HOW TO DEFINE THE QUALITY OF AN ECONOMIC SCENARIO GENERATOR TO ASSESS THE BEST-ESTIMATE OF A FRENCH SAVINGS CONTRACT IN €?

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Applying a Mark-to-Market approach to evaluate the fair value of the insurer's commitment (best-estimate) for a saving French contract in €, implies having the prices of options and guarantees of insurance policies. Since this information is not observable on an organized and liquid market, the calculation is made in a Mark-to-Model framework.

The calibration and validation of the economic scenario generator (ESG), used to evaluate the best-estimate, by comparing the simulations to the observed data as part of a statistical approach, cannot be considered.

The ESG is then calibrated and validated with reference to the financial instruments (caps, floors, swaptions, etc.), derived from the modelled risk factors, without justifying a direct link or a bijection between these financial instruments and the liability options (see for example Laurent & al. [2014], Planchet & al. [2009], Armel & Planchet [2018]).

The purpose of this paper is to examine how we can define the quality of an economic scenario generator to evaluate the best-estimate of French savings contracts in €.

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1 Preamble

1.1 French savings contracts in €

The savings market in € has grown in France in a favourable institutional and fiscal context. Savings contracts benefit from reduced taxation on income and inheritance which encourages households to hold these products over the long term. In the absence of pension funds in France, life insurance provided an accounting and taxation framework for households, employers and protection institutions to organize retirement and savings in addition to the mandatory pay-as-you-go system.

French savings contracts offer a capitalization of the investment and the possibility of buying back the contract at any time (C. ass., Article R-132-5-3). Premiums collected by insurers are invested in financial markets, real estate and infrastructure.

For the policyholder, the capital loss can occur only in case of bankruptcy of the insurer. In this case, the "*fonds de garantie des assurances de personnes (FGAP)*" can be seized. The amount guaranteed is up to € 70,000.

The technical interest constitutes a minimum contractual revaluation of outstanding amounts (C. ass., Article A-132-1). This revaluation is completed by additional remuneration: Profit Sharing (PS). This represents the remainder of the technical and financial income after taking into account the technical interests.

The PS is regulated by the insurance code (C. ass., Articles A331-3 and following) and gives no individual rights to the policyholder. The PS is either distributed immediately or allocated to the profit-sharing provision, which must be distributed within eight years from its allocation to the fund.

Therefore, policyholders have two acquired provisions:

- The mathematical provisions which are determined individually and correspond to the acquired savings.
- The provision for profit sharing, which is global: its redistribution depends on the policy of the insurer.

The profit-sharing mechanism allows the insurer to smooth the remuneration in time and among policyholders, and to monitor the activity according to the commercial constraints and the conditions of the financial market. In addition to the PS, other provisions are constituted by the insurer implying a smoothing of the book performance of the asset over time. In particular, the "*réserve de capitalisation*", the "*provision pour aléas financiers*" and the "*provision pour risque d'exigibilité*".

The insurer also has a latent wealth (difference between the market value and the book value of the asset) giving him the capacity of managing the revaluation of savings.

The options included in classic savings contracts can be summarized into three categories:

- Financial options: the insurer commits to a minimum remuneration of savings by guaranteeing a minimum revaluation rate or a guaranteed PS.
- Behavioural options: the insurer offers options for surrender, arbitrage, free or scheduled payment, fidelity bonus ... The activation of these options is at the discretion of the policyholder.
- Biometric options: are the options depending on the risk of mortality (or longevity) like a guarantee of a table if the policyholder transforms his savings into an annuity.

From the point of view of the policyholder (see Brys and de Varenne [1994]):

- The option of technical rate or guaranteed PS can be likened to a European vanilla option;
- The surrender option can be likened to a US put option;
- The option of guaranteed rate on free or scheduled payments, can be likened to a swaption.

1.2 Evaluate savings contracts liabilities in €

Established by the European Commission, the Solvency 2 Directive requires the calculation of insurance liabilities in a “fair value” framework. This valuation involves the assessment of the options and guarantees offered by the insurer in its contracts.

Article 77 of the Solvency 2 Directive introduces the concept of the best-estimate for the economic valuation of the insurer's liabilities and defines it as “the best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure”.

The assessment of the best-estimate implies taking into account two sources of risks:

- Hedgeable risks related to the financial markets.
- Non-hedgeable risks related to technical risks: biometric risks, customer behaviour…

The management actions of the insurer intervene in particular in the management of the book yield distributed to the policyholders. They are functions of risk factors.

The cash flows projections must also include, in the contract limits: future premiums, reinsurance, future benefits (deaths, disposals, annuities, etc.), future expenses (administrative expenses, management fees). ...) and taxes.

The calculation of the best-estimate must take into account:

- Financial options and guarantees of contracts;
- The biometric structure;
- The behaviour of the policyholder;
- The impact of management actions on the options of the contract;
- An appropriate modelling of the underlying risks and their dependency structure.

Also, the construction assumptions of cash-flows must be defined with a sufficiently fine granularity and the aggregation of the contracts must be done in groups presenting homogeneous risks.

The evaluation of financial options and guarantees, the diffusion of future cash flows of savings contracts in € and the construction of the risk-free interest rate curve require the establishment of a stochastic Economic Scenario Generator (ESG).

1.3 Economic scenario generator to assess the best-estimate

ESGs can produce, over several horizons, scenarios showing the impact of various economic and financial risk factors on asset prices, such as interest rates, inflation rate, equity yield, real estate yield.

These economic scenarios must be consistent with the prices observed (Market Consistent). A mark-to-market valuation consists of assessing interest variables by referring to the values of the assets and liabilities really exchanged. The objective is to produce a fair value that is consistent with observable and measurable market prices and risks.

Applying a Mark to Market approach to evaluate the best-estimate at fair value implies having the prices of options and guarantees of insurance policies. Since this information is not observable in an organized and liquid market, the calculation is carried-out in a Mark-to-Model framework. In this context, the ESG is not calibrated with reference to the options and guarantees of the insurance contract but with reference to some financial products (caps, floors, swaptions ...). Its quality is appreciated by its ability to reproduce the prices of these financial products.

The guidelines developed by EIOPA (see ACPR [2015] - guidelines 55 to 60) present a certain number of constraints that an ESG must satisfy. It states in particular that:

- The financial instruments used to calibrate the ESG must be relevant given the characteristics of the financial options and guarantees offered by the insurer;
- The data must come from financial markets that are deep, liquid and transparent. The results provided by the ESG must be consistent with the financial market data (Article 76 delegated regulation).

1.4 The problem

The ability of a model to best represent the financial instrument it models is a criterion for ensuring consistency with market values. Models should therefore be chosen and calibrated to best represent the prices of financial instruments used in the modelling process. They are also not intended to correctly represent the prices of other instruments of different structure. The ESG is in fact specific to the objective for which it is built (see Félix & Planchet [2015]).

In this logic, the economic scenario generation process should be validated by appreciating its ability to replicate the price of savings contract options in €. These values are not observable. Validating an ESG by comparing model results with observed prices cannot be considered. Then one appreciates the ESG by comparing it to the financial instruments that were used to construct it without justifying a correspondence or a bijection between these financial instruments and the liability options.

The financial literature is rich in examples confronting the model with its destination or its use. In life insurance, for example, Armel & al. [2011] present the impact of the choice of the dependency structure on the market-SCR and Laïdi and Planchet [2015] propose an alternative calibration method of the LMN model for credit bonds.

The process of generating economic scenarios for the valuation of the best-estimate in a mark-to-model frame work can be summarized in three steps (see Armel and Planchet [2018]):

1. The modelling environment: it consists in choosing the economic variables to model. Classically, the measure chosen is a neutral risk probability.
2. Models: it is a matter of constructing the mathematical models of the variables of interest. This consists in choosing the models that will represent the individual dynamics of these variables and the choice of the model that represents the dependence structure.
3. Parameters and calibration: this involves choosing financial derivative products for calibrations, data, methods for statistical estimation of model parameters and validation methods.

We will discuss here steps 2 and 3. This paper is organized as follows:

- In section 2 we propose a formalization of the theoretical framework for calculating the best-estimate and we propose a quantitative reformulation of the problem;
- In section 3 we propose to study the sensitivity of the best-estimate to the choice of interest rate models and their calibration. The objective is to illustrate, on the basis of an ALM model and real data, the variability of the best-estimate according to the choice of the model of risk-free rate and / or of its calibration. All models are chosen in a family respecting all the valuation constraints imposed by the supervisor;
- In section 4 we propose an analysis of the optional structure of the best-estimate and we seek to build a link between this structure and the economic scenario generation model.

Issues related to risk margin calculations will not be addressed here.

2 Quantitative reformulation

2.1 Introduction

Two sources of randomness, represented by two filtered probabilistic spaces, are distinguished:

- $\left(\Omega^f, \left(F_t^f\right)_{t\geq 0}, Q\right)$: for financial risks that are hedgeable[3];
- $(\Omega^a, (F_t^a)_{t\geq 0}, P)$: for insurance risks that are not hedgeable. The use of this probability can be avoided by introducing the conditional expectation of cash flows knowing the financial risk factors.

The best-estimate of contracts in € calculated at a time t is written (Laurent & al., 2016):

$$BE(t) = E^{P\otimes Q}\left(\sum_{i=t}^{+\infty} F_i . \exp(-i . r_i)\right)$$

where r_i is the risk-free rate at maturity i.

The cash-flow F_i is the sum of payments to policyholders and fees less premiums and loadings:

$$F_i = Payements_i^{gross} - Premiums_i + Expenses_i - Loadings_i$$

In practice, the evaluation of the best-estimate is done in a Monte-Carlo calculation framework and the computation of cash-flows stops at a projection horizon T. The best-estimate at time 0 is therefore written as an average of M x N simulated paths; M representing the paths of the financial variables under the probability Q and N the number of paths of the non-hedgeable risks under the probability P:

$$BE(0) = E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} F_i . \exp(-i . r_i)\right) = \frac{1}{\mathrm{MN}} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\sum_{i=0}^{\mathrm{T}} F_{i,m,n} . \exp(-i . r_{i,m})\right)$$

For non-hedgeable risks, the calculation of the expectation is generally simple. The use of simulations in simulations can be avoided by introducing average flows as shown in Section 2.2.3.

In the absence of future premiums the best-estimate is written:

$$BE(0) = E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Payements_i^{gross} . \exp(-i . r_i)\right) - E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Loadings_i . \exp(-i . r_i)\right) + E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Expenses_i . \exp(-i . r_i)\right)$$

Without loss of generality, we assume in the following that loadings are equal to expenses. The general case where expenses are different from loadings can be deduced directly from the following by decomposing the loadings rate into expenses rate on outstanding amounts and on a margin on loadings (positive or negative). We can write then:

[3] Or replicable, the 2 terms are used here without distinction.

$$E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Loadings_i . \exp(-i . r_i)\right) = E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Expenses_i . \exp(-i . r_i)\right)$$

and:

$$BE(0) = E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Payements_i^{gross} . \exp(-i . r_i)\right)$$

Then, by decomposing the revaluation rate into a net rate of revaluation and a loadings rate we can wright:

$$BE(0) = \underbrace{E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Payements_i^{net} . \exp(-i . r_i)\right)}_{Terme\ 1\ :\ net\ payements} + \underbrace{E^{P\otimes Q}\left(\sum_{i=0}^{\mathrm{T}} Expenses_i . \exp(-i . r_i)\right)}_{Terme\ 2\ :\ contrat\ price/expenses}$$

We can write the best estimate as the sum of two terms:

- Term 1: representing the future cash flows paid to the policyholder including the options and guarantees offered by the insurer;
- Term 2: representing loadings collected by the insurer that are equal to expenses. This term can be interpreted as the price of the contract and implicitly incorporates the price of options and guarantees.

The best estimate is then: $BE(0) = BE^{net}(0) + BE^{expenses}(0)$. In the following we are firstly interested in the best-estimate calculated on the basis of a net revaluation rates. The best estimate of loadings/expenses is then explained.

It is noted that the calculation of the best-estimate in the Solvency 2 standard is carried out on runoff portfolios (future contracts are outside the scope).

2.2 Theoretical framework

2.2.1 The surrender value

At each time u, the insurer revaluates the savings in € by an instantaneous rate of revaluation net of loadings noted c_u. This rate of remuneration is the result of a management decision considering, in particular, the rate of return on assets, the available wealth of the insurer and the guaranteed minimum rate.

The surrender value of a savings contract, denoted $VR(t)$, is written according to the mathematical provision at time 0 and the rates of revaluation net of loadings as follows (see Bonnin & al. [2014]):

$$VR(t) = PM(0) \times \exp\left\{\int_0^t c_u . du\right\}$$

The present value of the surrender value at time t is written:

$$VR(t) . \delta(t) = PM(0) \times \exp\left\{\int_0^t c_u . du - \int_0^t r_u . du\right\} = PM(0) . \psi(t)$$

where $\delta(t) = \exp\left(-\int_0^t r_u.du\right)$ and r_u is the instantaneous risk-free interest rate.

In case of exit from the contract, in case of death or surrender, the value paid by the insurer is equal to $VR(\tau)$ where τ denotes the exit-time (random) from the contract.

The discounted value of the outgoing flow is therefore:

$$\Lambda = VR(\tau).\delta(\tau)$$

The random variable τ is assumed to be the stopping time of the natural filtration associated with the surrender process $(VR(t), t \geq 0)$.

The best-estimate at time 0 for a savings contract in € is written:

$$BE^{net}(0) = E^{P\otimes Q}(\Lambda)$$

One can refer to Prudent [1996] which presents a similar framework for the valuation of early surrender clauses.

2.2.2 Continuous formula of best-estimate net of loadings

We can distinguish two sources of randomness under the probability P:

- A P^a component representing the technical risks related to insurance risks - structural surrender and mortality - that are assumed to be mutualized and independent of the financial markets. Let Θ_1 be the set of parameters defining these risks. Therefore, this parameter includes mortality tables and surrender curves.
- A P^h component representing the risks related to the dynamic behavior of policyholders, depending on revaluation rates. These behaviours are assumed to be independent of the technical risks (hence of P^a). The reaction of policyholders to the revaluation rates is assumed to be characterized by a function whose parameters are represented by the vector Θ_2;

The insurer's reaction to asset rates of return and expectations of policyholders' behaviour takes the form of management actions, on the basis of book rates. The vector of parameters is represented by Θ_3.

Let Θ_4 be the vector of parameters representing financial risks. This vector represents the parameters of the ESG (trend, volatilities, convergence factor ...).

We denote $\Theta = (\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ all the parameters of the calculation of the best-estimate.

Let h be the hazard function representing the instantaneous exit rate. This function is a function of $\mathbb{R}^+ \times (\mathbb{R}^+)^n$ satisfying for all $(t, u_t) \in \mathbb{R}^+ \times (\mathbb{R}^+)^n$:

$$S(t, u_t) = \exp\left(-\int_0^t h(t, u_t)\right)$$

$$dln\big(S(t, u_t)\big) = -h(t, u_t)dt$$

where:

- $S(t, u_t)$ is the survival function defined on $\mathbb{R}^+ \times (\mathbb{R}^+)^n$;
- t represents the time;
- u_t is a variable characterizing the dynamic surrenders. It can represent the last n revaluation-rates already distributed at t used in the calculation of the policyholders satisfaction function. The factor n can be interpreted as the reaction time on distributed revaluation-rates.

The hazard function and the survival function have as parameters Θ_1 and Θ_2.

We note: $BE^a(0) = E^{P^a}(\Lambda)$, so we can write:

$$BE^a(0) = E^{P^a}(\Lambda) = \int_0^{+\infty} \Lambda . dP^a = \int_0^{+\infty} -\Lambda . dS(t, u_t) = \int_0^{+\infty} \Lambda . S(t, u_t) h(t, u_t) dt$$

Therefore:

$$BE^a(0) = PM(0). \int_0^{+\infty} \psi(t). S(t, u_t) h(t, u_t) dt$$

It follows that:

$$BE^{net}(0) = E^{P \otimes Q}(\Lambda) = E^{P^h \otimes Q}\left(BE^a(0)\right)$$

$$BE^{net}(0) = PM(0). E^{P^h \otimes Q}\left(\int_0^{+\infty} \psi(t). S(t, u_t) h(t, u_t) dt\right)$$

Witch is written by making explicit the model's parameter vectors:

$$BE^{net}(0|\Theta) = PM(0). E^{P^h \otimes Q}\left(\int_0^{+\infty} \psi(t|\Theta_3, \Theta_4). S(t, u_t|\Theta_1, \Theta_2) h(t, u_t|\Theta_1, \Theta_2) dt\right)$$

If we note:

$$\alpha(\Theta) = E^{P^h \otimes Q}\left(\int_0^{+\infty} \psi(t|\Theta_3, \Theta_4). S(t, u_t|\Theta_1, \Theta_2) h(t, u_t|\Theta_1, \Theta_2) dt\right)$$

We establish then the following proportionality relationship:

$$BE^{net}(0|\Theta) = PM(0). \alpha(\Theta)$$

2.2.3 Discretized formula of the best-estimate net of loadings:

On a finite projection horizon denoted T we write the discretized best-estimate:

$$BE^{net}(0) = PM(0). E^{P^h \otimes Q}\left(\sum_{t=1}^{T} \frac{l_{t-1}}{l_0}. R(t-1). (q_{t-1} + v_{t-1} - q_{t-1}. v_{t-1}). \psi(t) + \frac{l_T}{l_0}. R(T). \psi(T)\right)$$

with:

- q_t: mortality rate between t and $t+1$.
- v_t: surrender rate between t and $t+1$. This rate includes dynamic surrender and structural surrender.
- $R(t) = \prod_{j=1}^{t-1}(1-v_j)$ and $R(0)=1$ is the non-redeemed part of savings between 0 and t.
- $\psi(t) = \exp\{\sum_{i=1}^{t} c_i - \sum_{i=1}^{t} r_i\}$ with:
  - c_i: the rate of revaluation of the savings net of loadings at time i. This rate must be higher than the guaranteed minimum rate;
  - r_i is the risk-free rate for the period between $i-1$ and i.

2.2.4 The best estimate of expenses

Let ι be the constant annual loadings rate applied to the savings earned by policyholders. As stated above, this rate also corresponds to the expenses rate. So we write the amount of expenses paid by the insurer between t' and t, as follow:

$$\mathrm{I(t,t')} = \mathrm{VR(t')}.\exp\left(\int_{t'}^{t} c_u.du + (\mathrm{t}-\mathrm{t'}).\iota\right) - \mathrm{VR(t')}.\exp\left(\int_{t'}^{t} c_u.du\right)$$

therefore:

$$\mathrm{I(t,t')} = \mathrm{VR(t)}.\left(\exp\left((\mathrm{t}-\mathrm{t'}).\iota\right)-1\right) \simeq (\mathrm{t}-\mathrm{t'}).\iota.\mathrm{VR(t)}$$

For $t' = t-1$ we can write:

$$\mathrm{I(t)} = \mathrm{I(t,t')} = \mathrm{VR(t)}.(\exp(\iota)-1) \simeq \mathrm{VR(t)}.\iota \simeq \iota.\mathrm{PM}(0).\exp\left(\int_{0}^{t} c_u.du\right)$$

Following the same reasoning as the previous section and keeping the same notations, the best estimate of loadings / expenses discretized on an annual basis is:

$$BE^{expenses}(0) = (\exp(\iota)-1).PM(0).E^{P^h\otimes Q}\left(\sum_{t=1}^{T}\frac{l_t}{l_0}.R(t).\psi(t)\right)$$

therefore:

$$BE^{expenses}(0|\Theta) = (\exp(\iota)-1).PM(0).\beta(\Theta) \simeq \iota.PM(0).\beta(\Theta)$$

2.2.5 The discretized formula of the best-estimate

Let's note:

- $f_t = \frac{l_{t-1}}{l_0}.R(t-1).(q_{t-1}+v_{t-1}-q_{t-1}.v_{t-1})$: the probability to exit between $t-1$ and t;
- $g_t = \frac{l_t}{l_0}.R(t)$: the probability to be under contract at time t.

The quantities f_t and g_t are stochastic since they integrate the dynamic surrender.

On a finite projection horizon denoted T the discretized best-estimate is written:

$$BE(0) = BE^{net}(0) + BE^{expenses}(0)$$

We prove therefore the following:

$$BE(0) = PM(0).E^{P^h \otimes Q}\left(\sum_{t=1}^{T}\left(f_t + \iota g_t\right).\psi(t) + g_T.\psi(T)\right)$$

2.3 Quantitative reformulation

To simplify the notation, let us take again the formula of proportionality demonstrated in the preceding sections:

$$BE(0|\Theta) = PM(0).\left(\alpha(\Theta) + \iota.\beta(\Theta)\right)$$

Assuming the following parameters are predefined:

- The vector Θ_1: life tables and structural surrender curves. These risks are supposed to be perfectly mutualised and do not depend on financial risks.
- The vector Θ_2: the set of parameters of the reaction function of the policyholder following revaluation rates (for example the function proposed in ACPR [2013]). The reaction function takes financial returns as arguments and the result is therefore random.
- The vector Θ_3: the set of parameters of the insurer's reaction function to the behaviour of policyholders and asset yield rates (management actions). The reaction function takes as arguments financial returns. The result is therefore random.

In a context where the reaction functions are deterministic, the stochastic structure of the best-estimate depends only on the ESG parameters vector Θ_4.

In this framework, we write for simplification:

$$BE(0|\Theta) = PM(0).\left(\alpha(\Theta_4) + \iota.\beta(\Theta_4)\right)$$

and we can denote:

$$\lambda(\Theta_4) = \left(\alpha(\Theta_4) + \iota.\beta(\Theta_4)\right) = \frac{BE(0|\Theta)}{PM(0)}$$

2.3.1 Standard calibration method

In general, the ESG should be chosen and calibrated to best represent the prices of the financial instruments selected in the modelling process. In this context, calibrating an ESG to calculate the best-estimate can be summarised as:

- Choose a measure of distance[4] noted d;
- Construct the vector of observations λ^{obs};

[4] The Minkovski distance of order p between two real vectors $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ is written: $d(X,Y) = \sqrt[p]{\sum_1^n |x_i - y_i|^p}$ and for $p = +\infty, d(X,Y) = \sup(|x_i - y_i|)$. We find a similar framework for measurable functions in Lebesgue spaces.

- Construct the theoretical price vector λ^{th};
- Calibrate the model by choosing Θ_4 such that: $\widehat{\Theta}_4 = \underset{\Theta_4}{\operatorname{argmin}}\left(d(\lambda^{obs}, \lambda^{th}(\Theta_4))\right)$.

Applications of this classical process in financial modelling can be found in Armel and Planchet [2018] and in reference publications such as Brigo and Mercurio [2001] or Hull [2007].

2.3.2 Conventional calibration method

In savings, the application of the standard process comes up against a major limit: the best-estimate is a price that is not observed. There is no market to exchange insurance liabilities for savings contracts in runoff.

In view of the need, including regulation, to calculate the best-estimate and the inability to calibrate ESGs by the standard process, a conventional valuation process has been established.

This conventional process consists in calibrating the ESG on the prices of financial products traded on the market acting as a substitute for the best-estimate. The parameters of the ESG are then deduced by the following optimization:

$$\widehat{\Theta}_4 = \underset{\Theta_4}{\operatorname{argmin}}\left(d(fin\_prod^{obs}, fin\_prod^{th}(\Theta_4))\right)$$

The assumption underlying this practice is the equivalence between the financial options and the best-estimate. This hypothesis cannot be verified, however, since the best-estimate is not observable.

In addition, Armel and Planchet [2018] present the conventional framework for calibrating interest rate models and explain its limits. They also present an approach to build a risk-neutral economic scenario generator.

The following section focuses on two conventional choices:

- The choice of the risk-free interest rate model;
- The choice of financial instruments to calibrate the ESG.

The objective is to study the impacts, on the best-estimate of a real-life savings portfolio, of choices of risk-free interest rate models and calibration data.

3 Impact of the choice of interest rate models and calibration data on the best-estimate

3.1 ESG models used

In Armel and Planchet [2018] we present an approach of building an ESG to produce economic scenarios which are consistent with an economic environment characterized by negative rates. They apply this approach to generate economic scenarios for the valuation of euro-savings liabilities in the solvency 2 framework. In addition, they analyse the calibration convention for interest rate models, explain its limits and propose a sensitivity study, of the ESG parameters and simulations, to the displacement factor.

In the following, the study of the best-estimate sensitivity to the choice of interest rate models and their calibration is based on the theoretical framework and the applications that we present in Armel and Planchet [2018].

The asset is assumed to be composed of equities, real estate investment and sovereign zero-coupon bonds. Stock prices and investments in real estate are assumed to follow a Brownian motion under the neutral risk probability.

Three interest rate models are tested:

- A mono-factorial model: Hull & White calibrated on caps and swaptions;
- A two-factor model: Gaussian model G2 ++ calibrated on caps and swaptions;
- A market model: Libor Market Model (LMM) calibrated on swaptions.

Three models combined with 2 types of financial products (caps and swaptions) are proposed in the following. These models respect the constraints of the regulator and are used by the market.

In addition, the following choices are retained:

- The options chosen for the calibration of the different models (call, caps and swaption) are ATM;
- The market volatilities of the caps and swaptions used in the calibration process are non-shifted ATM log-normal volatilities observed on January 02, 2018 and provided by Bloomberg (the displacement factor is equal to 0);
- The risk-free interest rate curve used for the calibration and simulation processes is the yield curve provided by EIOPA on December 31, 2017;
- The use of the EIOPA curve implies the need to introduce a non-zero displacement factor to calibrate and project the interest rate models. The introduction of this non-zero factor (while the extraction of volatilities is carried out with a zero-displacement factor) induces a certain bias in the model:
    - For Hull & White and G2 ++ models: different calibrations have been performed corresponding to different levels of the displacement factor. These models are normal and do not require the introduction of a displacement factor for simulations;
    - The LMM model has been calibrated on Black's volatilities on maturities with no negative rates: no displacement factor has been introduced in the calibration process;
    - The diffusion of the LMM model requires the definition of a displacement factor. Several factors were tested;
    - Three levels of the displacement factor are tested: 0.4%, 1% and 2%. The value of 0.4% corresponds to the rounded absolute value of the minimum interest rate of the EIOPA risk-free rate curve observed on December 31, 2017.

- For the projection of investments in equities and real estate, we selected models of Black-Scholes with constant volatilities:
    - The implied volatility of an equity investment is calibrated on the implied volatility of the ATM call on the CAC 40 with a maturity of 3 years;
    - The volatility of an investment in real estate corresponds to the historical volatility of the returns of the house price index published by INSEE[5].

In the following, we present the parameters of the interest rate models used to evaluate the sensitivity of the best-estimate to the choice of interest rate models and their parameterization. The correlation matrices, the calibration results of equity and real estate investment models and a complete analysis of the results are proposed in Armel and Planchet [2018].

Table 1 and Table 2 present respectively the results of the Hull & White and G2 ++ model calibrations on cap and swaption data.

Table 1: calibration results of the Hull & White model

Index	Black shift	a	σ	Total relative square error
HW 1 (Caps)	0.40%	0.97%	0.53%	2.71%
HW 2 (Caps)	1.00%	1.00%	0.80%	2.98%
HW 3 (Caps)	2.00%	1.04%	1.25%	2.92%
HW 4 (Swaption)	0.40%	0.07%	1.27%	7.32%
HW 5 (Swaption)	1.00%	0.10%	1.62%	8.48%
HW 6 (Swaption)	2.00%	0.01%	2.20%	10.43%

Table 2: calibration results of the G2 ++ model

Index	Black shift	a	b	σ	η	ρ	Total relative square error
G2 1 (Caps)	0.40%	1.48%	0.50%	26.02%	25.64%	-99.99%	0.009%
G2 2 (Caps)	1.00%	5.14%	0.03%	7.50%	7.07%	-100.00%	0.005%
G2 3 (Caps)	2.00%	8.39%	4.36%	17.46%	16.76%	-99.99%	0.012%
G2 4 (Swaption)	0.40%	11.47%	9.04%	24.20%	23.22%	-99.99%	0.17%
G2 5 (Swaption)	1.00%	11.49%	8.90%	26.50%	25.67%	-99.8%	0.21%
G2 6 (Swaption)	2.00%	11.71%	8.38%	25.59%	24.97%	-99.97%	0.29%

Table 3 presents the results of the calibration of the LMM model on swaptions data. This calibration is carried out without introducing a displacement factor by retaining only the market volatilities corresponding to the positive rates.

Table 3: calibration results of the LMM model

LMM shift	a	b	c	d	*Beta*	Total relative square error
0.00%	18.85%	0.19%	7.45%	0.01%	0.10%	1.19%

The calibrated LMM model on swaptions diverges, even with the smallest allowable shifting factor to circumvent the constraint of negative rates. This calibration, although

[5] https://www.insee.fr/fr/statistiques/series/102770558

market-consistent, cannot be retained in the state for the valuation of liabilities of savings contracts in €.

We relied on this market-consistent calibration to propose a convergent parameterization of the LMM model. This setting is not market-consistent.

As shown in Table 4, we increased the factor of convergence of Rebonato volatility and the level of the asymptotic limit.

Table 4: Convergent parameterization of the LMM model

a	*b*	*c*	*d*	*Beta*
18.85%	0.19%	**20.00%**	**1.00%**	0.10%

3.2 Impact of the choice of interest rate model on the best-estimate

3.2.1 Best-estimate calculation model

In order to evaluate the best-estimate we used the R package SimBEL[6]. This valuation incorporates tables of economic scenarios that we generated and uses real-life modified data from an insurer.

The SimBEL tool calculates the best-estimate provisions and the SCRs of the standard formula.

The ESG used is based on the theoretical framework and the applications that we present in detail in Armel and Planchet [2018][7].

3.2.2 Parameters and results

Table 5 presents some input data of the best-estimate valuation model. Assets are essentially composed of sovereign bonds. The unrealized gain is € 6 million. The mathematical provision is 70 M € and the projection horizon is 20 years.

Table 5: asset description

Assets	*Market value (M €)*	*Book value (M €)*	*Strategic allocation*
Equity	20.00	18.54	20%
Real estate	10.00	9.88	10%
Sovereign Bonds	63.00	58.85	63%
Cash	7.00	7.00	7%
Total	*100.00*	*94.26*	*100%*

Table 6 and Table 7 present the sensitivity of the best-estimate to the choices and calibrations of Hull & White and G2 ++ market-consistent risk-free rate models.

We notice that:

[6] See: http://www.ressources-actuarielles.net/C1256F13006585B2/0/C5542E1CF549F21FC12581680046FD2E
[7] See: http://www.ressources-actuarielles.net/C1256CFC001E6549/0/709DE72DB6128DBDC12582700071015B

- Best-estimates, evaluated using the Hull & White rate model, are insensitive to displacement factors and financial instruments chosen for calibration;
- The best-estimates, evaluated using the G2 ++ rate model, are more sensitive to displacement factors (maximum impact of 4%) and the choice of financial instruments for calibration (maximum impact of 4.4%);
- The impact of the choice of Hull & White and G2 ++ rate models is at most 4.3% of the best-estimate average (Table 9).

Table 6: best-estimate by the Hull & White market-consistent model

Amounts in € millions	*HW 1*	*HW 2*	*HW 3*	*HW 4*	*HW 5*	*HW 6*
Best-estimate net of expenses	82.92	82.92	82.64	82.89	82.63	82.45
Expenses	7.95	7.94	7.85	7.86	7.76	7.6
Best-estimate	90.88	90.86	90.49	90.75	90.39	90.05

Table 7: best-estimate by the G2 ++ market-consistent model

Amounts in € millions	G2 1	G2 2	G2 3	G2 4	G2 5	G2 6
Best-estimate net of expenses	80.58	83.18	81.42	82.75	82.84	83.6
Expenses	6.90	7.79	7.4	7.88	7.77	7.76
Best-estimate	87.48	90.97	88.82	90.63	90.61	91.36

Table 8 presents the best-estimates evaluated using the non-market-consistent LMM model derived from the LMM market-consistent model by adjusting the volatility parameter (see section 3.1). We find that best-estimates are insensitive to displacement factors and are comparable to quantities presented in Table 6.

Table 8: best-estimate by non-market-consistent LMM model

Amounts in € millions	*LMM 1*	*LMM 2*	*LMM 3*
Best-estimate net of expenses	82.87	82.62	82.78
Expenses	7.94	7.92	7.94
Best-estimate	90.81	90.54	90.72

Table 9: comparison of best-estimates

Best-estimate	*Standard deviation*	*Min*	*Max*	*Difference (Max-Min) / Average*
Only market-consistent models (HW & G2 ++)	1.12%	87.48	91.36	4.3%
All models	1.08%	87.48	91.36	4.29%

In addition, the ratios between the mathematical provision and the best-estimates vary between 77% and 80%. These ratios are lower than the ratios observed for *BNP Paribas Cardif*[8] (88%) and *AXA France Vie*[9] (83%) but remain in the same order of magnitude. The difference between the best estimate and the mathematical provision can be explained by an initial wealth at the discretion of the insurer (see section 4.1.1).

Although the models used to evaluate the sensitivity of the best-estimate have different characteristics, the impact on the value of the latter remains fairly contained compared to

[8] The data are taken from the SFCR report available on BNP Paribas Cardif's institutional website: link.
[9] The data are extracted from the SFCR report available on the AXA France corporate website: link.

the best-estimate[10]. The difference between the minimum and maximum value is indeed 4.30%. It is although significant compared to own-funds: the mean of own-funds of French savings insurers is around 5% in 2016 according to FFA [2017].

In addition, a particular attention must be given to the LMM model. This model, calibrated on the data observed on January 02, 2018, cannot be retained for the evaluation of the best-estimate because of its divergence.

4 Can we build an ESG consistent with the optional structure of the best-estimate?

4.1 Qualitative analysis of the construction of the best-estimate

4.1.1 Analysis of the initial wealth of the insurer

Future contracts are excluded from the best-estimate scope. In addition, if the savings contracts do not contain predetermined financial guarantees for all future payments, which is generally the case for classical savings contracts, future premiums cannot be taken into account in the scope of valuation (ACPR [2013]). The valuation of the best-estimate of savings contracts in € is therefore carried out within a framework assuming that the insurer stops its commercial activity and merely manages its outstanding.

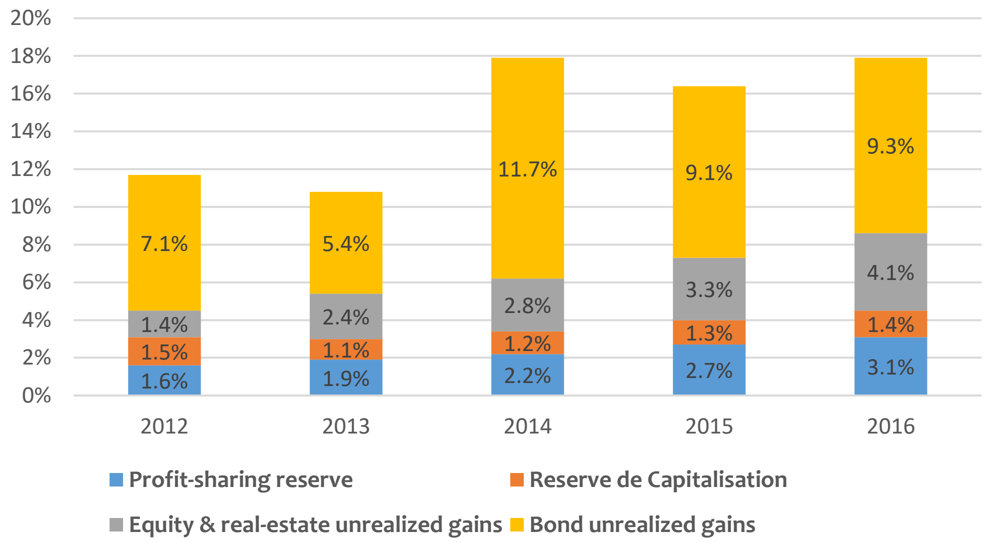
A policyholder, whose acquired savings on valuation date 0 is the mathematical provision, $PM(0)$, can benefit from an initial wealth cumulated by the insurer. The latter is as follows:

- The provision for profit-sharing which is totally acquired to the policyholder. Its distribution is at the discretion of the insurer. The initial value of this provision is denoted $PPB\ (0)$;
- The unrealized gains or losses generated by the accounting management of revaluation rates already distributed. The initial value is denoted $PMVL\ (0)$;
- Provisions for steering book return such as : "*la réserve de capitalisation*", "*la provision pour aléas financiers*" and "*la provision pour risque d'exigibilité*". The management of these provisions is at the discretion of the insurer. Their initial value is denoted $PR(0)$.

Figure 1 illustrates the importance of the initial wealth cumulated by companies depending on the French Insurance Code, on the French market (see FFA [2017]).

[10] The study of the sensitivity of the best-estimate to interest rate models realized by Bollotte [2018] shows similar results.

Figure 1 : initial wealth of insurers on the French market from 2012 to 2016



In 2016, the maximum wealth is at least 17.9%. The initial wealth acquired to policyholders is 3.1%. The initial wealth that can be distributed by the insurer can theoretically vary in a range whose diameter represents 14.8% of the mathematical reserves in 2016.

In 2016, the mathematical provisions for savings contracts in € amounted to € 1,286.9 billion. The own funds of life, capitalization and joint companies before appropriation of the results are 67.5 billion euros (FFA [2017]).

An additive uncertainty of 5% on the integration of initial wealth and its distribution in future cash flows represents a best-estimate variability of € 64.3 billion. This is comparable to the accumulated own funds of life insurers.

In a runoff frame work, assets will be sold gradually to serve an accounting revaluation rates or to honour outgoing cash flows. A part of initial latent value will be gradually distributed to policyholders and integrated into future cash flows at the discretion of the insurer. The latter also shares book profits made on the sale of assets within the regulatory and contractual limits of profit-sharing clauses.

The initial and maximum discounted latent value that can be distributed to policyholders is $PMVL_{max} = PMVL(0)$[11]. The minimum value depends on several parameters such as the book performance management policy, the asset performance policy, the profit sharing distribution clauses and the policy for endowment / reversal of provisions for steering book returns. If, for example, the insurer decides to realise all gains or losses to distribute them to the policyholder, then the regulatory minimum to be distributed is:

$$PMVL_{min} = 85\%\ PMVL(0)$$

[11] The market value of the asset is equal to the book values plus unrealized gains or losses. The sum of the discounted future cash flows of the asset at market value is equal to the present value (arbitrage free). In runoff framework, all the asset will be distributed. The sum of the discounted future cash flows is equal to the current asset at market value and therefore equal to the current book value of the asset plus the unrealized gains or losses.

In all cases, there are $PMVL_{min} \leq PMVL(0)$ such as:

$$PMVL_{min} \leq PMVL_{distributed} \leq PMVL_{max}$$

Provisions for steering the book returns will be endowed or reversed as the asset is disposed of and the latent values are realized. Their distribution is discretionary and depends on the insurer's investment policy. These provisions modulate the financial returns of the assets.

If certain steering provisions are non-zero at the end of the projected cash flows, the insurer has the right not to pay them to the policyholders (example: *"la réserve de capitalisation"*). The current maximum amount that can be distributed to policyholders is $PR_{max} = PR(0)$. The current minimum amount that can be distributed depends on the insurer's policy and can be nil: $0 \leq PR_{min} \leq PR(0)$.

Let's note:

$$Wealth_{min}^{init}(0) = PPB(0) + PMVL_{min}(0) + PR_{min}(0)$$

$$Wealth_{max}^{init}(0) = PPB(0) + PMVL_{max}(0) + PR_{max}(0)$$

Thus, the sum of discounted future cash flows that the insurer must distribute is at least (arbitrage free):

$$PM(0) + Wealth_{min}^{init}(0)$$

Policyholders benefit or suffer from a memory effect on the assets managed by the insurer. If the initial wealth embedded in future cash flows is positive, the present value of these cash flows is greater than the initial investment ($PM\ (0)$). If this wealth is negative (in the case of significant unrealised losses, for example), rational policyholders can:

- Keep their contracts if the guarantee offered by the insurer is significant. In this case, the insurer may be forced to use its own funds to serve the guaranteed rates;
- Arbitrate or redeem their savings.

It can therefore be assumed that the sum of discounted future cash flows distributed by the insurer is in all cases greater than or equal to the $PM(0)$.

In addition, the rational insurer will seek to maximize its expected utility and avoid paying more than the maximum initial wealth to avoid using its own funds. To do this, the economic value of its minimum commitment to policyholders must be less than the mathematical provision plus the maximum initial wealth.

This economic value consists of evaluating the sum of guaranteed minimum future cash flows and is necessary to assess the economic equilibrium of managed savings contracts. This is what we call in the following: the minimum contractual best-estimate.

4.1.2 Valuation of the minimum contractual best-estimate

We define the contractual best-estimate as the sum of discounted guaranteed future cash flows. It corresponds to the scenario where the insurer pays the minimum rates on which it has a commitment (or it intends to commit).

Let $TMG = \{tmg_i\}_{i \in [\![1,T]\!]}$ be the anticipated curve of guaranteed minimum rates such that tmg_i is the expected deterministic guaranteed rate between time $i-1$ and i. Let us consider the notations of section 2.2.3. By definition, we have: $c_i = tmg_i$ for each time i.

Since revaluation rates are deterministic, f_t and g_t are also deterministic[12].

Note also that:

$$E^{P^h \otimes Q}(\psi(t)) = exp\left(\sum_{i=1}^{t} tmg_i\right) E^Q\left(\exp\left(-\sum_{i=1}^{t} r_i\right)\right) = exp\left(\sum_{i=1}^{t} tmg_i\right).P(0,t)$$

Therefore we prove that:

$$BE_{contr}(0) = PM(0).\sum_{t=1}^{T}\left(f_t^{(tmg_i)} + \iota g_t^{(tmg_i)}\right) P^M(0,t).exp\left(\sum_{i=1}^{t} tmg_i\right)$$
$$+ g_T^{(tmg_T)}.P^M(0,T).exp\left(\sum_{i=1}^{T} tmg_i\right)$$

where $P^M(0,t)$ is the price of a risk-free zero-coupon bond observed on the market paying a monetary unit at maturity t and ι is the loading rate.

Note that:

- This formulation of the contractual best-estimate does not depend on ESG models;
- The contractual best-estimate represents the minimal commitment of the insurer. It indicates in particular the level of the guarantee when the promised rates are significant;
- The contractual best-estimate is completely determined by the policy and the contract price of the insurer (tmg_i and ι), the biometric structure and the behavioural structure (f_t and g_t);
- The contractual best-estimate represents the mandatory minimum value of discounted future cash flows. If the best-estimate is strictly lower than the guaranteed best-estimate, two cases occur:
    - At least one revaluation rate is lower than the guaranteed rate, which means the insurer's default;
    - There is no revaluation rate lower than the guaranteed rate: this means that the origin of the lowering lies in the behaviour of policyholders (dynamic surrender). But this behaviour is not rational in the sense that it does not maximize the expected utility[13].

[12] Assuming that dynamic lapse is a deterministic function of the revaluation rate which is a stochastic variable.
[13] If the average revaluation rates are higher than the guaranteed rates, policyholders are certain to have a best-estimate higher than the guaranteed best-estimate while keeping the same behaviour: $f_t^{(tmg_i)}$ and $g_t^{(tmg_i)}$.

- The contractual best-estimate can be interpreted as the minimum amount that the insurer can invest in risk-free assets while being certain to honour its contractual commitments;
- The contractual best-estimate allows to study the moneyness of the contract and to evaluate the difference between the guarantee and the surrender value (see section 8).

Note that the contractual best-estimate is different from the guaranteed best-estimate (noted BEG for "*best-estimate garanti*") as it is defined in ACPR [2013]. The contractual best-estimate is indeed a possible scenario, especially when the returns of the assets are not sufficient to cover the guaranteed rates. The BEG is a calculated value, corresponding to no possible scenario. It indicates the guaranteed part of the regulatory best-estimate. Indeed, the method of calculation of BEG proposed in ACPR [2013] is presented in four steps:

- Step 1: extraction of non-revaluated cash flows used in the calculation of the best-estimate for each scenario and for each time step (deaths, structural surrenders, dynamic surrenders ...) [14];
- Step 2: revaluation of these cash flows by guaranteed returns. These guaranteed returns incorporate the technical rates and the profit-sharing reserve. Savings are reduced annually by any contractual loadings;
- Step 3: discounting the guaranteed cash flows by the interest rates specific to each scenario;
- Step 4: the BEG is equal to the average of the values obtained on all the scenarios.

This construction of the BEG is therefore based on cash flows extracted from the scenarios used for the calculation of the best-estimate before the integration of guaranteed returns. In this case the series f_t and g_t are dependent on the stochastic paths. The BEG is an indication of the guaranteed part of the best-estimate and not a measure of the insurer's minimum commitment.

In conclusion, the minimum value of the best-estimate is written:

$$BE_{min}(0) = max\big(PM(0), BE_{contr}(0), PM(0) + Wealth_{min}^{init}(0)\big)$$

Assuming that the insurer maximizes its utility and is careful to preserve its own funds, the maximum value of the best-estimate is written as:

$$BE_{max}(0) = max\big(PM(0), BE_{contr}(0), PM(0) + Wealth_{max}^{init}(0)\big)$$

If $BE_{max}(0) = BE_{min}(0) = BE_{contr}(0)$ this means that the insurer will mobilize a part of its own funds to honour its commitments conditionally to the state of the world at time 0.

Note that these two best-estimate bounds are independent of the choice of ESG models.

[14] This corresponds to series f_t and g_t introduced in this article and which are stochastic variables.

4.1.3 The best-estimate depends on the policy of the insurer

The policy of the insurer impacts the level of the best-estimate.

The following example illustrates how steering the initial wealth can have a significant impact on the best-estimate.

Let A and B be two insurers with identical characteristics:

- They have the same assets and liabilities;
- They operate in the same economic environment;
- They have enough wealth to serve the guaranteed rates.

Having the same portfolio structure, the best-estimates of both insurers have the same maximum and minimum limits.

Let $BE^A(0) < BE_{max}(0)$ be the best-estimate of the company A.

Whatever the state of the economy, the insurer B can choose a target best-estimate such as: $BE^A(0) < BE_{target}(0) < BE_{max}(0)$ by serving a spread a which is the solution of the following equation (see section 7):

$$BE_{target}(0) = PM(0)E^{P^h \otimes Q}\left(\sum_{t=1}^{T}(f_t^a + \iota g_t^a).\psi_a(t) + g_T^a.\psi_a(T)\right)$$

with: $\psi_a(t) = \exp\{\sum_{i=1}^{t} c_i - \sum_{i=1}^{t} r_i + t.a\}$ and:

- c_i: revaluation rate of the savings net of loadings at time i paid by insurer A. This rate must be higher than the guaranteed minimum rate;
- r_i: is the risk-free rate for the period between $i-1$ and i.

Insurer B, by a discretionary decision, shares more initial wealth than insurer A. B's best-estimate is different from A's best-estimate, whereas the exposure of both insurers to risks is identical.

This example illustrates that a management action on wealth-sharing policy can have a significant impact on the best-estimate, function of the spread level a.

The difference between the best-estimate of A and B can be significant and vary on average between 0% and 17.6% of the mathematical provision based on the 2016 market statistics. This difference, which is significant, can be defined independently of the ESG models.

4.1.4 Conclusion

We have shown in this section that the insurer's policy can have a significant impact on the best-estimate regardless of the universe of biometric, behavioural or economic risks.

In a predefined risk environment, the best estimate of a savings portfolio in € is not unique. The instantaneous transfer of the portfolio from one company to another instantly changes the amount of the best-estimate if the management policies are different, even if the economic environment remains unchanged.

There is therefore no economic value of an insurance liability but an infinity of values each representing a subjective policy of managing savings contracts in € by the insurance company. The best estimate is not the economic value of the liability. It best represents the economic value of the insurance commitments conditionally on the insurer's management policy.

The process of calibration and validation of the economic scenario generator, used to evaluate the best-estimate, by comparing the simulations to the data observed in a statistical approach, cannot be considered. In fact:

- The best-estimate is not observed in a deep and liquid market;
- We can obtain different best-estimate values for the same underlying risks: a complete market for the best-estimate as defined in Solvency 2 framework cannot exist.

Moreover, the best-estimate is bounded. Its boundaries do not depend on ESG models.

Thus, whatever the choice of the ESG, it is possible to control the revaluation rates to reach a predefined target-best-estimate between the minimum value and the maximum value of the best-estimate.

It is therefore questionable to link an ESG to a best-estimate without specifying the revaluation policy of the insurer.

It is also questionable to use certain interest rate derivatives to calibrate the ESG without linking these derivatives to: the insurer's optional structure of liabilities, its assets and its policy.

Table 10 shows the allocation of insurers' assets at market value at the end of 2016 (see FFA [2017]).

Table 10: investments of insurance companies at the end of 2016

Investments of insurance companies at the end of 2016	*€ Billions*	*Allocation*
Equity	401	17%
Corporate Bonds	907	39%
Bonds issued or guaranteed by States	773	33%
Real estate	97	4%
Monetary	123	5%
Other	49	2%
Total	*2,350*	*100%*
Life and mixed societies	2,114	90%
Non life / Casualty	236	10%

An ESG that prices bonds, equities, real estate investments and monetary assets covers 98% of the assets of insurance companies and allows to simulate risk-free rates.

Then a question arises: an ESG built and calibrated to reproduce prices of derivatives such as caps, floors and swaptions, is-it relevant for the valuation of the best-estimate?

The ESG designed to evaluate the best-estimate is therefore intended to evaluate the options and guarantees of the savings contract. The underlying of these options is the insurer's assets. We seek in the following to explain the nature of these options and guarantees in order to establish a link between the ESG and the best-estimate.

4.2 The optional structure of savings liabilities in € and its link with the ESG

4.2.1 Definition of scope and analytical framework

As explained in section 2.2.2, the best-estimate evaluation incorporates four components:

- Component 1: definition of the mortality and surrender tables;
- Component 2: definition of the behavioural structure of policyholders;
- Component 3: definition of the behavioural structure of the insurer;
- Component 4: construction of an economic scenario generator.

Without making any assumptions on points 1, 2 and 4, we have shown in section 4.1.3 that the best-estimate may vary significantly depending on the insurer's revaluation policy and that this variation may be totally independent of the ESG model.

More generally, if the components 1, 2 and 4 are completely defined, then we can build a correspondence between the revaluation rates and target best-estimates (see section 4.1.3).

We can generalize this observation as follows:

- The best-estimate can only be defined if the four components above are defined;
- To build a correspondence between a component and the best-estimate it is necessary to define the 3 other components;
- In a calibration process (of components 2, 3 or 4) it is necessary (1) to define three components and (2) to have best-estimate values to deduce the parameters of the component that we are trying to calibrate.

As presented in section 2, in the absence of best-estimate observations, the calibration of interest rate models is conventional and consists of using interest rate derivatives (caps, swaption, etc.) to infer the parameters.

The link between these derivatives and the best-estimate is questionable. Indeed, the function that links the best-estimate to the parameters of the ESG depends on the components: 1, 2 and 3. The ESG settings may be inconsistent with the objective of evaluating the best-estimate if these components are not involved in the calibration process.

In this section we seek to build a link between the optional structure of the best-estimate and the ESG. Our analysis is placed in a framework defined as follows:

- Insurance risks (structural surrender and mortality): are assumed to be mutualized and independent of the economic and financial environment;
- Customer behaviour: the dynamic surrender is assumed to be nil;
- Insurer's policy:
    - Investment policy: the insurer transfers the policyholder investment directly to the financial market. Every year, the company sells and buys

instantaneously its assets. The composition and risk profile of the asset are stable over the entire projection period;

- The initial wealth consists only of the provision for profit sharing and is invested on non-risky assets. The initial wealth that the insurer decides to distribute will be liquidated at a fixed rate noted a;
- Revaluation rate: each year the insurer revaluates the savings with the financial rate of return and a spread (a) defined on the basis of the initial wealth;
- Loadings and expenses: the insurer applies a loadings rate on the savings acquired. The rate of loadings is equal to the rate of expenses;
- The guaranteed minimum rates are deterministic and denoted $TMG = \{tmg_i\}_{i\in[\![1,T]\!]}$. The minimum rates of return on the assets, that the insurer must have, to honour its obligations are therefore $K = \{k_i\}_{i\in[\![1,T]\!]} = \{tmg_i - a\}_{i\in[\![1,T]\!]}$.

4.2.2 The financial optional structure implicit to a savings contract in €

Let s_t be the financial yield of the insurer's asset. At each time $t+1$ we have:

$$\begin{aligned} VR(t+1) &= VR(t).\exp(max(k_{t+1}, s_{t+1}) + a) \\ &= VR(t).\mathrm{e}^{a}.e^{s_{t+1}} + VR(t).\mathrm{e}^{a}.max(0, e^{k_{t+1}} - e^{s_{t+1}}) \end{aligned}$$

At each time $t+1$ the payoff is similar to a vanilla put option whose nominal value is the surrender value at time t, revaluated with the spread a.

The payoff depends on the savings acquired at time t. The guarantee relates to a stochastic nominal.

The financial optionality that is implicit to the best-estimate is therefore different from that of the floor guarantee. The latter relates to a constant nominal that is known at subscription. The assessment of the best-estimate of a savings contract in € by the puts method as it is the case in floor guarantee is inappropriate. Indeed, for a time $0 < t$ the guaranteed value is not deterministic. It depends on the financial returns distributed over the entire path:

$$VR(t) = PM(0).\mathrm{e}^{a.t}.\exp\left(\sum_{i=1}^{t} max(k_i, s_i)\right)$$

The financial optionality implied to the surrender value has a cliquet structure (also known as ratchet structure). It is therefore path-dependent.

4.2.3 What is a ratchet-option?

Ratchet options (also called "cliquet" options) are financial derivative contracts that offer an annual guaranteed minimum return each year during the life of the contract. These are exotic options consisting of a series of consecutive forward start options.

These options reduce the risk of losses while offering the opportunity to profit from a potential increase.

The yield s_t of an asset, whose price process is S_t, over a period from $t-1$ to t is:

$$s_t = \frac{S_t}{S_{t-1}} - 1$$

The yields defined by: $\bar{s}_t = max\,(min\,(s_t, C_t), F_t)$ are truncated yields. They are capped at a level C_t and are at least equal to F_t ($F_t < C_t$). The absence of floor and / or cap corresponds to $F_t = -1$ and $C_t = +\infty$.

The general expression of the payoff Z_T of a ratchet option of maturity T indexed on S_t is:

$$Z_T = B.\exp\left(min\left(max\left(\sum_{t=1}^{T} \bar{s}_t\,, F_g\right), C_g\right)\right)$$

where :

- F_g is the global floor representing the overall minimum yield over the entire period from 0 to T;
- C_g is the overall cap representing the overall maximum yield over the entire period from 0 to T;
- B is the notional.

For $C_g = +\infty$, the general expression of the payoff is written:

$$Z_T = B.\exp\left(max\left(\sum_{t=1}^{T} \bar{s}_t\,, F_g\right)\right)$$

By posing: $F_g = \sum_{t=1}^{T} F_t$ and $C_t = +\infty$ we can write:

$$Z_T = B.\exp\left(max\left(\sum_{t=1}^{T} \bar{s}_t\,, F_g\right)\right) = B.\exp\left(\sum_{t=1}^{T} \max(s_t, F_t)\right)$$

This formula is similar to the one presented for the surrender value in the previous section.

The expected present value of the payoff of a ratchet option with maturity T whose minimum guaranteed rate at t is F_t is written:

$$Y_T = E^Q\left(\exp\left(\sum_{t=1}^{T} \max(s_t, F_t) - \sum_{t=1}^{T} r_t\right)\right)$$

We will not present in this paper a review of the financial literature on the valuation of ratchet options. We just present some observations:

- Sensitivity of the price of the ratchet option to the pricing model: Wilmott [2002] shows that the ratchet options are sensitive to the models and parameters of the underlying asset dynamics;
- The price sensitivity of the option to the calibration data: Windcliff and al. [2006] explores various modelling alternatives of the underlying of ratchet options. They find that a model calibrated on vanilla options does not correctly price the exotic

options. Assuming that the underlying follows a Merton jump model, Windcliff and al. [2006] show that a correction of the vanilla implied volatility may lead to more precision when valuing ratchet options;

- The interest rate models used in consulted papers are essentially deterministic. Ahlip and Rutkowski [2008] propose a forward start option valuation approach in a framework where the interest rate follows a CIR model and volatility follows a Heston model;
- The volatility model of the underlying is a fairly recurrent concern of the articles consulted;
- Kjaer [2004] presents a particular case of ratchet options and develops closed formulas to value them under certain hypotheses in the case where the strikes are constant;
- Windcliff and al. [2006] propose numerical methods to evaluate ratchet options for a set of mathematical models of the underlying.

4.2.4 The expression of the best-estimate according to ratchet options

Let Y_t be the price of a ratchet option with:

$$Y_t = E^Q\left(\exp\left(\sum_{i=1}^{t}\max(s_i, F_i) - \sum_{i=1}^{t} r_i\right)\right)$$

As presented in section 4.2, the total return on the asset is distributed. We can write then:

$$F_i = k_i + \iota$$

$$c_i = \max(s_i, F_i) + a - \iota$$

The distributed yield c_i is therefore greater than or equal to the guaranteed rate tmg_i net of loadings.

The probability P^h is independent of the financial randomness since the dynamic surrenders are assumed to be nil and the policy of the insurer does not affect the financial return. So we have:

$$E^{P^h \otimes Q}\big(\psi(t)\big) = E^Q\left(\exp\left\{\sum_{i=1}^{t} c_i - \sum_{i=1}^{t} r_i\right\}\right)$$

$$= E^Q\left(\exp\left\{\sum_{i=1}^{t}\max(s_i, F_i) + a.t - \iota.t - \sum_{i=1}^{t} r_i\right\}\right)$$

then:

$$E^Q\big(\psi(t)\big) = \mathrm{e}^{a.t}.e^{-\iota.t}.Y_t$$

therefore:

$$BE_a(0) = PM(0).\left(\sum_{t=1}^{T}\left(f_t + \iota g_t\right).\mathrm{e}^{at}.e^{-\iota.t}.Y_t + g_T.\mathrm{e}^{aT}.e^{-\iota.T}.Y_T\right)$$

The best-estimate is written as a sum of ratchet options:

- whose nominal values are: $N_t = PM(0)\left(f_t + \iota g_t\right).\mathrm{e}^{a.t}.e^{-\iota.t}$ if $t < T$ and $N_T = PM(0)\left(f_T + (\iota+1)g_T\right)\mathrm{e}^{a.t}.e^{-\iota.T}$;
- whose underlying is the insurer's asset.

Note also that the best-estimate net of loadings is written:

$$BE_a^{net}(0) = PM(0).\left(\sum_{t=1}^{T} f_t.\mathrm{e}^{a.t}.e^{-\iota.t}.Y_t + g_T.\mathrm{e}^{a.t}.e^{-\iota.T}.Y_T\right)$$

To hedge its commitments in order to serve the guaranteed rates, the insurer can buy ratchet options with the following characteristics:

- Strikes are equal to $\{F_i\}_{i\in[\![1,T]\!]}$;
- Each notional is defined as: $N_t^{net} = PM(0).f_t.\mathrm{e}^{a.t}.e^{-\iota.t}$ if $t < T$ and $N_T^{net} = PM(0)\left(f_T + g_T\right).\mathrm{e}^{a.t}.e^{-\iota.T}$.

The total cost of these options is:

$$Garantee\_cost = \sum_{t=1}^{T} N_t^{net}.(Y_t - 1)$$

Assuming that the loadings on outstanding amounts are only used to cover the costs of the asset management (the costs of structure, administrative management, etc., are nil) then, the total cost of the hedge is equal to the best-estimate of expenses:

$$BE^{expenses}(0) = PM(0).\left(\sum_{t=1}^{T} \iota g_t.\mathrm{e}^{a.t}.e^{-\iota.t}.Y_t\right) = \sum_{t=1}^{T} N_t^{net}.(Y_t - 1)$$

4.2.5 Characterization of implied financial options of the best-estimate

The optional financial structure of the best-estimate is totally characterized by the ratchet options composing it. It is sufficient to describe the elementary optional structure of a ratchet option to deduce that of the best-estimate.

In the following we present three analysis frameworks. From this we deduce that the optional structure of the best-estimate is consistent with:

- An optional start-forward vanilla structure on risky equity-like assets;
- An optional structure by floorlets on bond assets.

Therefore, in the framework presented in section 4.2.1:

- The calibration of interest rate models intended for the valuation of the best-estimate may be consistent with a calibration on floorlets;

- Calibration of the equity-like models can be consistent with calibration on vanilla options.

4.2.5.1 *The insurer's assets are treated as an equity index*

In the case where the insurer's assets are treated as an equity index, the valuation of ratchet options can be carried out using mathematical equity-like models. The parameters of these models must represent the characteristics of the insurer's assets. A classic example is the Black-Scholes model.

The financial literature is rich in references dealing with the issues of valuation of ratchet options whose underlying is an equity-like index. The reader can refer, for example, to Kjaer [2004] who presents closed formulas for the valuation of ratchet options on an equity index.

In practice, the calibration of such models in a market-consistent framework by considering market expectations cannot be realized. Derivatives whose underlying asset is the insurer's asset are not traded in liquid and deep markets.

4.2.5.2 *The insurer's assets are treated as a basket of bonds*

Let's assume that the insurer's assets consist of a basket of risk-free bonds with floating coupons indexed to risk-free interest rates.

Let T be a maturity of interest and let Y_T be a ratchet option with a strike vector $\{F_t\}_{1\leq t\leq T}$. This option corresponds to the best-estimate cash flow at time T.

In order to replicate the payoff of this ratchet option, the insurer can:

- Buy a variable coupon bond with maturity T indexed on the risk-free interest rate of which the one-year forward rate curve is given by $\{b_t\}_{1\leq t\leq T}$ [15];
- Reinvest coupons on variable risk-free rates and the residual maturity;
- Buy floorlets whose strikes are $\{F_t\}_{1\leq t\leq T}$ and whose nominal is variable: a kind of "ratchet floor".

Indeed, at each time $t \leq T$, the value of the asset backed by the maturity T is written:

$$B_{t+1} = B_t.(1 + max(b_{t+1}, F_{t+1})) = B_t(1 + b_{t+1}) + B_t(F_{t+1} - b_{t+1})^+$$

and: $B_0 = 1$ is the initial investment in a risk-free floating rate bond.

The second term of the formula above corresponds to the payoff of a floorlet whose nominal is B_t.

This expression shows that:

- The optionality of the liability is consistent, under certain conditions, with floorlets;
- The payoffs of cliquet options on an asset, consisting of a basket of bonds, can be evaluated by recurrence based on the valuation of floorlets. Some models, like the Black model or the LMM model, allow the pricing of floorlets by closed formulas.

[15] The annual interest rate between t and t+1 is: b_{t+1}.

4.2.5.3 *The insurer's assets consist of bonds and risky assets*

Let T be a maturity of interest and let Y_T be the price of a ratchet option on the insurer's assets whose strikes are $\{F_t\}_{1\leq t\leq T}$.

Without loss of generality, let's assume that the insurer's asset consists of two families of financial assets:

- A risky equity-like asset (equity, real estate, infrastructure ...) whose initial price is noted $A_0 = 1$ and whose initial allocation is x_1. The annual yield between t and $t + 1$ is denoted a_{t+1};
- A risk-free asset (bond-like) whose initial price is $B_0 = 1$ and whose initial allocation is x_2. The rate of return at each time is the one year forward risk-free interest rate denoted $\{b_t\}_{1\leq t\leq T}$.

In order to replicate the payoffs of the ratchet option, the insurer can buy:

- A risky equity asset and a ratchet option on this asset with strikes: $\{F_t^A\}_{1\leq t\leq T}$;
- A variable coupon bond indexed on the risk-free interest rate associated with floorlets whose strikes are $\{F_t^B\}_{1\leq t\leq T}$ and whose nominal values are variable.

At each time $t \leq T$, the prices of the assets are written then:

$$A_{t+1} = A_t + A_t.max(a_{t+1}, F_{t+1}^A)$$

$$B_{t+1} = B_t.(1 + max(b_{t+1}, F_{t+1}^B)) = B_t(1 + b_{t+1}) + B_t(F_{t+1}^B - b_{t+1})^+$$

The total asset is written:

$$S_{t+1} = S_t + S_t.max(x_1.a_{t+1} + x_2.b_{t+1}, F_{t+1})$$

At each time, the exercise prices F_t^A and F_t^B can be defined so that the asset is revaluated at least by F_{t+1}. Several solutions are possible. They depend essentially on the nature of the assets, their dependency structure and the profitability objectives for each asset. This issue is out of the scope of this paper.

In the case where $F_t = x_1.F_t^A + x_2.F_t^B$, the replication of payoffs is conservative. Indeed, we can note that:

$$S_t.max(x_1.a_{t+1} + x_2.b_{t+1}, F_{t+1}) \leq x_1.A_t.max(a_{t+1}, F_{t+1}^A) + x_2.B_t.max(b_{t+1}, F_{t+1}^B)$$

Equality can be obtained in the example where $a_t = b_t$ and $F_t^A = F_t^B$.

We thus note that the implicit optional structure of the best-estimate can be decomposed into vanilla options on equity-like assets and floorlets on bonds.

4.2.6 Equilibrium equations in an arbitrage free framework

In the following, we assume that expenses rate ι only reflects the cost of the asset management of savings contract options and guarantees in order to ensure the company's ability to meet its obligations.

In a market without arbitrage opportunities, the sum of future cash flows distributed by the insurer must be equal to the initial investment.

Policyholders invest at $t = 0$ the mathematical provision $PM(0)$ and do not benefit from any initial wealth: $a = 0$. So, the best-estimate is equal to the $PM(0)$:

$$BE(0) = PM(0) = PM(0).\left(\sum_{t=1}^{T}\left(f_t + \iota g_t\right).e^{-\iota.t}.Y_t + g_T.e^{-\iota.T}.Y_T\right)$$

In order to have a polynomial writing in e^{ι} in the following developments, let us take again the approximation of the section 2.2.4: $e^{\iota} - 1 \simeq \iota$. We can write then:

$$BE(0) = PM(0) = PM(0).\left(\sum_{t=1}^{T}\left(f_t + (e^{\iota} - 1).g_t\right).e^{-\iota.t}.Y_t + g_T.e^{-\iota.T}.Y_T\right)$$

The best-estimate is a polynomial function of order T of $e^{-\iota}$.

In addition to the output factors related to mortality and surrender, we note three vector variables in the formula above:

- ESG parameters: θ_4;
- Guaranteed rates K ;
- The price of the hedge (what we can also call the contract price): ι.

Let's note:

- $\Gamma(\theta_4, k, \iota) = PM(0)\left(\sum_{t=1}^{T}\left(f_t + (e^{\iota} - 1)g_t\right).e^{-\iota.t}.Y_t + g_T.e^{-\iota.T}.Y_T\right) - PM(0)$;
- $\Delta(\theta_4, k, \iota) = PM(0).\left(\sum_{t=1}^{T}(e^{\iota} - 1)g_t.e^{-\iota.t}.Y_t\right) - \sum_{t=1}^{T} N_t^{net}.(Y_t - 1)$.

The following system of equations reflects two properties: the best-estimate is equal to the mathematical provision in the absence of arbitrage opportunity and the cost of the guarantee is equal to the prices of the ratchet options. It is written:

$$\begin{cases}\Gamma(\theta_4, k, \iota) = 0\\ \Delta(\theta_4, k, \iota) = 0\end{cases}$$

This system cannot characterize all the parameters used in the calculation of the best-estimate because the number of these parameters is greater than the number of equations to solve.

If we have an ESG and want to price a savings contract with an initial investment of $PM(0)$ at $t = 0$ then:

- The guaranteed minimum rates K and the expense rate ι are the solutions of the equation system presented above;
- If all guaranteed rates of the vector K are equal (the same guaranteed rate k over the entire lifetime of the contract) then the system of equations may admit a solution (k, ι). This pair of guaranteed rate (k) & cost (ι) characterizes the economic balance of the savings contract in an arbitrage free framework.

Moreover, in a hypothetical[16] framework with no arbitrage opportunity, where the economic price ι of a contract with a guaranteed rate k, is available, the parameters of the ESG verify the two-equation system above. It is clear that if the number of parameters of

[16] This framework does not exist in practice because there are no markets to trade best-estimates.

the ESG is greater than or equal to 3 parameters, the system of equation can admit an infinity of solutions.

We conclude then that in an arbitrage free framework, there may be an infinity of possible parameters of the ESG that are consistent with the economic characteristics of a savings contract in € (the couple guaranteed rate / cost rates).

Therefore, a direct link between the ESG parameters and the best-estimate cannot be established, even if the insurer's policy is predefined.

5 Conclusion

Applying a Mark-to-Market approach to evaluate the fair value of the insurer's commitment (best-estimate) for a saving French contract in €, implies having the prices of options and guarantees of insurance policies. Since this information is not observable on an organized and liquid market, the calculation is made in a conventional Mark-to-Model framework.

We are interested in this paper in this conventional framework. Our work focused on three points:

- An analysis of the best-estimate sensitivities to the choice of rate models and the choice of data (section 3);
- An analysis of the link between the best-estimate, the insurer's revaluation policy and the ESG (section 4.1);
- An analysis of the elementary financial optional structure of the best-estimate and its link with the ESG (section 4.2).

It appears from the analysis of the sensitivities of the best-estimate to the choice of rate models and the choice of data that the impact on the value of the latter remains fairly contained. The difference between the minimum and maximum value is 4.30%. It is although significant compared to own-funds: the mean of own-funds of French savings insurers is around 5%.

Particular attention must be given to the LMM model. This model, calibrated on the data observed on January 02, 2018, cannot be retained for the evaluation of the best-estimate because of its divergence.

The analysis of the link between the best-estimate, the insurer's revaluation policy and the ESG concluded that:

- The best estimate of a savings portfolio in € is not unique. The best estimate is not the economic value of the liability. It represents at best the economic value of the insurance commitments conditionally on the insurer's revaluation policy;
- The process of calibrating and validating the ESG, by comparing the simulations with the data observed in the context of a statistical approach, cannot be considered since the best-estimate is not observed in a deep and liquid market. Such a market cannot exist in the absence of the uniqueness of the best-estimate conditionally to risks;
- The best-estimate is bounded. Its boundaries do not depend on ESG models;

- Whatever the choice of the ESG, it is possible to monitor the revaluation rates to reach a target best-estimate.

In the analysis framework presented in section 4.2.1, the study of the elementary financial option structure of the best-estimate and its link with the ESG concluded that:

- The financial optional structure of the best-estimate is fully characterized by ratchet options;
- The calibration of the interest rate models intended for the valuation of the best-estimate is consistent, under certain conditions, with a calibration on floorlets;
- The calibration of the equity-like models is consistent, under certain conditions, with a calibration on vanilla options;
- In a hypothetical arbitrage free framework where we can observe the cost of guarantees, there is infinite choices of ESG parameters that reproduces this cost of guarantees;
- A direct link between the parameters of the ESG and the best-estimate cannot be established, even if the insurer's revaluation policy is predefined.

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7 Appendix 1: best-estimate and initial wealth distribution

Let’s denote:

$$B_{sto}(a) = PM(0)\left(\sum_{t=1}^{T}(f_t^a + \iota g_t^a).\psi_a(t) + g_T^a.\psi_a(T)\right)$$

Where: $\psi_a(t) = \exp\{\sum_{i=1}^{t} c_i - \sum_{i=1}^{t} r_i + t.a\}$ and:

- c_i: the rate of revaluation net of loadings at time i. This rate must be higher than the guaranteed minimum rate;
- r_i is the risk-free rate for the period between $i-1$ and i;
- a is a real number.

Let's denote:

$$BE_a(0) = E^{P^h \otimes Q}\left(B_{sto}(a)\right)$$

In particular, for $a = 0$ we find the best-estimate $BE(0)$.

<u>Proposition:</u>

Let x be a positive real number such that $BE\ (0) < x$. Then there is a positive real number a such that $BE_a(0) = x$.

Indeed, the function $BE_a(0)$ is continuous and its limit when a tends positively to infinity is infinite.

<u>Solution by approximation of the equation $BE_a(0) = x$:</u>

Let's get close to $BE\ (0)$ and look for an approximation of a such that $BE_a(0) = x$.

Assume that the variability of f_t and g_t with respect to a is negligible when the variation of a is small:

$$dBE_a(0) = PM(0).E^{P^h \otimes Q}\left(\sum_{t=1}^{T}(f_t + \iota g_t).d\psi_a(t) + g_T.d\psi_a(T)\right)da$$

then:

$$dBE_a(0) = PM(0).E^{P^h \otimes Q}\left(\sum_{t=1}^{T}t(f_t + \iota g_t).\psi_a(t) + Tg_T.\psi_a(T)\right)da$$

therefore:

$$\frac{dBE_a(0)}{BE_a(0)} = \frac{E^{P^h \otimes Q}\left(\sum_{t=1}^{T}t(f_t + \iota g_t).\psi_a(t) + Tg_T.\psi_a(T)\right)}{E^{P^h \otimes Q}\left(\sum_{t=1}^{T}(f_t + \iota g_t).\psi_a(t) + g_T.\psi_a(T)\right)}da$$

The duration of the best-estimate is written:

$$D_{BE}(a) = \frac{E^{P^h \otimes Q}\left(\sum_{t=1}^{T}t(f_t + \iota g_t).\psi_a(t) + Tg_T.\psi_a(T)\right)}{E^{P^h \otimes Q}\left(\sum_{t=1}^{T}(f_t + \iota g_t).\psi_a(t) + g_T.\psi_a(T)\right)}$$

so:

$$\frac{dBE_a(0)}{BE_a(0)} = D_{BE}(a)da$$

Assuming that the duration variability with respect to a is small near $BE\ (0)$ then:

$$\frac{dBE_a(0)}{BE_a(0)} \backsimeq D_{BE}(0)da$$

and:

$$BE_a(0) \backsimeq BE_0(0)\exp(D_{BE}(0)a)$$

Thus, for small variations of a we can have an indication on the solution of $x = BE_a(0)$:

$$a \backsimeq \frac{ln(x) - ln(BE(0))}{D_{BE}(0)}$$

This solution is particularly useful for initiating the optimization algorithm in case of numerical resolution of the equation: $x = BE_a(0)$.

This expression can be interpreted as follows: the distribution rate a of a wealth A corresponding to the difference between a calculated best-estimate $BE(0)$ and a target best-estimate x corresponds to this wealth A, divided by the duration of the portfolio.

8 Appendix 2: Moneyness of a savings contract liability in €

In the financial literature, the definitions of the moneyness agree to take into account, in its valuation, the relationship between the strike and the market price of the underlying asset at the maturity of the option. For example:

- Goncalves & Guidolin [2006] and Bernales & Guidolin [2014] propose the following definition of the moneyness denoted M for the maturity T: $M = \frac{\ln\left(\frac{K_T}{S_T}\right)}{\sqrt{T}}$ where S_T is the price of the underlying at time T.
- In other references, the moneyness is simply defined as a relationship between the strike and the market price of the underlying (Cont & Fonseca [2001] and Christoffersen & Jacobs [2002]): $M = \frac{K_T}{S_T}$.

In the case where the underlying pays dividends, one can write:

$$S_T = \exp(r_T T)\,.\,S_0 - FVD_T$$

The term FVD_T corresponds to future dividends until the expiry of the option and therefore corresponds to the capitalized value of the dividends paid between 0 and T.

If the underlying does not pay dividends between 0 and T then: $S_T = \exp(r_T T)\,S_0$.

Note that in the latter case:

$$\frac{K_T}{S_T} = \frac{K_T}{\exp(r_T T)\,S_0} = \frac{K_T\exp(-r_T T)}{S_0} = \frac{K_0}{S_0}$$

The moneyness can be defined as the ratio between the discounted value of the strike and the initial investment.

Note also that an insurance contract is a capitalization contract. It does not pay dividends.

The sum of discounted guaranteed future cash flows (the strikes being materialized by the guaranteed rates) is the $BE_{contr}(0)$ (see section 4.1.2).

For the policyholder, the initial value of the investment is the $PM(0)$. We define the moneyness of an insurance contract as the ratio between the contractual best-estimate and the mathematical provision at $t = 0$:

$$Moneyness_{policyholder}(0) = \frac{BE_{contr}(0)}{PM(0)}$$

The insurer must distribute an initial wealth at its discretion. This wealth can be added to the $PM(0)$ and plays the role of the initial investment. The moneyness is written in this case:

$$Moneyness_{insurer}(0) = \frac{BE_{contr}(0)}{PM(0) + Wealth_{init}}$$