Building Models for Credit Spreads

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Abstract *: We present and study a modelling framework for the evolution of credit spreads. The credit spreads associated with a given rating follow a multidimensional jump-diffusion process while the movements from a given rating to another one are modelled by a continuous time Markov chain with a stochastic generator. This allows for a comprehensive modelling of risky bond price dynamics and includes as special features the approaches of Jarrow, Lando and Turnbull (1997), Longstaff and Schwartz (1995 and, Duffie and Kan (1996)*[2]*. The main appealing feature is the ability to get explicit pricing formulas for credit spreads, thus allowing easier implementation and calibration. We present examples based on market data and some empirical assessment of our model specification with historical time series.*

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[2] Duffie and Kan (1996) is not directly dedicated to the modelling of risky rates. However, it suffices to consider their short rate dynamics as a risky short rate dynamics to obtain tractable models of risky bonds.

1. Introduction

This paper presents a modelling framework for the evolution of the credit risk spreads which are driven by an underlying credit migration process plus some multidimensional jump-diffusion process[3]. This framework is appropriate for pricing credit derivatives such as risky bonds, default swaps, spread options, insurance against downgrading etc. These instruments therefore have payoffs that depend on various things such as default events, credit spreads and realised credit ratings. It is also possible to look for the effect of default or downgrading on the pricing of convertible bonds, bonds with call features, interest rate or currency swaps.

In order to design a model that can fulfil the above objectives it is necessary to consider the evolution of the risk free interest rates and of the credit spreads. In this analysis we will concentrate on developing a model for credit spreads, which can be coupled with any standard model for the risk free term structure such as Ho-Lee (1986), Hull-White (1990) or Heath, Jarrow and Morton (1992). To simplify the analysis we impose the restriction that the evolution of the credit spreads is independent of the interest rates[4].

Typically, the credit spread for a specific risky bond exhibits both a jump and a continuous component. The jump part may reflect credit migration and default, i.e. a discontinuous change of credit quality. Meanwhile, credit spreads also exhibit continuous variation so that the spread on a bond of a given credit rating may change even if the riskless rates remain constant. This may be due to continuous changes in credit quality, stochastic variations in risk premia (for bearing default risk) and liquidity effects.

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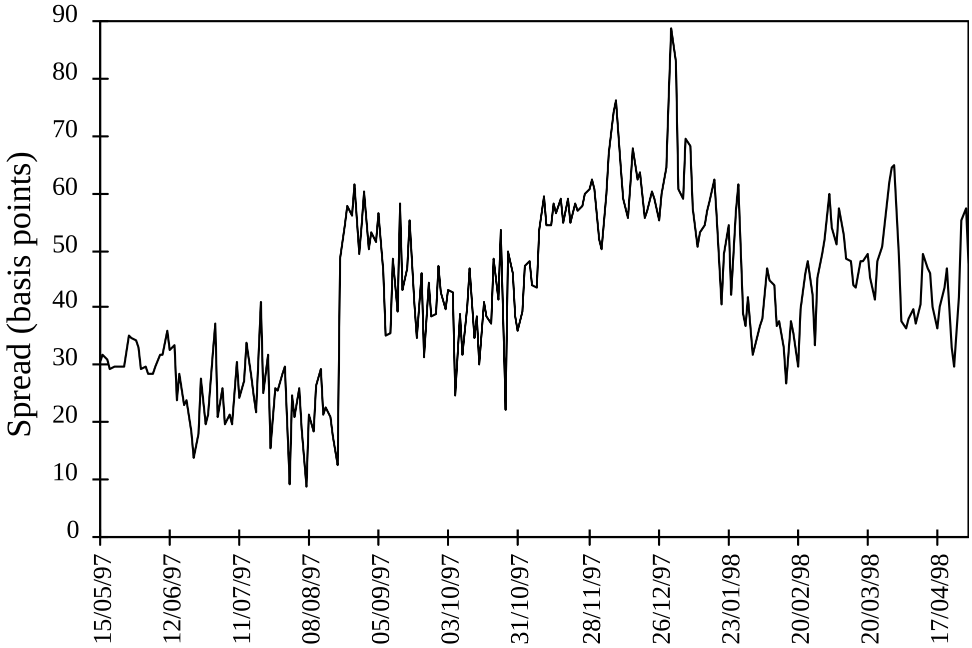
[3] We rely on hazard-rate models ; this allows to handle a wide variety of dynamics for credit spreads in a tractable way. The so-called structural approaches where default is modelled as the first hitting time of some barrier by the process of assets' value leads to some practical difficulties. It may be cumbersome to specify endogenously the barrier, to handle jumps in credit spreads or non zero-short spreads (see Duffie and Lando (1998) for a discussion).

[4] Our analysis can be expanded when there is some correlation between credit ratings and riskless rates. We have simply to assume that $r(s)E+\Lambda(s)$, where E is a square matrix with unit elements, has constant eigenvectors ; see further.

Figure 1. Credit spread for a AA rated bond.



We consider a model that takes into account these two effects. In that sense, it is a natural extension of the Jarrow, Lando and Turnbull (1997, JLT thereafter) model where the spreads for a given rating are constant and of models like Longstaff and Schwartz (1995), Duffie and Kan (1996) where the credit spread follows a diffusion or a jump-diffusion process. A similar model is also presented in Lando (1998)[5]. In this framework, it is possible to get some explicit pricing formulas for the prices of risky bonds. Duffie and Singleton (1998) propose a related model, but in their approach, simulation of the credit rating is required. In these Markovian models, the credit spreads and risk neutral default probabilities are uniquely determined by the state variables, some of them being discrete, i.e. credit ratings and following a Markov chain, while the others follow jump-diffusion processes. In addition, the credit spreads depends on the recovery rate in the event of default, that will be assumed to be constant for the sake of simplification[6].

As usual, calibration to market data is an important issue. It is simplified since we deal with explicit pricing formulas but still have the problem that market data can be sparse and there are a relatively large number of unknown parameters. We adopt a

[5] This model expands on a previous less general model of Lando (1994).

Bayesian approach where the prior is provided by historical information on credit migration and is marginally modified to fit prices of coupon bonds across different credit classes observed in the market. Thus, we are able to estimate a risk-neutral process. The inputs into the calibration algorithm are the prices of coupon bonds observed in the market across different credit classes and historical information on credit migration. The model can be used as a powerful stripping algorithm to generate yield curves consistently across asset classes by imposing an underlying economic structure. This is particularly useful in markets with sparse data.

In section 2, we present Markovian models of credit spreads dynamics. We start from the standard textbook example, where the credit spread is constant up to default-time. This model can be extended to allow credit spreads to be piece-wise constant as in JLT. We also present a state space extension of this model, in order to take into account credit rating time dependency (see Moody's (1997)). This allows a firm that has been recently upgraded to be assigned an upward trend and to therefore exhibit a lower credit spread than a firm with the same credit rating that has been recently downgraded. The previous models can be extended by considering a stochastic generator of the Markov chain, that depends on other state variables; in order to keep tractability, we consider a special family of generators where only the eigenvalues are stochastic. This framework allows explicit computation of credit spreads.

In section 3, we focus on implementation issues. We present a calibration algorithm in the JLT framework and provide some examples of fitted curves. In the more general model where the credit spreads have a diffusion component, we discuss calibration to bond prices, look for the dimension required to explain the credit spreads and consider the modelling assumption that the eigenvectors of the generator remain constant through time.

Section 4 describes the conclusions.

2. Modelling the credit spreads

[6] This assumption can be relaxed in different ways ; see further for a discussion.

In this section, we consider some approaches to the modelling of credit spreads, starting from the simplest case and developing the model in order to incorporate more realistic features of the dynamics of credit spreads.

One may first consider a model where there are only two states, default and no default. A risky discount bond promises to pay 1 unit at maturity T if there is no default ; in the event of default, the bond pays a constant recovery rate (δ) at maturity T[7]. Let us denote by $v(t,T)$ the price at time t of this risky bond, $B(t,T)$ the price of the risk free bond, and $q(t,T)$ the (risk-neutral) probability of default before time T as seen from time t. It is assumed that the default event is independent of the level of interest rates. This leads to the standard equation :

$$v(t,T) = B(t,T)\left[1 - q(t,T) + q(t,T)\delta\right] \tag{1}$$

Equivalently, the implied risk-neutral default probabilities are given by:

$$q(t,T) = \frac{1 - \dfrac{v(t,T)}{B(t,T)}}{(1-\delta)} \tag{2}$$

In this framework, the first time to default can be represented by the first jump of some non homogeneous Poisson process. This simple model is useful for pricing default swaps. To be practical, it requires the knowledge of the prices of risky zero coupon bonds issued by the counterparty on which the default swap is based, whose maturities equal the payment dates of the default swaps. This approach may be difficult to implement since a given counterparty has usually only a few outstanding coupon bonds traded and so it is not possible to know the prices of the risky zero coupon bonds directly.

[7] There are several standard ways to model the recovery. We follow here the presentation of JLT where in case of default, the holder of the risky bond receives a fraction δ of the riskless bond (with the same maturity). In Duffie and Singleton (1997), the recovery rate has a different meaning since, in case of default, the holder of a risky bond receives a fraction of the value just prior to default. At last, the holder of the risky bond may receive a fraction of par in case of default. The consequences of these assumptions are discussed below.

To be able to get over this requirement, one can make an important economic assumption such as “all firms in the same credit rating are on the same risky yield curve”. This allows us to take into account bonds issued by different firms in the same risky class as if there were a single issuer.

Instead of using a standard stripping procedure that deals with bonds within each credit class separately, we may look in greater detail at the changes in credit quality that lead to default. This more structural modelling approach will guarantee that the different risky curves will be consistently estimated ; moreover, it will be possible to use information coming from bonds in different credit classes to build up the risky curves.

The simplest model that considered only two possible states, default and no default can be expanded by introducing more states, such as credit ratings. The state dynamics can be represented by a continuous, time-homogenous Markov chain on a finite state space, $S = \{1,2,\cdots K\}$. This means that there are a finite number of possible states (K) and that being in a given state gives all the information relevant in the pricing of structures involving credit risk. In this modelling, the probability to go from one state to another depends only on the two states themselves (the so-called Markov property) and is assumed to be independent of time (time homogeneity). Such a model has been introduced by JLT and for the paper to be self contained, we briefly recall their presentation.

The first state is the best credit quality and the $(K-1)$ state the worst (before default). The (K) state represents default, which is an absorbing state and provides a payment of δ at maturity.[8] Once in this state, we impose the simplifying assumption that there is no chance of moving to a higher state, i.e. for the bankrupted firm to recover. For the purposes of the present paper, we will consider the states in the model to be equivalent to credit ratings although we emphasise that other descriptions are possible.

The transition matrices for any period from t to T, $Q(t,T)$ characterise the Markov chain. Its elements $q_{ij}(t,T)$ represent the probability to go from state i at time t and be in state j at time T. We will further make the modelling assumption that the transition matrices can be written as :

$$Q(t,T) = \exp\left(\Lambda(T-t)\right), \tag{3}$$

where Λ is the generator matrix Λ and is assumed to be diagonalisable[9] (i.e. $\Lambda = \Sigma D \Sigma^{-1}$ where D is a diagonal matrix)[10] ; the previous expression can be computed as :

$$Q(t,T) = \Sigma \exp\left(D(T-t)\right)\Sigma^{-1}, \tag{4}$$

where the diagonal terms in D and the columns of Σ represent respectively the eigenvalues and (right) eigenvectors of Λ[11]. Most importantly, we see the generator matrix now defines all possible default probabilities.

The generator matrix is the continuous time analogue of a discrete time, finite state transition matrix. It describes the evolution of the Markov chain during an infinitely small time period dt .

[8] For a coupon bond the assumption is that in the case of default the holder receives δ at maturity and $\delta\times$coupon at each coupon payment date.

[9] It is possible to consider transition matrices which do not admit generator matrices, which may not be diagonalisable and transition matrices which admit a generator which is not diagonalisable. This model thus restricts to tractable transition matrices.

[10] JLT allow for some time dependence in Λ , while keeping it deterministic.

[11] Since the rows of Λ sum up to zero, the unit vector is a right eigenvector of Λ associated to eigenvalue $d_K = 0$. The rows of Σ^{-1} are the left eigenvectors of Λ . There is one left eigenvector corresponding to $d_K = 0$ (the last row of Σ^{-1}) which can be interpreted as the invariant (or stationary) measure of the Markov chain. Since default is an attainable absorbing state, we can readily construct the invariant measure and deduce that the last row of Σ^{-1} is equal to $(0,0,\cdots,0,1)$. Since all last column terms are positive, it can be shown that this invariant measure is unique. In the long-term, regardless of their initial rating all firms go to default.

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & . & \lambda_{1,K-1} & \lambda_{1K} \\ \lambda_{21} & \lambda_{22} & . & \lambda_{2,K-1} & \lambda_{2K} \\ . & . & . & . & . \\ \lambda_{K-1,1} & \lambda_{K-1,2} & . & \lambda_{K-1,K-1} & \lambda_{K-1,K} \\ 0 & 0 & . & 0 & 0 \end{pmatrix}. \tag{5}$$

The probability of going from state i to state j $(i \neq j)$ between the dates t and $t + dt$ is $\lambda_{ij} dt$. The probability of staying in state i is $1 - \lambda_{ii} dt$. The transition matrix over a small period dt is $I + \Lambda dt$ (I is the $K \times K$ identity matrix).

In the modelling framework some extra constraints are imposed on the generator matrix used to ensure that the evolution of the credit spreads is appropriate. These constraints are summarised below.

1) The transition probabilities are non-negative : $\lambda_{ij} \geq 0 \quad \forall \; i, j, i \neq j$.
2) The sum of transition probabilities from any state is unity : $\sum_{i=1}^{K} \lambda_{ij} \; = 0, \; \forall \; i = 1, \ldots, K$.
3) The last state (default) is absorbing : $\lambda_{Kj} \; = 0, \quad \forall \; j = 1, \cdots K$.
4) A state i+1 is always more risky than a state i : $\sum_{j \geq k} \lambda_{ij} \leq \sum_{j \geq k} \lambda_{i+1,j}, \quad \forall i, k \quad k \neq i + 1$.

Using the previous properties, it is possible to write more explicitly the probabilities of default. If we denote respectively by σ_{ij} and $\tilde{\sigma}_{ij}$ the i,j elements of Σ and Σ^{-1}, and by d_j the eigenvalues of Λ, we can obtain :

$$q_{iK}(t, T) = \sum_{j=1}^{K-1} \sigma_{ij} \tilde{\sigma}_{jK} \left[\exp\left(d_j (T - t) \right) - 1 \right], \; 1 \leq i \leq K - 1. \tag{6}$$ [12,13]

[12] This can readily be shown by noting that $Q(t, T) - I = \Sigma\left(\exp D(T - t) - I\right)\Sigma^{-1}$ and $d_K = 0$.

[13] Let us remark that some eigenvalues may be complex $d_j = \mathrm{Re}\left(d_j\right) + i\,\mathrm{Im}\left(d_j\right)$. Thus terms like $\exp\left[\mathrm{Re}\left(d_j\right)(T - t)\right]\cos\left[\mathrm{Im}\left(d_i\right)(T - t)\right]$ appear in the expression of $q_{iK}(t, T)$, implying some cycles in the default probabilities. As a by-product, we notice that $\mathrm{Re}\left(d_j\right) \leq 0$, since the $q_{iK}(t, T)$

We can now express the price of a risky zero-coupon bond, $v^i(t, t+h, \Lambda)$, of any maturity *h* for any credit class *i* according to equation (1) which we re-write:

$$v^i(t, t+h, \Lambda) = B(t, t+h)\big((1-q_{iK}(t, t+h)) + q_{iK}(t, t+h)\delta\big) \tag{7}$$

As can be seen from the previous equation the credit spread for any given risky class and given maturity remains constant. The only changes in credit spreads occur when there are changes in credit ratings.

Though we have emphasised on the use of credit ratings to represent the state space some other approach can be used as detailed below. Credit ratings can exhibit memory in that a firm that has been recently upgraded is in an upward trend and therefore exhibits a lower credit spread than a firm with the same credit rating that has been recently downgraded.

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should stay in $[0,1]$. Indeed, a stochastic matrix has a spectral radius equal to one (see Horn and Johnson (1985), p. 493). Since the eigenvalues of the generator are obtained as logarithms of the eigenvalues of the transition matrix, we get $\mathrm{Re}(d_j) \le 0$. More can be said about the eigenvalues of $Q(t,T)$: Though this matrix is not positive nor irreducible (due to the absorbing state), we can show that 1 ($= \exp 0$) is a simple (algebraically) eigenvalue and that all other eigenvalues have modulus strictly smaller than 1. Moreover, the second eigenvalue (in modulus), $\left(= \exp\left[d_{K-1}(T-t)\right]\right)$ shares similar properties to the first one ; it is real, simple, associated to a non negative right eigenvector and all further eigenvalues have smaller modulus.

Proof : Firstly, the matrix $Q(t,T)$ can be written as $\begin{pmatrix} Q_{K-1\times K-1}(t,T) & . \\ 0_{1\times K-1} & 1 \end{pmatrix}$; for sufficiently large T, the matrix $Q_{K-1\times K-1}(t,T)$ has positive coefficients, meaning that all credit ratings are strongly connected (every rating is attainable with positive probability, whatever the initial rating). $\det\left(I - \lambda Q(t,T)\right) \equiv (1-\lambda)\det\left(I - \lambda Q_{K-1\times K-1}(t,T)\right)$ and Perron theorem applies for $Q_{K-1\times K-1}(t,T)$. Moreover since the rows of $Q_{K-1\times K-1}(t,T)$ sum up to quantities less than one (the last column of $Q(t,T)$ has positive terms), the spectral radius of $Q_{K-1\times K-1}(t,T)$ is strictly less than one. The associated positive eigenvector completed with 0 provides a non negative eigenvector of $Q(t,T)$.

Now, it can be proven that d_{K-1} is related to the long-term credit spread.

A tractable way to incorporate this information into the model is to split some or all credit states into two new states; in one state the company is in an improving trend (X^+) and in the other in a deteriorating trend (X^-). Note that X^+ and X^- are not themselves standard credit ratings but different versions of the rating X, according to the outlook for a particular X rated company. This new approach requires estimation of more independent parameters although, since a downgraded issuer cannot be in a positive trend and an upgraded issuer cannot be in a negative trend, some terms in the generator matrix are constrained to be zero. We refer to this model as pseudo non-Markovian since the population of one of the new states gives additional information about the previous state in the process[14].

In order to illustrate these ideas, we have calibrated both Markovian and pseudo non-Markovian models to the same set of data. Tables 2a and 2b show calibrated generator matrices for the general and pseudo non-Markovian models, using US bank data with ratings AAA, A and BBB. Note that in the non-Markovian model, it is not possible to migration from BBB to A^- or from AAA to A^+.

Table 1a. Generator matrix for Markovian model.

	AAA	A	BBB	D
AAA	-0.046	0.042	0.003	0.002
A	0.027	-0.111	0.081	0.003
BBB	0.012	0.025	-0.047	0.011
D	0.000	0.000	0.000	0.000

Table 1b. Generator matrix for pseudo non-Markovian model.

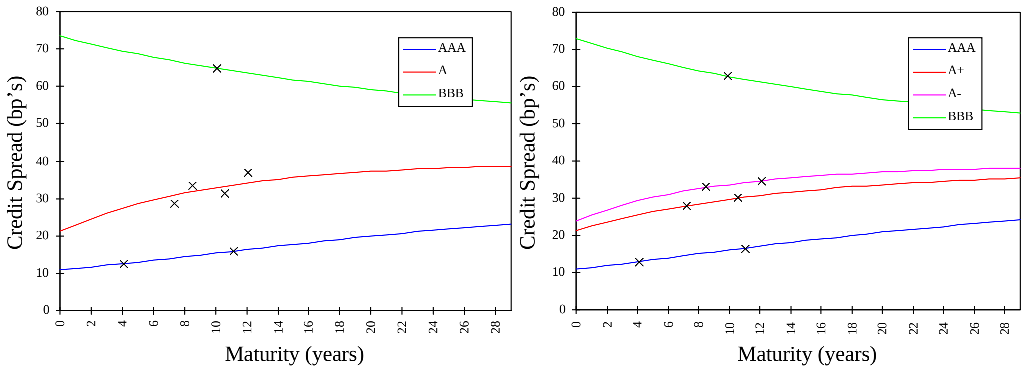
	AAA	A^+	A^-	BBB	D
AAA	-0.055	0.000	0.051	0.003	0.001
A^+	0.035	-0.157	0.070	0.050	0.002

[14] An alternative way to expand the state space would be to make use of extra-information provided by the rating agencies ; for example, we might distinguish between a AA bond with negative perspective and a AA bond with a positive outlook.

A^-	0.027	0.065	-0.180	0.081	0.002
BBB	0.012	0.039	0.000	-0.058	0.007
D	0.000	0.000	0.000	0.000	0.000

Figure 2 shows the calibrated credit spread curves. The non-Markovian extension enables us to estimate the yield curves corresponding to the positive and negative trends and fit the observed prices exactly. In class A, there are 4 bonds which cannot be fitted with the basic model because they come from 2 companies, one in a negative trend and one in a positive trend. In the pseudo non-Markovian model, the bonds can be fitted well since there are now two states of A rating.

Figure 2. Illustration of the credit risky spreads using the Markovian (left) and the pnon-Markovian (right) models.



Up to now the credit spread for a given credit rating class has been assumed non-stochastic and therefore the credit spread for a given credit class is constant. As market participants are constantly exposed to ever changing market conditions they require different compensation in order to bear default risk. An intuitive interpretation would be that the risk premia are stochastic. A more realistic model would be for the credit spread to move even if the credit rating does not change. We will show that this credit spread volatility can be modelled by introducing a random generator matrix $\Lambda(t)$ as follows :

$$\Lambda(t) = \Sigma\left(DU(t)\right)\Sigma^{-1}. \tag{8}$$

where $U(t)$ follows some scalar jump-diffusion process. Let us notice that in this modelling framework, the eigenvectors of the generator $\Lambda(t)$ remain unchanged. This allows to get some simple expressions of the conditional probabilities to default :

$$q_{iK}(t,T)|U(t) = \sum_{j=1}^{K-1} \sigma_{ij} \tilde{\sigma}_{jK} \left(E_t \left[\exp d_j \int_t^T U(s)ds \right] - 1 \right), \quad 1 \le i \le K-1. \tag{9}$$ [15]

Further simplifications arise when the computation of the Laplace transform of $\int_t^T U(s)ds$ is explicit[16]. These are very similar to those involved in bond pricing and $U(t)$ can be chosen to follow a square root or an Ornstein-Uhlenbeck process in order to get exponential affine type functions of the state variable $U(t)$.

Let us remark that the short spreads are of the form :

$$(1-\delta)\lambda_{iK}(t) = (1-\delta)U(t)\sum_{j=1}^{K-1} \sigma_{ij} \tilde{\sigma}_{jK} d_j \ . \tag{10}$$

They are thus proportional to $U(t)$ which can for example be taken as a square root process to guarantee that the credit spreads are positive.

We now deal with a model where the credit spreads associated to a given rating are stochastic. However, movements in credit spreads across rating classes are perfectly correlated which is not supported by empirical evidence.

The model can be easily expanded in order to get more complex dynamics of the credit spreads by assuming that the stochastic generator $\Lambda(t)$ is of the form :

$$\Lambda(t) = \Sigma D(t) \Sigma^{-1}, \tag{11}$$

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[15] Where E_t denotes conditional expectation.

where $D(t)=\mathrm{diag}\big(d(t)\big)$ is a diagonal matrix such that the vector of non positive eigenvalues $d(t)$ follows a $K-$dimensional jump-diffusion process[17,18]. The transition probabilities may be computed as :

$$q_{iK}(t,T)\big|d(t)=\sum_{j=1}^{K-1}\sigma_{ij}\tilde{\sigma}_{jK}\left(E_t\left[\exp\int_t^T d_j(s)ds\right]-1\right),\ \ 1\le i\le K-1. \tag{12}$$

The short spreads take the form :

$$(1-\delta)\lambda_{iK}(t)=(1-\delta)\sum_{j=1}^{K-1}\sigma_{ij}\tilde{\sigma}_{jK}d_j(t). \tag{13}$$

A simple modelling that includes the scalar case arises when $d(t)$ can be expressed in linear form, i.e. $d(t)=Mf(t)$, where M is a $K\times N$ matrix and $f(t)$ is a $N-$dimensional diffusion process with orthogonal components. The short spreads then appear to be linear combinations of orthogonal factors.

We can consider (as an example) that the factors $f(t)$ follow some multidimensional Ornstein-Uhlenbeck process, i.e. $df(t)=A\big(B-f(t)\big)dt+CdW_t$, where A,C are diagonal $N\times N$ square matrices and B a N- dimensional vector, the expressions $E_t\left[\exp\int_t^T d_j(s)ds\right]$ can be computed as $\exp\left[B_j(t,T)+\sum_{n=1}^{N}A_{jn}(t,T)f_n(t)\right]$ where A_{jn},B_j are deterministic functions.

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[16] We know that the Laplace transform is defined on an interval including 0 ; Thus the $\mathrm{Re}\big(d_j\big)$ must belong to this interval for the $q_{iK}(t,T)$ to be well defined.

[17] This model also appears in Lando (1998) ; in Duffie-Singleton (1998), the $K-1\times K-1$ matrix governing the transitions between credit ratings (excluding default) is constant. The last column of $\Lambda(t)$ governing transition to default can be made more general. Since the eigenvectors of $\Lambda(t)$ are no more constant, numerical simulation of transition times is required to get the default probabilities.

[18] An interesting special case arises when $D(t)$ is deterministic. We then get a non homogeneous Markov chain that can be seen as an extension of the non homogeneous Poisson model that is often used to calibrate separately risky yield curves, and as a tractable modification of the non homogeneous

It is important to note that analytical tractability comes from the explicit computation of the terms $E_t\left[\exp\int_t^T d_j(s)ds\right]$. This question is addressed for instance in Duffie-Kan (1996), and thus we can allow for jumps in the $d(t)$ and keep exponential affine form for the previous quantities. In that case, the prices of zero-coupon bonds appear as linear combinations of exponential affine terms. Practically, it means that we can handle jumps in credit spreads, even if the credit rating remains unchanged.

The knowledge of the matrix $Q(t,T)$ allows a direct price computation at time t of contracts contingent of the realised credit rating at time T. Let us denote by α_j the amount received at time T if we are in credit class j and by α the vector of α_j. The pricing formula[19] of such a contract is given by $B(t,T)Q(t,T)\alpha$. If α_j=1 when $j = j_0$ and 0 otherwise, we have a contingent zero-coupon. These form a basis of payoffs. Another basis is made of the eigenvectors of $Q(t,T)$, i .e. the columns of Σ. These former payoffs share the property such that the associated pricing formula is proportional to the payoff itself whatever the payment dates.

The model degenerates when there are only two states in the Markov chain (default and no default) to the Longstaff and Schwartz (1995) or Duffie and Kan (1996) type models where the credit spread follows a diffusion or a jump-diffusion model.

Let us remark that the model can quickly be expanded in various ways while keeping analytical tractability. One way is to consider both stochastic recovery rates $\delta(t)$ in the form of Duffie and Singleton (1997) and correlation between riskless interest rates and default events. We can start from the general representation of the price of the risky bond $v(t,T)$ as :

$$v(t,T) = E_t\left[\exp-\int_t^T \left(r(s)+\lambda(s)(1-\delta(s)\right)ds\right]$$

Markov chain of JLT, especially when there are few credit classes and a relatively large number of bonds.

where $r(s)$ is the riskless short rate and $\lambda(s)$ is the hazard rate. We can make $r(s)$, $\lambda(s)$, $\delta(s)$ depend on some continuous state variables $f(t)$ and $\lambda(s)$, $\delta(s)$ of the current credit rating *i*. Let us denote by $\Xi(t)$ the matrix whose general term is $\Xi_{ij}(t) = r\big(f(t)\big) + \lambda_{ij}\big(f(t)\big)\big(1-\delta_i\big(f(t)\big)\big)$ and let us assume that this matrix can diagonalised with constant eigenvectors, $\Xi(t) = \Sigma D(t)\Sigma^{-1}$. Then we can readily apply the previous computations of the default probabilities and get explicit pricing formulas for the risky bonds[20]. It can be noticed that $R(s) = r(s) + \lambda(s)(1-\delta(s))$ is the risky short rate and that one may go for a direct specification of this rate (as the last column of matrix $\Xi(t)$) without regarding it as the sum of a riskless rate plus some spread[21]. Such a modelling might also be used for the riskless short rate, for instance in order to take into account switching regimes in the monetary policy.

3. Model implementation.

The simplest approaches to the pricing of structures involving credit risk makes few modelling assumptions, but needs a lot of input market prices. More structured approaches based on the dynamics of credit ratings can be implemented even when the observed market provides only sparse data.

We first present the calibration of the model with constant generator, from observed risky bond prices of different maturities and credit ratings, and from an historical

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[19] By pricing formula, we mean the price expressed as a function of the current state. See Darolles and Laurent (1998) for a more systematic use of eigenvectors of pricing operators.

[20] Another way to deal with correlated interest rates and credit risk is to make use of the forward measure. The price of the risky bond can be written as if interest rates where not correlated with default risk, provided that the expectations are taken under the forward measure Q_T: $v(t,T) = B(t,T)E_t^{Q_T}\left[\exp -\int_t^T \lambda(s)\big(1-\delta(s)\big)ds\right]$. Now, let us consider the matrix $\Psi(t)$ whose general term is $\Psi_{ij}(t) = \lambda_{ij}(t)\big(1-\delta_i(t)\big)$. If this matrix can be written with constant eigenvectors, $\Psi(t) = \Sigma D(t)\Sigma^{-1}$ (this property is purely algebraic and does not depend on the choice of measure, it is only the distribution of $D(t)$ that changes), we can again obtain explicit computations of the risky bond prices. Lando (1998) proposes a third way to handle correlation between riskless rates and default. He conditions first on the paths of the state variables and then takes the expectation of the explicit expression obtained.

[21] This can lead to some simplification and may be the only sensible approach in markets where government bonds are illiquid and risky.

generator matrix. We then examine the implementation issues when the generator of the Markov chain is itself stochastic.

3.1 Model with constant generator

For risky bond pricing, the effective use of the model requires the estimation of default probabilities. We want the model to be both able to match observed market prices of risky bonds and be similar to historical data on transition probabilities provided by ratings agencies. We will denote by Λ^{hist} a generator estimated from historical data[22] (see Table 1a). The next step is a calibration procedure to estimate a generator matrix Λ. In order to perform a stable calibration we use both current bond prices and historical transition probabilities Λ^{hist}.

The price of a bond $j,\ j \in \{1, \cdots, J\}$ in credit class *i* can be expressed as :

$$P^i_{j,\text{model}}(\Lambda) = \sum_{h=1}^{T} F^i_j(h) v^i(h, \Lambda), \tag{14}$$

where $F^i_j(h)$ is the coupon of bond *j* in state *i* at date *h* and $v^i(h, \Lambda)$ is the price of a zero coupon bond in state *i* with maturity *h*.

A least squares optimisation can be used to calibrate the model as shown,

$$\min_{\tilde{\Lambda}} \left\{ \sum_{i=1}^{k} \sum_{j=1}^{J} \left(P^i_{j,\text{model}}(\Lambda) - P^i_{j,\text{market}} \right)^2 + \sum_{i,j=1}^{k} \left(\frac{\left(\lambda_{ij} - \lambda^{hist}_{ij} \right)^2}{\beta_{ij}} \right) \right\}^{23}, \tag{15}$$

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[22] A historical generator matrix can be easily computed from a historical transition matrix. The rating agencies do not reproduce historical generator matrices directly.

[23] Once the matrix of eigenvectors Σ is fixed, the eigenvalues d_k must satisfy the following linear constraints, $\lambda_{ij} = \sum_{k=1}^{K-1} \sigma_{ik} \tilde{\sigma}_{kj} d_k \geq 0,\ \forall i, j,\ i \neq j$. The extra constraints take the same form : $\sum_{j \geq k} \lambda_{ij} \leq \sum_{j \geq k} \lambda_{i+1,j},\ \ \forall i, k\ \ k \neq i+1$. Thus, the eigenvalues must belong to a closed convex cone.

where we minimise the deviation between the model and market prices while keeping the generator matrix close to historical data. Normally the calibrated Λ is not very different form the historical Λ^{hist}. This is a desirable property of the calibration procedure because it implies risk premia close to one. The term $\sum_{i,j=1}^{k}\left(\frac{\left(\lambda_{ij}-\lambda_{ij}^{hist}\right)^2}{\beta_{ij}}\right)$ can be interpreted as a the square of a norm distance[24] between the risk-neutral measure Λ and an a priori measure Λ^{hist}. Thus this approach is related to the Bayesian approach recently introduced for model calibration.

This has nice consequences when we consider the spectra of the estimated generator Λ. In the general case the spectra of Λ may not be real (since Λ is not a symmetric matrix). However, it happens that the historical generator provided by Moody's has a real spectra with disjoint eigenvalues. It can be shown that there is exists a neighbourhood (for instance in the sense of the norm distance introduced before) such that all matrices in it have real and distinct eigenvalues[25].

Other kinds of matching procedures may be used ; one of particular interest consists in looking for :

$$\min_{\Lambda}\left\{\sum_{i=1}^{k}\sum_{j=1}^{J}\left(P^{i}_{j,\text{model}}(\Lambda)-P^{i}_{j,\text{market}}\right)^2+\alpha\left\|\Lambda\Lambda^{hist}-\Lambda^{hist}\Lambda\right\|^2\right\},$$

where α is a positive number. A standard result in matrix algebra indeed states that two matrices with distinct eigenvalues commute if and only if they share the same eigenvectors. Thus our penalty term does not constrain the eigenvalues of the estimated matrix, while keeping the eigenvectors close to their historical counterparts. One may use matrix norms that lead to fast numerical computations.

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[24] Of Hilbert-Schmidt type.

[25] The reason for this is the continuity of the roots of the characteristic polynomial with respect to its coefficients plus the fact that complex roots are conjugate. In other words the space of matrices with distinct real eigenvalues is an open subspace of the space of matrices. This guarantees in turn that our estimated generators, that are close to the historical one, have real eigenvalues. Now, why Moody's generator has real eigenvalues ? It is very close to a diagonal matrix since the most likely event by far is to remain in the same rating. Moreover, the eigenvalues are likely to be distinct, since the space of real matrices with multiple eigenvalues has a zero-measure. From before, it is not surprising that Moody's generator has real and distinct eigenvalues.

If the number of bonds is relatively large compared to the number of parameters then the bond prices will probably not be matched exactly. However, since the number of independent parameters[26] in the generator matrix is $(K\text{-}1)^2$, it is likely that the number of bonds is considerably smaller than this. This means that there may be more than one solution to the calibration. By using historical data, our procedure will give a solution that is close to the historical one.

The model has been calibrated on data from the US telecommunications industry covering all ratings from CCC to AAA. Table 1a and 1b show the historical generator matrix estimated from Moody's data and the generator matrix calibrated to the market prices.

Let us emphasise that we have been able to use all prices of bonds, including bonds with different ratings, to estimate jointly the risky term structures.[27] We are also guaranteed that a more risky curve will be above a less risky curve[28]. It is even possible, due to the richness of the default dynamics, to estimate risky curves with no or very few price observations within this risky class (of course, such a result should be treated with caution). Although 49 parameters may seem like a lot, quite a large number of bonds have been used in the estimation procedure and we are now able to derive thousands of default probabilities (corresponding to different current ratings and different time horizons).

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[26] This is due to the fact that the bottom row is identically zero and each row sums to zero.

[27] Unlike the usual stripping procedure.

[28] Provided that $\sum_{j \geq k} \lambda_{ij} \leq \sum_{j \geq k} \lambda_{i+1,j}$, $\forall i,k \quad k \neq i+1$ (see JLT or Anderson (1991)) and that the recovery rate does not depend on credit rating prior to default.

Table 2a. Historical generator matrix based on Moody's data.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.0683	0.0615	0.0066	0.0000	0.0002	0.0000	0.0000	0.0000
AA	0.0169	-0.0993	0.0784	0.0027	0.0009	0.0001	0.0000	0.0002
A	0.0007	0.0237	-0.0786	0.0481	0.0047	0.0012	0.0001	0.0000
BBB	0.0005	0.0028	0.0585	-0.1224	0.0506	0.0075	0.0008	0.0016
BB	0.0002	0.0005	0.0045	0.0553	-0.1403	0.0633	0.0026	0.0138
B	0.0000	0.0004	0.0014	0.0059	0.0691	-0.1717	0.0208	0.0741
CCC	0.0000	0.0000	0.0000	0.0074	0.0245	0.0488	-0.3683	0.2876
DEF	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2b. Calibrated generator matrix.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.1322	0.1265	0.0033	0.0012	0.0005	0.0004	0.0002	$\approx 10^{-5}$
AA	0.0507	-0.1267	0.0588	0.0141	0.0002	0.0023	0.0002	0.0004
A	0.0046	0.0385	-0.0825	0.0281	0.0061	0.0033	0.0005	0.0015
BBB	0.0006	0.0103	0.0323	-0.0998	0.0437	0.0101	0.0009	0.0019
BB	0.0006	0.0023	0.0088	0.0663	-0.2093	0.0817	0.0232	0.0264
B	0.0006	0.0020	0.0041	0.0068	0.0294	-0.2473	0.1768	0.0277
CCC	0.0005	0.0001	0.0017	0.0111	0.0294	0.0188	-0.1632	0.1016
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

The estimated credit spread curves are shown by Figure 3. The crosses show the bonds used in the calibration procedure plotted according to the difference between the yield corresponding to the market price and the model price. The accuracy of the fit is indicated by the fact that all crosses lie on the actual credit spread curves. Figure 3 illustrates that the model has adequate underlying economic structure in order to estimate curves even with few or no data inputs for certain credit classes.

Figure 3. Illustration of the credit spreads as predicted by the model calibrated to risky bonds issued by US telecommunications companies.

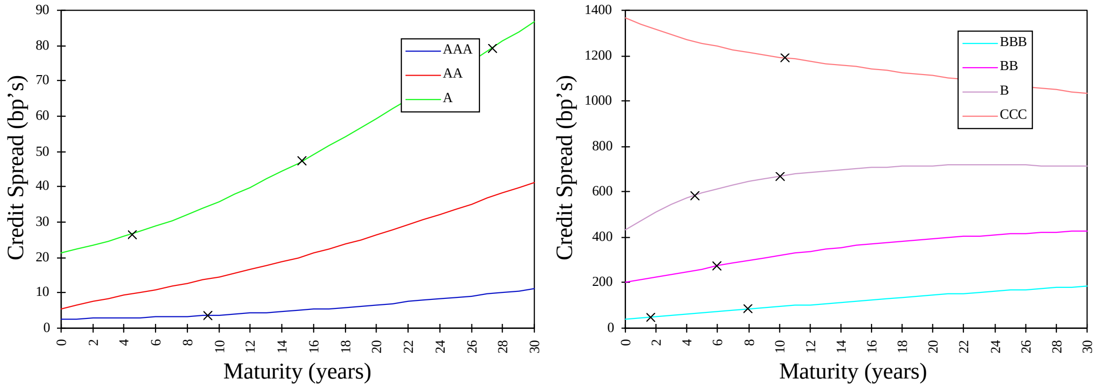
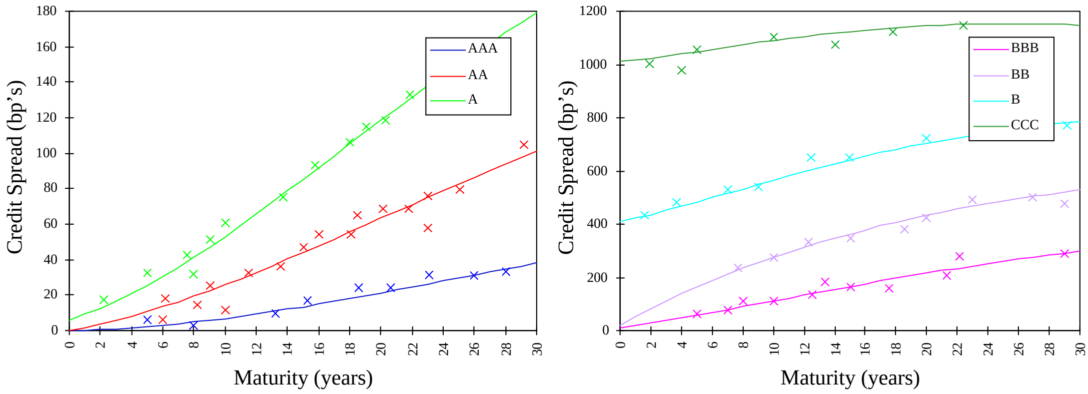


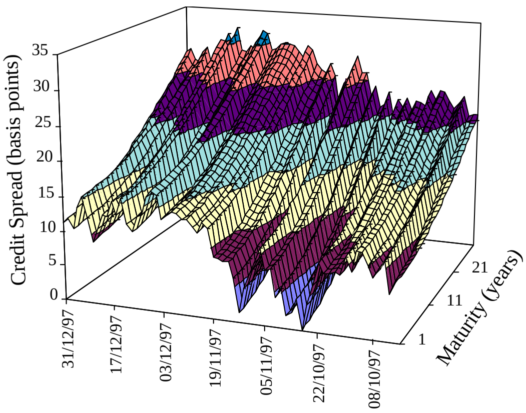
Figure 4 represents an estimation of generic credit spread curves for the US industrial sector using 73 bonds in total.

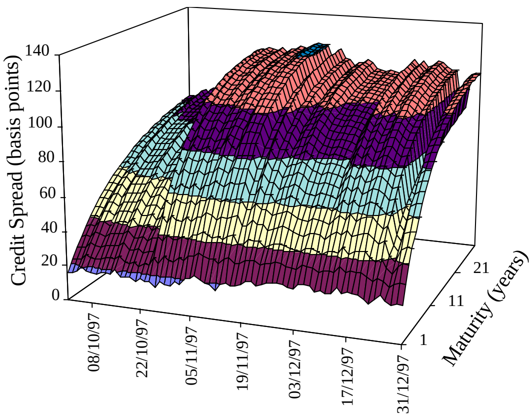
Figure 4. Illustration of the credit risky spreads calibrated for the US industrial sector.



We have also performed the calibration over time for many days of bond data. Figure 5 provides estimated credit spread curves for AAA and BBB ratings. The period includes the Asian crisis. We can see that long-term BBB bonds are very affected by the crisis while short-term BBB bonds are not. AAA bonds are almost unaffected, even showing a decreasing tendency illustrating a "flight to quality". This means that the sensitivity to some state variables might be opposite reflecting portfolio rebalancing between high and low quality bonds.

Figure 5. Evolution of credit spread curves with time for AAA (left) and BBB (right) ratings.





3.3 Model with stochastic generator.

As we have seen, a model with constant generator is appropriate to strip risky bonds. However, it will clearly lead to inappropriate bond option prices or options on credit spreads, since spreads are constant unless the credit rating changes. The model with stochastic premia better accounts for the dynamics of credit spreads. We will now discuss about estimation of the dynamics of the underlying processes, the dimensionality required and specification assessment of our modelling assumption.

Firstly, we intend to show that cross-calibration of the dynamics of the underlying diffusion processes from only risky bond prices is almost impossible and that some bond option prices are required for such a task.

If we consider (as in the example considered before) that the factors $f(t)$ follow some multidimensional Ornstein-Uhlenbeck process, then we obtain the constant generator case by letting $C = 0,\ B = f(0)$.

Let us assume that we try to estimate the unknown parameters, A, B, C, M, Σ from some standard cross calibration on observed bond prices procedure. We may think of minimising a least square distance between the theoretical and observed prices. Since we get obtain almost perfect calibration with a constant generator, this implies that an iterative calibration algorithm starting from the estimated constant generator will not move any further.

There is not a unique solution to the calibration problem. The parameters *A,B,C* have only some small influence of the shape of the risky yield curves and a rather wide range of parameters are likely to fit the risky bond prices. In other words, the information in risky bonds is too poor to properly estimate such parameters and extra prices more sensitive to the parameters[29] would be useful.

On the other hand, it is possible to get very good fit to bond prices for a variety of parameters. In particular, the constant generator model can provide good estimates of short term spreads (provided that we have included short term bonds in the estimation sample). Figure 4 shows that a reasonable range for short-term spreads obtained by a smooth model for risky curves is narrow.

The short spreads can be written as $(1-\delta)\sum_{n=1}^{N}\left(\sum_{j=1}^{K-1}\sigma_{ij}\tilde{\sigma}_{jK}m_{jn}\right)f_n(t)$ where m_{jn} are the elements of *M* and $f_n(t)$ are the components of $f(t)$. Figure A plots the dynamics of the short spreads.

We know that the dynamics of the short spreads under the historical probability may be different from what it is under its risk-neutral counterpart. But since there are equivalent and if we assume second order stationarity of $f(t)$ under the historical probability, we can get an idea of the number of factors required (provided it is less than *K*) through a simple PCA. The first three factors explain 99 % of the total variance and their dynamics is presented in Figure B.

At last, we can try to check the key assumption of the constancy of the eigenvectors. From the previous cross-calibration procedure, we have obtained day per day estimates of the generator matrix $\hat{\Lambda}(t)$ and we want to look for the constancy of the eigenvectors. Thus, we compute the quantities :

[29] Bond options for instance are likely to be very sensitive to the volatility parameters *C* since they create most of short term volatility.

$$r\left(\hat{\Lambda}(t), \hat{\Lambda}(t')\right) = \frac{\left\|\hat{\Lambda}(t)\hat{\Lambda}(t') - \hat{\Lambda}(t')\hat{\Lambda}(t)\right\|}{\left\|\hat{\Lambda}(t)\right\| \times \left\|\hat{\Lambda}(t')\right\|}$$ [30].

This quantities are equal to zero if $\hat{\Lambda}(t)$ and $\hat{\Lambda}(t')$ have the same eigenvectors, regardless of their eigenvalues and are always smaller than 2. Figure C shows that this indicator is small (less than 0.08 whatever *t* and *t'*) suggesting that the eigenvectors vary only by a small amount over time.

[30] $\|\Lambda\|$ is taken as the spectral norm of the matrix Λ which corresponds to the uniform norm of the linear operator associated to Λ. It is equal to the square root of the largest eigenvalue of ${}^t\Lambda\Lambda$.

4. Conclusions

We have described a general model for the evolution of credit spreads that may account for the memory observed in the changes in credit ratings. The model involves a jump component for sudden changes in the credit spread that result from a change in the credit rating of the issuer or default and a jump-diffusion component arising from the variability of the eigenvalues of the generator. Due to the analytical expressions of credit spreads, it is rather easy to implement the model. We have explored the dimension required to get an appropriate fit to the dynamics of credit spreads. We have also provided empirical evidence justifying the main modelling assumption, i.e. the constancy of the eigenvectors of the generator.

This model allows us to price default swaps with arbitrary payment dates, asset swaps taking into account credit risk and more complex kinds of derivative structures, such as payments contingent on a change of rating.

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Figure A : Estimated short term spread (The x-axis is in days).

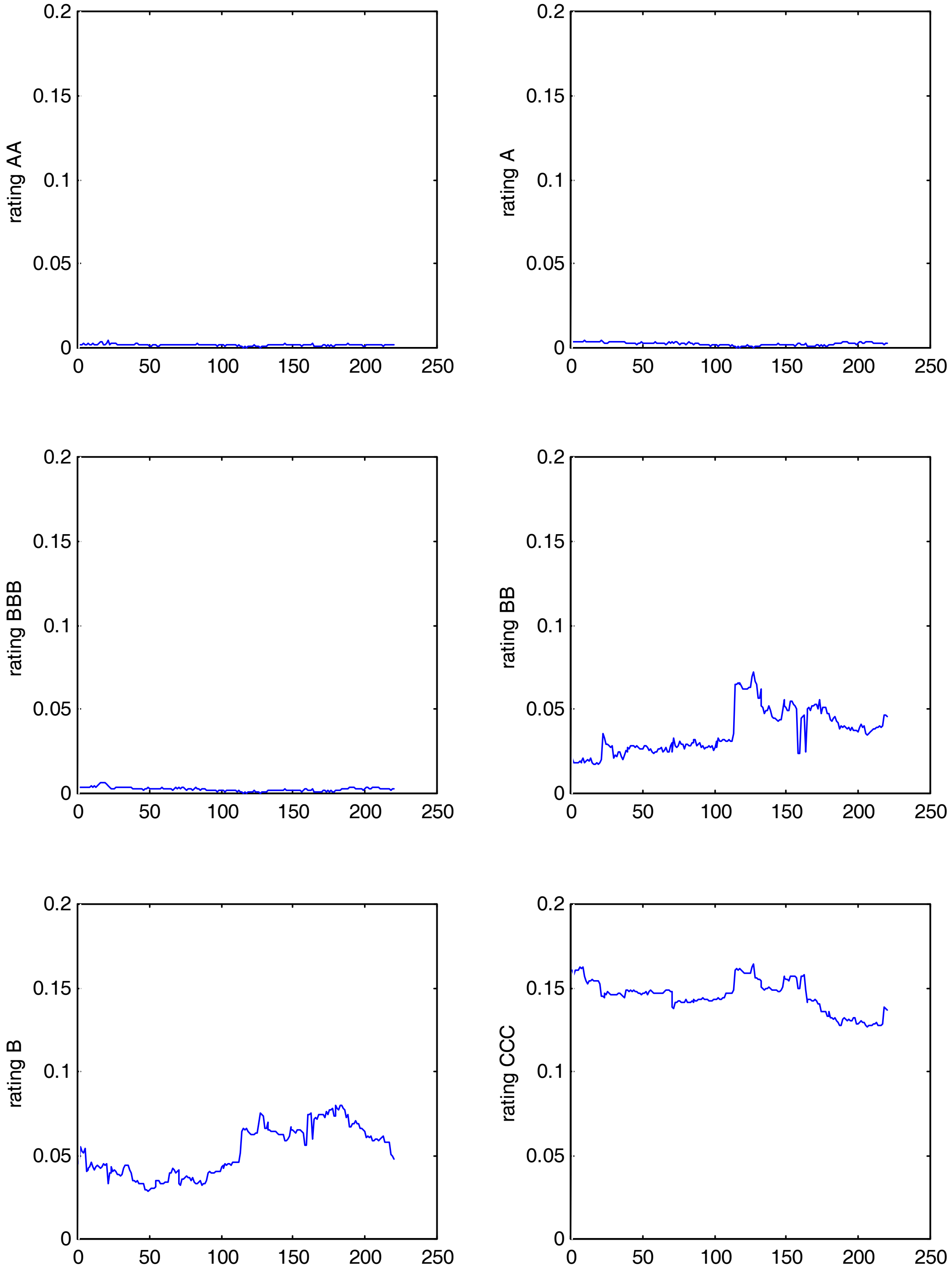
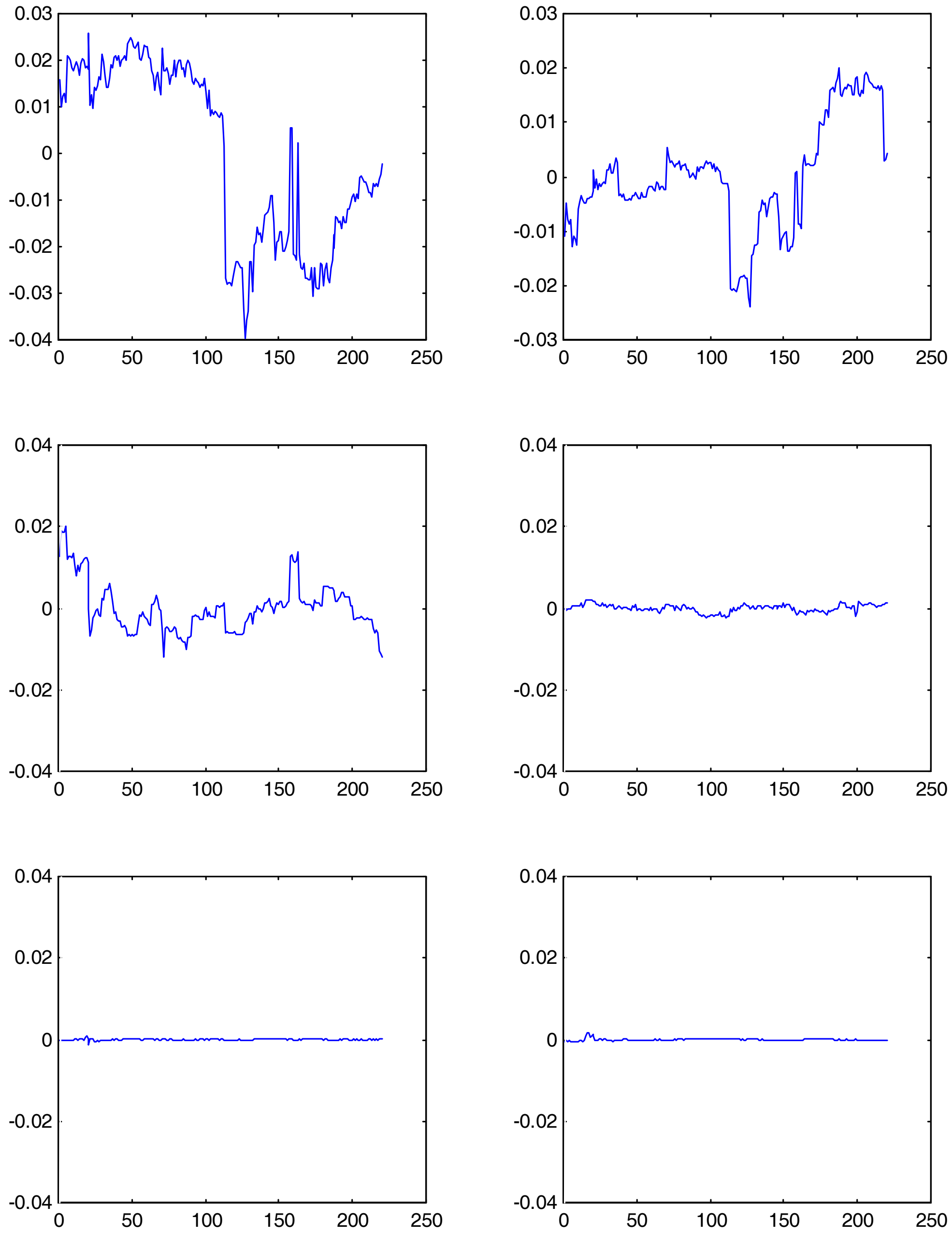


Figure B : Dynamics of the principal components



The x-axis is in days ; The PCA has been performed on the levels on the eigenvalues assuming second order stationarity.

Figure C : Analysis of the constancy of the eigenvectors hypothesis