4)}$$

$$V^{\gamma} \approx \theta^{\gamma}(1-\gamma) + \gamma \theta^{\gamma-1} V \qquad \text{(A: 5)}$$

The simple transformation shows that

$$\frac{1}{2} V \pi_{1LL} + [r - \lambda \mu_J + \frac{1}{2} V] \pi_{1L} + \rho \sigma_v [\theta^{\frac{\gamma+1}{2}} \frac{1-\gamma}{2} + \frac{\gamma+1}{2} \theta^{\frac{\gamma-1}{2}} V] \pi_{1LV}$$

$$+ \frac{1}{2} \sigma_v^2 [\theta^{\gamma}(1-\gamma) + \gamma \theta^{\gamma-1} V] \pi_{1VV} + [k(\theta - V) + \rho \sigma_v [\theta^{\frac{\gamma+1}{2}} \frac{1-\gamma}{2} + \frac{\gamma+1}{2} \theta^{\frac{\gamma-1}{2}} V]] \pi_{1V}$$

$$-\pi_{1\tau} - \lambda \mu_J \pi_1 + \lambda E_t((1+J) \pi_1(L + \ln(1+J), V, t) - \pi_1(L, V, t)) = 0 \qquad \text{(A: 6)}$$

and

$$\frac{1}{2} V \pi_{2LL} + [r - \lambda \mu_J - \frac{1}{2} V] \pi_{2L} + \rho \sigma_v [\theta^{\frac{\gamma+1}{2}} \frac{1-\gamma}{2} + \frac{\gamma+1}{2} \theta^{\frac{\gamma-1}{2}} V] \pi_{2LV}$$

$$+\frac{1}{2}\sigma_v^2[\theta^\gamma(1-\gamma)+\gamma\theta^{\gamma-1}V]\pi_{2VV}+k(\theta-V)\pi_{2V}-\pi_{2\tau}$$

$$+\lambda E_t(\pi_2(L+\ln(1+J),V,t)-\pi_2(L,V,t))=0 \qquad \text{(A: 7)}$$

Following Heston (1993), Bates (1996), Bakshi et al. (1997) and Scott (1997), the different moments generated by the probability π_j ($j=1,2$) also satisfy the same PDE's, in particular the corresponding characteristic functions for π_j denoted by $f_j(L,V,t,\phi)$ satisfy the above PDE's with the boundary conditions:

$$f_j(L,V,t,\phi)=e^{i\phi L_0} \qquad j=1,2 \qquad \text{(A: 8)}$$

The PDE's for the characteristics functions have an exact solution given by :

$$f_1(L,V,t,\phi)=\exp(u(\tau)+x(\tau)V(t)+i\phi L(t)) \qquad \text{(A: 9)}$$

with the boundary conditions:

$$u(0)=x(0)=0$$

and

$$f_2(L,V,t,\phi)=\exp(z(\tau)+y(\tau)V(t)+i\phi L(t)+r\tau) \qquad \text{(A: 10)}$$

with the terminal conditions:

$$z(0)=y(0)=0$$

By substituting (A: 9) for $f_1(L,V,t,\phi)$ in (A: 2) and separating the terms we obtain the first term $x(\tau)$:

$$x(\tau)=\frac{i\phi(1+i\phi)(1-e^{\Delta_x\tau})}{2\Delta_x-(\Delta_x+\eta_x)(1-e^{\Delta_x\tau})} \qquad \text{(A: 11)}$$

where $\Delta_x=\sqrt{\eta_x^2-\sigma_v^2 i\phi(1+i\phi)\gamma\theta^{\gamma-1}}$ and $\eta_x=\rho\sigma_v\frac{\gamma+1}{2}\theta^{\frac{\gamma-1}{2}}(i\phi+1)-\kappa$.

When substituting (A: 10) for $f_2(L,V,t,\phi)$ in (A: 3) and separating the terms we obtain an explicit form for $y(\tau)$:

$$y(\tau) = \frac{-i\phi(1-i\phi)(1-e^{\Delta_y\tau})}{2\Delta_y - (\Delta_y+\eta_y)(1-e^{\Delta_y\tau})} \qquad \text{(A: 12)}$$

where $\Delta_y = \sqrt{\eta_y^2 - \sigma_v^2 i\phi(i\phi-1)\gamma\theta^{\gamma-1}}$ and $\eta_y = \rho\sigma_v\frac{\gamma+1}{2}\theta^{\frac{\gamma-1}{2}}i\phi - \kappa$

The other term $u(\tau)$ in the characteristic function $f_1(L,V,t,\phi)$ is obtained by an integration of $x(\tau)$ and $x^2(\tau)$ as follows:

$$\begin{aligned}
u(\tau) &= [ri\phi - \lambda\mu_J(1+i\phi)]\tau \\
&-\frac{1}{\gamma\theta^{\gamma-1}\sigma_v^2}[\kappa\theta + \rho\sigma_v\frac{1-\gamma}{2}\theta^{\frac{\gamma-1}{2}}(i\phi+1)][2\ln(1-\frac{(\Delta_x+\eta_x)(1-e^{-\Delta_x\tau})}{2\Delta_x} + (\Delta_x+\eta_x)\tau] \\
&+\frac{\sigma_v^2(1-\gamma)\theta^{\gamma}}{2(\gamma\theta^{\gamma-1}\sigma_v^2)^2}[4\eta_x\ln(1-\frac{(\Delta_x+\eta_x)(1-e^{-\Delta_x\tau})}{2\Delta_x} \\
&+(\Delta_x+\eta_x)^2\tau - 2(\Delta_x+\eta_x)] \\
&+\lambda\tau(1+\mu_J)[(1+\mu_J)^{i\phi}e^{\sigma_v^2 i\phi(1+i\phi)} - 1] \qquad \text{(A:13)}
\end{aligned}$$

The other term $z(\tau)$ in the characteristic function $f_2(L,V,t,\phi)$ is obtained by an integration of $y(\tau)$ and $y^2(\tau)$ as follows:

$$\begin{aligned}
z(\tau) &= (r-\lambda\mu_J)\tau i\phi \\
&-\frac{1}{\gamma\theta^{\gamma-1}\sigma_v^2}[\kappa\theta + \rho\sigma_v\frac{1-\gamma}{2}\theta^{\frac{\gamma+1}{2}}i\phi][2\ln(1-\frac{(\Delta_y+\eta_y)(1-e^{-\Delta_y\tau})}{2\Delta_y} + (\Delta_y+\eta_y)\tau] \\
&-\frac{\sigma_v^2(1-\gamma)\theta^{\gamma}}{2(\gamma\theta^{1-\gamma}\sigma_v^2)^2}[4\eta_y\ln(1-\frac{(\Delta_y+\eta_y)(1-e^{-\Delta_y\tau})}{2\Delta_y} \\
&+(\Delta_y+\eta_y)^2\tau - 2(\Delta_y+\eta_y)] \\
&-\lambda\tau[(1+\mu_J)^{i\phi}e^{\sigma_v^2 i\phi(1+i\phi)/2} - 1] \qquad \text{(A:14)}
\end{aligned}$$

When studying the PDE's (A: 6) and (A: 7), it is easy to see that π_1 and π_2 are in fact the probabilities corresponding to the system described by the SDE:

$$\begin{aligned}
\frac{dS_t}{S_t} &= (r-\lambda\mu_J)dt + \sqrt{V_t}dZ_s(t) + J_t dq_t \\
dV_t &= k(\theta - V_t)dt + \sigma_v\sqrt{\theta^{\gamma}(1-\gamma) + \gamma\theta^{\gamma-1}V_t}dZ_v(t)
\end{aligned}$$

with

$$\ln(1+J(t)) \sim N(\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2)$$

and

$$Q(dq(t) = 1) = \lambda dt \quad \text{and} \quad cov(dZ_s(t), dZ_v(t)) = \widetilde{\rho}_t dt$$

with

$$\widetilde{\rho}_t = \frac{\rho(\theta^{\frac{\gamma+1}{2}}(\frac{1-\gamma}{2}) + \frac{\gamma+1}{2}\theta^{\frac{\gamma-1}{2}}V_t)}{\sqrt{V_t}\sqrt{\theta^\gamma(1-\gamma) + \gamma\theta^{\gamma-1}V_t}},$$

if at least $\theta^\gamma(1-\gamma)+\gamma\theta^{\gamma-1}V_t > 0$ and $\left|\rho(\theta^{\frac{\gamma+1}{2}}(\frac{1-\gamma}{2}) + \frac{\gamma+1}{2}\theta^{\frac{\gamma-1}{2}}V_t)\right| \leq \sqrt{V_t}\sqrt{\theta^\gamma(1-\gamma) + \gamma\theta^{\gamma-1}V_t}.$