Nonparametric Risk Management and Implied Risk Aversion*

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Abstract

Typical value-at-risk (VAR) calculations involve the probabilities of extreme dollar losses, based on the statistical distributions of market prices. Such quantities do not account for the fact that the same dollar loss can have two very different economic valuations, depending on business conditions. We propose a nonparametric VAR measure that incorporates economic valuation according to the state-price density associated with the underlying price processes. The state-price density yields VAR values that are adjusted for risk aversion, time preferences, and other variations in economic valuation. In the context of a representative agent equilibrium model, we construct an estimator of the risk-aversion coefficient that is implied by the joint observations on the cross-section of option prices and time-series of underlying asset values.

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1 Introduction

One of the most pressing economic issues facing corporations today is the proper management of financial risks. In response to a series of recent financial catastrophes,[1] regulators, investment bankers, and chief executive officers have now embraced the notion of *risk management* as one of the primary fiduciary responsibilities of the corporate manager. Because financial risks often manifest themselves in subtle and nonlinear ways in corporate balance sheets and income statements, recent attention has focused on quantifying the fluctuations of market valuations in a statistical sense. These value-at-risk (VAR) measures lie at the heart of most current risk management systems and protocols. For example, JP Morgan's (1995) *RiskMetrics* system documentation describes VAR in the following way:

> Value at Risk is an estimate, with a predefined confidence interval, of how much one can lose from holding a position over a set horizon. Potential horizons may be one day for typical trading activities or a month or longer for portfolio management. The methods described in our documentation use historical returns to forecast volatilities and correlations that are then used to estimate the market risk. These statistics can be applied across a set of asset classes covering products used by financial institutions, corporations, and institutional investors.

By modeling the price fluctuations of securities held in one's portfolio, an estimate and confidence interval of how much one can lose is readily derived from the basic principles of statistical inference.

However, in this paper we argue that statistical notions of value-at-risk are, at best, incomplete measures of the true risks facing investors. In particular, while statistical measures do provide some information about the range of uncertainty that a portfolio exhibits, they have little to do with the economic valuation of such uncertainty. For example, a typical VAR statistic might indicate a 5% probability of a \$15M loss for a \$100M portfolio over the next month, which seems to be a substantial risk exposure at first glance. But if this 15% loss occurs only when other investments of similar characteristics suffer losses of 25%, such a risk may seem rather mild after all.

[1] For example, the multimillion-dollar losses suffered by Gibson Greetings, Metallgesellschaft, Orange County, Proctor and Gamble, Barings Securities, etc.

This simplistic example suggests that a one-dollar loss is not always worth the same, and that circumstances surrounding the loss can affect its *economic* valuation, something that is completely ignored by purely statistical measures of risk.

In this paper, we propose an alternative to statistical VAR (henceforth S-VAR) that is based on economic valuations of value-at-risk, and which incorporates many other aspects of market risk that are central to the practice of risk management. Our alternative is based on the seminal ideas of Arrow (1964) and Debreu (1959), who first formalized the economics of uncertainty by introducing elementary securities each paying $1 in one specific state of nature and nothing in any other state. Now known as *Arrow-Debreu* securities, they are widely recognized as the fundamental building blocks of all modern financial asset-pricing theories, including the CAPM, the APT, and the Black and Scholes (1973) and Merton (1973) option-pricing models.

By construction, Arrow-Debreu prices have a probability-like interpretation—they are non-negative and sum to unity—but since they are *market* prices determined in equilibrium by supply and demand, they contain much more information than statistical models of prices. Arrow-Debreu prices are determined by the combination of investors' preferences, budget dynamics, information structure, and the imposition of market-clearing conditions, i.e., general equilibrium. Moreover, we shall show below that under certain special conditions, Arrow-Debreu prices reduce to the simple probabilities on which statistical VAR measures are based, hence the standard measures of value-at-risk are special cases of the Arrow-Debreu framework.

The fact that the market prices of these Arrow-Debreu securities need not be equal across states implies that a one-dollar gain need not be worth the same in every state of nature—indeed, the worth of a one-dollar gain in a given state is precisely the Arrow-Debreu price of that security. Therefore, we propose to use the prices of Arrow-Debreu securities to measure economic VAR (henceforth E-VAR).

Despite the fact that pure Arrow-Debreu securities are not yet traded on any organized exchange,[2] Arrow-Debreu prices can be estimated from the prices of traded financial

[2]This is changing as derivatives markets become more sophisticated. For example, it is now possible to construct a limited set of Arrow-Debreu securities by forming portfolios of "digital" or "binary" options.

securities using recently developed nonparametric techniques such as kernel regression, artificial neural networks, and implied binomial trees. Nonparametric techniques are particularly useful for value-at-risk calculations because departures from standard parametric assumptions, e.g., normality, can have dramatic consequences for tail probabilities. Using such techniques, we compare the performance of S-VAR and E-VAR measures and develop robust statistical methods to gauge the magnitudes of their differences.

Moreover, to provide an economic interpretation for the differences between S-VAR and E-VAR, we show how to combine S-VAR and E-VAR to yield a measure of the aggregate risk aversion of the economy, i.e., the risk aversion of the representative investor in a standard dynamic asset-pricing model. We propose to extract (unobservable) aggregate risk-preferences, what we call *implied risk aversion*, from (observable) market prices of traded financial securities. In particular, we are inferring the aggregate preferences that are compatible with the pair of option and index values.

When applied to daily S&P 500 option prices and index levels from 1993, our nonparametric analysis uncovers substantial differences between S-VAR and E-VAR (see Figure 2). A comparison of S-VAR and E-VAR densities shows that aggregate risk aversion is not constant across states or maturity dates, but changes in important nonlinear ways (see Figure 4).

In Section 2 we present a brief review of the theoretical underpinnings of Arrow-Debreu prices and their relation to dynamic equilibrium models of financial markets. In Section 3 formally introduce the notion of economic value-at-risk, describe its implementation, and propose statistical inference procedures that can quantify its accuracy and relevance over statistical VAR. An explicit comparison of E-VAR with S-VAR, along with the appropriate statistical inference, is described and developed in Section 4. We construct an estimator of implied risk aversion in Section 5 and propose tests for risk neutrality and for specific preferences based on this estimator. To illustrate the empirical relevance of E-VAR, we apply our estimators to daily S&P 500 options data in Section 6. We conclude in Section 7.

See, also, the "supershares" security proposed by Garman (1978) and Hakansson (1976) which has been test-marketed recently by Leland, O'Brien, and Rubinstein Associates, Inc.

2 DGP, SPD, MRS, and VAR

Denote by S_t the price at time t of a security or portfolio of securities whose risk we wish to manage and let $u_{t,\tau} \equiv \ln(S_{t+\tau}/S_t)$ denote its return between t and $t+\tau$. The usual statistical VAR measures are based on the probability distribution of $u_{t,\tau}$.[3] For example, one common VAR measure is the standard deviation of returns $u_{t,\tau}$. Another is the 95 percent confidence interval of $u_{t,\tau}$ centered at its historical mean. More sophisticated VAR measures incorporate conditioning information and dynamics in specifying and estimating the probability distribution of $u_{t,\tau}$, i.e., they are based on conditional probabilities obtained from the *data-generating process* (DGP) of $\{S_t\}$.

Although such VAR measures do capture important features of the uncertainty surrounding $u_{t,\tau}$, they fall short in one crucial respect: they are *statistical* evaluations of uncertainty, not *economic* valuations. In particular, one investor may be quite willing to bear a one-standard deviation drop in $u_{t,\tau}$, while another investor may be devastated by such an event. Therefore, although the dollar loss is the same for both investors, their personal valuations of such a risk can differ dramatically. More importantly, the *market* valuation of this risk—the value assigned by the interactions of many heterogeneous investors in a market setting—can differ substantially from statistical measures.

2.1 Dynamic Equilibrium Models

This distinction between the DGP and market valuations lies at the heart of dynamic equilibrium asset-pricing models in economics—beginning with Arrow (1964) and Debreu (1959)—in which the valuation of securities with uncertain payoffs is determined by the interaction and equilibration of market forces and market conditions.[4] In such models, the specific DGP for prices is not assumed, but rather is derived from first principles as the (stochastic) sequence of prices that equates supply and demand at each point in time.

More importantly, unlike a purely statistical model of prices, e.g., geometric Brownian motion, a DGP that is derived from equilibrium prices contains an enormous amount of

[3]See, for example, Smithson, Smith, and Wilford (1995).

[4]See Merton (1982, 1992) for a review of these and related models.

information about market conditions and investors' preferences that is critical for risk management. To see why, consider a standard dynamic exchange economy [see Lucas (1978) and Rubinstein (1976)] in which securities markets are dynamically complete, there is a single consumption good, no exogenous income, and all investors seek to maximize at date t a state-independent utility function, subject to the usual budget constraints. They can consume at date t and at some fixed future date T. There is one risky stock (the market portfolio, in total supply normalized to one share) and one riskless bond (in zero net supply) available for trading at any date between t and T. Under suitable assumptions for preferences and endowment shocks, it is well-known that market completeness allows us to introduce a representative agent with utility function U [see Constantinides (1982)] and the date-t equilibrium price S_t of a security with a single date-T liquidating payoff of $\psi(C_T)$—a function of aggregate consumption C_T—is given by:

$$S_t = \mathrm{E}_t\left[\psi(C_T)M_{t,T}\right], \qquad M_{t,T} \equiv \frac{U'(C_T)}{U'(C_t)} \tag{2.1}$$

where $M_{t,T}$ is the *stochastic discount factor* or *marginal rate of substitution* (MRS) between consumption at dates t and T. In equilibrium, the investor optimally invests all his wealth in the risky stock at every instant prior to T and then consumes the terminal value of the stock at T, $C_T = S_T$.

Assuming that the conditional distribution of future consumption has a density representation $f_t(\cdot)$, we can rewrite (2.1) as:

$$\begin{aligned} \mathrm{E}_t\left[\psi(C_T)M_{t,T}\right] &= \int_0^\infty \psi(C_T)\frac{U'(C_T)}{U'(C_t)}f_t(C_T)dC_T \\ &= e^{-r_{t,\tau}\tau}\int_0^\infty \psi(C_T)f_t^*(C_T)dC_T \\ &= e^{-r_{t,\tau}\tau}\mathrm{E}_t^*\left[\psi(C_T)\right] \end{aligned} \tag{2.2}$$

where $\tau \equiv T - t$ and

$$f_t^*(C_T) \equiv \frac{M_{t,T}f_t(C_T)}{\int_0^\infty M_{t,T}f_t(C_T)dC_T} \tag{2.3}$$

and $r_{t,\tau}$ is the continuously compounded net rate of return between t and T of a riskless bond promising one unit of consumption at T, assumed constant for simplicity.

This version of the Euler equation shows that the price of any asset can be expressed as a discounted expected payoff, discounted at the riskless rate of interest. However, the expectation must be taken with respect to f^*, an MRS-weighted probability density function, not the original probability density function f of future consumption. This density f^* is called the *state-price density* (SPD) and it is the continuous-state counterpart to the prices of Arrow-Debreu state-contingent claims that pay \$1 in a given state and nothing in all other states. Under market completeness, f^* is unique. In particular, Arrow (1964) and Debreu (1959) showed that if there are as many state-contingent claims as there are states, then the price of any security can be expressed as a weighted average of the prices of these state-contingent claims, now known as *Arrow-Debreu prices*. In a continuous-state setting, f^* satisfies the same property—any arbitrary security can be priced as a simple expectation with respect to f^*.

This underscores the importance of f^* for risk management: the SPD aggregates all economically pertinent information regarding investors' preferences, endowments, asset price dynamics, and market clearing, whereas purely statistical descriptions of the DGP of prices do not. It is possible in general to characterize the class of DGP of prices that are compatible with an equilibrium model [see for example Bick (1990), Wang (1993) and He and Leland (1993)]. Fixing the utility function, however, is not sufficient to identify uniquely the DGP of the price process. If parametric restrictions are imposed on the DGP of asset prices, e.g., geometric Brownian motion, the SPD may be used to infer the preferences of the representative agent in an equilibrium model of asset prices [see, for example, Bick (1987)]. Alternatively, if specific preferences are imposed, e.g., logarithmic utility, the SPD may be used to infer the DGP of asset prices. Indeed, in equilibrium, any two of the following imply the third: (1) the representative agent's preferences; (2) asset price dynamics; and (3) the SPD.

2.2 No-Arbitrage Models

The practical relevance of SPD's for derivative pricing and hedging applications has also become apparent in *no-arbitrage* or *dynamically-complete-markets* models in which sophisticated dynamic trading strategies involving a set of "fundamental"securities can perfectly replicate the payoffs of more complex "derivative" securities. For example, suppose that we observe a set of n_1 asset prices following Itô diffusions driven by n_2 independent Brownian motions:

$$dS_t \quad = \quad \mu_t dt \ + \ \sigma_t dW_t \tag{2.4}$$

with $n_1 \geq n_2$, and suppose that there exists a riskless asset with instantaneous rate of return r. Then path-independent derivative securities on an asset with payoff function $\psi(S_T)$ are spanned by certain dynamic trading strategies, i.e., derivatives are redundant assets hence they may be priced by arbitrage.[5] In such applications the asset price dynamics are specified explicitly and conditions are imposed to ensure the existence of an SPD and dynamic completeness of markets [see Harrison and Kreps (1979), Duffie and Huang (1985) and Duffie (1996)].

For example, the system of asset prices S in (2.4) supports an SPD if and only if the system of linear equations $\sigma_t \cdot \lambda_t = \mu_t$ admits at every date a solution λ_t such that

$$\exp\left[\int_t^T \lambda_\tau \cdot \lambda_\tau \, d\tau \Big/ 2\right]$$

has finite expectation, and

$$\exp\left[-\int_t^T \lambda_\tau dW_\tau - \int_t^T \ \lambda_\tau \cdot \lambda_\tau \, d\tau \Big/ 2\right]$$

has finite variance. In the presence of an SPD, markets are complete if and only if $\text{rank}(\sigma_t) \ = \ n_2$ almost everywhere. Then the SPD can be characterized explicitly with-

[5] Additional assumptions are, of course, required such as frictionless markets, unlimited riskless borrowing and lending opportunities at the same instantaneous rate $r_{t,\tau}$, a known diffusion coefficient, etc. See Merton (1973, 1992) for further discussion.

out reference to preferences—in the particular case of geometric Brownian motion, with constant volatility σ, interest rate $r_{t,\tau}$ and dividend yield $\delta_{t,\tau}$ over the period $(t, t+\tau)$, the SPD or risk-neutral pricing density is given by the conditional distribution of the risk-neutral stochastic process with dynamics

$$dS_t^* = (r_{t,\tau} - \delta_{t,\tau})S_t^* dt \ + \ \sigma S_t^* dW_t$$

which is a lognormal distribution with mean $((r_{t,\tau} - \delta_{t,\tau}) - \sigma^2/2)\tau$ and variance $\sigma^2\tau$.

More generally, denote by S_t the price of an underlying asset and by $f_t^*(S_t, S_T, \tau, r_{t,\tau}, \delta_{t,\tau})$ the SPD of the asset price S_T at a future date T, conditioned on the current price S_t. Consider now a European-style derivative security with a single liquidating payoff $\psi(S_T)$. To rule out arbitrage opportunities among the asset, the derivative and a risk-free cash account, the price of the derivative at t must be equal to:

$$e^{-r_{t,\tau}\cdot\tau} \int_0^{+\infty} \psi\left(S_T\right) f_t^*\left(S_t, S_T, \tau, r_{t,\tau}, \delta_{t,\tau}\right) dS_T \ . \tag{2.5}$$

For example, a European call option with maturity date T and strike price X has a payoff function $\psi(S_T) \ = \ \max[S_T - X, 0]$ hence its date-t price H_t is simply:

$$H\left(S_t, X, \tau, r_{t,\tau}\right) \quad = \quad e^{-r_{t,\tau}\tau} \int_0^{+\infty} \max\left[S_T - X, 0\right] f_t^*\left(S_t, S_T, \tau, r_{t,\tau}, \delta_{t,\tau}\right) dS_T \ . \tag{2.6}$$

Even the most complex path-independent derivative security can be priced and hedged according to (2.5).

3 Economic VAR

The relevance of the SPD for risk management is clear: the MRS-weighted probability density function f^* provides a more relevant measure of value-at-risk—*economic* value—than the probability density function f of the DGP. Therefore, we advocate the use of f^* in all VAR measures such as standard deviation, 95% confidence intervals, tail probabilities, etc. To distinguish the more traditional method of risk management from this approach, we shall refer to the statistical measure of value-at-risk as "S-VAR" since it is based on a purely statistical model of the DGP, and call the SPD-based measure "E-VAR" since it is based on economic considerations.

Now if the MRS in (2.1) were observable, implementing E-VAR measures and comparing them to S-VAR measures would be a simple matter. However, in practice obtaining f^* can be quite a challenge, especially for markets that are more complex than the pure-exchange economy described in Section 2.1. Fortunately, several accurate and computationally efficient estimators of f^* have been developed recently and we provide a brief review of these estimators in Section 3.1 and derive their asymptotic distributions in Section 3.2. With these estimators in hand, we show in Section 4 how to how to gauge the relative importance of E-VAR empirically by examining the ratio f^*/f.

3.1 Kernel Estimators of the SPD

Banz and Miller (1978), Breeden and Litzenberger (1978), and Ross (1976) were among the first to suggest that Arrow-Debreu prices may be estimated or approximated from the prices of traded financial securities. In particular, building on Ross's (1976) insight that options can be used to create pure Arrow-Debreu state-contingent securities, Banz and Miller (1978) and Breeden and Litzenberger (1978) provide an elegant method for obtaining an explicit expression for the SPD from option prices: the SPD is the second derivative (normalized to integrate to unity) of a call option pricing formula with respect to the strike price.

To see why, consider the portfolio obtained by buying two call options struck at X and selling one struck at $X - \epsilon$ and one at $X + \epsilon$. Consider $1/\varepsilon^2$ shares of this portfolio, often called a "butterfly" spread because of the shape of its payoff function $\psi(S_T)$ which pays nothing outside the interval $[X - \epsilon, X + \epsilon]$. Letting ϵ tend to zero, the payoff function of the butterfly tends to a Dirac delta function with mass at X, i.e., in the limit the butterfly becomes an elementary Arrow-Debreu security paying \$1 if $S_T = X$ and nothing otherwise. The limit of its price as ϵ tends to zero should therefore be equal to $e^{r_{t,\tau}\tau} f^*(X)$. Now denote by $H(S_t, X, \tau)$ the market price of a call option at time t with strike price X, time-to-maturity τ, and underlying asset price S_t. Then, by construction, the price of the butterfly spread must be:

$$\frac{1}{\varepsilon^2}\left[2H(S_t,X,\tau)-H(S_t,X-\epsilon,\tau)-H(S_t,X+\epsilon,\tau)\right] \tag{3.1}$$

which has, as its limit as $\epsilon \to 0$, $\partial^2 H(S_t,X,\tau)/\partial X^2$.

For example, recall that under the hypotheses of Black and Scholes (1973) and Merton (1973), the date-t price H of a call option maturing at date $T \equiv t+\tau$, with strike price X, written on a stock with date-t price P_t and dividend yield $\delta_{t,\tau}$, is given by:[6]

$$\begin{aligned} H_{\mathrm{BS}}(S_t,X,\tau,r_{t,\tau},\delta_{t,\tau};\sigma) &= e^{-r_{t,\tau}\tau}\int_0^\infty \max[S_T-X,0]f^*_{\mathrm{BS},t}(S_T)dS_T \\ &= S_t\Phi(d_1)-Xe^{-r_{t,\tau}\tau}\Phi(d_2) \end{aligned} \tag{3.2}$$

where

$$d_1 \equiv \frac{\ln(S_t/X)+(r_{t,\tau}-\delta_{t,\tau}+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 \equiv d_1-\sigma\sqrt{\tau}\ . \tag{3.3}$$

In this case the corresponding SPD is a log-normal density with mean $((r_{t,\tau}-\delta_{t,\tau})-\sigma^2/2)\,\tau$ and variance $\sigma^2\tau$:

$$\begin{aligned} f^*_{\mathrm{BS},t}(S_T) &= e^{r_{t,\tau}\tau}\frac{\partial^2 H_{\mathrm{BS}}}{\partial X^2}|_{X=S_T} \\ &= \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}}\ \exp\left[\ -\ \frac{[\ln(S_T/S_t)-(r_{t,\tau}-\delta_{t,\tau}-\sigma^2/2)\tau]^2}{2\sigma^2\tau}\ \right]\ . \end{aligned} \tag{3.4}$$

This expression shows that the SPD can depend on many quantities in general, and is distinct from but related to the PDF of the terminal stock price S_T. More generally, while sufficiently strong assumptions on the underlying asset price dynamics can often characterize the SPD uniquely, in most cases the SPD cannot be computed in closed form and numerically intensive methods must be used to calculate it. It is clear from

[6] Let $F_{t,\tau}$ denotes the value at t of a futures contract written on the asset, with the same maturity τ as the option. At the maturity of the futures, the futures price equals the asset's spot price. Thus a European call option on the asset has the same value as a European call option on the futures contract with the same maturity. As a result, we will often rewrite the Black-Scholes formula as $H_{BS}(F_{t,\tau},X,\tau,r_{t,\tau};\sigma) = e^{-r_{t,\tau}\tau}(F_{t,\tau}\Phi(d_1)-X\Phi(d_2))$, with $d_1 \equiv (\log(F_{t,\tau}/X)+(\sigma^2/2)\tau)/(\sigma\sqrt{\tau})$ and $d_2 \equiv d_1-\sigma\sqrt{\tau}$.

(3.4) that the SPD is inextricably linked to the parametric assumptions underlying the Black-Scholes option pricing model. If those parametric assumptions do not hold, e.g., if the dynamics of $\{S_t\}$ contain Poisson jumps, then (3.4) will yield incorrect prices, prices that are inconsistent with the dynamic equilibrium model or the hypothesized stochastic process driving $\{S_t\}$. Given the general lack of success in fitting highly parametric models to financial data (see, for example, Campbell, Lo, and MacKinlay [1997, Chapters 2 and 12]), combined with the availability of the data and the large effects of differences in specification, it is quite natural to focus on nonparametric methods for estimating SPD's.

Aït-Sahalia and Lo (1997) propose to estimate the SPD nonparametrically by exploiting Breeden and Litzenberger's (1978) insight that $f_t^*(S_T) = \exp(r_{t,\tau}\tau)\,\partial^2 H(\cdot)/\partial X^2$. They suggest using market prices to estimate an option-pricing formula $\hat{H}(\cdot)$ nonparametrically, which can then be differentiated twice with respect to X to obtain $\partial^2\hat{H}(\cdot)/\partial X^2$. They use kernel regression to construct $\hat{H}(\cdot)$.[7] Assuming that the option-pricing formula H to be estimated is a an arbitrary nonlinear function of a vector of option characteristics or "explanatory" variables, $\mathbf{Z} \equiv [\ F_{t,\tau} \quad X \quad \tau \quad r_{t,\tau}\ ]'$.

In practice, they propose to reduce the dimension of the kernel regression by using a *semiparametric* approach. Suppose that the call pricing function is given by the parametric Black-Scholes formula (3.2) except that the implied volatility parameter for that option is a nonparametric function $\sigma(X/F_{t,\tau},\tau)$:

$$H\left(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}\right) \quad = \quad H_{\,\mathrm{BS}}\left(F_{t,\tau}, X, \tau, r_{t,\tau}; \sigma(X/F_{t,\tau},\tau)\right) \tag{3.5}$$

We assume that the function H defined by (3.5) satisfies all the required conditions to be a "rational" option-pricing formula in the sense of Merton (1973, 1990).[8] In this semi-parametric model, we only need to estimate nonparametrically the regression of σ on a

[7]See Härdle (1990) and Wand and Jones (1995) for a more detailed discussion of nonparametric regression. There are other alternatives to that can be used to obtain option-pricing formulas: see Derman and Kani (1994), Dupire (1994), Hutchinson, Lo, and Poggio (1994), Jackwerth and Rubinstein (1996), and Rubinstein (1994). For an extension to American options and the nonparametric estimation of the early exercise boundary, see Broadie et al. (1996).

[8]See Merton (1990, Chapter 8.2). These conditions imply that $\sigma(X/F,\tau)$ cannot be an arbitrary function but must yield an $H_{\mathrm{BS}}(F_{t,\tau}, X, \tau, r_{t,\tau}; \sigma(X/F_{t,\tau},\tau))$ that satisfies all the conditions of a rational option-pricing formula.

subset $\tilde{\mathbf{Z}}$ of the vector of explanatory variables $\mathbf{Z}$. The rest of the call pricing function $H(\cdot)$ is parametric, thereby considerably reducing the sample size n required to achieve the same degree of accuracy as the full nonparametric estimator. We partition the vector of explanatory variables $\mathbf{Z} \equiv [\ \tilde{\mathbf{Z}}'\ F_{t,\tau}\ r_{t,\tau}\ ]'$ where $\tilde{\mathbf{Z}}$ contains $\tilde{d}$ nonparametric regressors. As a result, the effective number of nonparametric regressors d is given by $\tilde{d}$.

In our empirical application, we will consider $\tilde{\mathbf{Z}} \equiv [\ X/F_{t,\tau}\ \tau\ ]'$ ($\tilde{d} = 2$) and form the Nadaraya-Watson kernel estimator of $E\left[\sigma \,|\, X/F_{t,\tau}, \tau\right]$ as:

$$\hat{\sigma}(X/F_{t,\tau}, \tau) \quad = \quad \frac{\sum_{i=1}^{n} k_{X/F}\left(\frac{X/F_{t,\tau} - X_i/F_{t_i,\tau_i}}{h_{X/F}}\right) k_\tau\left(\frac{\tau - \tau_i}{h_\tau}\right) \sigma_i}{\sum_{i=1}^{n} k_{X/F}\left(\frac{X/F_{t,\tau} - X_i/F_{t_i,\tau_i}}{h_{X/F}}\right) k_\tau\left(\frac{\tau - \tau_i}{h_\tau}\right)} \tag{3.6}$$

where σ_i is the volatility implied by the option price H_i, and the univariate kernel functions $k_{X/F}$ and k_τ and the bandwidth parameters $h_{X/F}$ and h_τ are chosen to optimize the asymptotic properties of the second derivative of $\hat{H}(\cdot)$, i.e., of the SPD estimator. We then estimate the call pricing function as:

$$\hat{H}\left(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}\right) \quad = \quad H_{\mathrm{BS}}\left(F_{t,\tau}, X, \tau, r_{t,\tau}, \delta_{t,\tau}; \hat{\sigma}(X/F_{t,\tau}, \tau)\right) \ . \tag{3.7}$$

The SPD estimator follows by taking the second partial derivatives of $\hat{H}\left(\cdot\right)$ with respect to X:

$$\hat{f}_t^*(S_T) \quad = \quad e^{r_{t,\tau}\tau} \left[\frac{\partial^2 \hat{H}\left(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}\right)}{\partial X^2}\right]_{|X=S_T} \ . \tag{3.8}$$

3.2 Statistical Inference for E-VAR

We collect option prices in the form of panel data, consisting of N observation periods and J options per period. The sample size is $n = NJ$. We make the following assumptions on the data used to construct the nonparametric regression (3.6), i.e., $(\sigma, \tilde{\mathbf{Z}})$ where $\tilde{\mathbf{Z}} \equiv [\ X/F_{t,\tau}\ \tau\ ]'$. The nonparametric regression function is $\hat{\sigma}(\tilde{\mathbf{Z}})$, and we wish to estimate its m-th partial derivative with respect to the first component $X/F_{t,\tau}$ of the vector $\tilde{\mathbf{Z}}$.

Assumption 1

1. The process $\left\{Y_i \equiv (\sigma_i, \tilde{\mathbf{Z}}_i) : \; i = 1, \ldots, n\right\}$ is strictly stationary with $\mathrm{E}\left[\sigma_i^4\right] < \infty$ and $\mathrm{E}\left[\left\|\tilde{\mathbf{Z}}_i\right\|^2\right] < \infty$, and is β-mixing with mixing coefficients β_j that decay at a rate at least as fast as j^{-b}, $b > 4$, as $j \to \infty$. The joint density of (Y_1, Y_{1+j}) exists for all j and is continuous.

2. The density $\pi(\sigma, \tilde{\mathbf{Z}})$ is p-times continuously differentiable with respect to $\tilde{\mathbf{Z}}$, with $p > m$, and π and its derivatives are bounded and in $L_2(\mathbf{R}^{1+\tilde{d}})$. The marginal density of the nonparametric regressors, $\pi(\tilde{\mathbf{Z}})$, is bounded away from zero on every compact set in $\mathbf{R}^{\tilde{d}}$.

3. $\sigma(\tilde{\mathbf{Z}})\pi(\tilde{\mathbf{Z}})$ and its derivatives are bounded. The conditional variance

$$s^2(\tilde{\mathbf{Z}}) \equiv E\left[(\sigma - \sigma(\tilde{\mathbf{Z}}))^2 \,\middle|\, \tilde{\mathbf{Z}}\right] \tag{3.9}$$

is bounded and satisfies $s^4(\tilde{\mathbf{Z}}) \in L_2(\mathbf{R}^{\tilde{d}})$. The conditional fourth moment $\mathrm{E}\left[(\sigma - \sigma(\tilde{\mathbf{Z}}))^4 \,|\, \tilde{\mathbf{Z}}\right]$ is bounded.

Definition 1 *A kernel function k is of order q if:*

$$\int_{-\infty}^{+\infty} z^l k(z)\, dz \quad = \quad \begin{cases} 1 & \text{if} \quad l = 0 \\ 0 & \text{if} \quad 0 < l < q \\ (-1)^q \, q! \chi_q & \text{if} \quad l = q \end{cases}$$

where l is an integer and $\int_{-\infty}^{+\infty} |z|^l \, |k(z)| \, dz < \infty$ for all $0 \leq l \leq q$.

Assumption 2 *The kernel functions $k_{X/F}$ and k_τ are bounded, three-times continuously differentiable, and have derivatives which are bounded and in $L_2(\mathbf{R})$. $k_{X/F}$ is of order $q_{X/F}$ and k_τ is of order q_τ. The bandwidths are given by*

$$h_{X/F} = c_{X/F} \mathrm{s}(\mathbf{X/F}) n^{-1/(\tilde{d}+2(q_{X/F}+m))} \qquad , \qquad h_\tau = c_\tau \mathrm{s}(\boldsymbol{\tau}) n^{-1/(\tilde{d}+2q_\tau)} \tag{3.10}$$

where $\mathrm{s}(\mathbf{X/F})$ and $\mathrm{s}(\boldsymbol{\tau})$ are the unconditional standard deviations of the nonparametric regressors, $c_{X/F} \equiv \gamma_{X/F}/\ln(n)$, with $\gamma_{X/F}$ constant, and $c_\tau \equiv \gamma_\tau/\ln(n)$, with γ_τ constant.

In practice, we use the kernel functions

$$k_{(2)}(z) \equiv e^{-z^2/2} \Big/ \sqrt{2\pi} \quad , \quad k_{(4)}(z) \equiv \left(3 - z^2\right) e^{-z^2/2} \Big/ \sqrt{8\pi} \tag{3.11}$$

which are of order $q = 2$ and $q = 4$ respectively. We then obtain

Proposition 1 *Under Assumptions 1 and 2:*

$$n^{1/2}\, h_{X/F}^{(2m+1)/2}\, h_{\tau}^{1/2} \left[\frac{\partial^m \hat{\sigma}}{\partial X^m}(\tilde{\mathbf{Z}}) - \frac{\partial^m \sigma}{\partial X^m}(\tilde{\mathbf{Z}})\right] \quad \overset{d}{\rightarrow} \quad \mathcal{N}(0, \sigma^2_{d^m\sigma}) \tag{3.12}$$

where

$$\sigma^2_{d^m\sigma} \quad \equiv \quad \frac{s^2(\tilde{\mathbf{Z}}) \ \left(\int_{-\infty}^{+\infty} \left(k_{X/F}^{(m)}\right)^2 (\omega) d\omega\right) \left(\int_{-\infty}^{+\infty} k_{\ \tau}^{\ 2}(\omega) d\omega\right)}{\pi(\tilde{\mathbf{Z}}) \ F_{t,\tau}^{2m}} \ . \tag{3.13}$$

Therefore the E-VAR estimator is distributed asymptotically as

$$n^{1/2}\, h_{X/F}^{5/2}\, h_{\tau}^{1/2} \left[\hat{f}^*(S_T) - f^*(S_T)\right] \quad \overset{d}{\rightarrow} \quad \mathcal{N}(0, \sigma^2_{f^*}) \tag{3.14}$$

where $\partial H_{\mathrm{BS}}/\partial\sigma$ is the option's gamma evaluated at $\hat{\sigma}(\tilde{\mathbf{Z}})$ and

$$\sigma^2_{f^*} \quad \equiv \quad \left[e^{r_{t,\tau}\tau} \frac{\partial H_{\mathrm{BS}}}{\partial \sigma}(\hat{\sigma}(\tilde{\mathbf{Z}}), \mathbf{Z})\right]^2 \sigma^2_{d^2\sigma} \ . \tag{3.15}$$

Here $k_{X/F}^{(m)}$ denotes the m-th derivative of the univariate kernel function k. The term $F_{t,\tau}^{2m}$ in the denominator of (3.13) is due to

$$\frac{\partial^m \hat{\sigma}(\tilde{\mathbf{Z}})}{\partial X^m} \quad = \quad \frac{\partial^m \hat{\sigma}(\tilde{\mathbf{Z}})}{\partial (X/F_{t,\tau})^m} \frac{1}{F_{t,\tau}^m} \ .$$

We give in Table 1 the values of the integrals of the kernel functions that appear in the expressions above for the functions (3.11) to be used in our empirical estimation of the S&P 500 E-VAR.

This proposition follows from the functional delta method in Aït-Sahalia (1995): the expression $H_{\mathrm{BS}}(\hat{\sigma}(\tilde{\mathbf{Z}}), \mathbf{Z}) - H_{\mathrm{BS}}(\sigma(\tilde{\mathbf{Z}}), \mathbf{Z})$ behaves asymptotically like $\partial H_{\mathrm{BS}}/\partial\sigma\, (\hat{\sigma}(\tilde{\mathbf{Z}}) -$

$\sigma(\tilde{\mathbf{Z}})$); and because derivatives of $\hat{\sigma}(\tilde{\mathbf{Z}})$ converge at a progressively slower rate (by comparing (3.12) for $m = 0, 1$ and 2) the asymptotic distribution of $\partial^2 H_{\mathrm{BS}} / \partial X^2 \, (\hat{\sigma}(\tilde{\mathbf{Z}}), \mathbf{Z}) - \partial^2 H_{\mathrm{BS}} / \partial X^2 \, (\sigma(\tilde{\mathbf{Z}}), \mathbf{Z})$ is that of $\partial H_{\mathrm{BS}} / \partial \sigma \, (\partial^2 \hat{\sigma} / \partial X^2 - \partial^2 \hat{\sigma} / \partial X^2)$.

3.3 Other Estimators of the SPD

Several other estimators of the SPD have been proposed in the recent literature (see Aït-Sahalia and Lo [1997] for a more detailed discussion and an empirical comparison). Hutchinson, Lo, and Poggio (1994) employ several nonparametric techniques to estimate option-pricing models that they describe collectively as *learning networks*—artificial neural networks, radial basis functions, and projection pursuit—and find that all these techniques can recover option-pricing models such as the Black-Scholes model. Taking the second derivative of their option-pricing estimators with respect to the strike price yields an estimator of the SPD.

Another estimator is Rubinstein's (1994) *implied binomial tree*, in which the risk-neutral probabilities $\{\pi_i^*\}$ associated with the binomial terminal stock price S_T are estimated by minimizing the sum of squared deviations between $\{\pi_i^*\}$ and a set of *prior* risk-neutral probabilities $\{\tilde{\pi}_i^*\}$, subject to the restrictions that $\{\pi_i^*\}$ correctly price an existing set of options and the underlying stock, in the sense that the optimal risk-neutral probabilities yield prices that lie within the bid-ask spreads of the options and the stock (see also Jackwerth and Rubinstein (1996) for smoothness criteria).

This approach is similar in spirit to Jarrow and Rudd's (1982) and Longstaff's (1995) method of fitting risk-neutral density functions using a four-parameter Edgeworth expansion. However, Rubinstein (1994) points out several important limitations of Longstaff's method when extended to a binomial model, including the possibility of negative probabilities. Derman and Kani (1994) and Shimko (1993) have proposed related estimators of the SPD.

There are several important differences between kernel estimators and implied binomial trees. Implied binomial trees require a prior $\{\tilde{\pi}_i^*\}$ for the risk-neutral probabilities; kernel estimators do not. Implied binomial trees are typically estimated for *each* cross-section of options; kernel estimators aggregate options prices over time to get a single SPD. This implies that implied binomial tree is completely consistent with all option prices at each date, but is not necessarily consistent across time. In contrast, the kernel SPD estimator is consistent across time, but there may be some dates for which the SPD estimator fits the cross section of option prices poorly and other dates for which the SPD estimator performs very well.

Whether or not consistency over time is a useful property depends on how well the economic variables used in constructing the kernel SPD can account for time variation in risk-neutral probabilities. In addition, the kernel SPDs take advantage of the data temporally surrounding a given date. Tomorrow's and yesterday's SPDs contain information about today's SPD—this information is ignored by the implied binomial trees but not by kernel-estimated SPDs.

Finally, and perhaps most importantly, statistical inference is virtually impossible with learning-network estimators and implied binomial trees, because of the recursive nature of the former approach (see White (1992)), and the nonstationarities inherent in the latter approach (recall that implied binomial trees are estimated for each cross section of option prices). In contrast, the statistical inference of kernel estimators is well developed and computationally quite tractable.

4 Comparing S-VAR and E-VAR

Having obtained an estimator $\hat{f}^*$ of the SPD, we can now gauge its importance for risk management by studying the behavior of the ratio of $\hat{f}^*$ to $\hat{f}$, where $\hat{f}$ is an estimator of the conditional density of the DGP, or S-VAR. If the ratio $\hat{\zeta} \equiv \hat{f}^* / \hat{f}$ exhibits substantial variation over its domain, this indicates that E-VAR measures contain important economic information that is not captured by their S-VAR counterparts. Of course, because of estimation error, $\hat{\zeta}$ will never be constant in any given dataset even if ζ is. Therefore some measure of the statistical fluctuations inherent in $\hat{\zeta}$ is required, and we now propose estimators for f and f^* and describe how to conduct statistical inference for $\hat{\zeta}$.

4.1 Estimation and Statistical Inference for S-VAR

To estimate the actual statistical distribution of the future price value S_T conditioned upon the current value S_t, we collect the time series of the index values, calculate the τ-period continuously compounded returns, $u_\tau \equiv \log(S_T/S_t)$, and construct a kernel estimator of the density function $g(\cdot)$ of these returns:

$$\hat{g}(u_\tau) \equiv \frac{1}{NH_u} \sum_{i=1}^{N} k_u \left(\frac{u_\tau - u_{t_i,\tau}}{H_u} \right) . \tag{4.1}$$

We make the following assumptions:

Assumption 3 *The returns* $\{u_{t_i,\tau} : \; i = 1, \ldots, N\}$ *are strictly stationary, with* $E\left[u_{t_i,\tau}^4\right] < \infty$. *Their marginal density* g *admits* $p_u = 4$ *continuous derivatives, and is bounded away from zero on every compact set in* $\mathbf{R}$. *In addition, they are* β*-mixing at a rate* β_j *decaying as* $j \to \infty$ *at a rate at least as fast as* j^{-b}, $b > 4$. *The joint density of* $\left(u_{t_1,\tau}, u_{t_{1+j},\tau}\right)$ *exists for all* j *and is continuous.*

Assumption 4 *The kernel function* k_u *is in* $L_2(\mathbf{R})$ *and is of order* q_u. *The bandwidth* H_u *to estimate the* m_u*-th derivative of* g, $m_u \leq p_u$, *is given by*

$$H_u = c_u \mathrm{s}(u) N^{-1/(1+2(q_u+m_u))} \tag{4.2}$$

where $\mathrm{s}(u)$ *is the unconditional standard deviation of the returns,* $c_u \equiv \gamma_u / \ln(N)$, *with* γ_u *constant.*

From the density of the continuously compounded returns we can then calculate

$$\Pr(S_T \leq S) = \Pr(S_t e^{u_\tau} \leq S) = \Pr(u_\tau \leq \log(S/S_t)) = \int_{-\infty}^{\log(S/S_t)} g(u_\tau) du_\tau \tag{4.3}$$

and recover the price density $f_t(\cdot)$ corresponding to return density as

$$f_t(S) \quad = \quad \frac{\partial}{\partial S} \Pr(S_T \leq S) \quad = \quad \frac{g(\log(S/S_t))}{S} . \tag{4.4}$$

Our estimator of S-VAR will be

$$\hat{f}_t\left(S_T\right) \quad = \quad \frac{\hat{g}\left(\log(S_T/S_t)\right)}{S_T} \ . \tag{4.5}$$

which can be computed directly from the estimator (4.1) of the density function $g\left(\cdot\right)$. We then obtain

Proposition 2 *Under Assumptions 3 and 4:*

$$N^{1/2}H_u^{1/2}\left[\hat{g}\left(u\right)-g\left(u\right)\right] \quad \xrightarrow{d} \quad \mathcal{N}(0,\sigma_g^2) \tag{4.6}$$

where $\sigma_g^2 \equiv g\left(u\right)\left(\int_{-\infty}^{+\infty} k_u^2(\omega)d\omega\right)$. *Therefore*

$$N^{1/2}H_u^{1/2}\left[\hat{f}_t\left(S_T\right)-f_t\left(S_T\right)\right] \quad \xrightarrow{d} \quad \mathcal{N}(0,\sigma_f^2) \tag{4.7}$$

where

$$\sigma_f^2 \quad \equiv \quad \left(\int_{-\infty}^{+\infty} k_u^2(\omega)d\omega\right) f_t\left(S_T\right) \Big/ S_T \ .$$

The asymptotic distribution of the S-VAR estimator follows from the same technique as in Section 3.2. To obtain σ_f^2, simply multiply through (4.6) by $1/S_T$, and replace according to (4.4).

4.2 Statistical Inference for the Ratio $\hat{\zeta}$

At first glance, deriving the asymptotic distribution of the ratio $\hat{\zeta} \equiv \hat{f}^*/\hat{f}$ seems to involve a number of complex steps, due to the different sample sizes used to construct the numerator (based on the entire panel data of option prices) and the denominator (based only on the time series of index returns), as well as the cross-correlation between $\hat{f}^*$ and $\hat{f}$. Fortunately, this problem has a crucial characteristic that makes it quite simple: note from comparing (3.14) to (4.7) that the rates of convergence of the two estimators are different. In particular, the S-VAR estimator $\hat{f}$ converges faster than the E-VAR estimator $\hat{f}$.

Note that there are two effects partly offsetting each other: on the one hand, $\hat{f}^*$ is the product of differentiating twice a nonparametric regression function, while $\hat{f}$ is a direct nonparametric density estimator, and thus $\hat{f}^*$ converges slower; on the other hand, $\hat{f}^*$ is estimated using a larger sample size (based on the entire panel data of option prices, $n = NJ$) while $\hat{f}$ is estimated using only N observations (based only on the time series of index returns), and thus $\hat{f}^*$ converges faster. Note that our asymptotics have J fixed (the number of options trading on a given day, which is the result of institutional rules designed by the relevant market), and N (the number of days for which we collect data) increasing to infinity. Namely, we can see that the ratio of the convergence rates in (3.14) and (4.7) is given by

$$\begin{aligned} \frac{n^{1/2} h_{X/F}^{5/2} h_{\tau}^{1/2}}{N^{1/2} H_u^{1/2}} &= \mathbf{O}\left(\frac{\log(N) n^{(2p_{X/F}-5)/2(\tilde{d}+2p_{X/F})}}{\log(n) N^{p_u/(1+2p_u)}}\right) \\ &= \mathbf{O}\left(N^{\left(2p_{X/F}-5\right)/2(\tilde{d}+2p_{X/F})-p_u/(1+2p_u)}\right) \\ &= \mathbf{o}(1) \end{aligned}$$

by substituting in the specific bandwidth rules given in (3.10) and (4.2), and $\tilde{d} = 2$, $p_{X/F} = p_\tau = 5$ and $p_u = 5$.

As a result the asymptotic distribution of $\hat{f}^*/\hat{f}$ is identical to that of $\hat{f}^*/f$—that is, we can treat $\hat{f}$ as fixed at its true value f for the purpose of conducting inference on $\hat{\zeta}$. More specifically, the correlation between the two estimators induced by the use of the same data is of second order relative to the sampling variation of the slowly-converging E-VAR estimator, and is therefore discarded asymptotically. It follows immediately that:

Proposition 3 *Under Assumptions 1, 2, 3 and 4:*

$$n^{1/2}\, h_{X/F}^{5/2}\, h_{\tau}^{1/2} \left[\hat{\zeta}\left(S_T\right) - \zeta\left(S_T\right)\right] \quad \xrightarrow{d} \quad \mathcal{N}(0, \sigma_\zeta^2) \tag{4.8}$$

where

$$\sigma_\zeta^2 \quad \equiv \quad \sigma_{f^*}^2 \Big/ f_t^2\left(S_T\right) \ . \tag{4.9}$$

5 Implied Risk Aversion

It is apparent from (2.3) that the ratio of f^* to f is proportional to the MRS of the representative agent:

$$\zeta \;\equiv\; \frac{f_t^*(S_T)}{f_t(S_T)} \;\propto\; M_{t,T} \;=\; \frac{U_T'(C_T)}{U_t'(C_t)} \; . \tag{5.1}$$

Therefore, the ratio $\hat{\zeta}$ is an estimator—up to a scale factor—of the MRS itself. If $\hat{\zeta}(\cdot)$ is a nearly constant function over its domain (recall that both $\hat{f}^*(\cdot)$ and $\hat{f}(\cdot)$ are estimated *functions* with several arguments), this suggests that the representative agent of Section 2.1 is approximately risk neutral and that S-VAR and E-VAR measures will be close.[9] This remark will be the basis for a test of risk neutrality below. In interpreting the quantity $\hat{f}^*/\hat{f}$ as a scaled estimator of the MRS, we are implicitly assuming that an equilibrium asset-pricing model (such as the representative-agent model of Section 2.1) holds. Of course, the particular model of economic equilibrium used most often in these studies—the representative-agent model of Lucas (1978)—has been criticized on a number of theoretical and empirical grounds, e.g., Kirman (1992), Kocherlakota (1996) and Rogoff (1996).[10] Moreover, many of these models focus on aggregate consumption, whereas risk-management issues involve other quantities as well, e.g., the S&P 500 index.

Nevertheless, we can conduct the thought experiment of a simple economy in which market-wide financial aggregates like the S&P 500 proxy for aggregate consumption, as for instance in Brown and Gibbons (1995). In such a context, despite the fact that we do not know which model of economic equilibrium is the correct one, the relevance of $\hat{f}^*/\hat{f}$ for risk management can be motivated in another way: it is one measure of the risk preferences implicit in traded financial securities, and reflects the market's aggregation of such preferences in one particular model of economic equilibrium.

Although the differences between f and f^* can be quite large for certain types of preferences, implying important differences between S-VAR and E-VAR measures, there is

[9] Of course, if ζ is constant, then its only possible value is one, since both f^* and f integrate to one.

[10] See Constantinides (1982) for a discussion of the generality that the representative-agent model affords.

one set of preferences for which there is no difference: risk neutrality. If the representative agent is risk neutral, his utility function must be linear in C_T hence his MRS is unity and $f^*/f = 1$. Therefore, S-VAR measures are all special cases of the more general MRS-VAR measures, and the relevance of the latter over the former hinges on how risk averse the representative agent is in practice. In other words, the ratio f^*/f carries information that is relevant for risk management, but also more generally for the literature on tests of dynamic asset-pricing models, MRS bounds, the equity-premium puzzle, variance-bounds tests, etc.[11]

5.1 A Characterization of Implied Risk Aversion

Recall that in equilibrium any two of the following imply the third: (1) the representative agent's preferences U; (2) the asset price dynamics, or equivalently the conditional density function f; and (3) the SPD f^*. In this section, we make this statement practical by inferring from the market prices of options and index values the information on f^* and f respectively that is needed to characterize the representative agent's preferences. We have shown that

$$\zeta_t\left(S_T\right) \equiv \frac{f_t^*\left(S_T\right)}{f_t\left(S_T\right)} = \lambda\frac{U'\left(S_T\right)}{U'\left(S_t\right)}$$

where λ is a constant independent of the index level. Rather than extract from the ratio f^*/f information about the MRS (which would require knowledge of both the initial level of marginal utility $U'\left(S_t\right)$ and the constant λ), we can directly infer the Arrow-Pratt measure of (absolute) risk aversion $\rho_t\left(S_T\right)$ by noting that

$$\zeta_t'\left(S_T\right) = \lambda\frac{U''\left(S_T\right)}{U'\left(S_t\right)} \implies -\frac{\zeta_t'\left(S_T\right)}{\zeta_t\left(S_T\right)} = -\frac{U''\left(S_T\right)}{U'\left(S_T\right)} \equiv \rho_t\left(S_T\right) . \tag{5.2}$$

[11] See, for example, Hansen and Singleton (1983), Hansen and Jagannathan (1991), Mehra and Prescott (1985), and Shiller (1981).

This suggests a very natural estimation strategy for the measure of risk aversion $\rho(\cdot)$: we estimate the first derivatives of $f^*(\cdot)$ and $f(\cdot)$, and then calculate

$$\hat{\rho}_t(S_T) \equiv -\frac{\hat{\zeta}'_t(S_T)}{\hat{\zeta}_t(S_T)} = \frac{\hat{f}'_t(S_T)}{\hat{f}_t(S_T)} - \frac{\hat{f}^{*\prime}(S_T)}{\hat{f}^*(S_T)} . \tag{5.3}$$

Before we go any further, it is useful to examine what answer we would obtain if we were to apply our estimation strategy based on (5.2) to the case where the data were generated by the Black-Scholes model. It is well-known that Constant Relative Risk Aversion (CRRA) preferences sustain the Black-Scholes model in equilibrium [see for instance Rubinstein (1976), Breeden and Litzenberger (1978), Brennan (1979) and Bick (1987)], and we now show that our characterization (5.2) of the implied risk aversion through the ratio of E-VAR and S-VAR reproduces this result. In the Black-Scholes case, $f_t^*(S_T)$ is given by (3.4), while $f_t(S_T)$ is given by the same equation (3.4) with $(r-\delta)$ replaced by the actual drift of the price process under the actual measure, μ in

$$dS_t = \mu S_t dt + \sigma S_t dW_t . \tag{5.4}$$

Indeed, in the Black-Scholes model, S_T follows a geometric Brownian motion under both the actual and risk-neutral probability measures; the only difference lies in the expected return of the asset. It follows that

$$\begin{aligned}\zeta_{\mathrm{BS},t}(S_T) &= \left(\frac{S_T}{S_t}\right)^{-(\mu-r+\delta)/\sigma^2} \exp\left[\frac{(\mu-r+\delta)(\mu+r-\delta-\sigma^2)\tau}{2\sigma^2}\right] \\ &\equiv C\left(\frac{S_T}{S_t}\right)^{-a}\end{aligned} \tag{5.5}$$

where C denotes a constant and $a \equiv (\mu - r + \delta)/\sigma^2$. Depending upon the value of $a \geq 0$, we obtain (up to an irrelevant constant)

$$\begin{cases} a = 1: & U_{\mathrm{BS}}(S_T) = \log(S_T) \\ a \neq 1: & U_{\mathrm{BS}}(S_T) = S_T^{1-a}/(1-a) \end{cases} \tag{5.6}$$

$a = 0$ corresponds to a risk-neutral representative agent (in that case, the expected rate of return on the asset is $\mu = r - \delta$); $a = 1$ corresponds to logarithmic utility. More generally, the class of representative-agent utility functions which are implied by the Black-Scholes model all belong to the class,

$$\rho_{\mathrm{BS},t}\left(S_T\right) = \frac{a}{S_T} \tag{5.7}$$

with CRRA coefficient a.

This result can be linked quite naturally to the known necessary and sufficient condition for $\{S_t\}$ to be an equilibrium price process in a Black-Scholes economy [see Bick (1990), Wang (1993), and He and Leland (1993, Proposition 1)]: when σ is constant, the asset's expected return $\mu(S,t)$ in $dS_t = \mu(S_t,t)\,S_t\,dt + \sigma S_t dW_t$ must satisfy the partial differential equation

$$\frac{\partial \mu}{\partial t} + S\mu \frac{\partial \mu}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mu}{\partial S^2} = 0$$

with the boundary condition

$$\mu(S,T) = r - \delta - \sigma^2 \frac{SU''(S)}{U'(S)} .$$

In the Black-Scholes model, where μ is independent of the level of the stock price, this equation reduces to our characterization (5.7) of the representative agent's preferences jointly implied by f^*_{BS} and f_{BS}. Note that for each specification of the utility function (up to regularity conditions), a drift function can be found to construct an equilibrium price process with constant volatility. In other words,the equilibrium price dynamics of the underlying asset are not fully identified from the utility function alone. Knowledge of the SPD is required.

Of course, we wish to avoid relying on assumptions such as constancy of σ and/or μ and instead infer the representative agent's preferences directly from the market prices of options and index values in a nonparametric fashion—which is precisely what (5.3) allows us to do.

Using implied binomial trees, Jackworth (1996) also proposes to estimate aggregate risk aversion levels that are consistent with option prices. Indeed, any SPD estimator can be used to extract information about risk aversion. The relative strengths and weaknesses of these approaches are determined by the relative strengths and weaknesses of the corresponding SPD estimators on which they are based. See Section 3.3 and Aït-Sahalia and Lo (1997) for further details.

5.2 Statistical Inference for Implied Risk Aversion

We now derive the asymptotic distribution of our estimate of the Arrow-Pratt risk aversion measure $\hat{\rho}_t(\cdot)$. We are fortunate to be again in a context where one estimator, namely $\hat{f}_t^{*\prime}$, converges slower than every other one ($\hat{f}_t^*$, $\hat{f}_t'$ and $\hat{f}_t$) appearing in the ratio (5.3), all of which can then be taken as fixed for the purpose of computing the asymptotic distribution of $\hat{\rho}_t$. That is, $\hat{\rho}_t$ behaves asymptotically like $-\hat{f}^{*\prime}/\hat{f}^*$, which in turn behaves asymptotically like $-\hat{f}^{*\prime}/f^*$.

To estimate the first derivative of f^*, we select the bandwidth as in (3.10), except that we now assume $p_{X/F} = 5$ and have $m = 3$. The kernel choices are identical to those utilized to compute $\hat{f}^*$ itself, that is $k_\tau = k_{(4)}$ and $k_{X/F} = k_{(2)}$. We can apply (3.12)-(3.13) to $m = 3$ to obtain the distribution of $\partial^3\hat{\sigma}(\tilde{\mathbf{Z}})/\partial X^3 - \partial^3\sigma(\tilde{\mathbf{Z}})/\partial X^3$ with asymptotic variance $\sigma^2_{d^3\sigma}$. Under the same assumptions as Proposition 1, the asymptotic distribution of $\hat{f}^{*\prime}$ is given by that of $\partial H_{\mathrm{BS}}/\partial\sigma\,(\partial^3\hat{\sigma}/\partial X^3 - \partial^3\hat{\sigma}/\partial X^3)$, i.e.:

$$n^{1/2} h_{X/F}^{7/2} h_\tau^{1/2} \left[\hat{f}^{*\prime}(S_T) - f^{*\prime}(S_T)\right] \xrightarrow{d} \mathcal{N}(0, \sigma^2_{f^{*\prime}}) \tag{5.8}$$

where

$$\sigma^2_{f^{*\prime}} \equiv \left[\frac{\partial H_{\mathrm{BS}}}{\partial\sigma}(\hat{\sigma}(\tilde{\mathbf{Z}}), \mathbf{Z})\right]^2 \sigma^2_{d^3\sigma} . \tag{5.9}$$

To estimate f_t', we take the total derivative of (4.5) with respect to S_T, i.e., $f_t'(S_T) = (-1/S_T^2)\, g(\log(S_T/S_t)) + (1/S_T^2)\, g'(\log(S_T/S_t))$ and note that the leading term in the asymptotic distribution will be the slow-converging $(1/S_T^2)\, \hat{g}'(\log(S_T/S_t))$. We derive the distribution of $\hat{f}_t'$ for completeness only, since from what we have seen above it does not influence the distribution of $\hat{\rho}_t$. Set the bandwidth for $\hat{f}_t'$ as in (4.2), except that we now assume $p_u = 3$ and have $m_u = 1$. The kernel choice is $k_u = k_{(2)}$, and from

$$N^{1/2} H_u^{3/2} \left[\hat{g}'(u) - g'(u)\right] \quad \xrightarrow{d} \quad \mathcal{N}(0, \sigma_{g'}^2) \tag{5.10}$$

where $\sigma_{g'}^2 \equiv g(u) \left(\int_{-\infty}^{+\infty} (k_u')^2 (\omega) d\omega\right)$ we obtain, under the same assumptions as Proposition 2—simply multiply through (5.10) by $1/S_T^2$ and recall (4.4):

$$N^{1/2} H_u^{3/2} \left[\hat{f}_t'(S_T) - f_t'(S_T)\right] \quad \xrightarrow{d} \quad \mathcal{N}(0, \sigma_{f'}^2) \tag{5.11}$$

where

$$\sigma_{f'}^2 \quad \equiv \quad \frac{f_t(S_T)}{S_T^3} \left(\int_{-\infty}^{+\infty} (k_u')^2 (\omega) d\omega\right) \ . \tag{5.12}$$

We therefore obtain:

Proposition 4 *Under Assumptions 1–4:*

$$n^{1/2} h_{X/F}^{7/2} h_\tau^{1/2} \left[\hat{\rho}_t(S_T) - \rho_t(S_T)\right] \quad \xrightarrow{d} \quad \mathcal{N}(0, \sigma_\rho^2) \tag{5.13}$$

where

$$\sigma_\rho^2 \quad \equiv \quad \sigma_{f^{*\prime}}^2 \Big/ f^{*2}(S_T) \ . \tag{5.14}$$

These distributions will be used below to construct pointwise confidence intervals for the estimators.

5.3 Testing for Risk Neutrality

In our context, risk-neutrality of the representative agent can be characterized equivalently as $f^* = f$ or as $\rho = 0$. A test based on comparing either the $f^*(\cdot)$ to $f(\cdot)$ only involves the computation of the E-VAR and S-VAR, whereas comparing $\rho(\cdot)$ to 0 also involves calculating their first derivatives, and it can be shown that this results in a loss of power.

We therefore propose to test risk neutrality in the form of the hypothesis

$$\mathrm{H}_0 : \Pr\left(f_t^*\left(S_T\right) = f_t\left(S_T\right)\right) = 1 \quad \text{vs.} \quad \mathrm{H}_A : \Pr\left(f_t^*\left(S_T\right) = f_t\left(S_T\right)\right) < 1 \ . \tag{5.15}$$

A natural test statistic is:

$$R(f^*, f) \quad \equiv \quad E\left[(f_t^*\left(S_T\right) - f_t\left(S_T\right))^2 \omega(\tilde{\mathbf{Z}})\right] \tag{5.16}$$

where $\omega(\tilde{\mathbf{Z}})$ is a weighting function. An estimator for $R(f^*, f)$ is the sample analog of the right-hand-side of equation (5.16)

$$\hat{R}(\hat{f}^*, \hat{f}) \quad \equiv \quad \frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}_i^* - \hat{f}_i\right)^2 \omega_i \tag{5.17}$$

where $\hat{f}_i^* \equiv \hat{f}_{t_i}^*\left(S_{T_i}\right)$, $\hat{f}_i \equiv \hat{f}_{t_i}\left(S_{T_i}\right)$ and ω_i is a trimming index. Any other evaluation of the integral on the right-hand-side of (5.16) can be used. In practice, we evaluate numerically the integral on a rectangle of values of $\tilde{\mathbf{Z}}$ representing the support of the density π, and use the binning method to evaluate the kernels (see e.g., Wand and Jones (1995) for a description of the binning method). To construct the estimators involved in (5.16), we use the following bandwidth and kernel functions:

Assumption 5 *The bandwidths $h_{X/F}$ and h_τ to estimate the E-VAR $\hat{f}^*$ in (5.16) are given by:*

$$h_{X/F} \quad = \quad \eta_{X/F}\, n^{-1/\delta_{X/F}} \qquad , \qquad h_\tau = \eta_\tau n^{-1/\delta_\tau}$$

where $\eta_{X/F}$ and η_τ are constant, and $\delta_{X/F} = \delta_\tau + 4$ and δ_τ satisfy:

$$\left(\frac{3}{\delta_\tau + 6}\right) + \left(\frac{1}{\delta_\tau}\right) \quad < \quad \frac{1}{2} \quad < \quad \left(\frac{q + 9/2}{\delta_\tau + 4}\right) + \left(\frac{1/2}{\delta_\tau}\right) \tag{5.18}$$

in addition to

$$\left(\frac{3/2}{\delta_\tau + 4}\right) \quad > \quad \left(\frac{1/2}{\delta_\tau}\right). \tag{5.19}$$

Furthermore, $k_{X/F} = k_\tau = k$ is a kernel of order q. The bandwidth to estimate the S-VAR $\hat{f}$ in (5.16) is identical to H_u given by (4.2) with $m_u = 0$.

If the representative agent's preferences implied by the joint data on option prices (f^*) and index dynamics (f) are indeed risk-neutral, then the two density functions should be close to one another, and R close to zero. If we can take into account the sampling variation due to data noise, then we will be able to use (5.17) as the basis for a test statistic of (5.15):

Proposition 5 *Under Assumptions 1, 3 and 5, and under* H_0*:*

$$n\, h_{X/F}^{9/2}\, h_\tau^{1/2}\, \hat{R}(\hat{f}^*, \hat{f}) \; - \; h_{X/F}^{-1/2}\, h_\tau^{-1/2}\, B_R \;\; \stackrel{d}{\rightarrow} \;\; \mathcal{N}(0, \mathbf{\Sigma}_R^2) \tag{5.20}$$

where

$$B_R = \left(\int_{-\infty}^{+\infty} k''^2(w) dw \right) \left(\int_{-\infty}^{+\infty} k^2(w) dw \right) \int_{\tilde{\mathbf{Z}}} \sigma^2(\tilde{\mathbf{Z}}) \tilde{\omega}(\tilde{\mathbf{Z}}) d\tilde{\mathbf{Z}} \tag{5.21}$$

$$\begin{aligned} \mathbf{\Sigma}_R^2 = 2 \left[\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} k''(w) k''(w+v) dw \right)^2 dv \right] \times \\ \left[\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} k(w) k(w+v) dw \right)^2 dv \right] \int_{\tilde{\mathbf{Z}}} \sigma^4(\tilde{\mathbf{Z}}) \tilde{\omega}^2(\tilde{\mathbf{Z}}) d\tilde{\mathbf{Z}} \end{aligned} \tag{5.22}$$

where

$$\tilde{\omega}(\tilde{\mathbf{Z}}) \equiv \left(e^{r_{t,\tau}\tau} \frac{\partial H_{\mathrm{BS}}(\sigma(\tilde{\mathbf{Z}}), \mathbf{Z})}{\partial \sigma} \right)^2 \omega(\tilde{\mathbf{Z}}) \; . \tag{5.23}$$

We report in Table 2 the values of the new kernel integrals that appear in the expressions above for the actual choice $k = k_{(2)}$. The distribution (5.20) follows again from the functional delta method, applied now to derive the second order term in the expansion of the test statistic (under the null, the liner term is degenerate). It is remarkable that the estimation of S-VAR does not contribute any term to the asymptotic distribution of the test statistic. In addition, only the higher order derivative of the nonparametric regression function matter. This follows from the fact that $(e^{r_{t,\tau}\tau} \{\partial H_{\mathrm{BS}}(\sigma_{\mathrm{BS}}, \mathbf{Z})/\partial\sigma\} \{\partial^2 \hat{\sigma}(\tilde{\mathbf{Z}})/\partial X^2 - \partial^2 \sigma(\tilde{\mathbf{Z}})/\partial X^2\})^2$ is the leading term in the expansion of $(\hat{f}_t^*(S_T) - f_t^*(S_T))^2$, which is itself the leading term in the expansion of

$$\begin{aligned}
\left(\hat{f}_t^*\left(S_T\right)-\hat{f}_t\left(S_T\right)\right)^2 &= \left(\hat{f}_t^*\left(S_T\right)-f_t^*\left(S_T\right)+f_t\left(S_T\right)-\hat{f}_t\left(S_T\right)\right)^2 \\
&= \left(\hat{f}_t^*\left(S_T\right)-f_t^*\left(S_T\right)\right)^2-2\left(\hat{f}_t^*\left(S_T\right)-f_t^*\left(S_T\right)\right)\left(\hat{f}_t\left(S_T\right)-f_t\left(S_T\right)\right) \\
&\qquad +\left(\hat{f}_t\left(S_T\right)-f_t\left(S_T\right)\right)^2 \\
&= \left(\hat{f}_t^*\left(S_T\right)-f_t^*\left(S_T\right)\right)^2+\mathbf{o}\left(\left\|\hat{f}_t^*-f_t^*\right\|^2\right) .
\end{aligned}$$

To estimate consistently the conditional variance of the regression, $s^2(\tilde{\mathbf{Z}})$ we calculate the difference between the kernel estimate of the regression of the squared dependent variable σ^2 on $\tilde{\mathbf{Z}}$ and the squared of the regression $\sigma(\tilde{\mathbf{Z}})$ of the dependent variable σ. The regression $E[\sigma^2|\tilde{\mathbf{Z}}]$ is estimated with bandwidth $h_{cv} = \eta_{cv} n^{-1/\delta_{cv}})$, $\delta_{cv} = \delta_\tau$ and η_{cv} constant. Prior to computing the estimates, we standardize the regressors $\tilde{\mathbf{Z}}$ by removing their respective sample means, and dividing by their sample unconditional standard deviations.

The test statistics are formed by standardizing the asymptotically normal distance measure $\hat{R}$: we estimate consistently the asymptotic mean B_R and variance $\mathbf{\Sigma}_R^2$, then subtract the mean and divide by the standard deviation. The test statistic then has the asymptotic $N(0,1)$ distribution. Since the test is one-sided (we only reject when $\hat{R}$ is too large, hence when the test statistic is large and positive), the 10 percent critical value is 1.28, while the 5 percent value is 1.64. We fix the variables in $\mathbf{Z}$ that are excluded from $\tilde{\mathbf{Z}}$ at their sample means.

5.4 Testing for Specific Preferences

More generally, we can test whether the index and option data, as summarized by our estimated E-VAR and S-VAR, are compatible with the representative agent following a specific set of preferences. For instance, the null hypothesis could specify that the representative agent has CRRA preferences, $\rho_t\left(S_T\right) = a/S_T$. Let $\rho_t\left(S_T;\theta\right)$ be the Arrow-Pratt measure of risk aversion specified by the null hypothesis, where θ is a parameter vector in a compact subset of R^b. Note that we cannot test the specific preferences hypothesis at the level of f^*/f since that would involve an unknown factor (the initial value of the agent's marginal utility). However, the hypothesis is unambiguous if taken in the form of the risk aversion measure. We can test

$$\mathrm{H}_0 : \Pr\left(\rho_t\left(S_T\right) = \rho_t\left(S_T;\theta\right)\right) = 1 \quad \text{vs.} \quad \mathrm{H}_A : \Pr\left(\rho_t\left(S_T\right) = \rho_t\left(S_T;\theta\right)\right) < 1$$

using as a test statistic:

$$P(\rho_t, \rho_t\left(\cdot;\theta\right)) \quad \equiv \quad E\left[\left(\rho_t\left(S_T\right) - \rho_t\left(S_T;\theta\right)\right)^2 \omega(\mathbf{Z})\right]$$

where $\omega(\mathbf{Z})$ is a weighting function. The distribution of this statistic has the same form as in Section 5.3 , except that its convergence rate is slower due to the fact that $\hat{\rho}_t\left(\cdot\right)$ converges even slower that $\hat{f}_t^*\left(\cdot\right)$. Just as the estimation of $\hat{f}_t\left(\cdot\right)$ did not affect the distribution of the statistic in the previous case, the estimation of the (a priori unknown) parameter vector θ will not affect the distribution of this statistic. Any consistent estimator θ of θ may be used for the purpose of calculating

$$\hat{P}\left(\hat{\rho}_t, \hat{\rho}_t\left(\cdot;\hat{\theta}\right)\right) \quad \equiv \quad \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\rho}_t, \rho_{i,\hat{\theta}}\right)^2 \omega_i \ .$$

6 An Empirical Example

To gauge the empirical relevance of E-VAR, we compare it to S-VAR in the case of S&P 500 index options using data obtained from the CBOE for the sample period from January 4, 1993 to December 31, 1993. In particular, we estimate $\zeta \equiv f^*/f$ by taking the ratio of nonparametric estimators of f^* and f, where f^* is estimated according to Aït-Sahalia and Lo (1997) (see Section 3) and f is estimated by standard kernel density estimation techniques. We also estimate the coefficient of risk aversion ρ as described in Section 5.1, and test the null hypothesis of risk neutrality as proposed in Section 5.3.

6.1 S&P 500 Index Options

We use the same dataset of option prices and characteristics as Aït-Sahalia and Lo (1997) hence our discussion of its properties shall be brief. The dataset contains 16,923 daily pairs of call- and put-option prices for S&P 500 Index Options (symbol: SPX), traded on the

Chicago Board Options Exchange between January 4, 1993 and December 31, 1993. We take averages of bid- and ask-prices as our raw data. Observations with time-to-maturity less than one day, implied volatility greater than 70%, and price less than 1/8 are dropped, which yields a final sample of 14,431 observations and this is the starting point for our empirical analysis.

To address problems of infrequent trading, nonsynchronous prices, and dividends, we process the raw data using the following procedure. Since all option prices are recorded at the same time on each day, we require only one temporally-matched index price per day. To circumvent the unobservability of the dividend rate $\delta_{t,\tau}$, we infer the futures price $F_{t,\tau}$ for each maturity τ. By the spot-futures parity, $F_{t,\tau}$ and S_t are linked through:

$$F_{t,\tau} = S_t e^{(r_{t,\tau}-\delta_{t,\tau})\tau} . \tag{6.1}$$

To derive the implied futures, we use the put-call parity relation which must hold if arbitrage opportunities are to be avoided, independently of any parametric option pricing model:[12]

$$H(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}) + Xe^{-r_{t,\tau}\tau} = G(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}) + F_{t,\tau}e^{-r_{t,\tau}\tau} . \tag{6.2}$$

where G denotes the put price. To infer the futures price $F_{t,\tau}$ from this expression, we require reliable call and put prices—prices of actively traded options—at the *same* strike price X and time-to-expiration τ. To obtain such reliable pairs, we must use calls and puts that are closest to at-the-money [recall that in-the-money options are illiquid relative to out-the-money ones, hence any matched pair that is not at-the-money would have one potentially unreliable price]. On every day t, we do this for all available maturities τ to obtain for each maturity the implied futures price from put-call parity.

Given the derived futures price $F_{t,\tau}$, we then replace the prices of all illiquid options, i.e., in-the-money options, with the price implied by put-call parity at the rele-

[12] Put-call parity is a pure arbitrage relationship, and as a result is almost never violated in practice. See also Black and Scholes (1972), Harvey and Whaley (1992), Kamara and Miller (1995) and Rubinstein (1985).

vant strike prices. Specifically, we replace the price of each in-the- money call option with $G(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}) + F_{t,\tau}e^{-r_{t,\tau}\tau} - Xe^{-r_{t,\tau}\tau}$ where, by construction, the put with price $G(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau})$ is out-of-the-money and therefore liquid. After this procedure, all the information contained in liquid put prices has been extracted and resides in corresponding call prices via put-call parity, therefore put prices may now be discarded without any loss of reliable information.

To estimate S-VAR, we collect the time series of S&P 500 index returns from the Center for Research in Security Prices (CRSP) up to December 31, 1993. We calculate the continuously compounded returns corresponding to holding the S&P 500 index for τ days, and use them as the inputs $\{u_\tau\}$ to construct the estimator (4.1) for each horizon τ. Each series contains $N = 1008$ observations, i.e., four years of daily observations.

6.2 The E-VAR and S-VAR of the S&P 500

We focus our empirical analysis on the six-month ($\tau = 126$ days) horizon. We report in Figure 1 the estimated implied volatility curve $X/F \mapsto \hat{\sigma}(X/F, \tau)$ and its first three derivatives with respect to moneyness, estimated from the semiparametric model. The bandwidth values are given in Table 3. The curves in Figure 1 are the basic inputs to our subsequent estimators of $\hat{f}^*$, $\hat{\zeta}$ and $\hat{\rho}$. The confidence intervals are calculated based on Proposition 1.

Figure 2 plots the estimated E-VAR and S-VAR, in addition to a 95% confidence interval for E-VAR, for the four traded option maturities. The main difference between E-VAR and S-VAR is the difference in skewness between the two densities: E-VAR is strongly negatively skewed, while S-VAR is slightly positively skewed.

These facts combine to make the estimated ratio $\hat{\zeta}$ in Figure 3 generally decreasing as a function of the S&P 500 value, and statistically different from one (which would have corresponded to risk neutrality of the representative agent). That $\hat{\zeta}$ is decreasing in S_T can be understood intuitively by noting that the expected value of S_T under the actual measure is empirically higher than its expected return under the risk-neutral measure, i.e.,

$$E[S_T|S_t] = \int_0^{+\infty} S_T f_t(S_T)\, dS_T \quad > \quad \int_0^{+\infty} S_T f_t^*(S_T)\, dS_T = E^*[S_T|S_t] \tag{6.3}$$

which in terms of rates of return corresponds to $\mu > r - \delta$. Therefore, for (6.3) to hold, we expect $f_t(S_T) > f_t^*(S_T)$ for the high values of S_T and conversely for the low values of S_T; that is, $\hat{\zeta}(S_T) > 1$ for low values of S_T and $\hat{\zeta}(S_T) < 1$ for large values of S_T. This downward-slopping pattern for $\hat{\zeta}$ is confirmed by Figure 3.

To quantify the relative preferences of the representative agent between a \$1 payoff in different states, consider the future value of the inverse of the ratio $\hat{\zeta}$ plotted in Figure 3:

$$\frac{1}{e^{-r_{t,\tau}\tau}\zeta_t(S_T)} = \frac{f_t(S_T)}{e^{-r_{t,\tau}\tau} f_t^*(S_T)}. \tag{6.4}$$

The denominator of the right-hand side is the price at t of an Arrow-Debreu security paying \$1 at $T = t+\tau$ if the S&P 500 state is between $\$S_T$ and $\$(S_T+1)$. The numerator is the probability of that event actually being realized, that is, the expected payoff from buying the Arrow-Debreu security. Hence their ratio is one plus the expected rate of return from buying the security at t and selling it at T. We report these returns in Table 5 for different states S_T, and contrast them with those of the Black-Scholes model. Not surprisingly, the Black-Scholes model makes the negative states substantially less valuable than the nonparametric estimates do, reflecting the lack of skewness in its E-VAR for continuously-compounded returns.

We then implement empirically the test for risk neutrality of the representative agent that was developed in Proposition 5. The null hypothesis of risk neutrality ($f^* = f$) is rejected with a p-value of 0.00, which confirms Figure 3 (recall that risk neutrality corresponds to $\zeta = 1$, i.e., a horizontal line in Figure 3).

6.3 The Implied Risk Aversion of S&P 500 Options

To estimate the coefficient of risk aversion, we need to select the bandwidth values to estimate the third derivative of the call pricing function, and the first derivative of S-VAR. Bandwidth values to estimate $\hat{f}^{*\prime}$ and $\hat{f}'$ are reported in Table 6, and we plot the implied risk aversion in Figure 4. The confidence interval is constructed from the asymptotic distribution theory derived in Proposition 4.

The notable feature of Figure 4 is that $\hat{\rho}$ is decreasing as a function of S_T. In other words, the market prices of S&P 500 options and the market returns on the S&P 500 index are such that the representative agent becomes more averse as the index goes down in value. This phenomenon is more pronounced than under CRRA preferences [see (5.7)], providing yet another characterization of the differences between market prices and the Black-Scholes model. Note also that $\hat{\rho}$ in Figure 4 is everywhere positive, thereby implying a concave utility function.

Fitting the curve (5.7) to the nonparametric estimate of $\hat{\rho}$ in Figure 4 provides an estimate of the coefficient of risk aversion under the null hypothesis of CRRA preferences. The best fit curve yields in turn an estimate of the CRRA a (which in the Black-Scholes model is equal to $(\mu - r + \delta)/\sigma^2$). An interesting comparison is whether the implied value of a is "reasonably" compatible with the values of μ, δ and σ given by the S&P 500 returns and the riskfree rate r. In other words, is there an equity-premium-like puzzle at the levels of option prices, or do they imply coefficients of risk aversion that are less extreme than those typically found in the equity-premium literature?[13] We find that the implied value of the coefficient a is 25.5, which is substantially higher than the typical values used in theoretical models (where the range is typically 1-10), yet comparable to the estimates exhibited by studies of the Euler equation in consumption asset pricing models: see Table 7 for the range of values of the constant CRRA that have been reported in the literature. While the implied CRRA coefficient is informative, it is important to keep in mind that Figure 4 shows that CRRA preferences are quite misspecified: the CRRA best fit curve is quite far from the nonparametrically-estimated risk aversion curve.

7 Conclusion

Risk management has become a first-order concern for financial managers and in this paper, we argue that economic value-at-risk is a more relevant quantity for risk managers than the more traditional statistical value-at-risk. The difference lies in the fact that E-VAR incorporates and reflects the combined effects of aggregate risk preferences, supply and demand, and probabilities; S-VAR involves only one of these effects. Moreover, if

[13] See Renault and Garcia (1996) for a different attempt to confront option data with the Euler Equation.

aggregate preferences were risk neutral, E-VAR reduces to S-VAR as a special case, hence no information is lost in using E-VAR as a starting point for the risk management process.

However, E-VAR is computationally more demanding, particularly for the nonparametric estimators that we have proposed. Although a parametric version of E-VAR is readily available (as in our Black-Scholes case), there is so much mounting empirical evidence against the standard parametric models for so many types of assets that we are reluctant to propose any approach other than a nonparametric one. Therefore, a distinct disadvantage of E-VAR is its computational complexity. Nevertheless, given the very nature of the risk management function, the potential benefits of E-VAR would seem to outweigh the computational costs.

In our empirical example, we have demonstrated that S-VAR and E-VAR yield considerably different risk assessments for the S&P 500 index. This suggests that E-VAR is capturing aspects of market risk that S-VAR is not (recall that S-VAR is a special case of E-VAR). However, conclusive evidence of the superiority of E-VAR must lie in its applications to specific risk-management processes, and we hope to collect such evidence in the near future.

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m	0	1	2	3
$\int_{-\infty}^{+\infty}\left(k_{(2)}^{(m)}\right)^2(w)dw$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}}$	$\frac{3}{8\sqrt{\pi}}$	$\frac{31}{16\sqrt{\pi}}$
$\int_{-\infty}^{+\infty}\left(k_{(4)}^{(m)}\right)^2(w)dw$	$\frac{27}{32\sqrt{\pi}}$	$\frac{175}{64\sqrt{\pi}}$	$\frac{273}{128\sqrt{\pi}}$	$\frac{2025}{256\sqrt{\pi}}$

Table 1: Kernel Constants in Asymptotic Distributions

Kernel constants that characterize the asymptotic variances of the nonparametric estimators in Propositions 1, 2, 3 and 4. The kernel functions $k_{(2)}$ and $k_{(4)}$ are defined in (3.11). $k^{(m)}$ denotes the m-th derivative of k.

m	0	1	2
$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}k_{(2)}^{(m)}(w)k_{(2)}^{(m)}(w+v)dw\right)^2dv$	$\frac{1}{2\sqrt{2\pi}}$	$\frac{3}{32\sqrt{2\pi}}$	$\frac{105}{512\sqrt{2\pi}}$

Table 2: Kernel Constants in Test Statistic

Kernel convolution constants that characterize the asymptotic variance of the test statistic in Proposition 5. The kernel function $k_{(2)}$ is defined in (3.11), and $k_{(2)}^{(m)}$ denotes its m-th derivative.

Variable	Estimator	Kernel	n	N	q	p	m	d	s	h	h_{cv}	H
X/F	$\hat{f}^*$	$k_{(2)}$	$14,431$		2	5	2	2	0.074	0.040	0.021	
τ	$\hat{f}^*$	$k_{(2)}$	$14,431$		2	5	0	2	72.32	20.52	20.52	
u	$\hat{f}$	$k_{(2)}$		$1,008$	2	5	0	1	0.071			0.0196

Table 3: Bandwidth Values for E-VAR and S-VAR

Bandwidth selection for the second derivative of the estimator $\hat{\sigma}(X/F_{t,\tau},\tau)$ required in addition to the previous one to compute $\hat{f}^{*\prime}$, and bandwidth selection for $\hat{f}$. The rules for $h_{X/F}$ and h_τ are $h_{X/F} = c_{X/F}\mathrm{s}(\mathbf{X/F})n^{-1/(\tilde{d}+2(q_{X/F}+m))}$and $h_\tau = c_\tau \mathrm{s}(\boldsymbol{\tau})n^{-1/(\tilde{d}+2q_\tau)}$ where $k_{X/F}$ is of order $q_{X/F}$ and k_τ is of order q_τ, $\mathrm{s}(\mathbf{X/F})$ and $\mathrm{s}(\boldsymbol{\tau})$ are the unconditional standard deviations of the nonparametric regressors, $c_{X/F} \equiv \gamma_{X/F}/\ln(n)$ with $\gamma_{X/F}$ constant, and $c_\tau \equiv \gamma_\tau/\ln(n)$, with γ_τ constant. h_{cv} denotes the bandwidth used to estimate the conditional variance of the nonparametric regression. The rule for H_u is $H_u = c_u s(u)N^{-1/(1+2(q_u+m_u))}$ where q_u is the order of the kernel k_u to estimate the m_u -th derivative of g, $\mathrm{s}(u)$ is the unconditional standard deviation of the returns, and $c_u \equiv \ \gamma_u/\ln(N)$ with $\gamma_u$constant.

	Kernel	δ	η	h	h_{cv}	Trim	$H_0 : f^* = f$
X/F	$k_{(2)}$	8.75	0.48	0.011856	0.01496	50 / 5	
τ	$k_{(2)}$	4.75	0.48	4.62255	14.6577	50 / 5	
Integral							54.5
Test Statistic							19.7
p-value							0.00

Table 4: Test of Risk Neutrality

Nonparametric tests of the null hypothesis of risk neutrality of the representative agent, based on comparing globally E-VAR to S-VAR. The bandwidths are $h_{X/F} = \eta_{X/F}\, n^{-1/\delta_{X/F}}$ and $h_\tau = \eta_\tau n^{-1/\delta_\tau}$ with $\eta_{X/F}$ and η_τ constant, $\delta_{X/F} = \delta_\tau + 4$ and δ_τ satisfying the inequalities (5.18)-(5.19). Further, $k_{X/F} = k_\tau = k$ is a kernel of order $q = 2$. The bandwidth to estimate the S-VAR $\hat{f}$ in (5.16) is H_u given by the rule (4.2) with $m_u = 0$. The average value of H_u across all maturities is 0.015833. The bandwidths to estimate the conditional variance of the nonparametric regression are $h_{X/F,cv}$ and $h_{\tau,cv}$ respectively, and are optimally smoothed to produce consistent estimates (rate $\delta_{X/F,cv} = (2q_{X/F} + d) = 6 = \delta_{cv,\tau}$). The weighting function ω is a trimming index, i.e., only observations with estimated density above a certain level, and away from the boundaries of the integration space, are retained. The two numbers in the column "Trim" refer respectively to the trimming level (as a percentage of the mean estimated density value), and the percentage trimmed at the boundary of the integration space when calculating the test statistics. For instance, if the latter is 5 percent, the trimming index retains the values between 1.05 times the minimum evaluation value and 0.95 times the maximum value. "Integral" refers to the percentage of the estimated density mass on the integration space that is kept by the trimming index, i.e., $\int \pi(\tilde{Z})\omega(\tilde{Z})d\tilde{Z}$, where π is the marginal density of the nonparametric regressors $\tilde{Z} = [\, X/F_{t,\tau}\ \tau\,]'$. "Test Statistic" refers to the standardized distance measure between the E-VAR and S-VAR estimates (remove the bias term, divide by the standard deviation). The integral defining R is calculated over the integration space given by the rectangle $[0.85, 1.10] \times [10, 136]$ in the moneyness $\times$ days-to-expiration space. The kernel weights are constructed using the binning method with 30 bins in the moneyness dimension and 20 in the days-to-expiration dimension.

State: S_T	375	400	425	450	475	500
Nonparametric Rate of Return	-4.65	-0.17	-0.93	-0.51	0.75	0.85
Black-Scholes Rate of Return	-3.09	-2.09	-1.34	-0.27	0.56	1.35

Table 5: Expected Rates of Returns of S&P 500 States

Expected rate of return (annualized and continuously compounded over a horizon of $\tau = 126$ days) from an investment in an Arrow-Debreu claim that pays if the state at T falls between $\$S_T$ and $\$(S_T + 1)$. The nonparametric estimates are obtained with the bandwidths given in Table 3. The Black-Scholes estimate is based on $\hat{\mu} = 7.64\%$, $\hat{\sigma} = 9.74\%$, $r_{t,\tau} = 3.10\%$ and $\delta_{t,\tau} = 2.78\%$. The resulting Black-Scholes value of the CRRA coefficient is 7.72. Negative values in the Table correspond to states which are more likely to occur under the risk-neutral than the actual probabilities; they are correspondingly ”expensive” based on the expected rate of return measure.

Variable	Kernel	Estimate	n	N	q	p	m	d	s	h	H
X/F	$k_{(2)}$	$\hat{f}^{*\prime}$	$14,431$		2	5	3	2	0.074	0.047	
τ	$k_{(2)}$	$\hat{f}^{*\prime}$	$14,431$		2	5	0	2	72.324	20.52	
u	$k_{(2)}$	$\hat{f}^{\prime}$		$1,008$	2	5	1	1	0.071		0.029

Table 6: Bandwidth Values for the Risk Aversion Coefficient

Bandwidth selection for the third derivative of the estimator $\hat{\sigma}(X/F_{t,\tau},\tau)$ required in addition to the previous two to compute $\hat{f}^{*\prime}$, and bandwidth selection for $\hat{f}^{\prime}$. The rules for $h_{X/F}$ and h_τ are $h_{X/F} = c_{X/F}\mathrm{s}(\mathbf{X/F})n^{-1/(\tilde{d}+2(q_{X/F}+m))}$ and $h_\tau = c_\tau \mathrm{s}(\boldsymbol{\tau})n^{-1/(\tilde{d}+2q_\tau)}$ where $k_{X/F}$ is of order $q_{X/F}$ and k_τ is of order q_τ, $\mathrm{s}(\mathbf{X/F})$ and $\mathrm{s}(\ \boldsymbol{\tau})$ are the unconditional standard deviations of the nonparametric regressors, $c_{X/F} \equiv \gamma_{X/F}/\ln(n)$, with $\gamma_{X/F}$ constant, and $c_\tau \equiv \gamma_\tau/\ln(n)$, with γ_τ constant. The rule for H_u to estimate the m_u-th derivative of g is $H_u = c_u\mathrm{s}(u)N^{-1/(1+2(q_u+m_u))}$ where q_u is the order of the kernel k_u, $\mathrm{s}(u)$ is the unconditional standard deviation of the returns, and $c_u \equiv \gamma_u/\ln(N)$, with γ_u constant.

	CRRA Range
Arrow (1971)	1
Friend and Blume (1975)	2
Hansen and Singleton (1982,1984)	$0 - 1$
Mehra and Prescott (1985)	55
Ferson and Constantinides (1991)	$0 - 12$
Cochrane and Hansen (1992)	$40 - 50$

Table 7: Estimated Values of the Constant Coefficient of Relative Risk Aversion

Representative values of the CRRA coefficient reported in the literature, to be compared .to then the fitted CRRA found here. The value in Arrow (1971) is based on a summary of a number of studies as well as theoretical considerations. Friend and Blume (1975) study individual portfolio holdings. Other values are based on Table I in Hansen and Singleton (1984); Footnote 5 in Mehra and Prescott (1985); Table 4 in Ferson and Constantinides (1991); and in Cochrane and Hansen (1992), the range 40-50 is required to fit the Hansen-Jagannathan (1991) bound (their Figure 1), i.e., values of a of at least 40 are required to generate a variance of the stochastic discount factor implied by the equity premium region on the graph. Even at that value, the mean-standard deviation pairs do not lie inside the Hansen-Jagannathan "cup" however.

Figure 1a
Implied Volatility Curve

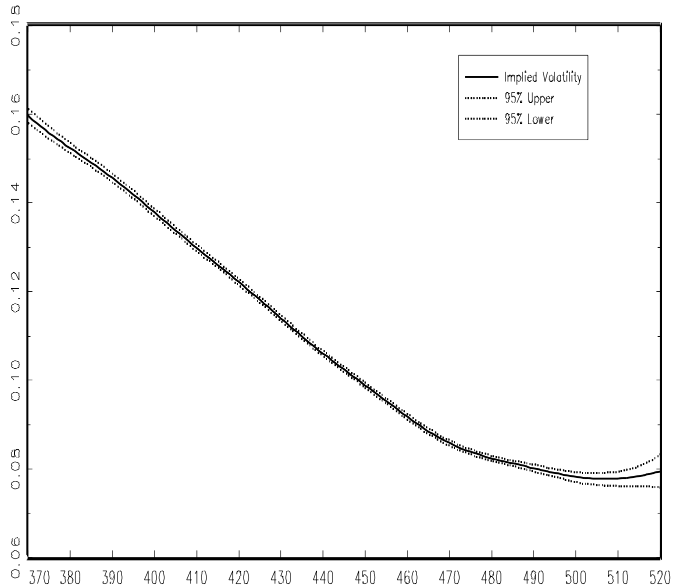

Figure 1b
First Derivative

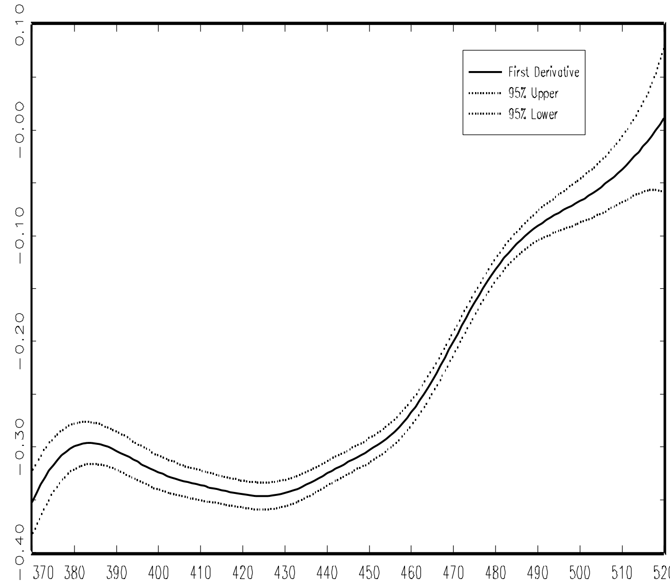

Figure 1c
Second Derivative

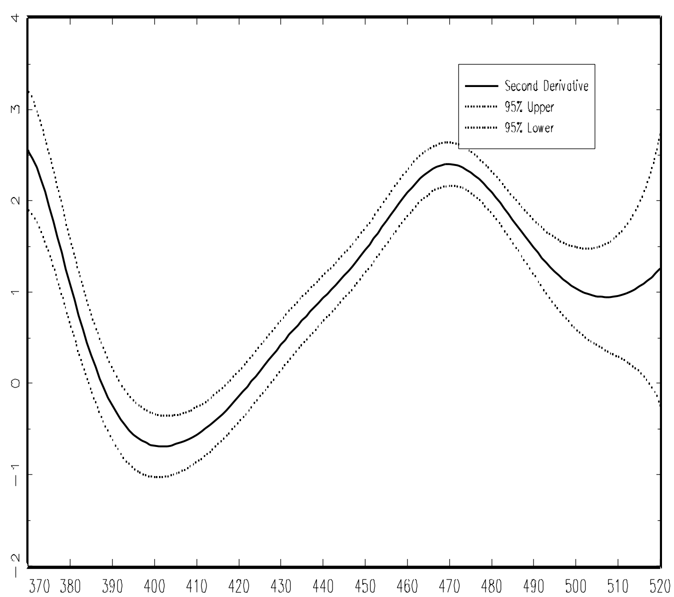

Figure 1d
Third Derivative

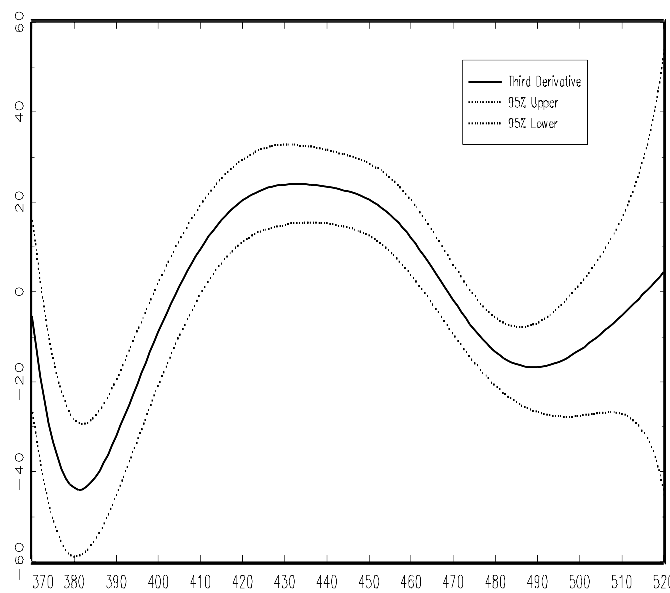

Figure 2
Comparison of E-VAR and S-VAR

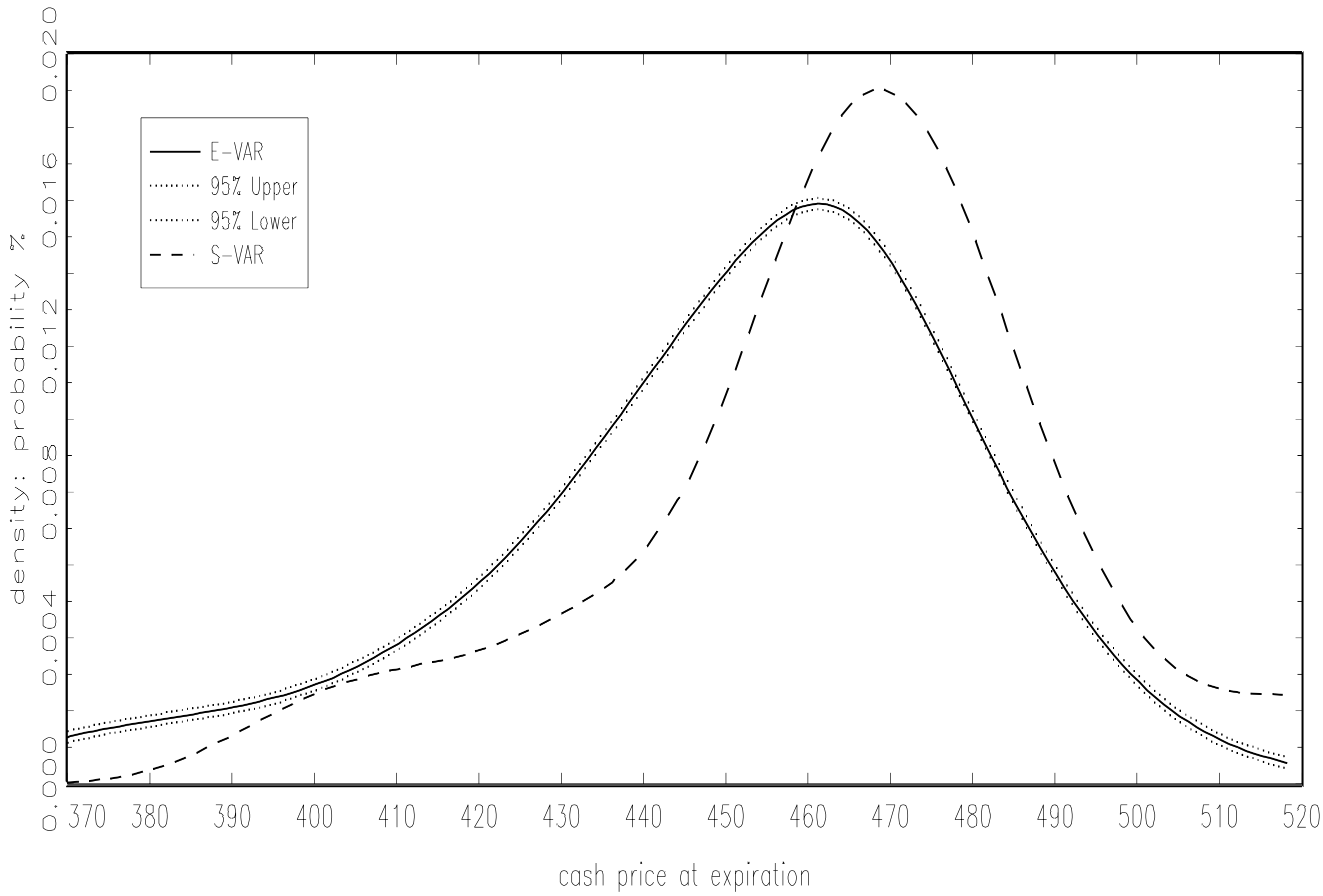

Figure 3

Scaled Marginal Rate of Substitution

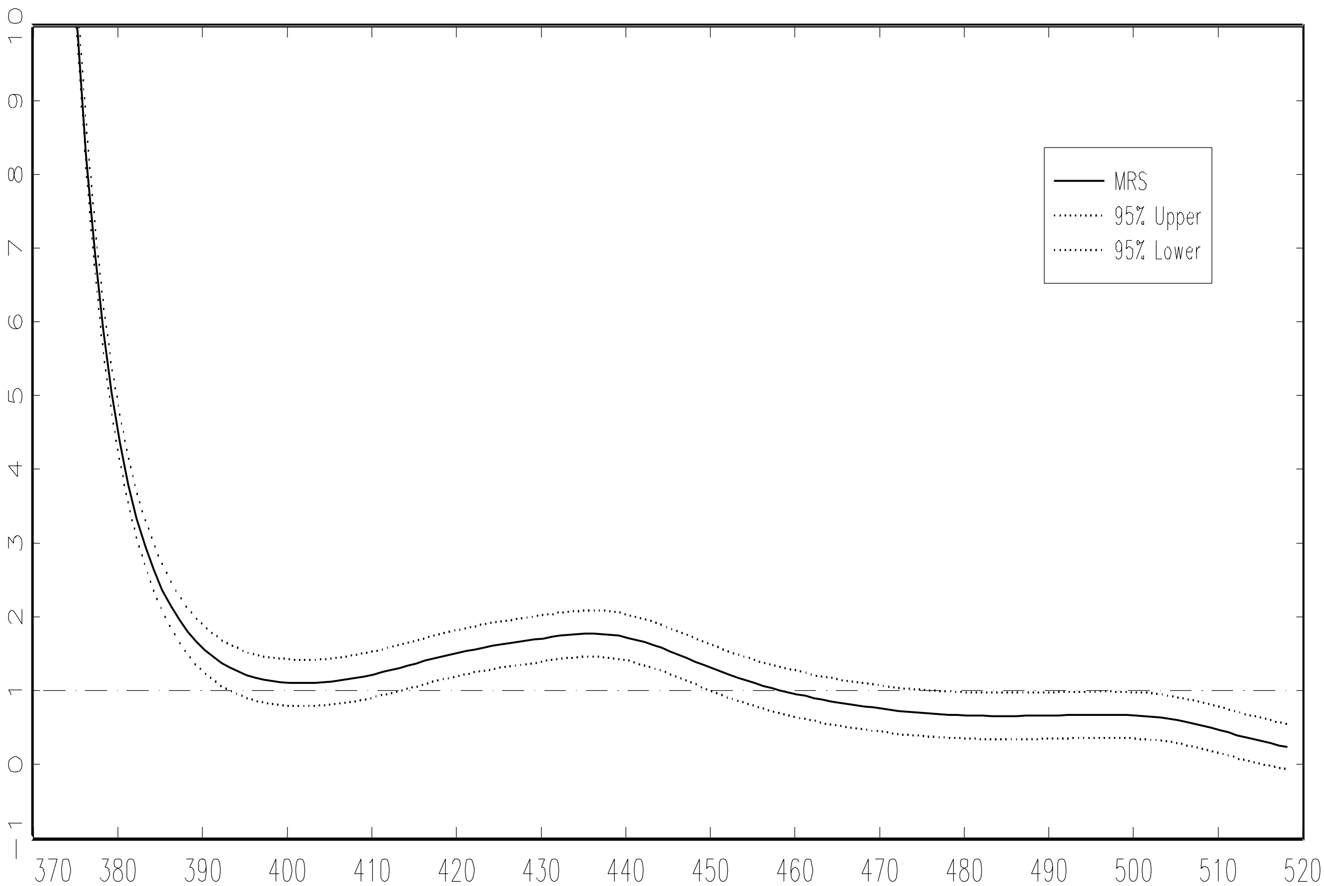

Figure 4
Implied Risk Aversion