International Journal of Theoretical and Applied Finance

Empirical evaluation of the market price of risk using the CIR model

Luca Torosantucci, Adamo Uboldi, Massimo Bernaschi
Istituto per le Applicazioni del Calcolo "Mauro Picone" – CNR
Viale del Policlinico, 137, 00161 Roma, Italy

We apply the classical model proposed by Cox-Ingersoll-Ross (CIR) to the European Fixed Income Market following two different approaches. In the first case, we apply the non-linear least squares method to cross section data (*i.e.*, all the rates of a single day). In the second case, we consider the short rate obtained by means of the first procedure as a *proxy* of the real market short rate. Starting on this new proxy, we evaluate the parameters of the CIR model by means of martingale estimation techniques.
An estimate of the market price of risk is provided by comparing the results obtained with these techniques.

1. Introduction

The Cox-Ingersoll-Ross (CIR model) model proposed in [3] and [4] is based on the stochastic differential equation:

$$dr_t = k(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t, \tag{1.1}$$

where W_t is a standard Brownian motion; k is the speed of adjustment, μ is the long-term average rate (the mean-reverting level) and $\sigma\sqrt{r_t}$ is the implied volatility. The Local Expectation Hypothesis states that:

$$\mathbb{E}(\frac{dP}{P}) = r_t + \lambda r_t \frac{P_r}{P}, \tag{1.2}$$

where $P(t,T)$ is the price at time t of a zero-coupon with maturity T. It is possible to prove that, in the framework provided by the CIR model, $P(t,T)$ can be written as:

$$p(t,T) = F(t,T)e^{G(t,T)r_t}, \tag{1.3}$$

where

$$\begin{cases} F(t,T) &= \left[\frac{\phi_1 e^{\phi_2(T-t)}}{\phi_2(e^{\phi_1(T-t)}-1)+\phi_1}\right]^{\phi_3}, \\ \\ G(t,T) &= \left[\frac{(e^{\phi_1(T-t)}-1)}{\phi_2(e^{\phi_1(T-t)}-1)+\phi_1}\right]. \end{cases} \tag{1.4}$$

$\phi_i,\ i = 1,2,3$ depends on the parameters of the model:

$$\begin{cases} \phi_1 &= \sqrt{(k+\lambda)^2 + 2\sigma^2}, \\ \\ \phi_2 &= \frac{k+\lambda+\phi_1}{2}, \\ \\ \phi_3 &= \frac{2k\mu}{\sigma^2}. \end{cases} \tag{1.5}$$

where $-\lambda$ represents the market price of risk.
In [13] we applied the CIR model and the Nelson-Siegel model (see [12]) to the Italian Treasury fixed income securities market. In that case we employed the price of bonds quoted from November 1999 to November 2000 that, hereafter we denote as "Dataset-A". In this paper we apply the CIR model to another dataset with the main goals of getting an estimate of the market price of risk.

The present dataset, that we call "Dataset-B", is composed by daily Euribor and swaps rates in the period from January 1999 to December 2002 (the whole archive is composed by 1042 days). More to the point, for each day we have a set of 16 maturities: Euribor rates for 3 and 6 months and swap rates from 1 to 10 plus 15, 20, 25 and 30 years.

The main reason of such choice is that the corresponding securities are very liquid.

We implemented the CIR model following two approaches:

1. in the first case, we apply the non-linear least squares method by cross section to each day of our archive. We denote this method as **static**, to highlight that data used to implement the model correspond to the rates of a single day;

2. in the second case, we consider the short rate obtained following the first approach as a "proxy" of the real market short rate. Starting on this new proxy, we evaluate the parameters of the CIR model by means of martingale estimation techniques. We denote this method as **dynamic**, to highlight that the set of data used to implement the model is composed by the last n daily values of the short rate.

We estimate the short rate r_t by applying the static method to the Dataset-B. The resulting value is very close to the 3- and 6-month Euribor rate and to the 1-year swap rate. This confirms that the CIR model provides a good description of the market. Due to the theoretical features of the CIR model, the description of the term structure is more precise for the medium and long term maturities than for the short term maturities (we discuss this issue in section), especially when there is an inversion of the curve on the very short term (for example, when the 3-month Euribor rate is higher than the 6-month Euribor rate).

The original part of the present paper is the evaluation of the market price of risk. Following the static approach, it is not possible to separate the market price of risk ($-\lambda$) from the speed of adjustment k (see (1.5)), since we can only obtain $k_{\texttt{Static}} = (k+\lambda)$. By comparing this value with the speed of adjustment k obtained following the dynamic approach, we can separate the market price of risk. This is possible thanks to the different financial meaning of the two approaches and to a careful choice of the number of days used to calibrate the model. In particular, the static approach represents the situation of a single day, whereas the dynamic approach represents the average situation of the term structure.

Finally, we evaluate, according to the Local Expectations Hypothesis, the risk premium of the market.

2. Static implementation of the CIR model

The original method to implement the CIR model (see [3] and [4]) is based on formula 1.3. By using the non linear least squares method it is possible, starting on the market price (or the return) for each maturity, to obtain the parameters of the model from (1.4) and (1.5).

In this section we show the results for r_t, m, k and σ obtained from the Dataset-B following this approach. Then, we compare these results with those obtained in [13] and [1].

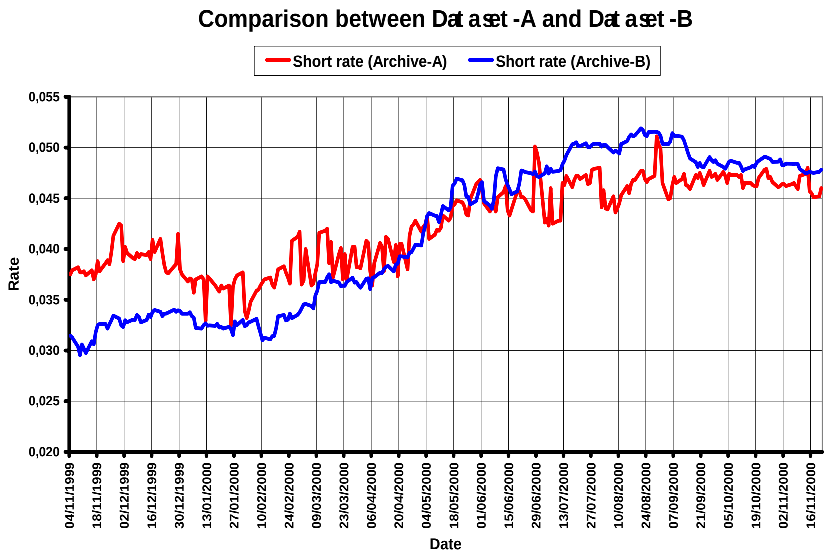

Figure 1:

Short rate r_t					
Dataset and Period	Mean	St. Dev.	Rel. Var.	Min	Max
Archive-B 1999-2002	3.309	0.878	26.5%	1.838	5.189
Archive-B 11/99-11/00	4.162	0.738	17.7%	2.952	5.189
Archive-A 11/99-11/00	4.232	0.421	9.9%	3.255	5.114
BCZ 1984-1989	11.519	3.156	27.4%	8.071	16.129

Table 1: Short rate r_t.

2.1. *Evolution of the short rate* r_t

In the second row of table 1, we report the results for r_t obtained with the whole Dataset-B. In the third row we show the results obtained with the subset of the same dataset corresponding to the period November 1999-November 2000. In the fourth row, we report the results of [13] obtained with the same approach applied to the Dataset-A (which includes data in the period November 1999-November 2000). In the last row we report the results of Barone, Cuoco, Zautik ([1]) for the period 1984-1989 obtained by using Italian Treasury bonds. Note that the Reference Rate was greater than 10% at that time.

For the period November 1999-November 2000, Dataset-B and Dataset-A provide a similar short rate (see figure 1 and table 1). This is an indication that, if there are data for all maturities, it is possible to obtain a reasonable description of the term structure from either the quoted bonds or the Swap and Euribor rates.
However, figure 1 shows that the fluctuations of the short rate obtained by the Dataset-B are smaller than those obtained by the Dataset-A.

This is likely due to the greater liquidity of the Euribor and Swap rates with

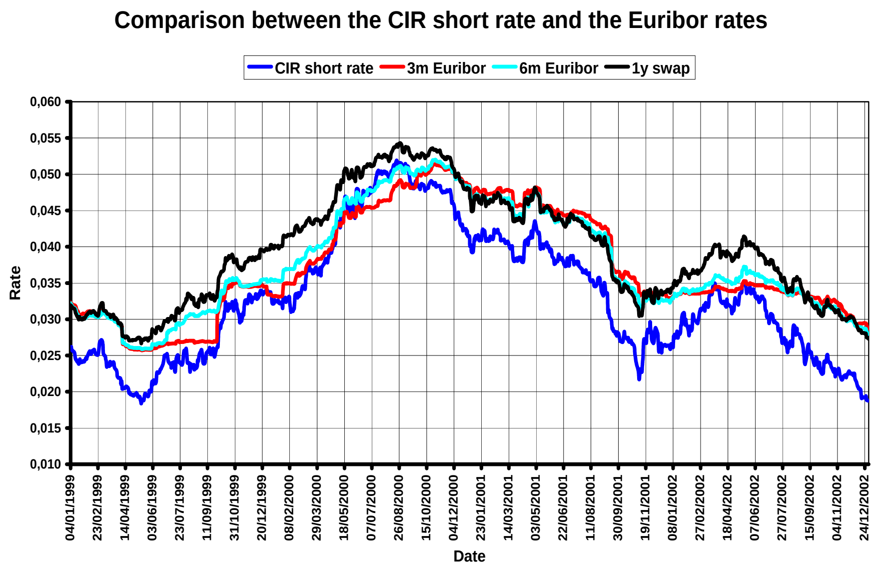

Figure 2:

Implied volatility $\sigma\sqrt{r_t}$					
Model and Period	Mean	St. Dev.	Rel. Var.	Min	Max
Archive-B 1999-2002	1.1522	0.2648	22.9%	0.6594	1.7872
Archive-B 11/99-11/00	1.4092	0.1704	12.1%	1.1579	1.7872
Archive-A	0.8098	0.3753	46.4%	0.2217	1.4807
BCZ 1984-1989	2.1004	0.3321	15.8%	1.1606	3.5880

Table 2: Implied volatility $\sigma\sqrt{r_t}$.

respect to the quoted bonds. For this reason the Dataset-B appears more suitable than Dataset-A to the calibration procedure.

Figure 2 shows the behavior of the short rate obtained from Dataset-B with respect to the behavior of 3- and 6-month Euribor rate for the period January 1999-December 2002. The estimated short rate appears very close but lower than the Euribor rates. This is coherent with the definition of short rate, that is the rate of a security with “instantaneous maturity”.

2.2. *Evolution of the implied volatility* $\sigma\sqrt{r_t}$

The implied volatility describes the behavior of the stochastic part of the model. The results in table 2 are in agreement with the expectation that to a more stable market situation corresponds a lower implied volatility (1999-2002 vs. 1984-1989). It is also interesting to note that the implied volatility for Dataset-A is very similar to that for Dataset-B. This is a confirmation of the model's capability of describing the real fluctuations regardless of the liquidity of the securities in the dataset.

Speed of adjustment k under the hypothesis $\lambda = 0$					
Model and Period	Mean	St. Dev.	Rel. Var.	Min	Max
Archive-B 1999-2002	0.3603	0.1301	36.1%	0.1791	0.6956
Archive-B 11/99-11/00	0.3913	0.1187	30.3%	0.2311	0.6956
Archive-A	0.2804	0.1192	42.5%	0.0386	0.4653
BCZ 1984-1989	0.2433	0.0031	1.3%	0.2269	0.2524

Table 3: Speed of adjustment k.

Long term average rate μ under the hypothesis $\lambda = 0$					
Model and Period	Mean	St. Dev.	Rel. Var.	Min	Max
Archive-B 1999-2002	5.910	0.353	5.9%	5.149	6.565
Archive-B 11/99-11/00	6.187	0.115	1.8%	5.923	6.565
Archive-A	7.470	1.230	16.5%	6.192	14.454
BCZ 1984-1989	11.128	1.802	17.2%	4.261	16.837

Table 4: Long term average rate μ.

2.3. *Evolution of the speed of adjustment k and of the average rate μ*

Note that with the static procedure it is not possible to separate k from the market price of risk $-\lambda$ (see 1.5). *

Only when the market price of risk, $-\lambda$, is equal to 0, it is possible to study independently the parameters k and μ. This is a severe restriction that we eliminate in the following sections.

Looking at table 3, it is clear that the values obtained in [13] and [1] are very similar to the results for Dataset-B. In table 4, we report the long-term average rate μ. For this parameter the results are very similar to the results obtained in [13] and very different from the results of [1]. This is an obvious consequence of the completely different macroeconomic situation of Italy in the two periods.

3. Dynamic implementation of the CIR model

It is possible to calibrate the CIR model by resorting to the time series of a bond price (or return) as a proxy for the short rate. This approach entails three "critical issues":

1. It is not clear what is the best possible proxy of the short rate.
2. The model is, by construction, unable to describe dramatic changes in the market (*shock phenomena*). If such changes appear in the time series, we can expect that the model fails (and it does). We are going to come back to this crucial point later.
3. If we use the historical series of a benchmark, we miss the information about all the other securities.

*From an practical point of view this is not a serious problem, since it is possible to obtain the term structure directly from the parameters ϕ_1, ϕ_2, ϕ_3 and r, if one is not interested in the values of the model parameters.

Very often these reasons are sufficient for a practitioner to prefer a static procedure.

3.1. *The choice of the proxy*

The natural choices for the proxy of the short rate are:

- The shortest maturity rate in the dataset: in our case the 3-month Euribor rate
- The short rate obtained by the static implementation of the CIR model (we denote it as "static short rate").

The 3-month Euribor rate shows anomalous characteristics because in different periods it is the only element of the term structure to remain very stable (figure 2). For instance, during the period between June 1999 and November 1999, the 3-month Euribor rate is almost constant even though the 6-month Euribor rate shows a clear up-trend. After this period, suddenly, the 3-month Euribor rate grows up until it reaches the same value of the 6-month Euribor rate. A similar situation occurs between mid November 2001 and mid April 2002.
In these periods, the short rate obtained by means of the static procedure follows the same trend of the other longer maturities present in dataset-B (see figure 2).
Another problem is the so-called "inversion" on short term maturities (it occurs when the 3-month Euribor rate is greater than the 6-month Euribor rate and, more rarely, than the 1-year Swap rate). Since the CIR model can not describe term structures with this characteristic, using the 3-month Euribor rate as a proxy of the short rate could introduce a bias in the implementation of the model. For all these reasons, the 3-month Euribor rate does not appear suitable for the description of the term structure according to the Local Expectation Hypothesis.
On the other hand, figure 2 shows that the behavior of the static short rate is very close to the behavior of both (3-month and 6-month) Euribor rates and the 1-year Swap rate. Moreover, the behavior of the static short rate is determined by using information coming from the whole term structure. For all these reasons, we believe that the static short rate is a better proxy of the real short rate than the 3-month Euribor rate.

We can estimate the parameters of the CIR model by means of martingale estimations techniques (see [6] and [9]). We denote this method as "dynamic", to highlight that the data belongs to the time series of the short rate proxy.

3.2. *The Martingale Estimation*

For the sake of simplicity, we formulate the CIR model (1.1) as follows:

$$dr_t = (a + br_t)dt + \sigma\sqrt{r_t}dW_t, \tag{3.6}$$

where:

$$\begin{cases} b &= -k, \\ \\ \frac{a}{b} &= -\mu. \end{cases} \tag{3.7}$$

It is useful to remember that $b < 0$, $a > 0$ and $\sigma > 0$. To apply the martingale estimation technique it is necessary to discretize the model (3.6). Since we add more terms from the Ito-Taylor expansion to the Milstein scheme we end up with the following formulation (see [6] and [9]):

$$\begin{aligned} r_{t+1} \quad = \quad & r_t + \Delta(a + br_t) + \sigma\sqrt{r_t}dW + \tfrac{\sigma^2}{4}(dW^2 - \Delta) \\ & + b\sigma\sqrt{r_t}dZ + \tfrac{1}{2}\Delta^2 b(a + br_t) + \tfrac{\sigma}{2\sqrt{r_t}}(a + br_t - \tfrac{\sigma^2}{4})(dW\Delta - dZ), \end{aligned} \tag{3.8}$$

where:

$$\begin{cases} dW = U_1\sqrt{\Delta}, \\ dZ = \Delta^{3/2}(U_1 + \frac{U_2}{2}). \end{cases} \tag{3.9}$$

Here U_1 and U_2 are independent $N(0,1)$-distributed random variables and Δ is the discretized time step. This approach guarantees the existence of closed formulas for a and b. In [6] and [2] it is proved that:

$$\begin{cases} b &= \frac{1}{\Delta} ln\Big[\frac{n\sum_t(\frac{r_t}{r_{t-1}})-\sum_t r_t \sum_t(\frac{1}{r_{t-1}})}{n^2-\sum_t r_{t-1}\sum_t(\frac{1}{r_{t-1}})}\Big], \\ a &= \frac{b}{1-e^{b\Delta}} \frac{ne^{b\Delta}-\sum_t(\frac{r_t}{r_{t-1}})}{\frac{1}{r_{t-1}}}. \end{cases} \tag{3.10}$$

Moreover, the estimator for σ^2 is:

$$\sigma^2 = \frac{\sum_t \Big[r_t - \frac{(a+br_{t-1})e^{b\Delta}-a}{b}\Big]^2 \frac{1}{r_{t-1}}}{\sum_t \Big[\frac{(a+2br_{t-1})e^{2b\Delta}-2(a+br_{t-1})e^{b\Delta}+a}{2r_{t-1}b^2}\Big]}. \tag{3.11}$$

To assess the reliability of the martingale estimators, we studied, by means of simulations of CIR paths, their convergence properties with respect to the number of data used in (3.10) and (3.11).

From a theoretical viewpoint, it would be better to use a very large set of data in order to obtain a reliable estimate of the parameters but, from a practical point of view, this could be misleading: the market is not stable and it can suddenly change its characteristics. The CIR model describes the *average* behavior of the market in the period used to calibrate the parameters. If a shock occurs in that period, the resulting parameters are a mix of two different market situations (before and after the shock). As a consequence, we need to find out what is the number of data that represents the best tradeoff between the following opposite requirements:

- use a large set of data to obtain reliable estimates;
- use a pretty limited set of data to reduce the impact of the market instability.

Another issue is the choice of the time step parameter Δ. This choice influences the number of data required to ensure a reliable convergence of the martingale estimate. In our case, with a dataset of 1042 daily data spanning a period of 4 years (from 1999 to 2002), we use $\Delta = 1/250 = 0.004$. In [14], we prove that, for $\Delta = 0.004$, at least 550 daily data are necessary. In particular, if Δ is equal to 1/250, the better choice is $N = 600$ daily data.

3.3. *Results of the dynamic implementation*

Our dynamic procedure is applied to the time series of the short rate proxy and works as follows:

1. starting on the 601^{st} day we estimate the CIR parameters k, μ, σ by using the martingale estimation formulas, applied to the last 600 data.
2. we repeat the procedure starting on the 602^{nd} day using only the last 600 data;

Dynamical Approach vs Static Approach From 2/1/2001 to 31/12/2002, time step Δ=1/250						
Parameter	Method	Mean	St. Dev.	Rel. Var.	Min	Max
Speed of adjustment	Dynamic	0.7449	0.2812	37.8%	0.1975	1.6179
k	Static	0.4078	0.1248	30.6%	0.1791	0.5937
Long term average rate	Dynamic	3.5882	1.0179	28.4%	0.9159	6.6734
μ	Static	5.7112	0.3441	6.0%	5.1494	6.4191
Implied volatility	Dynamic	0.7798	0.0569	7.3%	0.6462	0.9016
σ	Static	1.1552	0.2067	17.9%	0.8632	1.5650

Table 5: Dynamical Approach vs Static Approach.

3. the procedure continues until we reach the last day of our archive;

In the end, we obtain 3 new time series, respectively for the parameters k, μ, and σ. Obviously, each triple $\{k_t, \mu_t, \sigma_t\}$ describes the average market situation of the last 600 days. The number of data for each time series is equal to 442 (1042, the total number of days in our archive, - 600).

The results are reported in table 5 and 6. It is interesting to note that:

- The implied volatility of the dynamic procedure is lower than that of the static procedure. The reason is probably that the dynamic method is influenced mainly by the fluctuations of the short term maturities, whereas the static method is based on the whole term structure and it may be influenced by the fluctuations of each maturity. In other words, temporary fluctuations on the medium or long term can influence more significantly the static method than the dynamic one.

- The ratio between the standard deviation and the mean value of the implied volatility is lower for the dynamic procedure (see table 6). In [14], we prove the ability of the dynamic method to capture the volatility of the CIR model. It appears that, as to the implied volatility, the results of the dynamic procedure may be considered more reliable.

- The implied volatility obtained by means of the dynamic procedure is very close to the historical volatility of the short rate proxy.

In (3.8) the implied volatility represents the annual volatility more than the daily volatility. This becomes apparent if we consider in 3.8 only the terms up to first order in Δ:

$$r_{t+1} = r_t + \Delta(a + br_t) + \sigma\sqrt{r_t}\sqrt{\Delta}dU, \tag{3.12}$$

where U is a $N(0,1)$-distributed random variable. The discretization introduces a reduction of the implied volatility of a factor $\sqrt{\Delta}$. In table 6 it is possible to compare the implied volatility obtained by the static implementation with that obtained by the dynamic implementation and the historical annual volatility of the short rate proxy. We highlight that the dynamic and the historical volatility are very close to each other.

3.4. *Comparison between the short rate proxy and the Euribor rates*

We selected the static short rate as a proxy of the real short rate for three reasons:

Comparison between the volatilities From 23/04/2001 to 31/12/2002, time step Δ=1/250				
Parameter	Method	Mean	St. Dev.	Relative Error
Implied (annual) volatility	Static	1.1552	0.2066	17.9%
Implied (annual) volatility	Dynamic	0.7798	0.05688	7.3%
Historical annual volatility	Short rate proxy	0.6577	0.5872	89.3%

Table 6: Comparison between the volatilities.

Comparison among implementations with different short rate proxy From 02/07/2001 to 31/12/2002						
Parameter	Method	Mean	St. Dev.	Rel.Var.	Min	Max
Speed of adjustment k	Dynamic	0.7449	0.2812	37.8%	0.1975	1.6179
	Euribor 3m	0.4036	0.4019	99.5%	0.0227	1.5002
	Euribor 6m	0.3632	0.3082	84.9%	0.0275	1.1609
Long term average rate μ	Dynamic	3.5882	1.0179	28.4%	0.9159	6.6734
	Euribor 3m	4.4346	2.0360	45.9%	0.2744	13.2694
	Euribor 6m	3.7452	1.7658	47.1%	0.0162	7.8324
Implied volatility $\sigma\sqrt{r_t}$	Dynamic	0.7798	0.0569	7.3%	0.6462	0.9016
	Euribor 3m	0.4386	0.0572	13.0%	0.3491	0.5226
	Euribor 6m	0.4251	0.0197	4.6%	0.3761	0.4639

Table 7: Comparison between short rate proxy and Euribor rates.

1. The 6-month Euribor or the 1-year Swap rate are too long, considering the existence of the 3-month Euribor rate and considering the implicit instantaneous nature of the short rate.

2. In some periods the 3-month Euribor rate shows an anomalous behavior with respect to other maturities, so it may be not suitable to describe the whole term structure.

3. The behavior of the CIR short rate seems to summarize quite well the information inside the whole term structure.

To double check these assumptions, we also followed the dynamic approach by using the 3-month Euribor rate and then the 6-month Euribor rate as a proxy of the short rate. Finally, we compared the results obtained by using the Euribor rates and the static short rate. The parameters obtained with the three different proxies are reported in table 7.

- **The speed of adjustment k:** the speed of adjustment is similar in every case. However, the relative errors obtained by using the 3-month and 6-month Euribor rates are about twice the relative error obtained by using the static short rate. So, from the stability viewpoint, the static short rate appears a more suitable choice.

- **The long term average rate μ:** each proxy provides, with very good approximation, the same value for the long term average rate μ.

- **The implied volatility $\sigma\sqrt{r_t}$:** the implied volatility obtained by using the static short rate is greater than the implied volatility obtained by using the other proxies. Such result is not surprising. In the dataset B, the Euribor rates (specially the 3-month) is very stable (almost constant) for pretty long periods. This behavior reduces the total volatility. However, the rate for other maturities behaves in a different way. (see, for instance, the 1 year swap rate in figure 2). If the CIR model really represents the market, the short rate behavior must reflect both the almost constant behavior of the short term Euribor rates and the fluctuations of the rest of the term structure. As a consequence, it is reasonable to obtain that the real short rate presents a greater volatility than the Euribor rates.

4. The market price of risk

We have seen that in the CIR framework the expected return of a bond is equal to (1.2), where the *elasticity* of the bond is given by:

$$el = r\frac{P_r}{P} = r\frac{1}{P}\frac{\partial P}{\partial r}.$$

In general, for a long term investment, an investor expects a greater return with respect to the short rate. For such reason, the second term of (1.2) is, almost always, positive.

The Local Expectation Hypothesis equation contains the derivative of the price P with respect to the short rate: it is well known that when the rates go down the prices increase and vice-versa so, on average, the derivative in (1.2) is negative.

But, if the derivative of P with respect to r is negative, the second term of the Local Expectation Hypothesis equation can be positive only if λ is negative. That is the reason why the market price of risk is defined as $-\lambda$ and not λ.

However, for very short periods, the market price of risk can be negative (that is $\lambda > 0$). Usually this happens when the market firmly believes in a significant decrease of the rates. In other words, the market price of risk provides a clue of the market expectations about the term structure evolution.

4.1. *Evaluation of the market price of risk*

Under the Local Expectation Hypothesis, the daily variations of the term structure are mostly due to the variations of the market price of risk. In the static implementation we can only obtain $(k + \lambda)$, whereas in the dynamic implementation we can directly obtain the parameter k. By comparing the daily value of k with $(k + \lambda)$ we can derive an indirect estimation of the market price of risk $-\lambda$. Since we have two time series, one for the speed of adjustment defined by the static procedure and one (restricted to 442 days) for the speed of adjustment defined by dynamic procedure, it is possible to obtain the value of λ for every day from July 2001 to December 2002.

We recall that the dynamic implementation provides a set of parameters that represent the market "on average", whereas the static implementation provides parameters that represent the actual situation of the market, *i.e.,* the term structure of a single day. The comparison of such different viewpoints helps in understanding the tensions and the expectations of the market.

The values of $(k+\lambda)$ obtained with the static procedure, of k obtained with the dynamic procedure and of the market price of risk are reported in table 8.

4.2. *Considerations about the market price of risk*

Estimation of the market price of risk $-\lambda$ From 02/07/2001 to 31/12/2002					
Parameter	Mean	St. Dev.	Rel.Var.	Min	Max
Static speed of adjustment k	0.7449	0.2812	37.8%	0.1975	1.6179
Dynamic speed of adjustment k	0.4078	0.1248	30.6%	0.1791	0.5937
Market price of risk $-\lambda$	0.3236	0.3397	105.0%	-0.2794	1.3493

Table 8: Comparison between short rate proxy and Euribor rates.

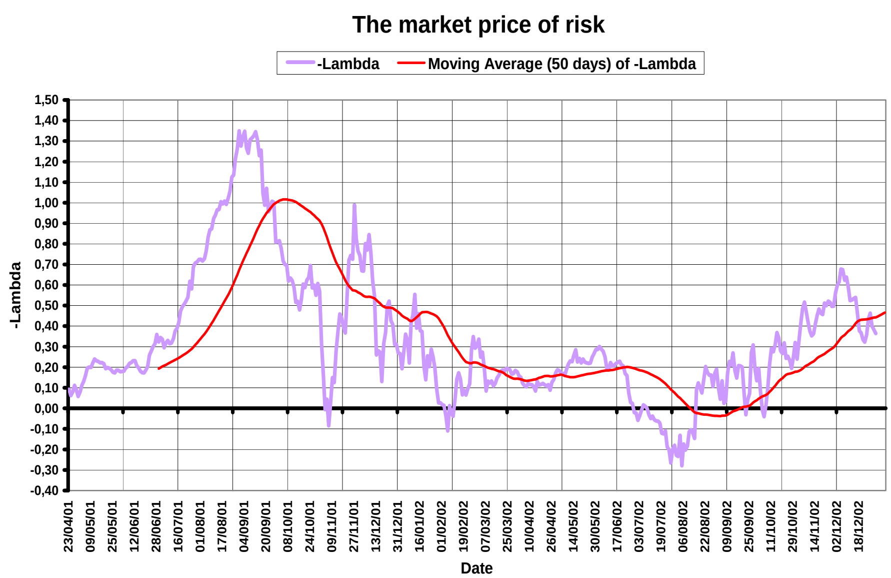

Figure 3:

The behavior of the market price of risk shown in figure 3, seems to give a good representation of the market situation in the period April 2001-December 2002. Indeed:

- After September 11 2001, $-\lambda$ goes down and shows a very unstable behavior for (about) 3 months. This is probably caused by the situation after the terroristic attacks with the expectation of a significant reduction of the Reference Rate by the European Central Bank (ECB) in order to help the economy.
- After January 2002, $-\lambda$ remains almost stable and pretty low according to the idea of a still long crisis and of the stability of the rates.
- After August 2002, $-\lambda$ becomes positive, but very volatile. In the end of November there is another peak that indicates new expectations of a reduction of the Reference Rate (that, actually, happened).

To verify if the market price of risk is able to represent the expectations of the market, it is interesting to compare the market price of risk with the Reference Rate of the ECB. The results are unexpected and interesting as shown in figure 4.

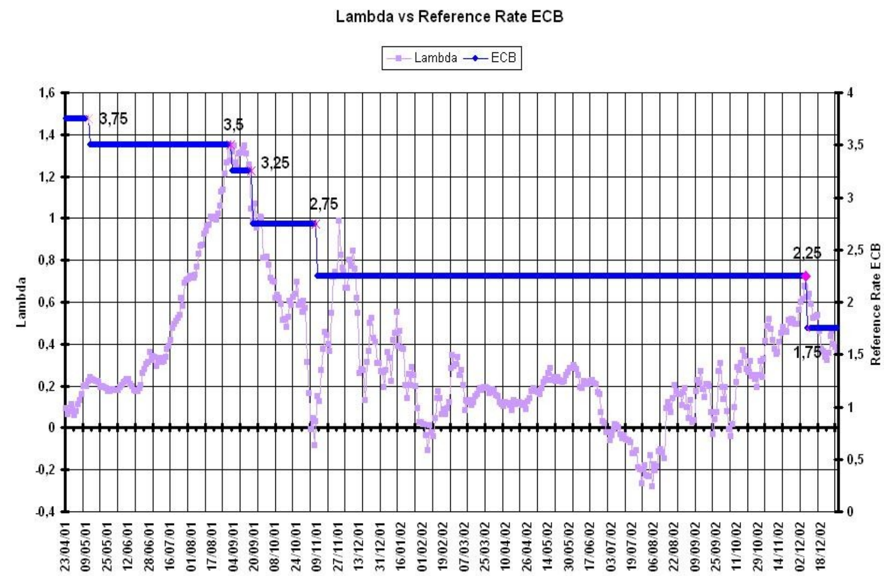

Figure 4:

It is apparent that each movement of the ECB rate generates a reaction in the market price of risk. Except in the first move, $-\lambda$ always follows the direction of the ECB rate. It is also interesting to note that during the period between December 2001 and November 2002, the market price of risk remains almost constant and low, according to the idea of a stable market. Also the peak in November 2002 seems to find a justification. From August 2002 the market price of risk increased, probably due to the forecast of a possible increase of the Reference Rate. The reduction of the ECB rate in November 2002 changed completely the expectations of the market, which reacted with a new decrement of the market price of risk.

From these results, it appears that the market price of risk, evaluated by means of the CIR model, is able to describe the reactions of the market to important events like the changes in the monetary policy.

Apparently, it is surprising that the CIR model is able to provide a good estimate of the market price of risk. However, we recall that Litterman and Scheinkman ([11]) shown that the parallel movements explain more than 80% of the yield curve movements. This means that, most of the time, a one factor model can describe the term structure. This observation is confirmed by Dybvig ([5]), who shown that one-factor models offer a reasonable first-order approximation of the term structure.

5. Conclusions

We estimated the market price of risk by comparing the results produced by two different approaches to the calibration of the Cox-Ingersoll-Ross model.

Our present results are limited to the European market for the period 1999-2002, but nothing prevents from using the same method with data coming from other markets (*e.g.,* the U.S. Treasury fixed income securities market).

The most critical issue remains the choice of the number of days to be used to implement the model since it can seriously influence the quality of results. In our experience it is possible to identify the minimum number of days by looking at the convergence of the martingale estimations. Finally, we shown that the best proxy of the short rate is the short rate obtained from a cross-section analysis of all the rates of a single day.

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