USE OF CIR-TYPE INTEREST RATE MODELS TO ASSESS THE ECONOMIC VALUE OF PARTICIPATING SAVINGS CONTRACTS?

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The price of savings contracts with options and guarantees (so called "with profit contracts or participating contracts") cannot be observed on an organized market. The justification of the choice and the calibration of the economic scenario generator by a direct comparison with observed prices cannot be considered. In this Mark-to-Model valuation framework, questions naturally arise about the relevance of model choices and the construction of quality criteria to validate these choices.

This article focuses on the problem of choosing interest rate models intended to assess the economic value (best estimate) of with-profit savings contracts in an economic environment characterised by negative interest rates. It follows on from previous work (see Armel and Planchet [2018] and Armel and Planchet [2019]). In this work, a construction of a risk-neutral economic scenario generator (ESG) used to assess the sensitivity of the best-estimate to the Hull & White, G2++ and LMM interest rate models as well as their calibration is proposed.

Here we introduce a third family of interest rate models. These are CIR (Cox-Ingersoll-Ross) type models, whose dynamics include a square-root component of the instantaneous short rate. The objective is to:

- Define the CIR++ and CIR2++ models and describe the methods for calibrating and simulating these models;
- Propose methods for constructing risk-neutral economic scenario generators whose interest rate models are CIR++ or CIR2++;
- Evaluate the sensitivity of the best-estimate of savings contracts with options and guarantees to the CIR++ and CIR2++ interest rate models and their calibration;
- Draw up a comparison of the CIR++, CIR2++, Hull & White, G2++ and LMM models intended to evaluate the best-estimate.

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1 Preamble

The price of with-profit savings contracts is not observable on an organized market. The justification for the model choices and calibration of the economic scenario generator (ESG) by direct comparison with observed data, using a statistical approach, cannot be considered. In this Mark-to-Model valuation framework, questions naturally arise about the relevance of the model choices and the construction of quality criteria to validate these choices (Armel and Planchet [2018] and Armel and Planchet [2019]).

This article focuses more particularly on the problem of choosing interest rate models to evaluate the best-estimate of participating savings contracts in an economic environment characterized by negative rates.

Armel and Planchet [2018] present an approach for building a risk-neutral economic scenario generator in an economic environment characterized by negative rates. This generator is intended for the valuation of the participating savings contract liabilities and complies with the normative framework of Solvency 2. Armel and Planchet [2019] use this ESG to assess the sensitivity, of the Solvency 2 best-estimate of participating savings contracts, to the choice of interest rate models and calibration data. Three interest rate models have been selected to achieve these sensitivities:

- Interest rate models following a normal distribution:
    - One-factor model: Hull-White model;
    - Two-factor model: two-factor Gaussian model denoted G2++.
- An interest rate model following a log-normal distribution: the LMM market model.

This paper introduces a third family of interest rate models. These are CIR (Cox-Ingersoll-Ross) type models, whose dynamics include a square-root component of the instantaneous short rate. The objective is to produce a comparative study of the following interest rate models: one-factor CIR, two-factor CIR, Hull & White, G2++ and LMM. In particular, this involves evaluating:

1. Simplicity of calibration and simulation;
2. The ability to replicate the market prices of caps and swaptions;
3. The relevance of the choice of the interest rate model with regard to the objective of valuing participating savings contracts.

In order to assess point 3, particular attention is paid to the ability of the studied interest rate models to reproduce the prices of caps (and therefore floors by symmetry). Indeed, the analysis of the elementary financial option structure of the best-estimate leads to observe that the calibration of the interest rate models intended to evaluate the best-estimate is consistent, under certain conditions, with a calibration on floorlets (or caplets by symmetry, see Armel and Planchet [2019]).

We focus here mainly on the one-factor CIR model (CIR++) and the two-factor CIR model (CIR2++), which had not been analysed in previous work. The analytical properties of these models are documented in a technical appendix attached to this paper.

The variables of interest studied in the following sections are risk-free interest rates, equities and real estate investments. Credit risk is not discussed here.

This paper is organized into three main sections:

- Section 2 defines the CIR++ and CIR2++ models and describes the methods for calibrating and simulating these models;
- Section 3 presents the models selected for equity and real estate investments and describes an approach for building economic scenario generators whose interest rate models are the CIR++ and CIR2++ models;
- Section 4 presents:
    - The results of the calibration of the CIR++ and CIR2++ models and an analysis of their quality;
    - A study of the sensitivities of the best-estimate of french participating savings contracts to the choice of CIR++ and CIR2++ models and their calibration.

2 CIR++ and CIR2++ models: definition, calibration and simulation

The success of models such as Vasicek [1977] and Cox, Ingersoll and Ross [1985] is mainly due to their ability to analytically evaluate bonds and bond options.

The dynamic of Vasicek [1977] model ($dr\ (t)\ =\ k\ [\theta - r(t)]dt + \sigma dW(t)$) is linear and can be solved explicitly. The distribution of the instantaneous short rate is Gaussian and prices of bonds and some options can be expressed in analytical form.

The general-equilibrium approach proposed by Cox, Ingersoll and Ross [1985] introduced a "square root" term of the short interest rate in the diffusion factor of the Vasicek [1977] model.

The resulting model has been a reference for many years because of its ease of analysis and the fact that, unlike Vasicek [1977] model, the instantaneous short rate is always positive, which has long appeared to be a desirable property for an interest rate model. The dynamic of the Cox, Ingersoll, and Ross [1985] model under the risk-neutral measure is written as follows:

$$dr(t) = k\big(\theta - r(t)\big)dt + \sigma\sqrt{r(t)}\,dW(t)$$

where: $r(0) = r_0$ and r_0, k, θ, σ are positive constants.

The instantaneous short rate remains strictly positive if the parameters of the model meet the following Feller condition:

$$2k\theta > \sigma^2$$

As specified in Section 3.1.2 of the technical appendix, the short rate process $r(t)$ conditionally to $r(s)$ follows the distribution of $\chi^2\big(v, \lambda_{t,s}\big)/c_{t-s}$:

$$r(t)|r(s) = \chi^2(v, \lambda_{t,s})/c_{t-s}$$

where:

- $\chi^2(v, \lambda_{t,s})$ is a non-central chi-square distribution with v degrees of freedom and non-centrality parameter $\lambda_{t,s}$;
- $c_{t-s} = \frac{4k}{\sigma^2\left(1-exp(-k(t-s))\right)}$;
- $v = 4k\theta/\sigma^2$;
- $\lambda_{t,s} = c_{t-s} r_s exp\left(-k(t-s)\right)$.

Although interesting from an analytical point of view, the Vasicek and CIR models do not replicate the term structure of interest rates observed on the market, regardless of the choice of parameters.

In order, for these models, to reproduce the term structure of interest rates, the financial literature offers at least two possibilities:

- Make the parameters time-dependent (Hull & White type extension, see section 2.1.8.1 of the technical appendix);
- Introduce additively a deterministic shift function that reproduces the initial yield curve (see section 2.1.8.2 of the technical appendix).

Other extensions, which we will not detail here, are proposed by the literature. We can cite, for example, the one presented in Shiu and Yao [1999] who propose closed formulas to value zero coupon bonds assuming that the instantaneous interest rate is described by the following equations:

$$dr(t) = \varphi(t)dt + k[\theta(t) - r(t)]dt + \sigma\sqrt{r(t)}\, dW(t)$$

$$d\theta(t) = \beta(r(t) - \theta(t))dt$$

The deterministic function $\varphi(t)$ allows to replicate the initial yield curve.

The Vasicek [1977] and CIR processes are models with an affine term structure. To facilitate the reading of this article, we present in section 2 of the technical appendix some generalities on the family of affine term-structure interest rate models and their extension by deterministic functions to take into account the initial yield curve observed on the market.

In the following, we define the CIR++ and CIR2++ interest rate models and present the calibration and simulation methods used. The properties of these models are detailed in the technical appendix. We mainly relied on Cox, Ingersoll and Ross [1985] and Brigo and Mercurio [2007] to write the following sections.

2.1 CIR++ model

2.1.1 Definition

The CIR++ model assumes that the instantaneous short rate process r is the sum of a deterministic function denoted φ and a CIR process denoted x whose parameter vector is denoted $\alpha = (k, \theta, \sigma)$ defined as follows:

$$dx(t) = k\big(\theta - x(t)\big)dt + \sigma\sqrt{x(t)}dW(t)\ ;\ x(0) = x_0$$

and we have:

$$r\ (t) = x(t) + \varphi(t)$$

where x_0, k, θ and σ are positive constants such as $2k\theta > \sigma^2$, thus ensuring that the origin is inaccessible for the variable x and therefore that this process remains positive.

The function $\varphi(t)$ is also denoted $\varphi^{CIR}(t; \alpha)$ in the following.

2.1.2 Price of a zero-coupon bond

The price at time t of a zero-coupon bond with a maturity T can be written as:

$$P(t,T) = \bar{A}'\ (t,T)e^{-B(t,T)x(t)}$$

where

- $\bar{A}'(t,T) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}A(t,T)$
- $P^M(0,T)$ is the market price of the risk-free zero-coupon bond observed at time 0 for maturity T.
- $A(t,T)$ and $B(t,T)$ are defined in section 2.1.4 of the technical appendix:

$$A\ (t,T) = \left[\frac{2h exp\left\{\frac{(k\ +\ h)(T\ -\ t)}{2}\right\}}{2h + (k+h)(exp\left\{(T-t)h\right\} - 1)}\right]^{\frac{2k\theta}{\sigma^2}}$$

$$B\ (\ t,T)\ = \frac{2(exp\left\{(T\ -\ t)h\right\} -\ 1)}{2h\ +\ (k\ +\ h)(exp\left\{(T\ -\ t)h\right\} -\ 1)}$$

$$h = \sqrt{k^2 + 2\sigma^2}$$

The compound interest rate at time t for the term T is therefore:

$$R\ (t,T) = \frac{1}{T-t}\left(\ln\left(\frac{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}{A(t,T)P^M(0,T)A(0,t)\ exp\{-B(0,t)x_0\}}\right) + B(t,T)x(t)\right)$$

The price $P(t,T)$ and the rate $R(t,T)$ are functions of the parameters of the one-factor CIR model x and the initial value x_0. We can therefore characterize the dynamic of $R\ (t,T)$ without calculating the function $\varphi(t)$.

Moreover, the diffusion of compound interest rates amounts to the diffusion of the process x. This process can be simulated by diffusing a non-central chi-square distribution (see section 4.2 of the technical appendix).

Also, the interest rate $R(t,T)$ is an affine function of $x(t)$ whose coefficients are deterministic, and is therefore, an affine function of a non-central chi-square distribution.

2.1.3 Price of an option on a zero-coupon bond

By taking up the developments presented in section 2.2.3 of the technical appendix, it can be noted that the price at time $t=0$ of a European call option, expiring at $T > t$ with a strike denoted K on a zero-coupon bond with a maturity of $\tau > T$ is equal to:

$$\begin{aligned} & ZBC(t=0,T,\tau,K) \\ & \qquad = P(0,\tau)F_{\chi^2}\left(2\hat{r}[\rho+\psi+B(T,\tau)];\frac{4k\theta}{\sigma^2},\frac{2\rho^2 x_0 exp\{hT\}}{\rho+\psi+B(T,\tau)}\right) \\ & \qquad - KP(0,T)F_{\chi^2}\left(2\hat{r}[\rho+\psi];\frac{4k\theta}{\sigma^2},\frac{2\rho^2 x_0 exp\{hT\}}{\rho+\psi}\right) \end{aligned}$$

where

- $\hat{r} = \frac{1}{B(T,\tau)}\left[ln\left(\frac{A(T,\tau)}{K}\right) - ln\left(\frac{P^M(0,T)A(0,\tau)exp\{-B(0,\tau)x_0\}}{P^M(0,\tau)A(0,T)exp\{-B(0,T)x_0\}}\right)\right]$;
- $\rho = \rho\,(T-t) = \frac{2h}{\sigma^2(exp[h(T-t)]-1)}$;
- $\psi = \frac{k+h}{\sigma^2}$;
- $h = \sqrt{k^2+2\sigma^2}$;
- $F_{\chi^2}(.;v,\lambda)$ is the cumulative distribution function of a non-central chi-square distribution with v degrees of freedom and non-centrality parameter λ.

The put option price is obtained by the *put-call* parity and is denoted ZBP:

$$ZBP(t=0,T,\tau,K) = ZBC(t=0,T,\tau,K) - P(0,\tau) + KP(0,T)$$

2.1.4 Calibration of the CIR++ model on caps

Let $\zeta = \{t_0, t_1, \dots, t_n\}$ be the set of all payment maturities of caps or floors and let t_0 be the initialization date of the option. Let τ_i be the difference between t_{i-1} and t_i.

The price at time $t = 0 < t_0$ of the cap with a strike X, a nominal value N and defined on the set $\zeta = \{t_0, t_1, \dots, t_n\}$ is given by:

$$Cap(t=0,\zeta,N,X) = N\sum_{i=1}^{n}(1 + X\tau_i) \times ZBP\left(0,t_{i-1},t_i,\frac{1}{1+X\tau_i}\right)$$

The price of the floor is given by:

$$Cap(t=0,\zeta,N,X) = N\sum_{i=1}^{n}(1 \; + \; X\tau_i) \times ZBC\left(0,t_{i-1},t_i,\frac{1}{1+X\tau_i}\right)$$

Let's note:

- $\beta = (k,\theta,\sigma)$: the parameters of the model;
- $Price_i^{black}$: the price of a cap i valued by the market using the Black formula[3];
- $Cap_i^{CIR++}(\beta)$: the price of a cap i valued by the CIR++ model.

The parameters of the CIR++ model are calculated by the following optimization over the set of caps selected for the calibration (d is a distance):

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\left(d\left(\left(Price_i^{black}\right)_i,\left(Cap_i^{CIR++}(\beta)\right)_i\right)\right)$$

The prices valued by the Black model using implied volatilities act as reconstructed market prices. This reconstitution by the Black model has at least two advantages. Indeed, it allows to:

- Retrieve the observed market prices used to calculate implied volatilities;
- Produce prices consistent with market prices, when the implied volatilities used are extracted from volatility models calibrated to market volatilities (e.g. reconstructing the volatility surface on additional dates from market volatilities).

If directly observable prices are available, the CIR++ model can be calibrated directly to these market prices by rewriting the optimization function as follows:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\left(d\left(\left(Price_i^{market}\right)_i,\left(Cap_i^{CIR++}(\beta)\right)_i\right)\right)$$

2.1.5 Calibration of the CIR++ model on swaptions

Let's consider a payer swaption with a strike rate denoted X, a maturity denoted T and a nominal value denoted N. It gives its holder the right to contract at time $t_0 = T$ an interest rate swap with payment dates equal to $= \{t_1,\dots,t_n\}$, $t_1 > T$ where he pays a fixed rate X and receives the variable rate.

Let τ_i be the fraction of a year from t_{i-1} to t_i, $i = 1,\dots,n$ and let $c_i = X\tau_i$ for $i = 1,\dots,n-1$ and $c_n = 1 + X\tau_n$.

The swaption valuation requires the evaluation of the deterministic sequence:

$$X_i = \bar{A}(T,t_i) \times e^{-B(T,t_i)r^*}$$

[3] Armel & Planchet [2018] detail the method for valuing caps, floors and swaptions using the Black model and presents a process for calibrating and simulating an ESG for valuing *the* best-estimate of a with-profit savings contract.

where:

- $\bar{A}(t,T) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}} A(t,T)e^{B(t,T)\varphi^{CIR}(t;\alpha)}$;
- r^* is the rate at time T solution of the equation: $\sum_{i=1}^{n} c_i \bar{A}(T,t_i) \times e^{-B(T,t_i)r^*} = 1$.

If we denote: $x^* = r^* - \varphi^{IRC}(T;\alpha)$, thus x^* is the solution of the equation $\sum_{i=1}^{n} c_i \overline{A}'(T,t_i) \times e^{-B(T,t_i)x^*} = 1$ and $X_i = \overline{A}'(T,t_i) \times e^{-B(T,t_i)x^*}$.

The calculation of r^* is only used to assess X_i, the introduction of x^* avoids the assessment of $\varphi^{CIR}(T)$.

The price of the payer swaption at the time $t < T$ is then given by:

$$PS(t=0,T,\zeta,N,X) = N \sum_{i=1}^{n} c_i \times ZBP(0,T,t_i,X_i)$$

Let's note:

- $\beta = (a,\theta,\sigma)$: the parameters of the model;
- $Price_i^{black}$: the price of a swaption i valued by the market using the Black formula;
- $f_i^{CIR++}(\beta)$: the price of a swaption i valued by the CIR++ model.

The model parameters are deduced by the following optimization on all swaptions selected for calibration (d is a distance):

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \left(d\left(\left(Price_i^{black}\right)_i, \left(f_i^{CIR++}(\beta)\right)_i \right) \right)$$

The prices valued by the Black model from the implied volatilities act here as reconstituted market prices. This reconstitution allows to find prices that are consistent with the observed prices.

If we have directly observable prices, the CIR++ model can be calibrated directly to these market prices by rewriting the optimization function as follows:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \left(d\left(\left(Price_i^{market}\right)_i, \left(f_i^{CIR++}(\beta)\right)_i \right) \right)$$

2.1.6 The CIR++ model and negative rates

The CIR++ model can generate negative rates and reproduce the expected yield curve without introducing a shift factor as in log-normal models.

If directly observable prices for caps, floors or swaptions are available, the CIR++ model can be calibrated directly to these market prices using the closed formulas presented above.

The use of market implied Black volatilities for calibrating the CIR++ model in an economic environment with negative rates requires the use of the shifted Black model. The calibration of the CIR++ model on caps, floors or swaptions evaluated by the Black model on the basis of implied volatilities can be achieved as follows (see Armel and Planchet [2018]):

- Step 1: calibration of the shifted Black model, which consists in defining a shift factor for a given risk-free yield curve and extracting the implied volatilities consistent with this factor;
- Step 2: valuation of the derivative using the shifted Black model. This step provides prices of the derivative assets consistent with the observed values;
- Step 3: calibrate the CIR++ model by minimizing the distance between Black prices and the theoretical prices of the CIR++ model as explained above.

The parameters of the CIR++ model calibrated in this way are consistent with observed values and with the risk-free yield curve in an economic environment with negative interest rates.

2.1.7 Simulation of the CIR++ model

We present here the simulation method of the CIR++ model that we have retained. We have chosen an exact diffusion of interest rates without going through Euler's discretization process, thus avoiding the problems associated to convergence.

The simulation method is an iterative process. This process is presented in the following with an annual frequency and can be adapted to other frequencies.

As presented in section 2.1.2 of the technical appendix, if x follows a CIR process, then for all $t \geq 1$ the random variable $x(t)$ conditionally to $x(t-1)$ has the same distribution as the variable $\chi^2(v, \lambda_{t,t-1})/c_1$ where $\chi^2(\mathrm{v}, \lambda_{\mathrm{t,t-1}})$ follows a non-central chi-square distribution with parameters $(\mathrm{v}, \lambda_{\mathrm{t,t-1}})$ such as:

- $c_1 = \frac{4k}{\sigma^2(1-exp(-k))}$;
- $v = 4k\theta/\sigma^2$;
- $\lambda_{t,t-1} = c_1 x_{t-1} exp(-k)$.

In order to simulate realizations of the continuous forward interest rate $R(t,T)$ evaluated at $t \geq 1$ for a maturity T we have retained the following algorithm (see Shao [2012] and Malham and Wiese [2013]):

- Defining an initial value x_0. This choice is arbitrary but has no impact on the reproduction of the observed term structure yield curve (see section 4.2.2 for an analysis of the impact of this initial value);
- Simulating a non-central chi-square distribution $\chi^2(v, \lambda_{t,t-1})$ following the method presented in section 3.2 of the technical appendix;
- Simulating $x(t)$ corresponding to the realizations of $\chi^2(v, \lambda_{t,t-1})$ divided by c_1 ;
- Simulating $R\ (t,T)$ using $x(t)$ simulations, the prices of observed zero coupon bonds and model parameters by the following formula:

$$R\ (t,T) = \frac{1}{T-t}\left(\ln\left(\frac{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}{A(t,T)P^M(0,T)A(0,t)\ exp\{-B(0,t)x_0\}}\right) + B(t,T)x(t)\right)$$

2.2 CIR2++ model

The CIR2++ model is a two-factor instantaneous short rate model that consists of adding a deterministic function to the sum of two independent CIR processes. This model can be viewed as the natural two-factor extension of the CIR++ model.

The CIR2++ model is of the form: $r_t = x_t + y_t + \varphi(t)$, where φ is a deterministic function allowing to reproduce the initial observed yield curve, and x and y are two independent CIR processes:

$$dx(t) = k_1(\theta_1 - x(t))dt + \sigma_1\sqrt{x(t)}dW_1(t)$$

$$dy(t) = k_2(\theta_2 - y(t))dt + \sigma_2\sqrt{y(t)}dW_2(t)$$

2.2.1 Price of a zero-coupon bond

Because of the independence of the factors, the price of a zero-coupon bond is directly derived from the analytical bond pricing formula of the reference one factor CIR model. The price at time t of a zero-coupon bond with a maturity T is (see section 2.3.3 of the technical appendix):

$$P(t,T; x(t), y(t), \alpha) = \Phi^{\xi}(t,T;\alpha) \times P^{\xi}(t,T; x(t), y(t), \alpha)$$

with:

- x_0 is the initial value of the process x;
- y_0 is the initial value of the process y;
- $\alpha = (k_1, \theta_1, \sigma_1, k_2, \theta_2, \sigma_2)$ represents the set of parameters of the CIR2++ model;
- $\Phi^{\xi}(u,v;\ \alpha) = \frac{P^M(0,v)P^{\xi}(0,u;\alpha)}{P^M(0,u)P^{\xi}(0,v;\alpha)}$;
- $P^M(0,T)$ is the zero-coupon market price observed at time 0 for the maturity T;
- $P^{\xi}(t,T;x(t),y(t),\alpha) = P^1(t,T;x(t),\alpha_1) \times P^1(t,T;y(t),\alpha_2)$;
- P^1 is the price of a zero-coupon bond valued by the one-factor CIR model. Recall that for a one-factor CIR process z with parameters $\alpha_i = (k_i, \theta_i, \sigma_i)$ the price of a zero-coupon bond is given by:

$$P^1(t,T; z(t), k_i, \theta_i, \sigma_i) = A_z(t,T)e^{-B_z(t,T)z(t)}$$

where

- $A_z\,(t,T) = \left[\frac{2h_i exp\left\{\frac{(k_i+h_i)(T-t)}{2}\right\}}{2h_i + (k_i+h_i)(exp\,\{(T-t)h_i\}-1)}\right]^{\frac{2k_i\theta_i}{\sigma_i^2}}$;
- $B_z\,(t,T) = \frac{2(exp\,\{(T-t)h_i\}-1)}{2h_i + (k_i+h_i)(exp\,\{(T-t)h_i\}-1)}$;
- $h = \sqrt{k_i^2 + 2\sigma_i^2}$;

- $z \in \{x, y\}$ and $i = 1$ if $z = x$, $i = 2$ otherwise.

The compound interest rate $R(t,T)$ is then:

$$R(t,T) = \frac{-\ln\big(P(t,T; x(t), y(t), \alpha)\big)}{T - t}$$

therefore:

$$R(t,T) = \frac{1}{T-t}\Big(B_x\,(t,T)x(t) + B_y\,(t,T)y(t) - \ln\Big(\Phi^{\xi}(t,T;\ \alpha) \times A_x\,(t,T) \times A_y(t,T)\Big)\Big)$$

Note that $R(t,T)$ is an affine function of x(t) and y(t) whose coefficients are deterministic and therefore is an affine function of two independent, non-central chi-square distributions. This property will be useful in the developments presented in section 2.2.4 this paper.

2.2.2 Calibration of the CIR2++ model on caps and swaptions

As mentioned in the technical appendix, section 2.3, the CIR2++ model does not allow the valuation of caps, floors and swaptions by closed formulas. However, we do have a semi-closed formula for valuing caps and floors by numerically calculating a double integral under the T-forward measure.

We have chosen here to value caps and swaptions using Monte Carlo methods. This consists in simulating interest rates, evaluating the pay-offs and calculating the expectation of these discounted pay-offs[4] in order to have the prices. The process of simulating compound interest rates is explained in section 2.2.4.

Let's note:

- $\beta = (k_1, \theta_1, \sigma_1, k_2, \theta_2, \sigma_2)$: the parameters of the model;
- $\mathrm{Price}_i^{\mathrm{black}}$: the price of a cap *i* (or a swaption, respectively) valued by the market via the Black formula;
- $\mathrm{f}_i^{\mathrm{CIR2++}}(\beta)$: the price of a *cap i* (respectively a *swaption* i) valued by the CIR2++ model.

The parameters of the model are evaluated by the following optimization over all the set of caps (respectively swaptions) selected for the calibration (d is a distance):

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\left(d\left(\left(Price_i^{black}\right)_i, \left(f_i^{CIR2++}(\beta)\right)_i\right)\right)$$

The prices valued by the Black model from the implied volatilities act here as reconstituted market prices. This reconstitution allows to assess prices that are consistent with the observed prices.

If directly observable prices are available, the CIR2++ model can be calibrated directly to these market prices by rewriting the optimization function as follows:

---

[4] The reader may refer to Chapter 1 of Brigo and Mercurio [2007] which presents the formulas of caps, floors and swaptions in the form of discounted pay-offs.

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \left(d \left(\left(Prices_i^{market} \right)_i, \left(f_i^{CIR2++}(\beta) \right)_i \right) \right)$$

2.2.3 The CIR2++ model and negative rates

The CIR2++ model can generate negative rates and reproduce the expected yield curve.

As for the CIR++ model, the process of calibrating the CIR2++ model on market prices valued by the Black model is the same as that presented in section 2.1.6.

2.2.4 Simulation of the CIR2++ model

As for the CIR ++ model, we have retained a simulation method that diffuses the model exactly without going through Euler's discretizations and thus avoiding the problems associated with convergence.

The independent variables x and y follow a simple CIR process. Then for all t ≥ 1 the random variables $x(t)$ conditionally on $x(t-1)$ and $y(t)$ conditionally on $y(t-1)$ have respectively the same distributions as the variables $\chi^2\left(v^{(x)}, \lambda_{t,t-1}^{(x)}\right)/c_1^{(x)}$ and $\chi^2\left(v^{(y)}, \lambda_{t,t-1}^{(y)}\right)/c_1^{(y)}$ where $\chi^2\left(v^{(z)}, \lambda_{t,t-1}^{(z)}\right)$ follows a non-central chi-square distribution with parameters $\left(v^{(z)}, \lambda_{t,t-1}^{(z)}\right)$ such that:

- $c_1^{(z)} = \frac{4k_i}{\sigma_i^2(1-exp(-k_i))}$;
- $v^{(z)} = 4k_i\theta_i/\sigma_i^2$;
- $\lambda_{t,t-1}^{(z)} = c_1^{(z)} z_{t-1} exp(-k_i)$;
- $z \in \{x, y\}$ and $i = 1$ if $z = x$, $i = 2$ otherwise.

In order to simulate the continuous compound interest rate $R\ (t, T)$ evaluated at time t $\geq$ 1 for maturity T we propose the following algorithm (see Shao [2012] and Malham and Wiese [2013]):

- Definition of initial values x_0 and y_0. The choices of these values are arbitrary but have no impact on the reproduction of the observed term structure (see section 4.2.2 for an analysis of the impact of these initial values);
- Simulation of two non-central chi-square distributions $\chi^2(v^z, \lambda_{t,t-1}^z)$, $z \in \{x, y\}$ following the method presented in section 3.2 of the technical appendix;
- Simulation of $x(t)$ and $y(t)$ corresponding, respectively, to $\chi^2(v^x, \lambda_{t,t-1}^x)$ simulations divided by c_1^x and $\chi^2(v^y, \lambda_{t,t-1}^y)$ simulations divided by c_1^y ;
- Zero-coupon price simulation $P(t, T; x(t), y(t), \alpha)$ using $x(t)$ and $y(t)$ simulations, the prices of the observed zero coupon bonds and model parameters (see section 2.2.1).
- Simulation of $R(t, T)$ using the following formula:

$$R(t,T) = \frac{-\ln\big(P(t,T;x(t),y(t),\alpha)\big)}{T-t}$$

In this section we defined the CIR++ and CIR2++ interest rate models and presented the calibration and simulation methods.

The following section presents the equity and real estate investments models that we have retained. It also presents the calibration process for these models and details the approach that we propose to model the dependency structure of interest rates, equity returns and real estate returns.

3 How to build an ESG with CIR++ and CIR2++ interest rate models?

The economic scenarios generation is a process that can be summarized in three steps (Armel and Planchet [2018]):

- Select the economic variables to be modelled. Here are interest rates, an equity index and a real estate investment;
- Build the mathematical models of the variables of interest. This consists in choosing the models that will represent the individual dynamics of these variables and choosing the model that represents the dependency structure;
- Parameterization: choose financial derivatives for calibrations, data, statistical estimation methods for models parameters and validation methods.

Two economic scenario generators are constructed and studied in this paper:

- **ESG1**: composed of a CIR++ model to model instantaneous risk-free interest rates, a Black-Scholes like model to represent the price of the equity portfolio and a Black-Scholes like model to represent the price of a real estate investment;
- **ESG2**: composed of a CIR2++ model to model instantaneous risk-free interest rates, a Black-Scholes like model to represent the price of the equity portfolio and a Black-Scholes like model to represent the price of a real estate investment.

Section 2 of this paper and the technical appendix present the technical characteristics of the CIR++ and CIR2++ models as well as calibration and simulation methods. The following sections present:

- The models used for the equity index and real estate investment and their calibration;
- The dependency structure modelling method.

3.1 Equity and real estate models

We use here the same modelling and calibration approach, for the equity index and the real estate index, which we have proposed in section 2.3 of the technical appendix of the article Armel and Planchet [2018]. These two indices are assumed to follow a Black-Scholes type model. By applying directly the Itô lemma, we can write the exact solution of the SDE as follows:

$$S_T = S_t \exp\left(\int_t^T r(u)du - \int_t^T \frac{\sigma_u^2}{2}du + \int_t^T \sigma_u dW_u\right)$$

In Armel and Planchet [2018], we proposed an approach for calibrating implied volatilities based on observed market call prices. This approach is used here to calibrate the volatility of the equity index. The volatility of a real estate investment is calibrated on historical prices.

3.2 Dependency structure modelling approach

3.2.1 Problematic

The compound interest rate $R(t,T)$ whose dynamic is described by a CIR++ model or a CIR2++ model is an affine function of non-central chi-square distributed random variables. The space generated by interest rates, the equity index, and real estate investments is therefore not Gaussian. Then the description of the dependency structure by correlations is not appropriate.

Also, the rationalization of the correlations between the Brownian noises of equity and real estate indices and the CIR++ and CIR2++ models remains complex. Indeed:

- The instantaneous short-term interest rate process is not observable on the market and the anticipated curve of the Brownian motion of the model is not accessible, even in the case of Euler's discretization;
- The market does not offer information on the future price of the underlying equity asset, but gives only access to information on its variability (implied volatility). The chronicle of the underlying Brownian motion is not accessible;
- Real estate investments are calibrated on historical data and not on anticipated prices;
- Estimating the implied correlation using derivatives (as in the BSHW model - see Laurent et al. [2016] and Haastrecht [2010]) is difficult in practice in a Gaussian multivariate framework. Indeed, it might be necessary to find at least one liquid derivative whose underlyings are the equity index and real estate investment to estimate their correlation.

The approach consisting in (1) constructing observations of Brownian noises in order to (2) deduce their correlations, which (3) are then used for the simulations, is not adapted to our problem.

Also, Armel and Planchet [2018] present an approach of modelling the dependency structure in order to generate risk-neutral economic scenarios. However, this approach is only appropriate for normal or log-normal interest rate models.

To date, we are not aware of any publications dealing with the problem of modelling and calibrating a non-linear dependency structure of a multivariate risk-neutral ESG. A detailed study of the introduction of non-linear dependency structures (copulas) in a multivariate model calibrated on historical data is proposed in Armel et al [2011].

Moreover, the generation of risk-neutral economic scenarios with a non-Gaussian dependency structure raises two problems:

- The choice of the dependency structure mathematical model;
- The calibration of this model.

In order to get around the complexity that could be generated by non-linear modelling of the dependency structure and in order to be able to calibrate our model on the only available observations (historical interest rates, the expected yield curve, the historical prices of the underlying assets and the prices of derivatives), we thought of constructing transformations, using bijective functions, of risk factors such that the probability distributions of the transformed uni-variate variables can be assimilated to Gaussian variables. The dependence of these transformed variables will be assumed to be linear. Our approach is presented in the following section.

3.2.2 Modelling of the dependency structure

The approach we propose to model the dependency structure is based on the idea of finding transformations of economic variables so that the transformed variables are observable and can be assimilated to Gaussian variables. This will allow to calibrate a historical correlation and then to simulate the transformed variables. By inverting the functions we will thus have a coherent diffusion of our economic and financial variables.

First note that the variable Λ_t defined below is Gaussian. Its characteristics are accessible and observable. They depend only on volatility (historical for real estate and implied for equities):

$$\Lambda_{t,T} = \ln\left(\frac{S_T}{S_t}\right) + \ln\big(D(t,T)\big) = -\int_t^T \frac{\sigma_u^2}{2} du + \int_t^T \sigma_u dW_u$$

$\mathrm{D}(\mathrm{t},\mathrm{T}) = \exp\left(-\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{r}(\mathrm{u})\mathrm{du}\right)$ is the stochastic discount factor.

Now suppose that there are two bijective functions H_{CIR++} and H_{CIR2++} such as $H_{CIR++}\big(R(t,T)\big)$ and $H_{CIR2++}\big(R(t,T)\big)$ are assimilable to Gaussian variables (see section 3.2.3).

Our approach proposes to inject into risk-neutral ESG the historical correlations observed among the following variables:

- $H_{CIR++}\left(R_1^{(t)}\right)$ the transformation of one-year historical risk-free interest rates observed on time t assuming that their dynamic follows the CIR++ model;
- $H_{CIR2++}\left(R_1^{(t)}\right)$ the transformation of one-year historical risk-free interest rates observed on time t assuming that their dynamic follows the CIR2++ model;
- $\Lambda_{t,t+1}^{equity}$ the annual logarithmic return observed at time $t+1$ of the historical equity index minus the one year risk-free interest rate;

- $\Lambda_{t,t+1}^{real\ estate}$ the annual logarithmic return observed at time $t+1$ of the historical real estate index minus the one year risk-free rate.

Two correlation matrices are calculated:

- $M_1 = correlation\left(H_{CIR++}\left(R_1^{(t)}\right), \Lambda_{t+1}^{equity}, \Lambda_{t+1}^{real\ estate}\right)$ the correlation matrix used for ESG 1 ;
- $M_2 = correlation\left(H_{CIR2++}\left(R_1^{(t)}\right), \Lambda_{t+1}^{equity}, \Lambda_{t+1}^{real\ estate}\right)$ the correlation matrix used for ESG 2.

We assume that these two matrices do not depend on time, maturity and the modelled variables.

The process of simulating correlated variables is as follows:

1. Simulation of compound interest rates independently of other economic variables;
2. At each projection step t, calculation of the distribution of the Gaussian noise, centred and reduced, underlying the compound rates transformation: $H_{CIR++}\big(R(t,t+1)\big)$ for ESG 1 and $H_{CIR2++}\big(R(t,t+1)\big)$ for ESG 2;
3. Simulation of two Gaussian noises correlated to the Gaussian noise calculated in step 2 according to the correlation matrix $\mathrm{M_i}$ (i=1 or 2) by the Cholesky method;
4. Use of these Gaussian noises for the diffusion of the two intermediate variables $\Lambda_{t,t+1}^{equity}$ and $\Lambda_{t,t+1}^{real\ estate}$;
5. Simulations of equity and real estate indices by adding $ln\big(D(t,t+1)\big)$ to the variables $\Lambda_{t,t+1}^{equity}$ and $\Lambda_{t,t+1}^{real\ estate}$.

This approach, consisting on simulating the equity and real estate indices conditionally to the interest rates, allows to have a risk-neutral economic scenarios generator whose dependency structure is consistent with the historical dependency structures observed.

The implementation of this approach is based on the existence of two bijective functions H_{CIR++} and H_{CIR2++} such that $H_{CIR++}\big(R(t,T)\big)$ and $H_{CIR2++}\big(R(t,T)\big)$ are assimilable to Gaussian variables. The following section presents twos methods for constructing such functions.

3.2.3 Approximation of the interest rate distribution by a Gaussian distribution

The interest rate $R(t,T)$ whose dynamic is the CIR++ model or the CIR2++ model is an affine function of non-central chi-square distributed random variables (see sections 2.1.2 and 2.2.1). Therefore:

- For the CIR++ model, there are two deterministic functions $\alpha(t,T)$ and $\beta(t,T)$ such as:

$$R_{CIR++}(t,T) = \alpha(t,T)\chi^2(v,\lambda_t) + \beta(t,T)$$

- For the CIR2++ model, there are three deterministic functions $\alpha_x(t,T)$, $\alpha_y(t,T)$ and $\beta_{xy}(t,T)$ such as:

$$R_{CIR2++}(t,T) = \alpha_x(t,T)\chi^2\left(v^{(x)}, \lambda_t^{(x)}\right) + \alpha_y(t,T)\chi^2\left(v^{(y)}, \lambda_t^{(y)}\right) + \beta_{xy}(t,T)$$

Recall that if the random variable z follows a CIR process with parameters (k, θ, σ) then z follows a $\chi^2(v, \lambda_t)/c_t$ distribution with:

- $\chi^2(v, \lambda_t)$ is a non-central chi-square distribution with v degrees of freedom and non-centrality parameter λ_t that is time-dependent;
- $c_t = \frac{4k}{\sigma^2(1-exp(-kt))}$;
- $v = 4k\theta/\sigma^2$;
- $\lambda_t = c_t z_0 exp(-kt)$.

In the following, we study two transformations of the non-central chi-square distributions allowing to assimilate interest rates to Gaussian variables: a linear transformation and a non-linear transformation. These two transformations were selected following the study of the transformations presented in section 3 of the technical appendix.

3.2.3.1 Linear transformation

This transformation is presented in point 1.a of section 3.3 of the technical appendix. It has a better quality than the linear approximation presented in point 1.b of the same section.

It allows the expression of a non-central chi-square distribution as a function of a Gaussien distribution. Indeed, the random variable

$$G_1\left(\chi^2(v,\lambda)\right) = \frac{\chi^2(v,\lambda) - v - \lambda}{\sqrt{2(v + 2\,\lambda)}}$$

has approximately the same distribution as a centred and reduced normal distribution (see technical appendix and Patel and Read [1982]).

Then: $\chi^2(v,\lambda) \approx \sqrt{2(v + 2\,\lambda)}N(0,1) - v - \lambda$.

The quality of this transformation is better when v or λ are significant.

We have studied the relevance of this approximation on the basis of the calibrations of the CIR++ and CIR2++ models presented in section 4. The following graphs and results are based on the calibration of the CIR++ model on caps with a 1% Black-shift and a 1% CIR++ model initial value.

We also validated the quality of this approximation over the entire projection horizon. The graphs, results and analyses presented in the following are made for two dates, a one-year projection date and a longer-term projection date of twenty years.

The calibrated parameters of the CIR++ model, for a Black model shift of 1% and a CIR++ model initial value of 1%, are : $k = 0{,}03$; $\theta = 1$ and $\sigma = 0{,}03$. In this case:

- The parameters of the non-central chi-square distribution for the first projection year are: $v = 133{,}26$ and $\lambda = 42{,}06$;
- The parameters of the non-central chi-square distribution for the 20th projection year are: $v = 133{,}26$ and $\lambda = 1{,}54$.

Let's denote: $X_t = \chi^2(v, \lambda_t)$ and $Y_t = \sqrt{2(v + 2\,\lambda_t)}N(0.1) - v - \lambda_t$.

In the following we present the Quantile to Quantile diagrams (QQ plot) of the variables X_t and Y_t (for *t=1* year and *t=20* years) as well as the relative differences of these quantiles[5].

Figure 1: Quantile to Quantile diagram of X_t and Y_t for: t=1 ; $v = 133{,}26$ and $\lambda = 42{,}06$

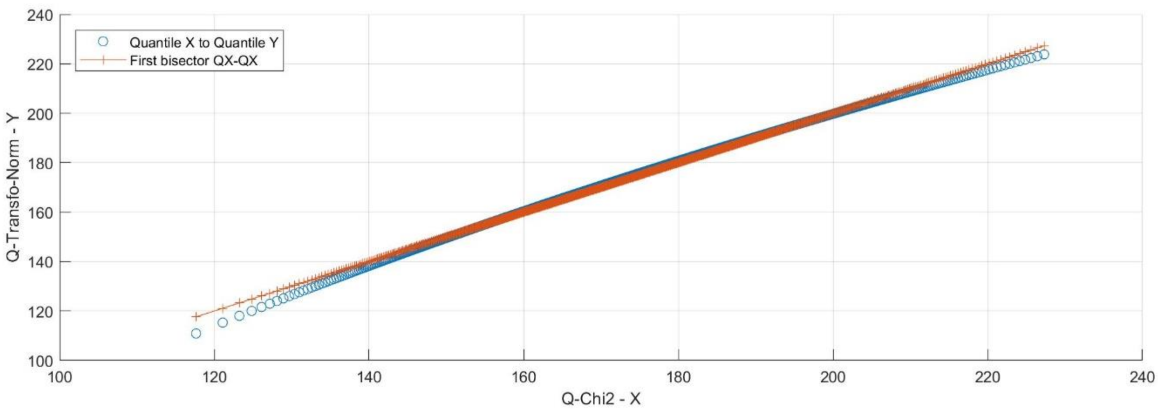
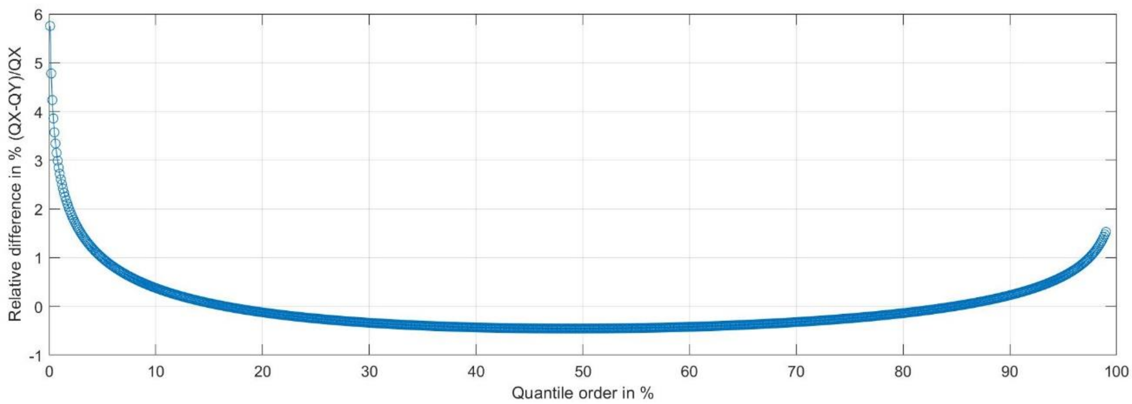


Figure 2: relative differences between the quantiles of X_t and Y_t for: t=1 ; $v = 133{,}26$ and $\lambda = 42{,}06$



We observe that the quality of the approximation of X_1 by Y_1 is good. We can indeed note that:

- The quantile-to-quantile diagram is almost aligned with the first bisector;
- All quantiles of X_1 can be assimilated to the quantiles of Y_1:
  - On all probability orders with an error less than 6%;
  - On all probability orders greater than 0.2% with an error less than 5%;
  - On all probability orders greater than 0.7% with an error less than 3%;
  - On all probability orders between 10% and 90% with an absolute value of the error less than 0.45%.

[5] Quantiles are evaluated on 990 points ranging from 0.1% to 99% order with a step of 0.1%. Relative differences are calculated as $\frac{(Q_X(p)-Q_Y(p))}{Q_X(p)}$ where $Q_X(p)$ (resp. $Q_y(p)$) is the p order quantile of X (resp. Y).

Figure 3: Quantile to Quantile diagram of X_t and Y_t for: t=20 ; $v = 133{,}26$ and $\lambda = 1{,}54$

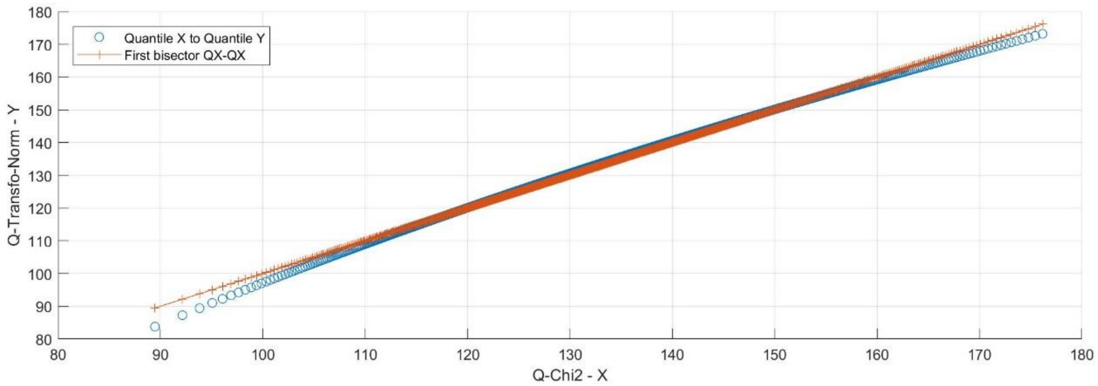
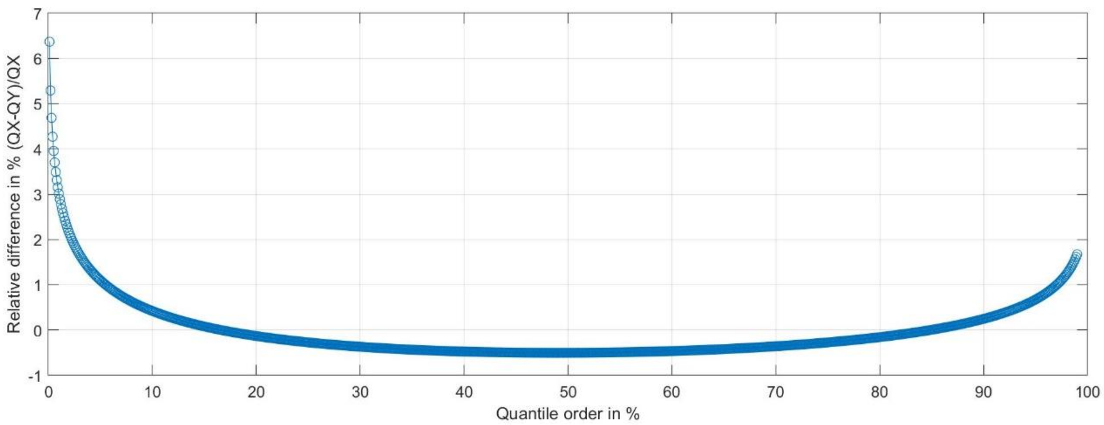


Figure 4: relative differences between the quantiles of X_t and Y_t for: t=20; $v = 133{,}26$ and $\lambda = 1{,}54$



As for the one-year projection horizon, we observe that the quality of the approximation of X_{20} by Y_{20} is good. We can indeed note that:

- The quantile-to-quantile diagram is almost aligned with the first bisector;
- All quantiles of X_1 can be assimilated to the quantiles of Y_1:
  - On all probability orders with an error less than 7%;
  - On all probability orders greater than 0.2% with an error less than 5%;
  - On all probability orders greater than 1% with an error less than 3%;
  - On all probability orders between 10% and 90% with an absolute value of the error less than 0.50%.

The quality of the approximation is good when the degree of freedom or the non-centrality parameter are significant (cf. Patel and Read [1982]). The calibration results of the CIR++ and CIR2++ models presented in section 4 show that the degrees of freedom or the non-centrality parameters of the chi-square distribution of these models are quite significant. The approximation errors remain therefore small and comparable to the errors presented above.

Also, the interest rate (CIR++ or CIR2++) is an affine function of non-central chi-square distributed random variables. Assuming that these variables can be assimilated to Gaussian distributions, allows the approximation of interest rates by Gaussian distributions.

For the CIR++ model, there are therefore two deterministic functions $\alpha'(t,T)$ and $\beta'(t,T)$ such that: $R_{CIR++}(t,T) \approx \alpha'(t,T) \times N(0,1) + \beta'(t,T)$ and $N(0,1)$ is a centered and reduced normal distribution.

For the CIR2++ model, there are thus three deterministic functions $\alpha'_x(t,T)$, $\alpha'_y(t,T)$ and $\beta'_{xy}(t,T)$ such that: $R_{CIR2++}(t,T) \approx \alpha'_x(t,T) \times N_1(0,1) + \alpha'_y(t,T) \times N_2(0,1) + \beta'_{xy}(t,T)$ with $N_1(0,1)$ and $N_2(0,1)$ are centered, reduced and independent normal distributions.

The major interest of this approximation is the possibility of approximating interest rates by Gaussian distributions. The interest rate can be assimilated to a Gaussian variable, which allows to be in a Gaussian vector space to calibrate correlation matrices and to simulate equity and real estate indices conditionally on interest rates as specified in section 3.2.2.

Note that interest rates are simulated following an exact discretization as described in sections 2.1.7 and 2.2.4. The linear approximation is used only to calibrate and take into account the co-movement of the modelled indices.

3.2.3.2 Non-linear transformation

This transformation is presented in point 2.c of section 3.3 of the technical appendix. It is of better quality than all the approximations presented in this section.

Let G_2 be a function defined for all $0 \leq z$ by:

$$G_2(z) = \frac{\left(\left(\frac{z}{v\ +\ \lambda}\right)^h - a\right)}{b}$$

with:

$$h \ = \ 1 \ - \frac{2(v+\lambda)(v+3\lambda)}{3(v+2\lambda)^2}$$

$$a \ = \ 1 + \frac{h(h-1)(v+2\lambda)}{(v+\lambda)^2} - \frac{h(h-1)(2-h)(1-3h)(v+2\lambda)^2}{2(v+\lambda)^4}$$

$$b \ = \frac{h\sqrt{2(v+2\lambda)}}{v+\lambda}\left(1 - \frac{(1-h)(1-3h)(v+2\lambda)}{2(v+\lambda)^2}\right)$$

Note that G_2 is continuous and strictly monotonous. It is therefore bijective from $\mathbb{R}^+$ to $G_2(\mathbb{R}^+)$.

The random variable $G_2\left(\chi^2(v,\lambda)\right)$ has approximately the same distribution as a centred and reduced normal distribution. No constraint is formulated on the parameters v and λ (cf. Patel and Read [1982]).

This transformation allows to express a non-central chi-square distribution as a function of a centred and reduced normal distribution: $\chi^2(v,\lambda) \approx G_2^{-1}\left(N(0,1)\right)$.

As in section 3.2.3.1, we studied the relevance of this approximation on the basis on the calibrations of the CIR++ and CIR2++ models presented in section 4. The following graphs and results are based on the same parameters presented in section 3.2.3.1

Let's denote: $X_t = \chi^2(v, \lambda_t)$ and $Y_t = G_2^{-1}(N(0,1))$.

In the following we present Quantile to Quantile (*QQ plot*) diagrams of the variables X_t and Y_t (for *t*=1 years and *t*=20 years) as well as the relative differences of these quantiles[6].

Figure 5: Quantile to Quantile diagram of X_t and Y_t for: t=1 ; $v = 133{,}26$ and $\lambda = 42{,}06$

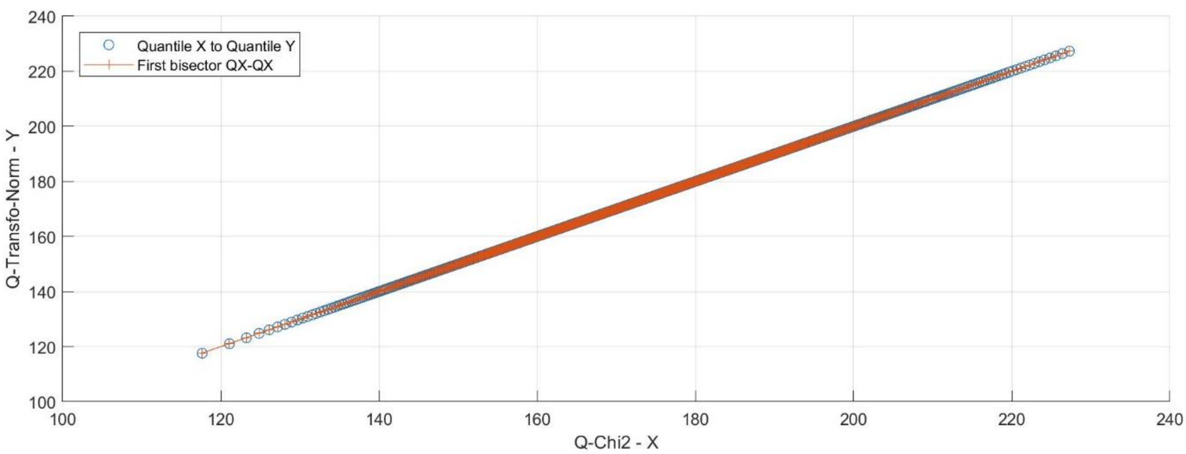
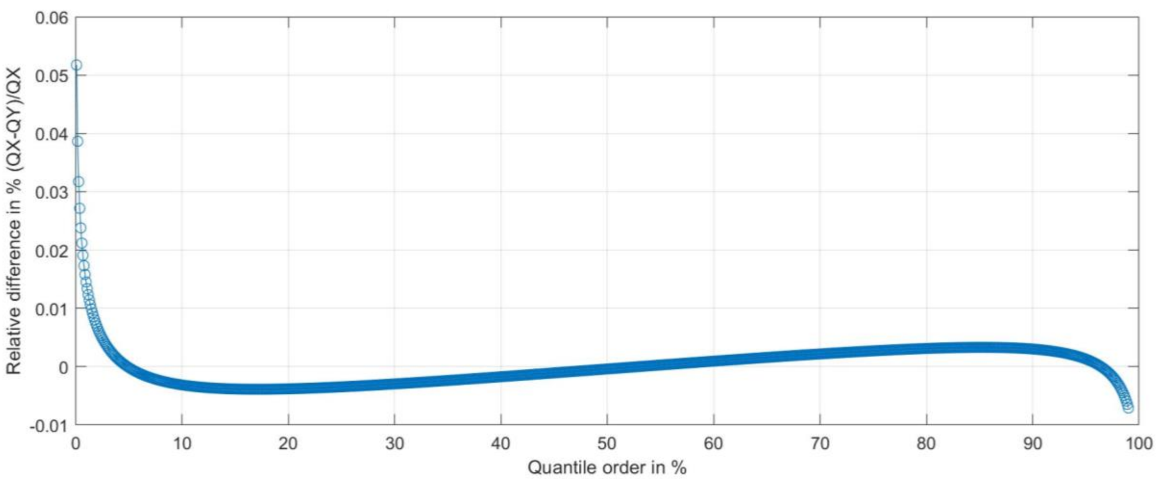


Figure 6: relative differences between the quantiles of X_t and Y_t for: t=1 ; $v = 133{,}26$ and $\lambda = 42{,}06$



We observe that the quality of the approximation of X_1 by Y_1 is very good. We can indeed note that:

- The quantile-to-quantile diagram is perfectly aligned with the first bisector;
- All quantiles of X_1 can be assimilated to the quantiles of Y_1:
    - On all probability orders with an error less than 0.06%;

[6] The quantiles are evaluated on 990 points ranging from 0.1% to 99% with a step of 0.1%. Relative differences are calculated as $\frac{(Q_X(p)-Q_Y(p))}{Q_X(p)}$ where $Q_X(p)$ (resp. $Q_y(p)$) is the p order quantile of X (resp. Y).

- On all probability orders greater than 0.2% with an error less than 0.04%;
- On all probability orders greater than 0.3% with an error less than 0.03%;
- On all probability orders between 1.5% and 100% with an absolute value of the error less than 0.01%.

Figure 7: Quantile to Quantile diagram of X_t and Y_t for: t=20 ; $\upsilon = 133{,}26$ and $\lambda = 1{,}54$

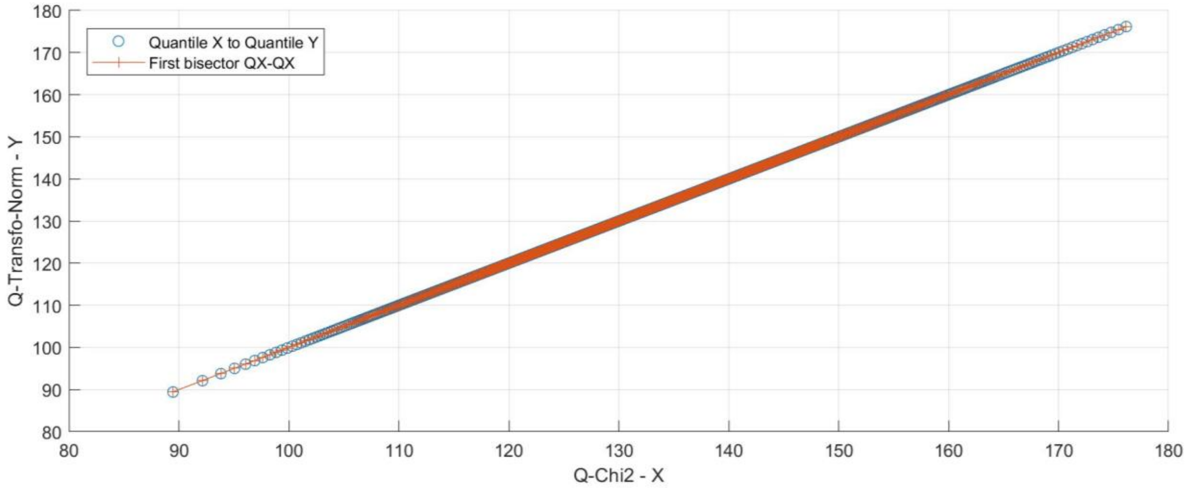
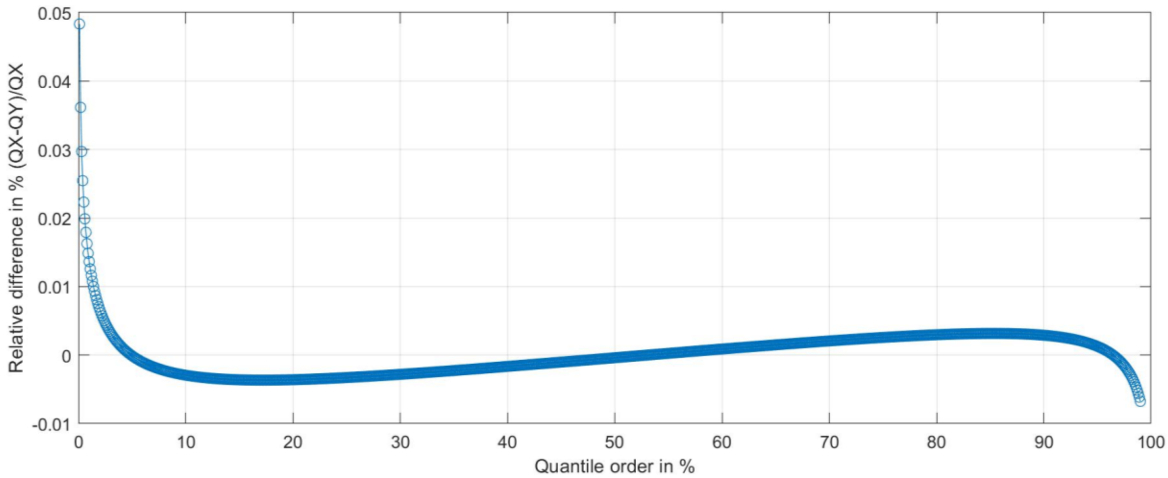


Figure 8: relative differences between the quantiles of X_t and Y_t for: t=20; $\upsilon = 133{,}26$ and $\lambda = 1{,}54$



As for the one-year projection horizon, we observe that the quality of the approximation of X_{20} by Y_{20} is very good. We can indeed note that:

- The quantile-to-quantile diagram is almost aligned with the first bisector;
- All quantiles of X_1 can be assimilated to the quantiles of Y_1:
  - On all probability orders with an error less than 0.05%;
  - On all probability orders greater than 0.3% with an error less than 0.03%;
  - On all probability orders between 1.5% and 100 % with an absolute value of the error less than 0.01 %.

The approximation presented in this section is the best approximation studied among those listed in Section 3.3 of the technical appendix. However, it is non-linear, which limits its use to the CIR++ model only.

Indeed, in case of CIR++ model we can write:

$$G_2\left(\frac{\left(R_{CIR++}(t,T) - \beta(t,T)\right)}{\alpha(t,T)}\right) = G_2\left(\chi^2(v,\lambda_t)\right) \approx N(0,1)$$

Therefore, we can find a bijection transforming the compound interest rate into a normal variable and apply the algorithm presented in section 3.2.2 to calibrate the dependency structure and simulate the equity and real estate indices.

In the case of the CIR2++ model, the compound interest rate is an affine function of two non-central chi-squared random variables. The use of the function G_2 is therefore unsuitable because it is non-linear.

Given the quality of the linear approximation presented in section 3.2.3.1 and the simplicity of its implementation, we have selected it to calibrate the dependency structure and simulate the equity and real estate indices of the two generators ESG1 and ESG2.

In sections2 and3 we have presented our approach to calibrate and simulate economic and financial variables within the framework of risk-neutral economic scenarios generation where interest rates follow CIR++ or CIR2++ models.

The following section presents the results of the calibrations of these models and focuses mainly on the study of rate models. A study of the sensitivity of the best-estimate to the choice of rate models and calibration data is also presented.

4 Result and sensitivity analysis of the best-estimate of French participating savings contracts to interest rate models

In this section we present the results of the practical implementation of the theoretical developments presented in sections 2 and 3. The objective is to present:

- The data and approach used to calibrate the economic scenario generators studied here (ESG1 and ESG2);
- Calibration results of the CIR++ and CIR2++ models and an analysis of their quality;
- Sensitivity studies of the best-estimate of participating savings contracts to the choice of CIR++ and CIR2++ rate models and their calibrations.

4.1 Approach and data

The data and approach used to calibrate the economic scenario generators studied here are similar to those presented in Armel and Planchet [2018]. We therefore use the same convention for calibrating the interest rate models. This convention can be summarized in four steps:

1. **Model and derivative instruments**: choice of interest rate model and choice of derivatives for its calibration: caps, floors, swaptions, etc.
2. **Strike prices and implied volatilities**: choosing a strike price and extracting market volatilities. These volatilities correspond to the implied volatilities of the derivatives chosen in step 1. They are consistent with the risk-free yield curve of the market.
3. **Valuation of derivatives** using the yield curve published by the EIOPA: use of the Black model (if volatilities are extracted by a log-normal model) or the Bachelier model (if volatilities are extracted by a normal model) to price derivatives using the risk-free rate curve published by the EIOPA. These prices will play the role of "market prices" to calibrate the selected interest rate model.
4. **Calibration of the selected interest rate model** by minimizing a distance between : (1) the prices re-evaluated using the EIOPA yield curve and market volatilities and (2) the theoretical prices given by the interest rate model.

We also made the following choices:

- The options selected for the calibration of the different models (calls, caps and swaptions) are ATM[7];
- The market volatilities of caps and swaptions used in the calibration process are lognormal ATM volatilities provided by Bloomberg (the shift factor is equal to 0);
- The risk-free yield curve used for the calibration and simulation processes is the yield curve provided by the EIOPA as at 12/31/2017;
- The use of the EIOPA yield curve implies the need to introduce a non-zero shift factor to evaluate the prices of caps and swaptions using the Black model;
- Different calibrations of the CIR++ and CIR2++ models have been performed corresponding to different levels of the Black model shift factor. Three levels of the shift factor are tested: 0.4%, 1% and 2%. The value of 0.4% corresponds to the rounded absolute value of the minimum interest rate of the EIOPA risk-free yield curve as at December 31, 2017;
- The calibration and simulation of the CIR++ and CIR2++ models requires the definition of initial values of the CIR processes. Several values are tested;
- For the projection of equities and real estate investments, we used Black-Scholes models with constant volatilities:
    - The implied volatility of an investment in equities is calibrated on the price of the ATM call on the CAC 40 with a maturity of 3 years;

[7] A cap, floor or swaption with payment dates set $\{T_i\}_{i\in[\![\alpha,\beta]\!]}$ is ATM (At The Money) if and only if the exercise price K_{ATM} is written $K_{ATM} = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{j=\alpha+1}^{\beta} \tau_j P(t,T_j)}$.

- The volatility of an investment in real estate corresponds to the historical volatility of returns extracted from the price index of selling of old housing published by INSEE[8].

The following sections present the calibration results and sensitivity studies of the best-estimate of participating savings contracts.

4.2 CIR++ model

4.2.1 Calibration results

Table 1 presents the calibration results of the CIR++ model (also noted CIR1F in the following) on ATM cap prices and Table 2 presents the results of the calibration on ATM swaptions prices.

The meta-parameter x_0 of the CIR++ model representing the initial value of the CIR process must be set upstream of the calibration process. This parameter has no impact on the reproduction of the initial yield curve but may have an impact on the dynamic of the simulated interest rate. As for the Black model shift factor, three levels of the meta-parameter x_0 are tested: 0.4%, 1% and 2%.

By the notation $CIR1F(i,j)$ we refer to the CIR++ model calibrated on caps or swaptions with the i^{th} Black shift factor and the j^{th} meta-parameter of the CIR++ model, both belonging to (0.4%; 1%; 2%; 0.4%; 1%; 2%).

Table 1: CIR++ model calibration results on Caps.

Parameters	CIR1F(1,1)	CIR1F(1,2)	CIR1F(1,3)	CIR1F(2,1)	CIR1F(2,2)	CIR1F(2,3)	CIR1F(3,1)	CIR1F(3,2)	CIR1F(3,3)
k	1.95%	2.91%	3.89%	2.2%	3.12%	4.09%	2.62%	3.45%	4.37%
θ	99.39%	99.22%	99.16%	98.77%	99.98%	99.24%	99.05%	99.34%	99.24%
σ	2.57%	2.1%	1.81%	3.68%	3.06%	2.68%	5.45%	4.69%	4.14%
Total relative squared error	**1.05%**	**1.19%**	**1.35%**	**1.00%**	**1.15%**	**1.34%**	**0.88%**	**1.03%**	**1.21%**

Table 2: CIR++ model calibration results on Swaptions

Parameters	CIR1F(1,1)	CIR1F(1,2)	CIR1F(1,3)	CIR1F(2,1)	CIR1F(2,2)	CIR1F(2,3)	CIR1F(3,1)	CIR1F(3,2)	CIR1F(3,3)
k	2.63%	2.99%	3.45%	3.52%	3.85%	4.31%	4.95%	5.19%	5.5%
θ	99.95%	99.99%	100,00%	99.97%	99.99%	100,00%	99.9%	99.96%	99.98%
σ	5.76%	5.31%	4.87%	7.05%	6.65%	6.2%	9.53%	9.16%	8.7%
Total relative squared error	**2.92%**	**3.19%**	**3.6%**	**3.45%**	**3.73%**	**4.15%**	**4.89%**	**5.16%**	**5.6%**

The total error is calculated as the sum of the squared errors (objective function) divided by the sum of the squared Black prices.

The calibrated parameters respect the Feller constraint.

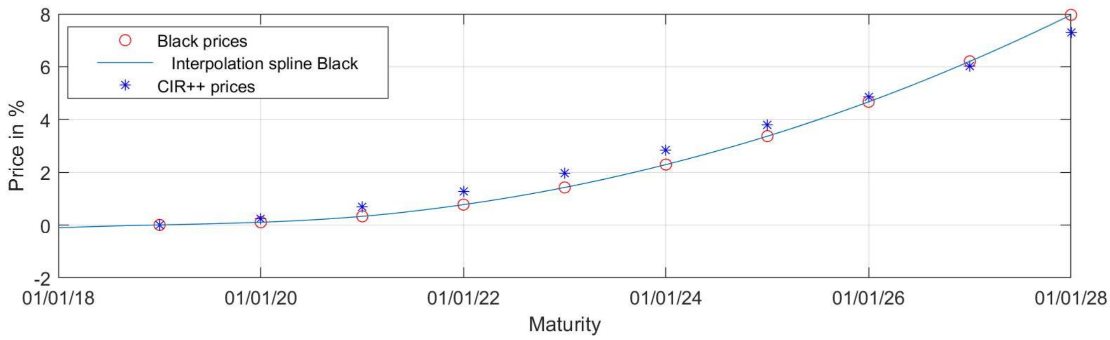
Analysing the calibration results of the Hull-White model and the G2++ model presented in Armel and Planchet [2018], it can be noted that the reproduction quality of caps and swaptions prices by the CIR++ model is better than that of the Hull-White model but less

[8] https://www.insee.fr/fr/statistiques/series/102770558

good than that of the G2++ model. This is mainly explained by the number of model parameters: the Hull-White model has two parameters, the CIR++ model has three parameters constrained by the Feller relation and the G2++ model has five parameters with a better ability to reproduce market volatilities (see section 4.3.1). Similar conclusions are presented in Brigo and Mercurio [2007].

The Figure 9 shows the quality of the CIR++ model calibration on cap prices evaluated by the 1% shifted Black model and using a 1% CIR meta-parameter. We observe that the CIR++ model reproduces prices fairly well.

Figure 9: reproducibility of cap prices by the CIR++ model



4.2.2 Impact analysis of the initial value of the CIR++ model on interest rate expectation and volatility

The meta-parameter x_0 of the CIR++ model represents the initial value of the CIR process. It must be set upstream of the calibration process. We have studied the impact of the choice of this parameter on interest rate means and their volatilities.

The volatility of instantaneous short rates in a CIR process can be written as[9] :

$$Var\{x(t)|F_s\} = \frac{x(s)\sigma^2}{k}\left(e^{-k(t-s)} - e^{-2k(t-s)}\right) + \theta\frac{\sigma^2}{2k}\left(1 - e^{-k(t-s)}\right)^2$$

Therefore the increase of x_0 systematically increases the volatility of compound interest rates[10]. The effect is much more pronounced for short-term volatilities. Indeed, x_0 does not affect the asymptotic variance of the instantaneous short rate since, for a positive k:

$$\lim_{t\to+\infty} Var\,\{x\,(t)\} \;=\; \theta\frac{\sigma^2}{2k}$$

Thus, long-term volatilities are theoretically less sensitive to changes in x_0 and the increase of x_0 has, as consequence, the reduction of the initial slope of the volatility curve.

[9] See section 2.1.2 of the technical appendix.
[10] Recall that CIR++ compound rates are affine functions of instantaneous short rates. The analysis of their volatility is the same as the analysis of the volatility of instantaneous short rates following a CIR model.

Also, during the calibration process, the increase of the initial value x_0 should be offset by the decrease in the dispersion σ or the increase of the mean reversion parameter k in order to maintain a level of volatility allowing to replicate the market prices set upstream. This theoretical observation is validated by the practical results presented in Table 1 and Table 2. Note that for a given Black model shift factor, the parameter σ is a decreasing function of x_0 and k is an increasing function.

In addition, the CIR++ model was calibrated over ten maturities ranging from 1 to 10 years. On these maturities, the volatility of interest rates, whose models are calibrated to the same prices, should be substantially equal.

The choice of the initial value will a priori have little impact on these volatilities (maturities of 1 to 10 years) because the calibration process will compensate the level of x_0 by σ and k. However, the initial value will have an indirect impact on the asymptotic variance via these parameters σ and k. When x_0 increases, the quantity $\frac{\sigma^2}{2k}$ is indeed decreasing. We can therefore expect a decrease in the asymptotic variance ($\theta \frac{\sigma^2}{2k}$) when x_0 increases and when θ remains stable. This theoretical observation is validated by the analysis of variance of the calibrated CIR++ models whose parameters are presented in Table 1 and Table 2 and illustrated in Figure 11.

The following figures illustrate the impact of the choice of the meta-parameter on the distributional characteristics of the CIR++ process for a calibration on the market prices of caps and swaptions evaluated with the 1% shifted Black model.

Figure 10: impact of the choice of meta-parameter on the expectation of compound rates, calibration on caps with a 1% Black shift.

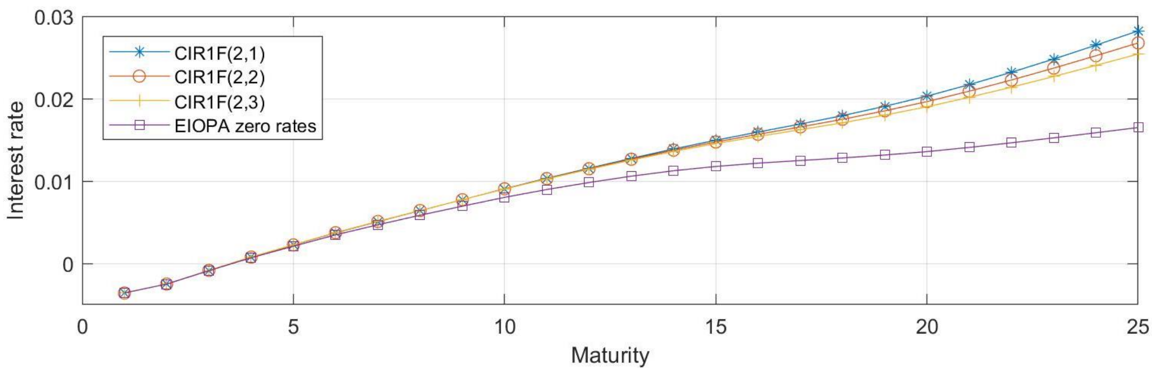


Figure 11: impact of the choice of meta-parameter on the volatility of compound rates, calibration on caps with 1% Black shift

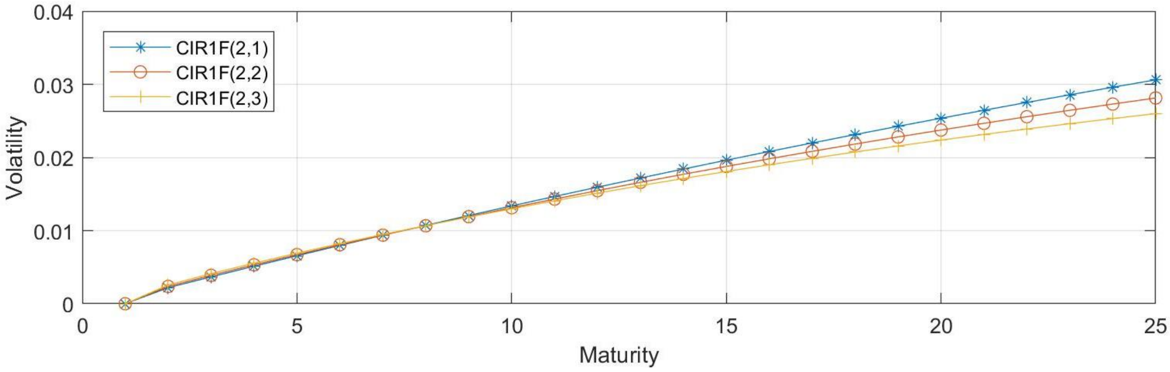


Figure 12: impact of the choice of meta-parameter on the discounting factor, calibration on caps with 1% Black shift

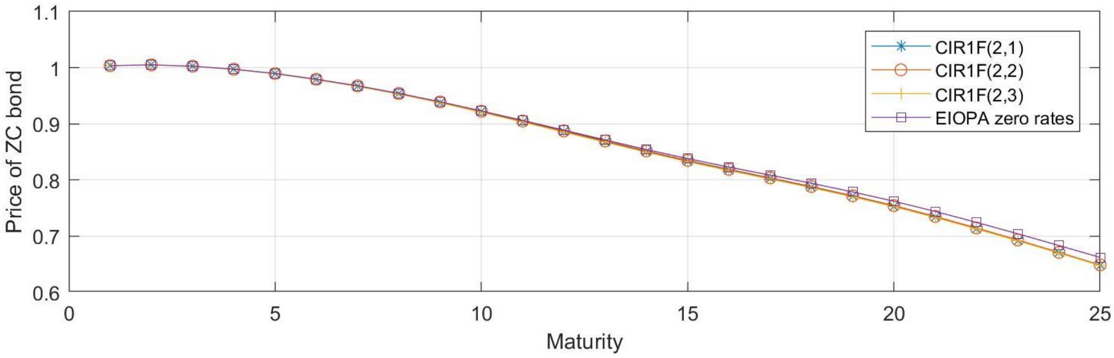


Figure 13: impact of the choice of meta-parameter on the expectation of compound rates, calibration on swaptions with 1% Black shift

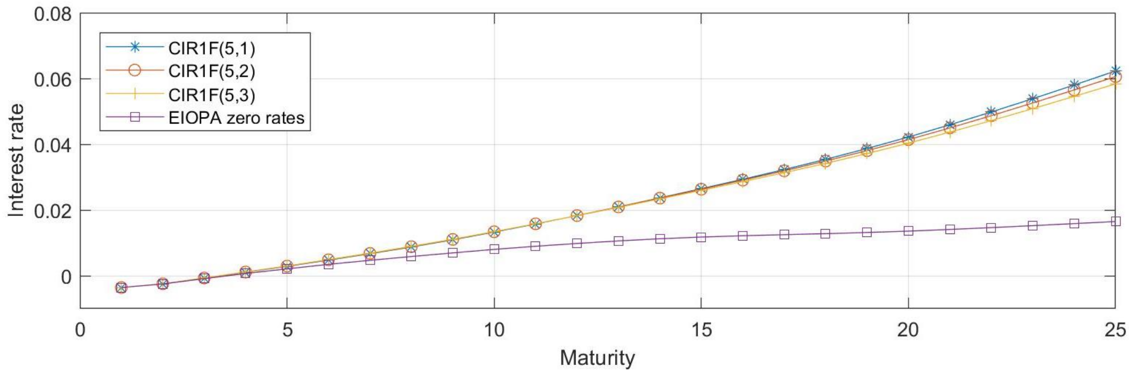


Figure 14: impact of the choice of meta-parameter on the volatility of compound rates, calibration on swaptions with 1% Black shift

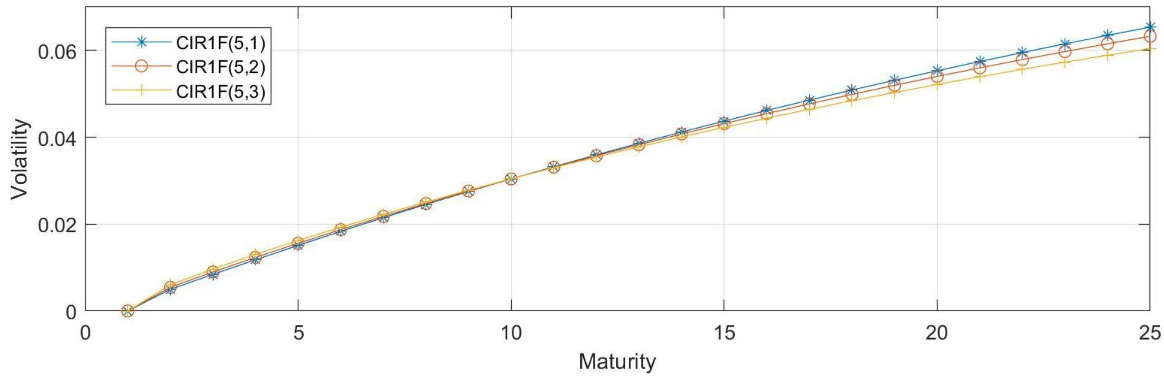
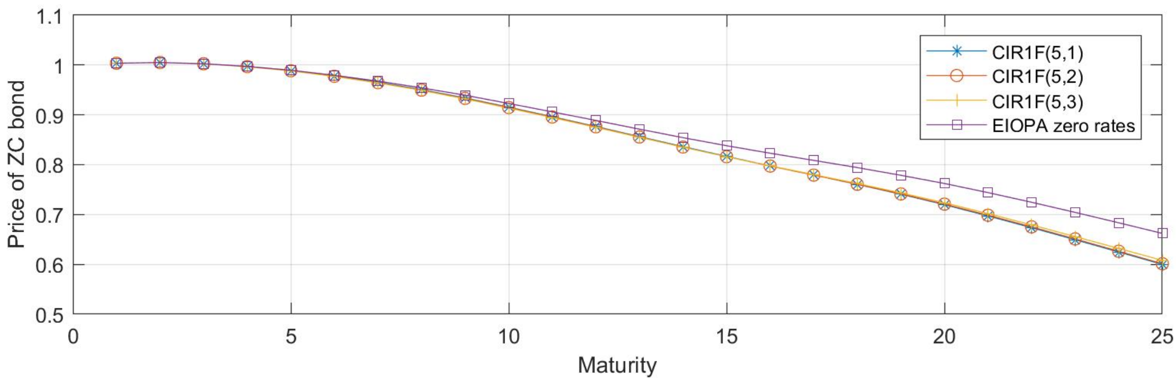


Figure 15: impact of the choice of meta-parameter on the discounting factor, calibration on swaptions with 1% Black shift



We observe that the expectation and the dispersion of the compound interest rate remain globally stable for the three initial values studied (0.4%; 1%; 2%). The average and dispersion of zero-coupon prices also remain stable. This observation is valid for all the Black model shift factors studied.

Given the low impact of the initial value of the CIR model, we focus in the following on studying the impact of the Black model shift factor and set the initial value of the CIR++ and CIR2++ models at 1%.

4.2.3 Sensitivity of the mean and the volatility of interest rates to the Black model shift factor

Figure 16 shows the expected yield curves generated by CIR++ models (with initial values equal to 1 %). The models are calibrated to cap and swaption prices evaluated with the Black model using the three shift factors shown above. These CIR++ models will be used to assess the sensitivity of the best-estimate presented in section 4.5.

We observe that expectations can deviate significantly from the central risk-free yield curve. This observation is consistent with the expression of the expectation of the interest rate presented in section 2.1.2.

Figure 16: dynamics of expectations - calibration of the CIR++ model on caps and swaptions, meta-parameter of 1%

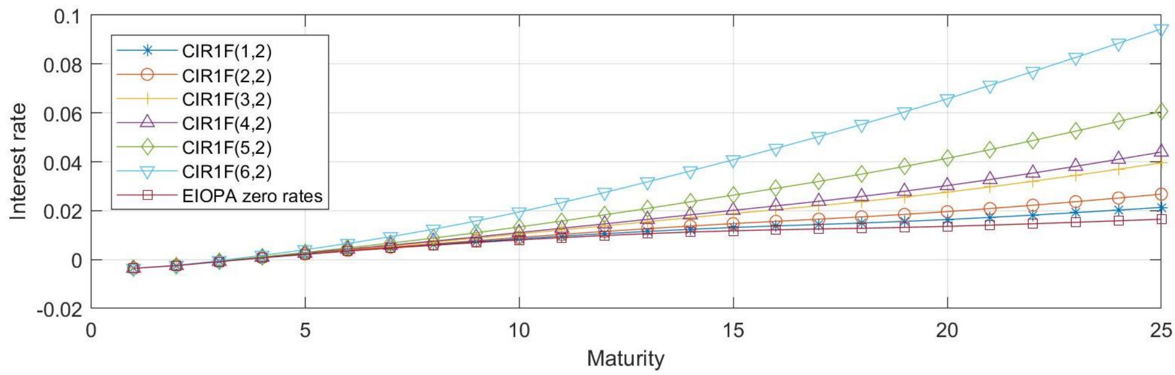
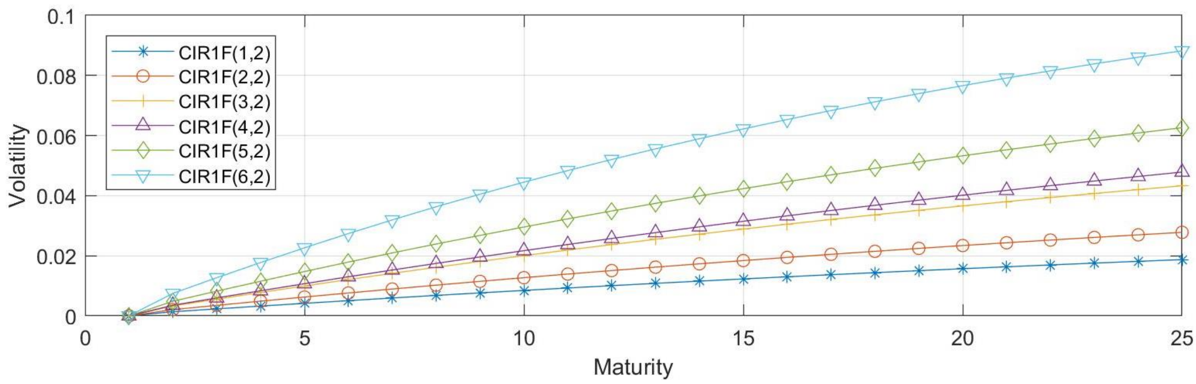


Figure 17 shows the evolution of interest rates volatilities as a function of maturities. We observe that the greater the volatility, the greater the differences between expectations and EIOPA risk-free rates.

For each type of financial instrument (caps and swaptions), we note that volatility is increasing as a function of the shift factor. Indeed, since the Black price increases with the shift factor, the CIR++ model "compensates" this price increase by increasing the volatility.

Figure 17: volatility curves - calibration of the CIR++ model on caps and swaptions, meta-parameter of 1%



Increasing the mean reversion parameter k decreases the volatility. The increase in parameters θ and σ increases this volatility.

4.3 CIR2++ model

4.3.1 Calibration results

The Table 3 presents the results of the calibration of the CIR2++ model (also noted CIR2F in the following) on the prices of ATM caps and ATM caps and swaptions. These CIR2++ models will be used to assess the sensitivity of the best-estimate presented in section 4.5.

The meta-parameters x_0 and y_0 of the CIR2++ model whose sum represents the initial value of the CIR2++ process are set upstream of the calibration process at $x_0 = 0.5\%$ and $y_0 = 0.5\%$. The initial value of the CIR2++ process is therefore equal to 1%.

By the notation CIR2F_i we denote the CIR2++ model calibrated on caps ($i \leq 3$) or swaptions ($4 \leq i$) with the i^{th} Black shift factor belonging to (0.4%; 1%; 2%; 0.4%; 1%; 2%).

Table 3: CIR2++ model calibration results on caps and swaptions

Parameters	*Caps*			*Swaptions*		
	CIR2F_1	CIR2F_2	CIR2F_3	CIR2F_4	CIR2F_5	CIR2F_6
k1	2.39%	2.38%	2.41%	3.00%	3.55%	2.93%
θ1	101.05%	98.49%	100.00%	100.39%	92.51%	97.08%
σ1	1.58%	1.62%	3.20%	3.99%	4.89%	7.72%
k2	2.41%	2.65%	2.41%	3.00%	3.51%	2.96%
θ2	101.05%	97.14%	100.00%	100.63%	98.29%	101.20%
σ2	1.58%	2.49%	3.20%	4.00%	5.01%	6.68%
Total relative squared error	**0.83%**	**0.94%**	**0.76%**	**3.25%**	**3.71%**	**5.47%**

The total error is calculated as the sum of the squared errors (objective function) divided by the sum of the squared Black prices.

The estimated parameters respect the Feller constraint.

Analysing the calibration results of the Hull-White model and the G2++ model presented in Armel and Planchet [2018], the CIR2++ model reproduces cap and swaption prices better than the Hull-White model and the CIR++ model.

The number of parameters in the CIR2++ model is six, constrained by Feller's formula. Consequently, the replication of market prices is generally better than the Hull-White model (two parameters) and the CIR++ model (three constrained parameters)[11].

However, it can be noted that the G2++ model (five parameters) reproduces prices better than the CIR2++ model (six constrained parameters).

Brigo and Mercurio [2007] make the same observation. They justify it by the presence of a correlation parameter ρ between the Brownian noises of the two factors in the G2++ model giving it more flexibility. When this correlation factor is strictly negative (which is the case in Armel and Planchet [2018]), the G2++ model allows a volatility hump of the instantaneous forward rate and reproduces market prices better. However, price reproduction is less good when the correlation parameter is zero: $\rho = 0$ because the model cannot reproduce the volatility hump.

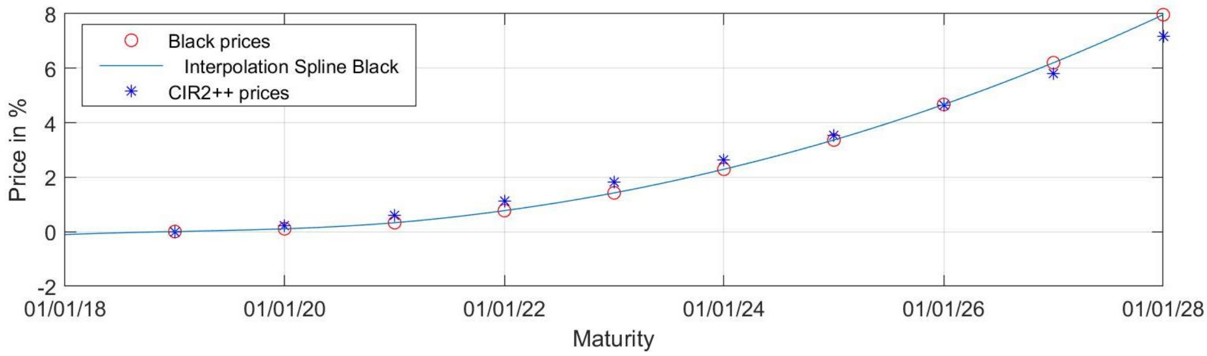
The situation is similar in the case of the CIR2++ model. The correlation between the Brownian noises of the two CIR factors in the CIR2++ model is zero and does not allow a volatility hump of the instantaneous forward rate. This hump shape is a desirable feature

[11] Brigo and Mercurio [2007] make the same observation.

of the model to better reflect market prices. However, a non-zero correlation factor would significantly limit the analytical properties of the CIR2++ model.

Figure 18 shows the quality of the CIR2++ model (initial value of 1%) calibrated on cap prices evaluated by the 1% shifted Black model. We observe that the CIR2++ model reproduces prices fairly well.

Figure 18: cap prices reproduction by the CIR2++ model



4.3.2 Sensitivity of the mean and the volatility of interest rates to the Black model shift factor

Figure 19 shows the expected yield curves generated by CIR2++ models whose calibrations are shown in the Table 3.

We observe that expectations can deviate significantly from the central risk-free yield curve. This observation is consistent with the expression of the expected interest rate presented in section 2.2.1

Figure 19: expected interest rates - calibration of the CIR2++ model on caps and swaptions

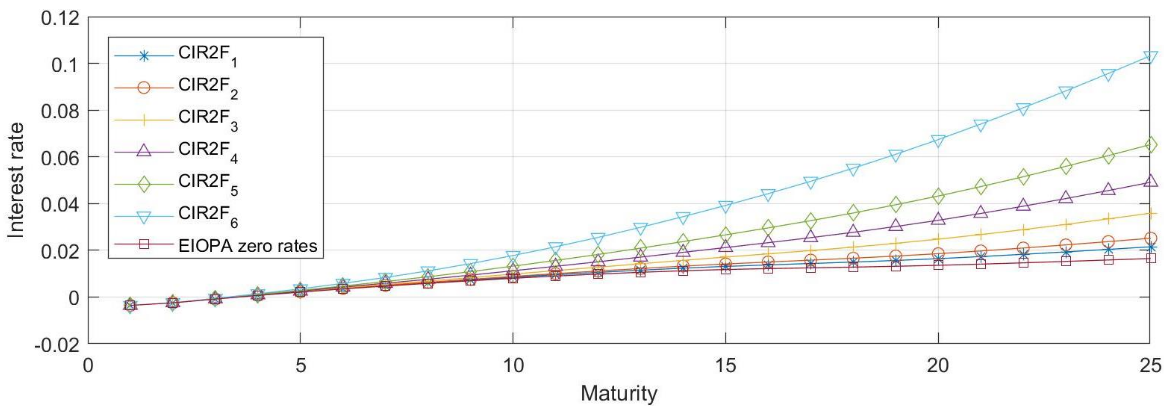
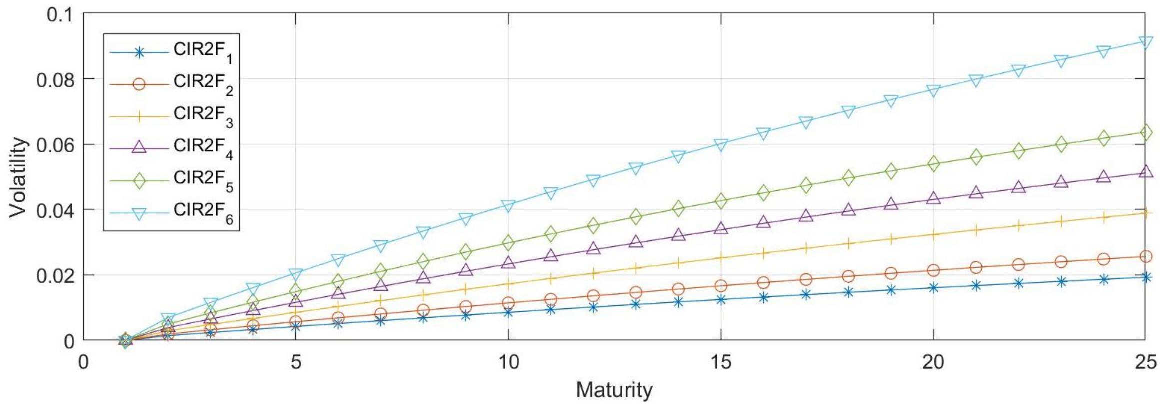


Figure 20 shows the evolution of volatilities as a function of maturities. We can see that the larger the volatilities, the greater the differences between model expectations and EIOPA risk-free rates.

Volatility increases with the shift factor. Indeed, since the Black price increases with the shift factor, the CIR2++ model "compensates" this price increase by increasing volatility.

Figure 20: volatility curves - calibration of the CIR2++ model on caps and swaptions



The increase of the mean reversion parameters k_1 and k_2 decreases the volatility. The increase in parameters θ_1, θ_2, σ_1 and σ_2 increases this volatility.

4.4 Dependency structure and equity and real estate models

In section 3.2.2 we presented a simulation approach for the equity index and the real estate index conditionally on compound interest rate paths using historical correlation matrices. The historical rates used to construct these matrices are the 12-month Euribor rates. The following table shows the historical correlation matrix used for the projection of our ESGs.

Table 4: correlation matrix

Correlation matrix	*Interest rate*	*Equity - Interest_rate*	*Real_Estate - Interest_Rate*
Interest rate	*100.00%*	*-53.71%*	*-3.91%*
Equity - Interest_rate	*-53.71%*	*100.00%*	*11.18%*
Real_Estate - Interest_Rate	*-3.91%*	*11.18%*	*100.00%*

The volatility of the real estate index corresponds to the historical volatility of returns extracted from the price index of selling of old housing published by INSEE[12] from December 2006 to December 2016. This volatility is 3.5%.

The volatility of the equity model is assumed to be independent of time and was calibrated to the price of a 3-year ATM call on the CAC 40 index. A Monte-Carlo approach was used, integrating the correlations presented above, to extract this volatility. The results are presented in Table 5 (the Black shift factor used to calibrate each interest rate model is equal to 1%).

[12] https://www.insee.fr/fr/statistiques/series/102770558

Table 5: implied volatility of the equity index

Interest rate model	*Equity implied volatility*
CIR++	10.34%
CIR2++	10.25%

The calibration of the CIR++ and CIR2++ interest rate models, as well as the calibration of the equity model, the real estate model and the dependency structure, allows to generate economic scenarios and thus achieve studies of the sensitivity of the best-estimate to the choice of the CIR++ and CIR2++ interest rate models and the calibration data. These studies are presented in the following.

4.5 Impact of the choice of interest rate model on the *best-estimate*

In Armel and Planchet [2019] we selected three interest rate models (Hull & White, G2++ and shifted LMM), calibrated on two types of financial products (caps and swaptions), to achieve sensitivity tests on the best-estimate of French participating savings contracts. These models comply in particular with the regulator's constraints.

The LMM model, calibrated on the data observed as of January 2, 2018, could not be retained, however, because of its divergence. The version used in the sensitivity tests is an adjusted and convergent model, which is therefore no longer "*market consistent*".

In this section, we complete these sensitivity tests by assessing the impact of the choice of the CIR++ and CIR2++ interest rate models and calibration data on the best-estimate of participating savings contracts.

In order to assess these impacts, we have used the same models and parameters to valuate liabilities as presented in Armel and Planchet [2019]. We have therefore used the SimBEL[13] R package fed with modified real data from an insurer. The market value of the assets is €100M, the mathematical reserve is €70M and the projection horizon is 20 years.

The Table 6 and Table 7 present the sensitivity of the best-estimate to the CIR++ and CIR2++ models and their market-consistent calibrations.

Table 6: best-estimate by the CIR++ market-consistent model

Amounts in € M	*Cap*			*Swaption*		
	CIR1F(1,2)	CIR1F(2,2)	CIR1F(3,2)	CIR1F(4,2)	CIR1F(5,2)	CIR1F(6,2)
Best-estimate net of expenses	82.62	82.66	83.03	83.58	83.94	84.67
Expenses	7.92	7.89	7.85	7.89	7.83	7.7
Best-estimate	90.54	90.55	90.89	91.47	91.78	92.37

Table 7: best-estimate by the CIR2++ market-consistent model

Amounts in € M	*Caps*			*Swaptions*		
	CIR2F_1	CIR2F_2	CIR2F_3	CIR2F_4	CIR2F_5	CIR2F_6
Best-estimate net of expenses	82.63	82.99	82.85	83.12	84.44	84.32
Expenses	7.93	7.94	7.89	7.82	7.86	7.74
Best-estimate	90.56	90.94	90.74	90.94	92.3	92.05

We find that the best-estimate evaluated are not very sensitive to the choice of interest rate model (CIR++ or CIR2++), to the Black model shift factors and to the choice of

[13] See http://www.ressources-actuarielles.net/C1256F13006585B2/0/C5542E1CF549F21FC12581680046FD2E

derivatives for calibration. The Table 8 shows that the difference between minimum and maximum values represents 2 % of the average value of the best-estimates evaluated using the CIR++ and CIR2++ models.

Table 8: comparison of best-estimates

Best-estimate in € M	Standard deviation	Min	Max	Difference (Max-Min) / Average
CIR ++ model and CIR2 ++ model	0.77%	90.54	92.37	2.00%
Market-consistent models: CIR ++, CIR2 ++, HW & G2 ++	1.13%	87.48	92.37	5.38%
All models including adjusted LMM	1.06%	87.48	92.37	5.38%

If we add the sensitivity results presented in Armel and Planchet [2019], the Table 8 shows that the impact on the value of the best-estimate may appear fairly limited at first: if we only use market consistent interest rate models, the difference between minimum and maximum values represents 5.4% of the average value of the best-estimates and 7.0% of the mathematical reserves. However, this impact is substantial when compared to shareholders' equity. Indeed, the latter represents in France on average 5.3% of the mathematical reserves at the end of 2016 (FFA [2017]).

The sensitivity of the best-estimate to interest rate models can be compared to the level of equity capital and there is no unquestionable criterion at this stage that would allow to prefer one or the other of the above models, once the LMM model has been excluded due to its lack of convergence.

5 Conclusion

This article focuses on the problem of choosing interest rate models to evaluate the best-estimate of participating savings contracts in an economic environment characterised by negative interest rates. It is in line with the articles that we have published: Armel and Planchet [2018] and Armel and Planchet [2019].

Here we have introduced a third family of interest rate models. These are CIR (Cox-Ingersoll-Ross) models, whose dynamic includes a square-root component of the instantaneous short rate. In this section, we present our conclusions and a comparison of the CIR++, CIR2++, Hull & White, G2++, and LMM interest rate models along three axes:

1. Simplicity of calibration and simulation;
2. The ability to replicate the market prices of caps and swaptions;
3. The relevance of the choice of the interest rate model with regard to the objective of valuing participating savings contracts.

Simplicity of calibration and simulation

The normal Hull and White and G2++ models allow the valuation of caps, floors and swaptions by closed formulas. This makes the calibration process of these models more efficient. Indeed, one can have a better accuracy of the calibration and a better optimization of the computing resources unlike a calibration process with valuations by Monte-Carlo methods.

The simulation of the normal Hull and White and G2++ models is relatively simple because interest rates can be discretized exactly and zero-coupon bonds can be evaluated by closed formulas.

The CIR++ model also allows the valuation of caps, floors and swaptions using closed formulas. The complexity of the calibration process is comparable to that of normal models and includes a constraint on the parameters (Feller constraint). The CIR++ model can be discretized exactly and the prices of zero-coupon bonds can be valued by closed formulas.

However, the CIR2++ model does not allow the valuation of caps, floors and swaptions by closed formulas. The calibration process necessarily uses alternative valuation methods such as Monte-Carlo or tree generation methods, which makes it complex. Also, if the diffusion is achieved by Euler discretizations (which is not the case in this paper), the complexity of the calibration process increases due to the additional convergence constraints. The CIR2++ model can be discretized exactly and zero-coupon bonds can be evaluated by closed formulas.

The LMM model also allows derivative valuation by closed formulas, which partly simplifies the calibration process. These formulas can be derived from approximations or from the Euler discretization of the model (see Armel and Planchet [2018]). The diffusion of the interest rate model is achieved by an Euler discretization.

The ability to replicate the market prices of caps and swaptions

The market-consistent calibration results presented here and in Armel and Planchet [2018] show that the G2++ model is the model that best reproduces the market prices of caps and swaptions.

The quality of price reproduction by the CIR++ model is better than that of the Hull-White model but less good than that of the G2++ model. This is mainly explained by the number of model parameters.

The quality of reproduction of caps and swaptions prices by the CIR2++ model is better than that of the Hull-White model and the CIR++ model, but remains less good than that of the G2++ model. The G2++ model allows a volatility hump of the instantaneous forward rate and reproduces market prices better. This property is desirable when the market is highly volatile (see Brigo and Mercurio [2007]).

The calibration of the LMM model shows that it reproduces market volatilities very well. However, the parameters derived from the market-consistent calibration cannot be retained due to the divergence of the model.

The relevance of the choice of the interest rate model with regard to the valuation objective of participating savings contracts

The impact of the choice of interest rate models on the value of the best-estimate may first appear rather contained: by retaining only market consistent interest rate models, the difference between minimum and maximum values represents 5.4% of the average value of the best-estimates and 7.0% of the mathematical reserves.

This impact is substantial when compared to equity capital and there is no unquestionable criterion at this stage to prefer one or the other of the above models, once the LMM model is excluded due to its lack of convergence.

Moreover, Armel and Planchet [2019] show that the ability of an interest rate model to reproduce floorlet prices and, by extension, cap prices, can be considered as a criterion for choosing interest rate models intended to value the liabilities of participating savings contracts.

CIR interest rate models (CIR++ and CIR2++) and Gaussian models (Hull and White and G2++) reproduce the market prices of caps with acceptable error. They also present a modelling framework that meets all the constraints imposed by the supervisor and allow negative rates to be managed without introducing arbitrary shift factors as in log-normal models.

Finally, the G2++ model is simple to calibrate and to simulate. It is the model that best reproduces the cap market prices. It also has more extensive analytical properties than the CIR2++ model.

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