

# OPTIMAL INVESTMENT FOR AN INSURER TO MINIMIZE ITS PROBABILITY OF RUIN

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## ABSTRACT

This paper studies optimal investment strategies of an insurance company. We assume that the insurance company receives premiums at a constant rate, the total claims are modeled by a compound Poisson process, and the insurance company can invest in the money market and in a risky asset such as stocks. This model generalizes the model in Hipp and Plum (2000) by including a risk-free asset. The investment behavior is investigated numerically for various claim-size distributions. The optimal policy and the solution of the associated Hamilton-Jacobi-Bellman equation are then computed under each assumed distribution. Our results provide insights for managers of insurance companies on how to invest. We also investigate the effects of changes in various factors, such as stock volatility, on optimal investment strategies, and survival probability. The model is generalized to cases in which borrowing constraints or reinsurance are present.

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## 1. INTRODUCTION

The optimal portfolio selection problem is of practical importance. Earlier work in this area can be traced back to Markowitz's mean-variance model (Markowitz 1959). Samuelson (1969) extends the work of Markowitz to a dynamic setup. By using a dynamic stochastic programming approach, he succeeds in obtaining the optimal decision for a consumption investment model. Merton (1971) uses the stochastic optimal control method in continuous finance to obtain a closed-form solution to the problem of optimal portfolio strategy under specific assumptions about asset returns and investor preferences. For recent developments on this subject, we refer readers to Campbell and Viceira (2001) and Korn (1997). In a paper in this journal, Gerber and Shiu (2000) consider the problem of optimal capital growth and dynamic asset allocation. In the case with only one risky asset and one risk-free asset, they show that the Merton ratio must be the risk-neutral Esscher parameter divided by the elasticity, with respect to current wealth, of the expected marginal utility of optimal terminal wealth. In the case where there is more than one risky asset, they prove the "two funds theorem" ("mutual fund" theorem).

Nowadays insurance companies invest not only in the money market, but also in stocks. Due to high risk in the stock market, investment strategies and risk management are becoming more and more important. The asset allocation problem for an insurance portfolio is different from that considered by Markowitz and others, since an insurer needs to pay claims. This paper considers the classical insurance risk model, but we also assume that the insurance companies invest in the money market and stocks. By providing some numerical results, we hope that the paper will provide insights to the practitioners.

In the actuarial science literature, an insurance company acts to maximize (or minimize) a certain objective function under different constraints (Borch 1967, 1969; Bühlmann 1970; Gerber 1972). Asmussen and Taksar (1997) reexamine this problem; they use results from the theory of controlled insurance diffusions and assume that the surplus process follows a Brownian motion with drift, where the control variable is the dividends. They are able to obtain closed-form solutions. Browne (1995)

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considers a model in which the aggregate claims are modeled by a Brownian motion with drift, and the risky asset is modeled by a geometric Brownian motion (see also Browne 1997, 1999). The compound Poisson model is the most popular model in risk theory; Hipp and Taksar (2000) use it in modeling the insurance business and consider the problem of optimal choice of new business to minimize the infinite time ruin probability. Hipp and Plum (2000) use the Cramér-Lundberg model to model the risk process of an insurance company and assume that the surplus of the insurance company can be invested in a risky asset (market index) that follows a geometric Brownian motion. When the claim size follows an exponential distribution, they are able to obtain explicit solutions. However, they do not incorporate a risk-free asset in their model. Taksar (2000) presents a short survey of stochastic models of risk control and dividend optimization techniques for a financial corporation. The objective in the models is to maximize the dividend payouts. Taksar shows that in most cases the optimal dividend distribution scheme is a barrier type, and the risk control policy depends substantially on the nature of reinsurance available. Recently many papers have been published on this topic: see, for example, Gerber and Shiu (2004), Højgaard and Taksar (1998a, 1998b, 2000), and Taksar and Markussen (2003).

This paper studies the portfolio selection problem for an insurance firm and aims to find an optimal investment strategy for the firm: that is, it decides how the firm should invest between a risk-free bond and a risky stock subject to its obligation to pay the policyholders when claims occur. Controlled diffusion models of stochastic control theory are used. The criterion for portfolio selection is the maximization of the survival probability (or minimizing the ruin probability) of the firm. The survival probability of a firm is defined as the probability that the surplus will not fall below zero at any time in the future for a given initial surplus. This definition implies that the portfolio selection problem is an infinite-horizon control problem. A continuous-time model is used to facilitate the mathematical treatment as the method of dynamic programming can be applied more efficiently. This study offers theoretical support to the optimal portfolio that should be held by the insurance companies and may serve as a guide for their real-world investment decisions. The model of this paper is an extension of the model in Hipp and Plum (2000) by including a risk-free asset. They obtain closed-form solutions for their model, but we are unable to obtain any closed-form solution. We rely on numerical methods for providing insights. After performing helpful transformations of our integro-differential equations, we are able to obtain numerical results.

The approach of this paper follows the standard line of control theory. The next section describes the model setup and states the Hamilton-Jacobi-Bellman (HJB) equation. We also state the existence and uniqueness of the solution to the Hamilton-Jacobi-Bellman equation and the verification theorem. Section 3 provides numerical results by assuming popular claim-size distributions. In Section 4 we study the effects of different factors, such as stock volatility, on the optimal investment strategies, and Section 5 extends the model to more general cases.

## 2. MODEL OF SURPLUS PROCESS AND THE HJB EQUATION

In this section the model of surplus process is specified, and the HJB equation is presented.

### 2.1 The Model

We assume that the standard assumptions of continuous-time financial models hold, that is, (1) continuous trading is allowed, (2) no transaction cost or tax is involved in trading, and (3) assets are infinitely divisible. For simplicity, we assume that there are only two assets in the financial market: a risk-free bond and a risky asset, namely stocks, in this section. The risk-free interest rate is assumed to be non-negative in the model. The price of the risk-free bond follows

$$dB(t) = r_0 B(t) dt,$$

where  $B(t)$  is the price of the risk-free bond at time  $t$ , and  $r_0$  is the risk-free interest rate, which is assumed to be constant ( $r_0 \geq 0$ ). The price of the stock follows

$$dP(t) = \mu P(t)dt + \sigma P(t)dW_t,$$

where  $P(t)$  is the price of the stock at time  $t$ ,  $\mu$  is the expected instantaneous rate of return of the stock ( $\mu \geq 0$ ),  $\sigma$  is the volatility of the stock price ( $\sigma > 0$ ), and  $\{W_t : t \geq 0\}$  is a standard Brownian motion defined on the complete probability space  $(\Omega, \mathcal{F}, P)$ , while  $\{\mathcal{F}_t\}$  is the  $P$ -augmentation of the natural filtration  $\mathcal{F}_t^W :=$  the sigma field generated by  $\{W_s; 0 \leq s \leq t\}$ .

Next, we assume that the risk process follows the Cramér-Lundberg model:

$$dR(t) = cdt - dS(t), \quad R(0) = s.$$

Here  $c$  is the constant premium rate, and  $S(t) = \sum_{i=1}^{N_t} Y_i$ , where  $N_t$  is the number of claims, which follows a Poisson process with intensity  $\lambda$ , and  $Y_i$ 's are individual claim sizes. We assume that  $\{Y_i\}$  is an i.i.d. sequence. As in the risk theory literature,  $c$  is set to equal  $(1 + \theta)E[S(1)]$ , where  $\theta(>0)$  is the relative security loading.

Now we can specify the surplus process  $X(t)$  as follows: let  $A(t)$  be the total amount of money invested in the stock market at time  $t$  by the insurer. We assume that  $A(t)$  is locally bounded and belongs to the set of admissible policies, that is,  $\{A(t), t \geq 0\}$  is a measurable and  $\mathcal{F}_t$ -adapted process, with

$$\int_0^T [A(t)]^2 dt < \infty$$

almost surely for every  $T < \infty$ . We can then specify the model of  $X(t)$  by

$$dX(t) = A(t) \frac{dP(t)}{P(t)} + (X(t) - A(t)) \frac{dB(t)}{B(t)} + dR(t),$$

or, more explicitly,

$$dX(t) = (\mu - r_0)A(t)dt + r_0X(t)dt + \sigma A(t)dW_t + cdt - dS(t), \quad X(0) = s. \quad (1)$$

## 2.2 The Hamilton-Jacobi-Bellman Equation

We use the survival probability as the objective function. This criterion is more objective than the expected discounted utility or other utility-related criteria because utility is unobservable and the specification of a utility function is somewhat arbitrary. First, we define the control policy as  $A$ , the amount of money invested in the stock market. It is the control variable to be adjusted so that the objective function is maximized.

Then we define the bankruptcy time or ruin time (with initial surplus  $s(\geq 0)$ ) to be the first time after time zero when the surplus becomes negative:

$$\tau_s := \inf\{t > 0 : X(t) < 0\}.$$

The survival probability with initial surplus  $s$  is the probability that no bankruptcy will ever occur:

$$\delta(s) := P\{\tau_s = \infty\}.$$

By using a standard method, we can obtain the following HJB equation. It is easy to check that the technical conditions that the equation has a unique solution are satisfied:

$$\max_A \{\lambda E[\delta(s - Y) - \delta(s)] + \delta'(s)[c + (\mu - r_0)A(s) + r_0s] + \frac{1}{2} \delta''(s) \sigma^2 A(s)^2\} = 0. \quad (2)$$

In the following we first assume that  $\delta(s)$  is strictly increasing. This is consistent with the smoothness assumption and the intuition that the more wealth there is, the higher the probability the insurer will survive. Another assumption is that  $\delta(s)$  is concave. However,  $\delta(s)$  is not assumed to have a second

derivative. To ensure smoothness and concavity, the claim density function must be locally bounded. The justification of these assumptions will be given later.

The HJB equation obtained in the previous section is a quadratic equation in  $A$ . The coefficient of the quadratic term is  $\frac{1}{2}\sigma^2\delta''(s)$ . Therefore, it should be that  $\delta''(s) < 0$ , which will also be proved later. (As the survival probability is bounded above by one, this is a reasonable assumption.)

Let  $A^*(s)$  be the optimal investment amount in stocks. For  $s > 0$ , the supremum is attained when

$$A^*(s) = -\frac{(\mu - r_0)\delta'(s)}{\sigma^2\delta''(s)}. \quad (3)$$

The term  $(\mu - r_0)/\sigma^2$  is often encountered in optimization problems in mathematical finance; see, for example, Merton (1971). It is called the price of risk, the return of the risky asset over the risk-free rate after adjustment for risk involved. Here risk is in the sense of the volatility of the risky asset.

### 2.3 Integro-Differential Equation

We assume that  $\mu > r_0$  (if  $\mu \leq r_0$ , the solution is trivial and is given by  $A^*(s) = 0$ ). Under the assumptions above,  $A^*(s)$  becomes positive for  $s > 0$ . Substitute  $A^*(s)$  in the HJB equation; after some algebra, we have, for  $s > 0$ ,

$$[(\mu - r_0)\delta'(s)]^2 = 2\sigma^2\delta''(s)\{(r_0s + c)\delta'(s) + \lambda E[\delta(s - Y) - \delta(s)]\}. \quad (4)$$

With the initial surplus  $s = 0$ , if we invest a positive amount in the risky asset, the ruin will occur immediately with probability one because of the fluctuation property of a Brownian motion. Therefore,  $A^*(0) = 0$ . Substituting this to the HJB equation yields

$$c\delta'(0) = \lambda\delta(0). \quad (5)$$

This is the initial condition for the HJB equation. Note that if  $\delta(s)$  is a solution, then  $\alpha\delta(s)$  is also a solution. Therefore, it is more convenient to set  $\delta'(0) = 1$  and adjust the multiple  $\alpha$  at the end to ensure  $\delta(\infty) = 1$ . The initial condition then becomes

$$c = \lambda\delta(0). \quad (6)$$

### 2.4 Verification Theorem

In this subsection we state the existence of solution for the HJB equation and its optimality. Because the proofs are similar to those in the literature, for example, Hipp (2000), Hipp and Plum (2000), and Schmidli (2002), we state the results without proofs.

The proof of Proposition 2.1 can be found in Hipp (2000). The proof of Proposition 2.2 is simple, and a similar result can be found in Hipp and Plum (2000).

#### Proposition 2.1

**(Existence of Solution)** Let the claim-size distribution have a locally bounded density. Then the HJB equation has a bounded twice continuously differentiable solution  $\delta \in C^2(0, \infty) \cap C^1[0, \infty)$ .

#### Proposition 2.2

**(Property of the Solution)** If  $\delta(s)$  is twice continuously differentiable and solves the HJB equation, then it is strictly increasing and strictly concave.

The next Proposition can be proved using an idea in Schmidli (2002).

#### Proposition 2.3

**(Verification Theorem)** If  $f(s) : \mathcal{R}_+ \rightarrow \mathcal{R}_+$  is a strictly increasing twice continuously differentiable function solving the HJB equation, then  $f(s)$  is bounded and  $\delta(s) = f(s)/f(\infty)$ .

### 3. NUMERICAL SOLUTION OF THE HJB EQUATION

This section presents the numerical solutions of the integro-differential equation under different assumptions on the claim-size distribution. First, the integro-differential equation is transformed into a form that is more readily solvable.

Let  $F(y)$  be the distribution function of the claim-size distribution. The integro-differential equation (4) can be written as

$$\lambda \int_0^{\infty} [\delta(s-y) - \delta(s)] dF(y) + (r_0s + c)\delta'(s) = \frac{1}{2} \frac{[\delta'(s)]^2}{\delta''(s)} \left( \frac{\mu - r_0}{\sigma} \right)^2.$$

Denoting  $(\mu - r_0/\sigma)^2$  by  $R$ ,  $(\lambda/R)$  by  $\bar{\lambda}$ ,  $(r_0/R)$  by  $\bar{r}_0$ , and  $(c/R)$  by  $\bar{c}$ , we have

$$\bar{\lambda} \int_0^{\infty} [\delta(s-y) - \delta(s)] dF(y) + (\bar{r}_0s + \bar{c})\delta'(s) = \frac{1}{2} \frac{[\delta'(s)]^2}{\delta''(s)}. \quad (7)$$

Let  $H(t) = 1 - F(t)$ , and by an integration by parts, we have

$$\int_0^{\infty} [\delta(s-y) - \delta(s)] dF(y) = \int_0^{\infty} H(y)d\delta(s-y) = -\delta(0)H(s) - \int_0^s \delta'(s-y)H(y) dy.$$

Therefore, the equation becomes

$$-\bar{\lambda}\delta(0)H(s) - \bar{\lambda} \int_0^s \delta'(s-y)H(y) dy + (\bar{r}_0s + \bar{c})\delta'(s) = \frac{1}{2} \frac{[\delta'(s)]^2}{\delta''(s)}.$$

From the initial condition,  $\bar{\lambda}\delta(0) = \bar{c}$ , and denoting  $u(s) = \delta'(s)$ , we have

$$-\bar{\lambda} \int_0^s u(s-y)H(y) dy + (\bar{r}_0s + \bar{c})u(s) - \bar{c}H(s) = \frac{1}{2} \frac{[u(s)]^2}{u'(s)}. \quad (8)$$

#### 3.1 Exponential Claim-Size Distribution

The following procedures are essentially the same as those proposed by Hipp and Plum (2000), except that an extra term  $r_0s$  is incorporated. A closed-form solution cannot be obtained.

Assume that  $f(y) = ke^{-ky}$ ; then  $F(y) = 1 - e^{-ky}$  and  $H(y) = e^{-ky}$ . Let  $v(y) = u(y)e^{ky}$ . Thus,

$$v'(y) = u'(y)e^{ky} + kv(y), \quad \frac{v'(y) - kv(y)}{v(y)} = \frac{u'(y)}{u(y)}.$$

Substituting them into equation (8), we have

$$-\bar{\lambda} \int_0^s v(y) dy + (\bar{r}_0s + \bar{c})v(s) - \bar{c} = \frac{1}{2} \frac{[v(s)]^2}{v'(s) - kv(s)}. \quad (9)$$

Let  $\omega(s) = [v(s)/(kv(s) - v'(s))]^2$ . Since  $u'(y) = \delta''(y) < 0$  and  $u(y) = \delta'(y) > 0$ , we have  $v'(y) = u'(y)e^{ky} + kv(y) < kv(y)$ . Thus,

$$\frac{1}{\sqrt{\omega(s)}} = \frac{kv(s) - v'(s)}{v(s)} = k - \frac{v'(s)}{v(s)}. \quad (10)$$

equation (9) becomes

$$-\bar{\lambda} \int_0^s v(y) dy + (\bar{r}_0 s + \bar{c})v(s) - \bar{c} = -\frac{1}{2}v(s)\sqrt{\omega(s)}.$$

Differentiate both sides with respect to  $s$  and divide both sides by  $v(s)$ , and we have

$$(-\bar{\lambda} + \bar{r}_0) + (\bar{r}_0 s + \bar{c}) \frac{v'(s)}{v(s)} = -\frac{1}{2} \frac{v'(s)}{v(s)} \sqrt{\omega(s)} - \frac{\omega'(s)}{4\sqrt{\omega(s)}}.$$

Using equation (10), we have

$$\omega'(s) = -2k\omega(s) - 4[(-\bar{\lambda} + \bar{r}_0) + k(\bar{r}_0 s + \bar{c}) - \frac{1}{2}]\sqrt{\omega(s)} + 4(\bar{r}_0 s + \bar{c}), \quad (11)$$

which is a nonlinear ordinary differential equation.

Now we provide the initial condition. Because

$$\sqrt{\omega(s)} = \frac{v(s)}{kv(s) - v'(s)} = -\frac{u(s)}{u'(s)} = -\frac{\delta'(s)}{\delta''(s)} = \frac{\sigma^2}{\mu - r_0} A^*(s),$$

we have

$$\omega(0) = 0. \quad (12)$$

We solve the ordinary differential equation by the finite-difference method. Let  $h$  be the length of the interval used in the numerical scheme. Denote  $\omega(nh)$  by  $\omega_n$ . Discretize the equation as

$$\omega_{n+1} = \omega_n + h\{-2k\omega_n - 4[(-\bar{\lambda} + \bar{r}_0) + k(\bar{r}_0 nh + \bar{c}) - \frac{1}{2}]\sqrt{\omega_n} + 4(\bar{r}_0 nh + \bar{c})\}. \quad (13)$$

By this recursive formula and the initial condition, the numerical solution of  $\omega(s)$  can be obtained.  $A^*(s)$  can then be found using

$$A^*(s) = \frac{\mu - r_0}{\sigma^2} \sqrt{\omega(s)}. \quad (14)$$

After finding  $A^*(s)$ ,  $\delta(s)$  can also be found as a by-product of the HJB equation.

The HJB equation in the exponential case is

$$\bar{\lambda} \int_0^s \delta(y)e^{k(y-s)} dy - \bar{\lambda}\delta(s) + (\bar{r}_0 s + \bar{c})\delta'(s) = -\frac{1}{2}\delta'(s)\sqrt{\omega(s)}.$$

Let  $\bar{g}(s) = \int_0^s \delta(y)e^{k(y-s)} dy$ , and we obtain

$$\bar{\lambda}\bar{g}(s) - \bar{\lambda}\delta(s) + (\bar{r}_0 s + \bar{c})\delta'(s) = -\frac{1}{2}\delta'(s)\sqrt{\omega(s)}, \quad \bar{g}'(s) = -k \int_0^s \delta(y)e^{k(y-s)} dy + \delta(s) = -k\bar{g}(s) + \delta(s), \quad (15)$$

and

$$\bar{g}(0) = 0. \quad (16)$$

Discretizing the systems of differential equations again and writing  $\delta_n = \delta(nh)$  and  $\bar{g}_n = \bar{g}(nh)$ , we have

$$\bar{\lambda}\bar{g}_{n+1} - \bar{\lambda}\delta_{n+1} + (\bar{r}_0(n+1)h + \bar{c}) \frac{[\delta_{n+1} - \delta_n]}{h} = -\frac{1}{2} \frac{[\delta_{n+1} - \delta_n]}{h} \sqrt{\omega_{n+1}},$$

$$\delta_{n+1} = \frac{[\bar{r}_0(n+1)h + \bar{c} + \frac{1}{2}\sqrt{\omega_n}]\delta_n - \bar{\lambda}h g_{n+1}}{\bar{r}_0(n+1)h + \bar{c} + \frac{1}{2}\sqrt{\omega_{n+1}} - \bar{\lambda}h}, \quad (17)$$

$$g_{n+1} = g_n + h(-k g_n + \delta_n). \quad (18)$$

From these equations and the initial conditions, the numerical value of  $\delta(s)$  can be computed. Finally, multiply this  $\delta(s)$  by an appropriate constant to ensure that  $\delta(\infty) = 1$ .

### Example 3.1

Let  $\mu = 0.1$ ,  $r_0 = 0.04$ ,  $\sigma = 0.3$ ,  $k = 1$ ,  $\lambda = 3$ ,  $\theta = 0.2$ , and  $h = 0.01$ . Then  $c = (1 + \theta)\lambda E(Y) = 1.2(3 \times 1) = 3.6$ . The result is shown in Figure 1. (Note that, in this paper, the x-axis denotes the surplus in all figures, and the y-axis denotes the amount invested in the stock in Figures 1–9, 16 and 19 and denotes the survival probability in Figures 10–15 and 17.)

It can be seen that when the surplus is small (smaller than three here) the insurer will invest more than its surplus in the stock market. This can be done by borrowing at the risk-free rate; that is, the insurer takes a leverage. (Our setup assumes that there is no difference between the borrowing and lending rates.) This means that the insurer is willing to achieve a higher return to prevent bankruptcy despite the higher risk involved. However, as the insurer's surplus increases, the proportion and even the amount invested in stocks in its portfolio decreases. The larger the surplus, the less immediate risk of bankruptcy that the insurer faces. It can withstand the impact of several claims without running into bankruptcy, meaning that it is more happy with a conservative strategy. Therefore, the insurer would rather choose to purchase risk-free bonds than to hold stocks to minimize losses because of the volatility of stocks.

The exponential distribution is light-tailed, which implies that the probability of an extraordinary event is very small. This is why it is safe to say that the insurer can withstand the risk from claims. To investigate the effect of claim-size distributions more comprehensively, fat-tailed distributions also need to be studied.

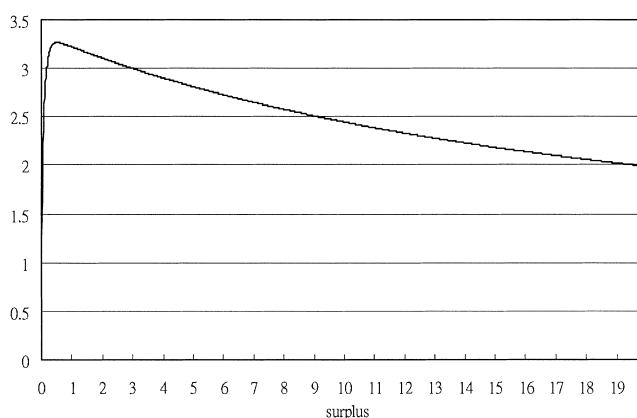
### 3.2 Gamma Claim-Size Distribution

The technique of transforming the integro-differential equation into an ordinary differential equation can be applied to the case when the claim-size distribution is a gamma distribution, as a gamma distribution also contains exponential factors.

Suppose that  $f(y) = [k^n y^{n-1}/(n-1)!]e^{-ky}$ . Then  $F(y) = \int_0^y [k^n t^{n-1}/(n-1)!]e^{-kt} dt = 1 - \sum_{i=1}^n [(ky)^{i-1}/(i-1)!]e^{-ky}$ . Therefore,  $H(y) = 1 - F(y) = \sum_{i=1}^n [(ky)^{i-1}/(i-1)!]e^{-ky}$ .

Figure 1

#### Optimal Investment Strategy for Exponential Distribution





Let  $S_n(t) = \sum_{i=1}^n [(t^{i-1}/(i-1)!)]$ . Then  $H(y) = S_n(ky)e^{-ky}$ . Note that  $S'_n(t) = \sum_{i=1}^{n-1} [t^{i-1}/i - 1!] = S_{n-1}(t)$ , so the HJB equation in the gamma case becomes

$$-\bar{\lambda} \int_0^s u(s-y)S_n(ky)e^{-ky} dy + (\bar{r}_0s + \bar{c})u(s) - \bar{c}S_n(s)e^{-ks} = \frac{1}{2} \frac{[u(s)]^2}{u'(s)}. \quad (19)$$

Again, let  $v(y) = u(y)e^{ky}$ . Transform the equation in the same way as in the exponential case, and we have

$$-\bar{\lambda} \int_0^s v(y)S_n(k(s-y)) dy + (\bar{r}_0s + \bar{c})v(s) - \bar{c}S_n(ks) = \frac{1}{2} \frac{[v(s)]^2}{v'(s) - kv(s)}. \quad (20)$$

When  $n = 2$ ,  $\omega(s) = [v(s)/(kv(s) - v'(s))]^2$ , the HJB equation becomes

$$-\bar{\lambda} \int_0^s v(y)[1 + k(s-y)] dy + (\bar{r}_0s + \bar{c})v(s) - \bar{c}(1 + ks) = -\frac{1}{2}v(s)\sqrt{\omega(s)}.$$

Similar to the exponential case, let  $z(s) = \int_0^s v(y) dy$  and  $m_1(s) = \bar{\lambda}z(s) + \bar{c}/z'(s)$ ; after some calculation we obtain

$$\omega'(s) = -2k\omega(s) - 4[-\bar{\lambda} + \bar{r}_0 - km_1(s) + (\bar{r}_0s + \bar{c})k - \frac{1}{2}]\sqrt{\omega(s)} + 4(\bar{r}_0s + \bar{c}). \quad (21)$$

Let  $m_{1_n} = m_1(nh)$ ; we can discretize the system of differential equations as

$$\begin{aligned} \omega_{n+1} &= \omega_n + h\{-2k\omega_n - 4[-\bar{\lambda} + \bar{r}_0 - km_{1_n} + (\bar{r}_0nh + \bar{c})k - \frac{1}{2}]\sqrt{\omega_n} + 4(\bar{r}_0nh + \bar{c})\}, \\ m_{1_{n+1}} &= m_{1_n} + h\left\{\bar{\lambda} - m_{1_{n+1}}\left[k - \frac{1}{\sqrt{\omega_{n+1}}}\right]\right\}, \\ \omega_0 &= 0, \quad m_{1_0} = c. \end{aligned} \quad (22)$$

Similar to the exponential case,

$$A(s) = \frac{\mu - r_0}{\sigma^2} \sqrt{\omega(s)}. \quad (23)$$

With  $g_2(s) = \int_0^s \delta(y)(1 + ky)e^{k(y-s)} dy$ , the HJB equation can be rewritten as

$$\bar{\lambda}g_2(s) - \bar{\lambda}\delta(s) + (\bar{r}_0s + \bar{c})\delta'(s) = -\frac{1}{2}\delta'(s)\sqrt{\omega(s)}.$$

Note that  $g_2'(s) = -kg_2(s) + \delta(s)(1 + ks)$  and  $g_2(0) = 0$ .

Let  $\delta_n = \delta(nh)$  and  $g_{2_n} = g_2(nh)$ ; the following discretized version of above differential equations can be used to obtain  $\delta(s)$ :

$$\begin{aligned} \bar{\lambda}g_{2_{n+1}} - \bar{\lambda}\delta_{n+1} + (\bar{r}_0(n+1)h + \bar{c}) \frac{[\delta_{n+1} - \delta_n]}{h} &= -\frac{1}{2} \frac{[\delta_{n+1} - \delta_n]}{h} \sqrt{\omega_{n+1}}, \\ \delta_{n+1} &= \frac{[\bar{r}_0(n+1)h + \bar{c} + \frac{1}{2}\sqrt{\omega_n}]\delta_n - \bar{\lambda}hg_{2_{n+1}}}{\bar{r}_0(n+1)h + \bar{c} + \frac{1}{2}\sqrt{\omega_{n+1}} - \bar{\lambda}h}, \end{aligned} \quad (24)$$

$$g_{2_{n+1}} = g_{2_n} + h(-kg_{2_n} + \delta_n(1 + knh)). \quad (25)$$



**Example 3.2**

For a gamma claim-size distribution, let  $\mu = 0.1$ ,  $r_0 = 0.04$ ,  $\sigma = 0.3$ ,  $n = 2$ ,  $k = 1$ ,  $\lambda = 3$ ,  $\theta = 0.2$ , and  $h = 0.01$ . Then  $c = (1 + \theta)\lambda E(Y) = 1.2(3 \times 2) = 7.2$ . The optimal investment strategy is shown in Figure 2.

**3.3 Pareto Claim-Size Distribution**

As shown by Embrechts, Klüppelberg, and Mikosch (1997), a heavy-tailed distribution should be used to model the claim distribution. In this subsection we discuss the Pareto distribution, which is often more appealing because it allows the occurrence of extreme events, an important feature of claims in the real world. Here the procedures employed in the case of a gamma distribution cannot be directly applied to transform the integro-differential equation into a system of ordinary differential equations, because the Pareto distribution function is not an exponential function. However, the procedures above do provide a way of stabilizing the integro-differential equation.

The probability density function of the Pareto distribution is  $f(x) = [\alpha\beta^\alpha/(x + \beta)^{\alpha+1}]$ , where  $\alpha > 0$  and  $\beta > 0$ , and the distribution function is  $F(x) = 1 - (\beta/(x + \beta))^\alpha$ . Hence,  $H(x) = (\beta/(x + \beta))^\alpha$ . Substitute them into equation (8), and we have

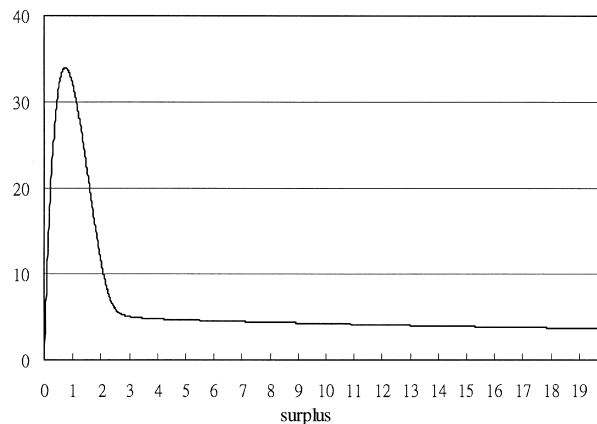
$$-\bar{\lambda} \int_0^s u(s-y) \left(\frac{\beta}{y+\beta}\right)^\alpha dy + (\bar{r}_0 s + \bar{c})u(s) - \bar{c} \left(\frac{\beta}{s+\beta}\right)^\alpha = \frac{1}{2} \frac{[u(s)]^2}{u'(s)}. \quad (26)$$

If we try to solve this equation directly by discretization, the algorithm obtained is not stable. Inspired by the procedures in the gamma case, we try to stabilize the equation by differentiating it with respect to  $s$ , which gives

$$\begin{aligned} \frac{1}{2} u(s) - \frac{1}{4} u(s) \frac{\omega'(s)}{\sqrt{\omega(s)}} = & -\bar{\lambda} \int_0^s u'(s-y) \left(\frac{\beta}{y+\beta}\right)^\alpha dy - \bar{\lambda} \left(\frac{\beta}{s+\beta}\right)^\alpha + (\bar{r}_0 s + \bar{c})u'(s) + \bar{r}_0 u(s) \\ & + \bar{c}\alpha \frac{\beta^\alpha}{(s+\beta)^{\alpha+1}}, \end{aligned}$$

where  $\omega(s)$  is the same as in the exponential and gamma cases. As in the gamma case, we divide both sides of the equation above by  $u'(s)$ . After some calculation, we obtain

Figure 2

**Optimal Investment Strategy for a Gamma Distribution**

$$\omega'(s) = 4 \left[ -\bar{\lambda} \int_0^s \frac{1}{\sqrt{\omega(s-y)}} \frac{u(s-y)}{u(s)} \left( \frac{\beta}{y+\beta} \right)^\alpha dy + \bar{\lambda} \left( \frac{\beta}{s+\beta} \right)^\alpha \frac{1}{u(s)} - \bar{c}\alpha \frac{\beta^\alpha}{(s+\beta)^{\alpha+1}} \frac{1}{u(s)} - \bar{r}_0 + \frac{1}{2} \right] \times \sqrt{\omega(s)} + 4(\bar{r}_0 s + \bar{c}). \quad (27)$$

Let

$$m(s) = \int_0^s \frac{1}{\sqrt{\omega(s-y)}} \frac{u(s-y)}{u(s)} \left( \frac{\beta}{y+\beta} \right)^\alpha dy. \quad (28)$$

Then

$$\omega'(s) = 4 \left[ -\bar{\lambda} m(s) + \bar{\lambda} \left( \frac{\beta}{s+\beta} \right)^\alpha \frac{1}{u(s)} - \bar{c}\alpha \frac{\beta^\alpha}{(s+\beta)^{\alpha+1}} \frac{1}{u(s)} - \bar{r}_0 + \frac{1}{2} \right] \sqrt{\omega(s)} + 4(\bar{r}_0 s + \bar{c}). \quad (29)$$

Together with  $\sqrt{\omega(s)} = -[u(s)/u'(s)]$ , we have a system of integro-differential equations for functions  $u(s)$ ,  $\omega(s)$ , and  $m(s)$ .

Now we discretize the system of equations and solve it numerically. Define  $\omega_n = \omega(nh)$ ,  $u_n = u(nh)$ ,  $m_n = m(nh)$ . Then

$$\omega_{n+1} = \omega_n + 4h \left[ -\bar{\lambda} m_n + \bar{\lambda} \left( \frac{\beta}{nh+\beta} \right)^\alpha \frac{1}{u_n} - \bar{c}\alpha \frac{\beta^\alpha}{(nh+\beta)^{\alpha+1}} \frac{1}{u_n} - \bar{r}_0 + \frac{1}{2} \right] \sqrt{\omega_n} + 4h(\bar{r}_0 nh + \bar{c}), \quad (30)$$

$$m_n = \int_0^{nh} \frac{1}{\sqrt{\omega(nh-y)}} \frac{u(nh-y)}{u_n} \left( \frac{\beta}{y+\beta} \right)^\alpha dy \approx \sum_{i=1}^n \frac{1}{\sqrt{\omega_{n-i}}} \frac{u_{n-i}}{u_n} \left( \frac{\beta}{ih+\beta} \right)^\alpha h, \quad (31)$$

$$\frac{1}{\sqrt{\omega_n}} = -\frac{u_{n+1} - u_n}{hu(s)}, \quad u_{n+1} = u_n \left[ 1 - \frac{h}{\sqrt{\omega_n}} \right]. \quad (32)$$

To find the optimal investment strategy, we use  $A_n = ((\mu - r_0)/\sigma^2)\sqrt{\omega_n}$ . To find the survival probability, we can use  $\delta(s) = \int_0^s u(y) dy$  or  $\delta_n \approx \sum_{i=1}^n u_i$ .

### Example 3.3

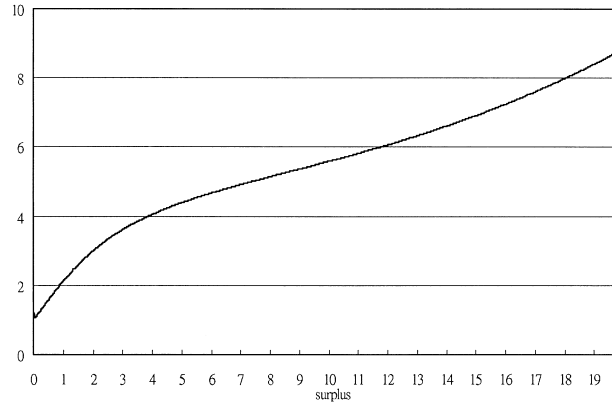
For a Pareto claim-size distribution, let  $\mu = 0.1$ ,  $r_0 = 0.04$ ,  $\sigma = 0.3$ ,  $\alpha = 3$ ,  $\beta = 2$ ,  $\lambda = 3$ ,  $\theta = 0.2$ , and  $h = 0.01$ . Then  $c = (1 + \theta)\lambda E(Y) = 1.2(3 \times [2/(3 - 1)]) = 3.6$ . The optimal investment strategy is shown in Figure 3.

Unlike the exponential and gamma cases, when the claim-size distribution is Pareto, the optimal investment strategy is to invest more in stocks when the initial surplus is large. Due to the heavy-tailed property of Pareto distribution, we cannot expect that the initial surplus will be enough to cover the claims, even when the initial surplus is relatively large.

## 4. EFFECTS OF UNDERLYING FACTORS

In this section we illustrate the effects of underlying factors, such as interest rate and stock price volatility, on optimal investment strategies and ruin probabilities. Although some of the properties can be proved analytically, we will focus on the numerical results here.

Figure 3  
**Optimal Investment Strategy for a Pareto Distribution with Support (0, ∞)**



#### 4.1 Effects of Underlying Factors on Optimal Investment Strategies

By substituting the solution of HJB equation into  $A^*(s)$ , we obtain

$$A^*(s) = \frac{2}{\mu - r_0} \left\{ \lambda \frac{\delta(s) - E[\delta(s - Y)]}{\delta'(s)} - r_0 s - c \right\}. \quad (33)$$

From this expression, it is clear that as  $r_0$  changes, all  $(2/(\mu - r_0))$ ,  $\delta(s) - E[\delta(s - Y)]$ ,  $\delta'(s)$ , and  $r_0 s$  will change. In particular, the relative ratio of changes in the terms  $\delta(s) - E[\delta(s - Y)]$  and  $\delta'(s)$  will depend on the claim-size distribution. This expression reveals that the effect of  $r_0$  on the optimal investment strategy is more complex than our intuition suggested. Changes in other parameters would also affect the degree of change in  $\delta(s) - E[\delta(s - Y)]$  and  $\delta'(s)$ . There may be two counterforces in determining the optimal investment strategy, and the actual effect of factors cannot be determined until we assume a claim-size distribution, plug in the actual values of parameters, and work out the result numerically. Here we use reasonable and realistic parameters and study the effects of each factor's change (in reasonable ranges) on the optimal investment strategy.

To illustrate, except for studying the effects of interest rate, we use a gamma distribution as the claim-size distribution when we study the influence of other factors. The effects for other claim-size distributions are similar. In the rest of this section we will give some numerical examples. The reference setup is given below. For each parameter, we consider its influence by varying it in a reasonable range, keeping other parameters constant. Note that these conclusions are only valid under the assumption that the parameters are in a reasonable range. Exaggerated parameters could give different results.

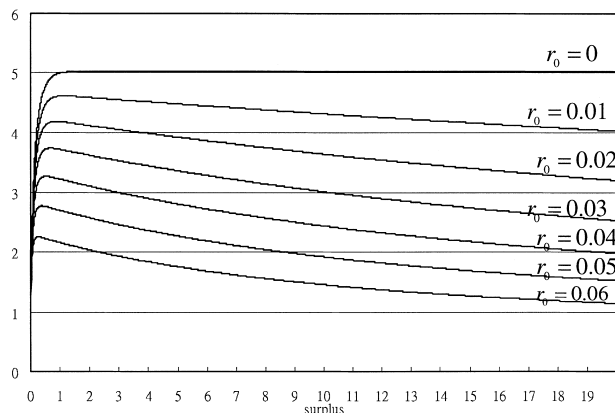
Reference setup:

- Exponential:  $\mu = 0.1, r_0 = 0.04, \sigma = 0.3, k = 1, \lambda = 3, \theta = 0.2, c = (1 + \theta)\lambda E(Y) = 1.2(3 \times 1) = 3.6$
- Gamma:  $\mu = 0.1, r_0 = 0.04, \sigma = 0.3, k = 1, \lambda = 3, \theta = 0.2, c = (1 + \theta)\lambda E(Y) = 1.2(3 \times 2) = 7.2$
- Pareto:  $\mu = 0.1, r_0 = 0.04, \sigma = 0.3, \alpha = 3, \beta = 2, \lambda = 3, \theta = 0.2, c = [1 + \theta)\lambda E(Y) = 1.2 \times 3 \times (2/3 - 1)] = 3.6$ .

##### (a) Risk-Free Interest Rate

We use exponential and Pareto distributions to illustrate the results: it is clear from Figures 4 and 5 that as the risk-free interest rate rises, investment in stocks decreases, and vice versa. This result seems intuitive. Keeping stock return and stock price volatility constant, a higher risk-free interest rate means that an insurer needs not invest as much in the stock market to survive because it can enjoy a high return on the riskless bond market. This can be viewed as a substitution effect. If the risk-free interest rate becomes lower, then an insurer will, in contrast, prefer to invest more in stocks.

Figure 4  
**Optimal Investment Strategy for an Exponential Distribution**



**(b) Stock Price Volatility**

Figure 6 shows that as stock price volatility increases, investment in stocks decreases, and vice versa. (Figure 6 is for a gamma distribution as mentioned before. For other distributions the results are similar.)

In a certain sense, volatility is a measure of the riskiness of a stock. If the volatility of the stock price increases but the expected rate of return of the stock stays the same, then the insurer will find the reward for accepting the risk unattractive and would rather invest less in stocks and invest more in bonds. Conversely, if the volatility of the stock price decreases, then the insurer will receive the same return but with a lower risk and will find it in its interest to invest in the stock.

**(c) Security Loading**

Figure 7 indicates that as security loading increases, investment in stocks decreases, and vice versa. Increasing the security loading means higher stable income from premiums, and so the insurer does not have to take the risk to invest a lot in stocks.

**(d) Expected Claim Frequency**

Figure 8 shows that as  $\lambda$  increases, investment in stocks increases, and vice versa. If  $\lambda$  decreases, then the expected claim frequency increases. This will not only increase the average shock from

Figure 5  
**Effect of  $r_0$  on the Optimal Investment Strategy for a Pareto Distribution**

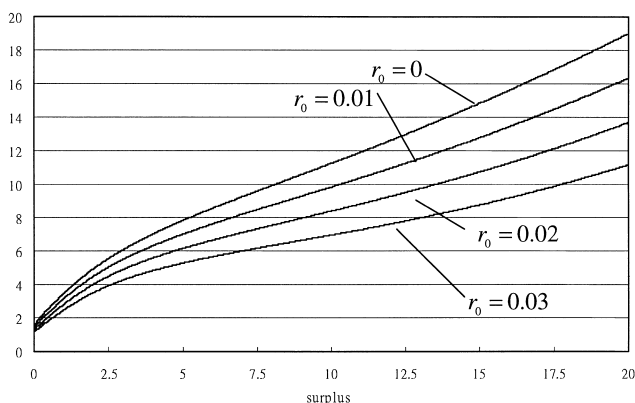
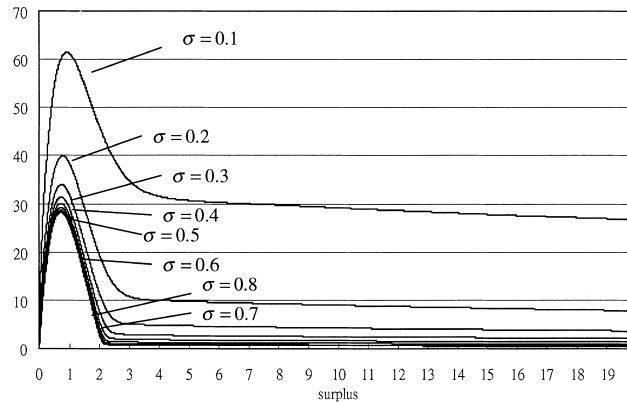


Figure 6  
Effect of  $\sigma$  on the Optimal Investment Strategy



claims, but also raise the premium rate. However, the effect of increase in the expected claim seems to be greater. Because bankruptcy can occur due to an above-average claim, a premium rate that reflects only an average claim is not sufficient to cover the additional bankruptcy risk. Therefore, the insurer must invest more in stocks to withstand the increase of the bankruptcy threat.

#### (e) Parameter of Gamma Distribution

As the parameter  $k$  of the gamma distribution decreases (i.e., expected claim size increases), investment in stocks increases, and vice versa (see Figure 9). The reason is the same as that given in (d). If  $k$  decreases, then the expected claim size increases. The difference between (d) and here is only the sensitivity of the investment in stocks to the changes of these two parameters.

#### REMARK

If we change our premium formula, then the result may be different. For example, if the premium rate is constant and independent of the claim-size distribution, then there is no so-called security loading.

In this case, as premium income increases, investment in stocks decreases, and vice versa. The reason is simple: higher-premium income means that an insurer needs not rely so much on the risky stock market to survive because of its premium income. If premium income decreases, it will rely more on the profit from

Figure 7  
Effect of  $\theta$  on the Optimal Investment Strategy

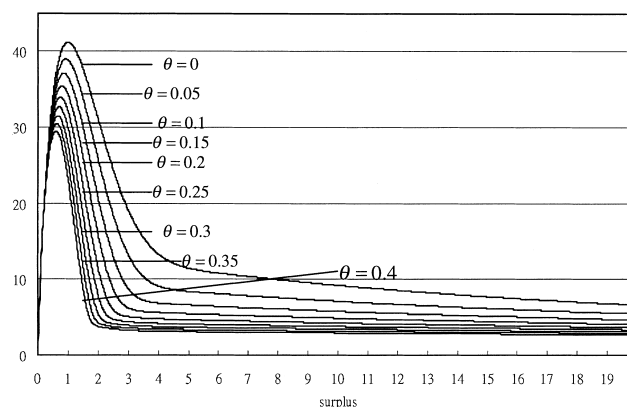
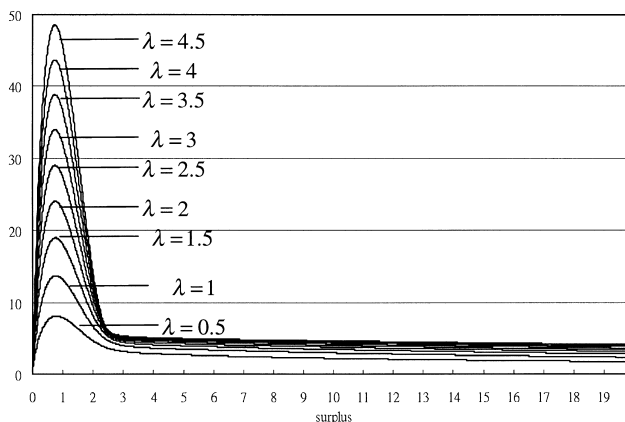


Figure 8  
**Effect of  $\lambda$  on Optimal Investment Strategy**



the stock market to prevent bankruptcy. This is similar to the case of increasing the risk-free interest rate, but there are two differences. First, premium income is fixed here, and return from risk-free interest rate is a multiple of surplus. Second, if premium income increases, then the total amount of surplus will increase at a fixed amount. In our model the objective is to increase the survival chance, instead of maximizing the expected return as in many economic models. In some cases this objective function makes the insurer regard stock as an inferior good; this may look quite ridiculous. However, if the objective function can be changed after attaining a sufficiently high survival probability, then the result will change.

**4.2 Effects of Underlying Factors on the Survival Probability**

The corresponding changes in the survival probability that are caused by the changes in parameters can also be computed. The result of this section is more intuitive than that of the previous section. Here only results for  $\text{gamma}(2, k)$  distribution will be shown, but the result is valid for the other distributions.

**(a) As the Risk-Free Interest Rate Increases, the Survival Probability Increases, and Vice Versa**

Figure 10 indicates that as the risk-free interest rate increases, the survival probability increases. This is because the return is higher if the interest rate is higher, thus the survival probability is higher.

Figure 9  
**Effect of  $k$  (of Gamma(2,  $k$ ) Distribution) on Optimal Investment Strategy**

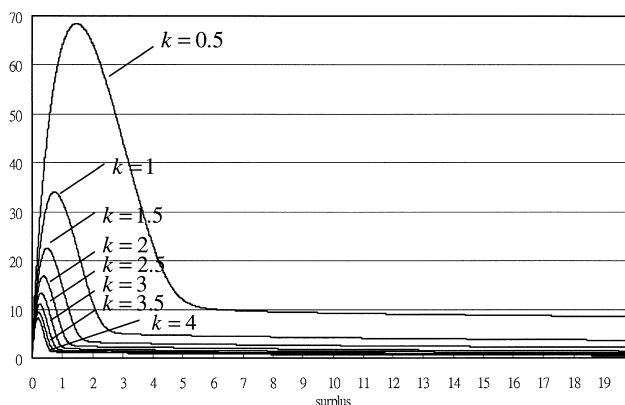
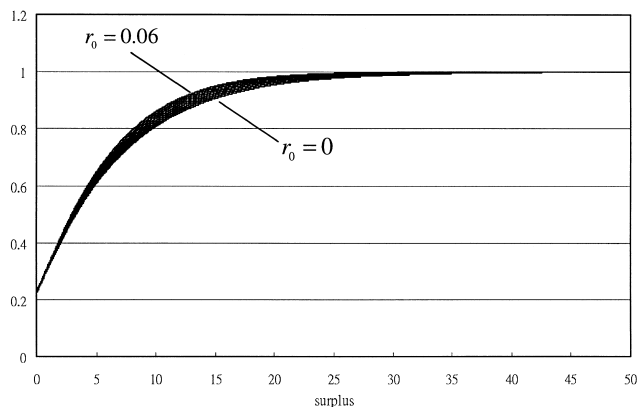


Figure 10  
Effect of  $r_0$  on the Survival Probability



**(b) As the Expected Rate of Return of Stocks Increases, the Survival Probability Increases, and Vice Versa**

See Figure 11. The reason is similar to (a).

**(c) As the Volatility of the Stock Price Increases, the Survival Probability Decreases, and Vice Versa**

Because all other factors remain the same, the risk from stocks increases, and the survival probability decreases (see Figure 12).

**(d) As the Security Loading Increases, the Survival Probability Increases, and Vice Versa**

When the security loading increases, the company receives more premiums. Because all other factors are the same, the survival probability will obviously be higher (see Figure 13).

**(e) As  $\lambda$  Increases, the Survival Probability Decreases, and Vice Versa**

The reason is the same as that given in (d) of the previous subsection (see Figure 14).

**(f) As  $k$  Decreases (i.e., Expected Claim Size Increases), the Survival Probability Decreases, and Vice Versa**

The reason is the same as that given in (e) of the previous subsection. Note that the survival probability is the same at  $s = 0$  for any level of  $k$ . This is consistent with the result from the classical model (see Figure 15).

Figure 11  
Effect of  $\mu$  on the Survival Probability

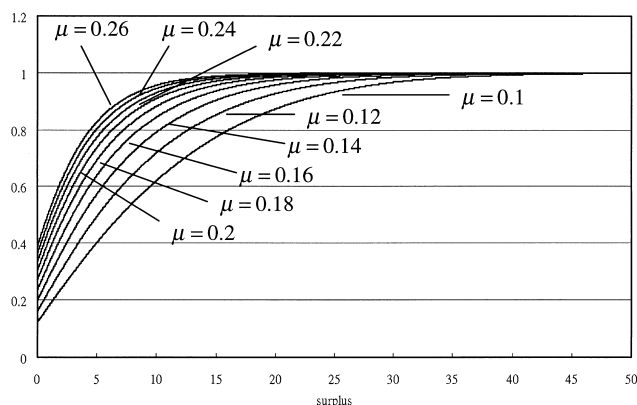




Figure 12  
**Effect of  $\sigma$  on the Survival Probability**

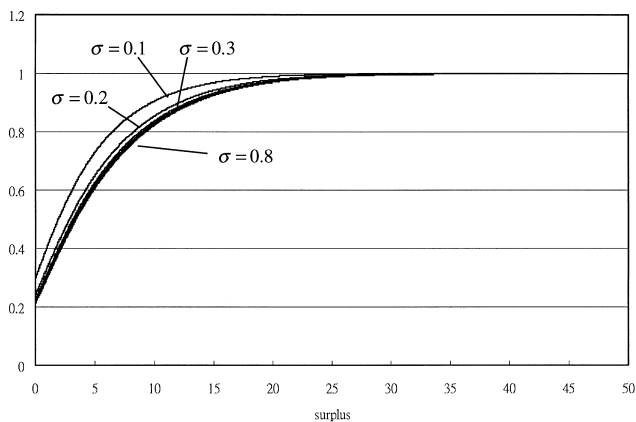


Figure 13  
**Effect of  $\theta$  on the Survival Probability**

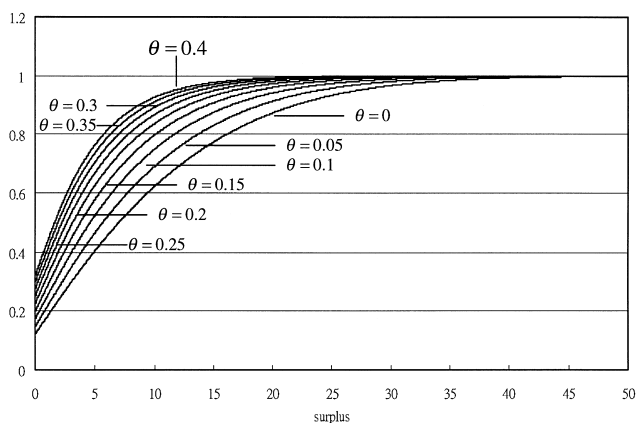


Figure 14  
**Effect of  $\lambda$  on the Survival Probability**

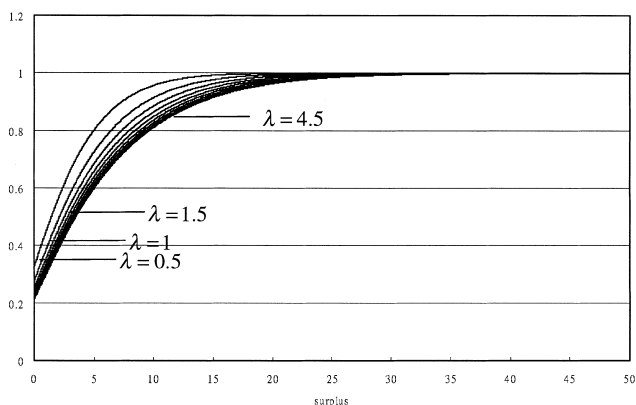
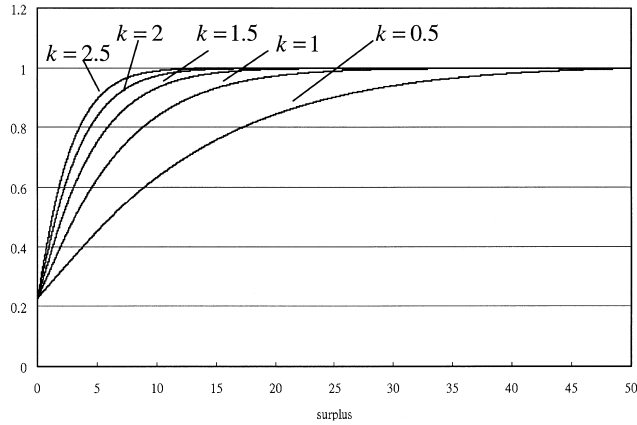


Figure 15  
Effect of  $k$  (of Gamma(2,  $k$ ) Distribution) on the Survival Probability



## 5. MODEL EXTENSIONS

### 5.1 Borrowing Constraint

In the previous sections we assumed there was no constraint on borrowing. However, in reality there may be. For simplicity, here we assume the insurer cannot borrow at all, that is,  $A(s) \leq s$ . Therefore, the HJB equation becomes

$$\max_{0 \leq A \leq s} \{\lambda E[\delta(s - Y) - \delta(s)] + \delta'(s)[c + (\mu - r_0)A + r_0s + \frac{1}{2}\delta''(s)\sigma^2A^2]\} = 0. \quad (34)$$

For  $s$ , such that

$$-\frac{(\mu - r_0)\delta'(s)}{\sigma^2\delta''(s)} > s,$$

$A^*(s)$  equals the upper bound  $s$ . Substituting  $A^*(s) = s$  into the equation above, we have

$$-\lambda \int_0^s u(s - y)H(y) dy - cH(s) + (\mu s + c)u(s) + \frac{1}{2}\sigma^2s^2u'(s) = 0. \quad (35)$$

The initial condition is  $\lambda\delta(0) = c$ .

For simplicity, we assume that the claim-size distribution is exponential with parameter 1, and  $v(x) = u(x)e^x$ ; then the HJB equation becomes

$$\frac{1}{2}\sigma^2s^2v'(s) + (\mu s + c - \frac{1}{2}\sigma^2s^2)v(s) - \lambda \int_0^s v(y) dy - c = 0. \quad (36)$$

Let  $z(s) = \int_0^s v(y) dy$ . Then  $z'(s) = v(s)$ ,  $z(0) = 0$ , and  $z'(0) = 1$ . The equation becomes

$$\frac{1}{2}\sigma^2s^2z''(s) + (\mu s + c - \frac{1}{2}\sigma^2s^2)z'(s) - \lambda z(s) - c = 0, \quad (37)$$

which can be readily solved by the finite-difference method. We use this equation to find the survival probability, until  $-[(\mu - r_0)\delta'(s^*)/\sigma^2\delta''(s^*)] = s^*$ . For  $s \geq s^*$ , we can use the HJB equation for the unconstrained case again, and its initial conditions can also be obtained by the principle of smooth fit.

**Example 5.1**

We provide the numerical solution of the optimal investment strategy and the survival probability for Exp (1) here. Let  $\mu = 1, r_0 = 0.1, \sigma = 1, \lambda = 1, \theta = 1,$  and  $h = 0.01.$  Then  $c = (1 + \theta)\lambda E(Y) = 2 \times 1 \times 1 = 2.$  The result is shown in Figure 16 and 17.

The survival probability with constraint is smaller than that without constraint because the optimal strategy cannot be adopted.

**5.2 Reinsurance**

It is a common for insurance companies to purchase reinsurance from other insurance companies to reduce the risk that they face. In this subsection we study the effect of reinsurance on the optimal investment strategy.

We consider the stop-loss reinsurance: in this case the insurance company needs to pay for the reinsurance. If the risk loadings for insurance and reinsurance companies are  $\theta_1$  and  $\theta_2,$  respectively, then the net rate of premium received by the insurance company is

$$c = \lambda[(1 + \theta_1)E(Y) - (1 + \theta_2)E(Y - Y_d)]. \tag{38}$$

After buying reinsurance, the net claim size becomes

$$Y_d = \begin{cases} d, & Y > d \\ Y, & Y \leq d. \end{cases} \tag{39}$$

For  $0 \leq s \leq d,$

$$E[\delta(s - Y_d) - \delta(s)] = -\delta(0)H_Y(s) - \int_0^s H_Y(y)\delta'(s - y) dy.$$

For  $s \geq d,$

$$E[\delta(s - Y_d) - \delta(s)] = -\int_0^d \delta'(s - y)H_Y(y) dy.$$

Figure 16

**The Optimal Strategy When Borrowing Constraint Exists**

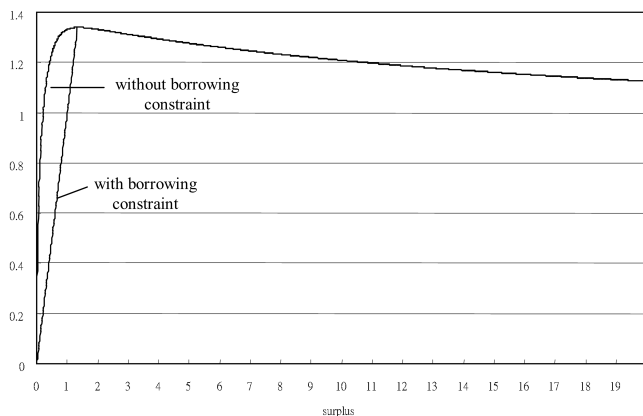
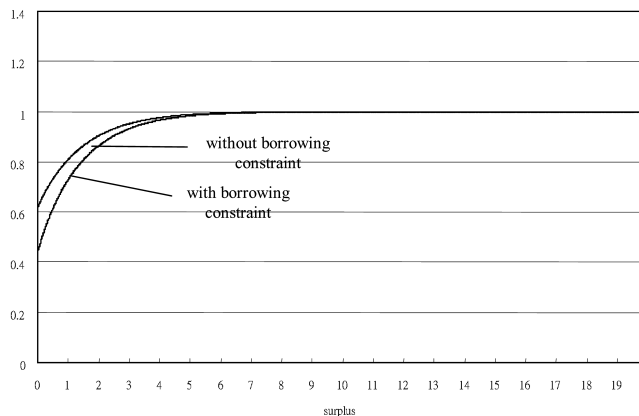


Figure 17

**The Survival Probability When Borrowing Constraint Exists**



The HJB equation becomes, for  $s < d$ ,

$$-\bar{\lambda} \int_0^s \delta'(s-y) H_Y(y) dy + (\bar{r}_0 s + \bar{c}) \delta'(s) - \bar{c} H_Y(s) = \frac{[\delta'(s)]^2}{2\delta''(s)},$$

and for  $s \geq d$ ,

$$-\bar{\lambda} \int_0^d \delta'(s-y) H_Y(y) dy + (\bar{r}_0 s + \bar{c}) \delta'(s) = \frac{[\delta'(s)]^2}{2\delta''(s)}.$$

As an illustration, we assume the claim-size distribution is exponentially distributed. For  $s < d$ , the HJB equation is the same as the one without reinsurance, that is,

$$\omega'(s) = -2k\omega(s) - 4[(-\bar{\lambda} + \bar{r}_0) + k(\bar{r}_0 s + \bar{c}) - \frac{1}{2}] \sqrt{\omega(s)} + 4(\bar{r}_0 s + \bar{c}), \quad (40)$$

and it can be readily solved numerically. For  $s \geq d$ , the HJB equation becomes

$$-\bar{\lambda} \int_0^d \vartheta(s-y) dy + (\bar{r}_0 s + \bar{c}) \vartheta(s) = -\frac{1}{2} \vartheta(s) \sqrt{\omega(s)}.$$

Differentiating both sides with respect to  $s$ , we have

$$\bar{\lambda} \vartheta(s-d) - \bar{\lambda} \vartheta(s) + (\bar{r}_0 s + \bar{c}) \vartheta'(s) + \bar{r}_0 \vartheta(s) = -\frac{1}{2} \left[ \vartheta'(s) \sqrt{\omega(s)} + \frac{\vartheta(s)}{2\sqrt{\omega(s)}} \omega'(s) \right].$$

Divide both sides by  $\vartheta(s)$ . After some calculation, we have

$$\omega'(s) = -2k\omega(s) - 4 \left[ \bar{\lambda} \frac{\vartheta(s-d)}{\vartheta(s)} - \bar{\lambda} + \bar{r}_0 + k(\bar{r}_0 s + \bar{c}) - \frac{1}{2} \right] \sqrt{\omega(s)} + 4(\bar{r}_0 s + \bar{c}). \quad (41)$$

Let  $\varepsilon(s) = \vartheta(s-d)/\vartheta(s)$ ; we have

$$\varepsilon'(s) = \varepsilon(s) \left[ \frac{1}{\sqrt{\omega(s)}} - \frac{1}{\sqrt{\omega(s-d)}} \right]. \quad (42)$$

By the principle of smooth fit,  $\delta(d-) = \delta(d)$ ,  $\delta'(d-) = \delta'(d)$ ,  $\delta''(d-) = \delta''(d)$ . Thus,

$$\omega(d-) = \omega(d), \quad \vartheta(d-) = \vartheta(d), \quad \varepsilon(d) = \frac{\vartheta(0)}{\vartheta(d)} = \frac{1}{\vartheta(d)}. \quad (43)$$

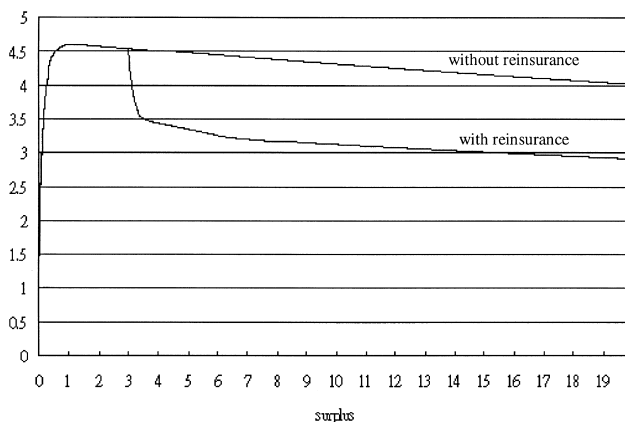
The numerical results can be obtained by discretizing this system of ordinary differential equations, that is, let ( $d = mh$ ):

$$\varepsilon_{n+1} = \frac{\varepsilon_n}{1 - h \left[ \frac{1}{\sqrt{\omega_{n+1}}} - \frac{1}{\sqrt{\omega_{n+1-m}}} \right]}, \quad (44)$$

$$\omega_{n+1} = \omega_n + h \{ -2k\omega(s) - 4[\bar{\lambda}\varepsilon_n - \bar{\lambda} + \bar{r}_0 + k(\bar{r}_0 nh + \bar{c}) - \frac{1}{2}] \sqrt{\omega_n} + 4(\bar{r}_0 nh + \bar{c}) \}, \quad (45)$$

$$\vartheta_{n+1} = \frac{\vartheta_n}{1 - h \left[ k - \frac{1}{\sqrt{\omega_{n+1}}} \right]}. \quad (46)$$

Figure 18  
**Optimal Strategy for Exponential Distribution with Reinsurance**



### Example 5.2

Suppose that the claim size is exponential with parameter 1; let  $\mu = 0.1$ ,  $r_0 = 0.01$ ,  $\sigma = 0.3$ ,  $\lambda = 3$ ,  $\theta = 0.2$ , and  $h = 0.01$ . Then  $c = (1 + \theta)\lambda E(Y) = 1.2 \times 3 \times 1 = 3.6$ . The level of reinsurance is  $d = 3$ . The result is shown in Figure 18.

Figure 18 indicates that the optimal investment drops rapidly once the surplus level exceeds the safety level, which is the level of reinsurance bought.

### ACKNOWLEDGMENTS

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### REFERENCES

- ASMUSSEN, SØREN, AND MICHAEL TAKSAR. 1997. "Controlled Diffusion Models for Optimal Dividend Payout," *Insurance: Mathematics and Economics* 20: 1–15.
- BORCH, KARL. 1967. "The Theory of Risk," *Journal of the Royal Statistical Society, Series B* 29: 432–52.
- . 1969. "The Capital Structure of a Firm," *Swedish Journal of Economics* 71: 1–13.
- BROWNE, SID. 1995. "Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin," *Mathematics of Operations Research* 20: 937–58.
- . 1997. "Survival and Growth with Liability: Optimal Portfolio Strategies in Continuous Time," *Mathematics of Operations Research* 22(2): 468–93.
- . 1999. "Beating a Moving Target: Optimal Portfolio Strategies for Outperforming a Stochastic Benchmark," *Finance and Stochastics* 3: 275–94.
- BÜHLMANN, HANS. 1970. *Mathematical Methods in Risk Theory*. Berlin: Springer.
- CAMPBELL, JOHN Y., AND LUIS M. VICEIRA. 2001. *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. New York: Oxford University Press.
- EMBRECHTS, PAUL, CLAUDIA KLÜPPELBERG, AND THOMAS MIKOSCH. 1997. *Modelling Extremal Events for Insurance and Finance*. New York: Springer.
- GERBER, HANS U. 1972. "Games of Economic Survival with Discrete and Continuous Income Processes," *Operations Research* 20: 37–45.
- GERBER, HANS U., AND ELIAS S. W. SHIU. 2000. "Investing for Retirement: Optimal Capital Growth and Dynamic Asset Allocation," *North American Actuarial Journal* 4(2): 42–62.
- . 2004. "Optimal Dividends: Analysis with Brownian Motion," *North American Actuarial Journal* 8(1): 1–20.
- HIPP, CHRISTIAN. 2000. "Optimal Investment for Investors with State Dependent Income, and for Insurers." Working paper No. 82, Centre for Actuarial Studies, University of Melbourne.
- HIPP, CHRISTIAN, AND MICHAEL PLUM. 2000. "Optimal Investment for Insurers," *Insurance: Mathematics and Economics* 27: 215–28.

- HIPP, CHRISTIAN, AND MICHAEL TAKSAR. 2000. "Stochastic Control for Optimal New Business," *Insurance: Mathematics and Economics* 26: 185–92.
- HØJGAARD, BJARNE, AND MICHAEL TAKSAR. 1998a. "Optimal Proportional Reinsurance Policies for Diffusion Models," *Scandinavian Actuarial Journal* 2: 166–80.
- . 1998b. "Optimal Proportional Reinsurance Policies for Diffusion Models with Transaction Costs," *Insurance: Mathematics and Economics* 22: 41–51.
- . 2000. "Optimal Dynamic Portfolio Selection for a Corporation with Controllable Risk and Dividend Distribution Policy." Preprint.
- KORN, RALF. 1997. *Optimal Portfolios: Stochastic Models for Optimal Investment and Risk Management in Continuous Time*. Singapore: World Scientific.
- MARKOWITZ, HARRY M. 1959. *Portfolio Selection: Efficient Diversification of Investment*. New York: John Wiley and Sons.
- MERTON, ROBERT C. 1971. "Optimum Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory* 3: 373–413; Erratum 6 (1973): 213–14.
- SAMUELSON, PAUL A. 1969. "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics* 51: 239–46.
- SCHMIDLI, HANSPETER. 2002. "On Minimizing the Ruin Probability by Investment and Reinsurance," *Annual Applied Probability* 12: 890–907.
- TAKSAR, MICHAEL. 2000. "Optimal Risk and Dividend Distribution Control Models for an Insurance Company," *Mathematical Methods of Operations Research* 1: 1–42.
- TAKSAR, MICHAEL, AND CHARLOTTE MARKUSSEN. 2003. "Optimal Dynamic Reinsurance Policies for Large Insurance Portfolios," *Finance and Stochastics* 7: 97–121.

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