Aggregating Economic Capital

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Abstract

In this paper we analyze and evaluate a standard approach financial institutions use to calculate their so-called total economic capital. If we consider a business that faces a total random loss S over a given one-year horizon then economic capital is traditionally defined as the difference between the 99.97% percentile of S and its expectation. The standard approach essentially assumes that the different components (risks) of S are multivariate normally distributed and this highly facilitates the computation of the total aggregated economic capital. In this paper we show that this approach also holds for a more general framework which encompasses as a special case the multivariate normal (and elliptical) setting. We question also the assumption of multivariate normality since for many risks one often assumes other than normal distributions (e.g. a lognormal distribution for insurance risk). Assuming that risks are either normal or lognormal distributed we propose, using the concept of comonotonicity, an alternative aggregation approach.

1 Introduction

In this paper we analyze and evaluate a standard approach financial institutions use to calculate their so-called total economic capital. Roughly speaking, the economic capital is the amount of capital a financial institution needs

in order to remain economically solvent. If we consider a business that faces a random loss S over a given one-year horizon then the (required) economic capital is most often defined as the difference between the 99.97% percentile of S and its expectation. In order to compute this required amount, the standard approach uses stand-alone economic capitals, which are essentially derived from the aggregate loss distribution, per type of risk (ALM, Credits,...) and per business line (Life Insurance, Retail Bank,...) as well as the Pearson correlations that exist between the different risks. Next, a straightforward formula enables to compute a closed-from expression for the total diversified economic capital and this in terms of the stand-alone economic capitals and the correlations.

It is well-known that this simple approach is correct when risks are assumed to be multivariate normally distributed. In this paper we show that this approach also holds for a more general framework which encompasses as a special case the multivariate normal (and elliptical) setting. We also prove that the simple approach can still be employed when one replaces in the definition of economic capital the quantile risk measure by a general distortion riskmeasure. Finally, we also question the assumption of multivariate normality since for the calculation of some of the stand-alone economic capitals, one often uses other than normal distributions (e.g. a beta distribution for the credit risk). We refer to Heckman and Meyers (1983), Panjer (1981) and Robertson (1992) amongst others for actuarial methods that can be used for modeling the aggregate loss distribution per type of risk and per business line. In Wang (1998) the author considers a variety of models of how to combine the different aggregate loss distributions and these methods could then be used to derive the total economic capital. Within this respect, our paper extends the list of methods that were mentioned in Wang (1998). Indeed, assuming that risks are either normal or lognormal distributed we propose, using the concept of comonotonicity, an easy to implement aggregation approach.

2   Economic capital and diversification

Consider a business that faces a random loss S over a given one-year horizon. We define the (required) economic capital as the difference between a given percentile of S (e.g. 0.9995) and its expectation:

$$\text{Economic Capital} = \text{EC}\,[S] \; = F_S^{-1}(1-p) - E(S). \tag{1}$$

2

We will call $F_S^{-1}(1-p)$ the 'total balance sheet capital requirement'. We define 'economic default' as the event 'S exceeds $F_S^{-1}(1-p)$'. Holding the total balance sheet capital $F_S^{-1}(1-p)$ means that if value of the available assets equals or exceeds this amount, then there is a probability of shortfall of at most p.

The first step in calculating the required amount of economic capital is to determine per risk type the required amount of economic capital for each business unit as if it existed on its own. The next and crucial step consists then in the aggregation of the different economic capitals from the different business lines and risk types, into one single aggregate capital amount.

Aggregation will lead to a diversification effect. Indeed, consider two business units with risks (losses) X_1 and X_2. Assume that the total balance sheet capital is determined by the risk measure ρ (e.g. $\rho[S] = F_S^{-1}(1-p)$). When each of the risks is considered on a stand-alone basis, i.e. each of the business units is not liable for the shortfall of the other one, the total balance sheet capital requirement for each portfolio is given by $\rho[X_j]$. When the two business units are viewed on an aggregate basis, the purpose is to avoid the eventual shortfall of the aggregate loss $X_1 + X_2$. As mentioned in Dhaene, Goovaerts & Kaas (2003), the following inequality holds with probability 1:

$$(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq \sum_{j=1}^{2} (X_j - \rho[X_j])_+ . \tag{2}$$

Here, the notation $(x)_+$ stands for $\max(x, 0)$. Inequality (2) states that the shortfall $(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+$ of the aggregated business units is always smaller than the sum of the shortfalls $(X_j - \rho[X_j])_+$ of the stand-alone business units, when adding the total balance sheet requirements. It expresses, that from the viewpoint of avoiding a shortfall, the aggregation is to be preferred in the sense that the shortfall decreases. The underlying reason is that within the aggregated portfolio, the shortfall of one of the entities can be compensated by the gain of the other one. This observation can be summarized as: '*merging decreases the shortfall risk*'. This also implies that the total balance sheet capital $\rho[X_1 + X_2]$ of the aggregated position can be chosen lower (to some extent) than the simple sum $\rho[X_1] + \rho[X_2]$ of the stand-alone total balance sheet capitals:

$$\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2] . \tag{3}$$

Since taking expectations is a linear operation the same observation holds for the economic capitals:

$$\text{EC}[X_1 + X_2] \leq \text{EC}[X_1] + \text{EC}[X_2].\tag{4}$$

For more details, see Dhaene, Laeven, Vanduffel, Darkiewicz & Goovaerts (2004).

3   Aggregation methods

3.1   Summing the individual capitals

Many financial institutions first determine economic capital for every risk type i within a given business unit j. Denoting the loss of risk type i in business unit j by X_{ij}, we have:

$$\text{Economic Capital of risk type } i \text{ within business unit } j \tag{5}$$
$$= EC_{ij} = F_{X_{ij}}^{-1}(1-p) - E[X_{ij}].$$

The aggregate loss S is the sum of the individual losses:

$$S = \sum_{i=1}^{a}\sum_{j=1}^{b} X_{ij}.\tag{6}$$

A first possible way to determine the aggregate economic capital EC corresponding with the aggregate loss S could be to take the sum of the individual economic capitals. We will denote the economic capital determined in this way by $EC^{(\text{sum})}$:

$$EC^{(\text{sum})} = \sum_{i=1}^{a}\sum_{j=1}^{b} EC_{ij}.\tag{7}$$

The aggregate economic capital $EC^{(\text{sum})}$ will be equal to the (1-p)-percentile of $S - E(S)$ provided the individual losses X_{ij} are all comonotonic:

$$\text{All } X_{ij} \text{ are comonotonic} \Rightarrow EC^{(\text{sum})} = F_S^{-1}(1-p) - E(S).\tag{8}$$

This result holds because for comonotonic random variables, the percentiles of the sum are given by the sum of the respective percentiles of the marginals.

Comonotonicity is an extreme positive dependency structure, where increasing one of the X_{ij} will lead to an increase of all the others too. Notice that the opposite implication does not necessary hold: the fact that the (1-p)-percentile of the sum equals the sum of the (1-p)-percentiles of the marginals will not necessary imply comonotonicity of the X_{ij}.

In the special case that the random vector $(X_{11}, X_{12}, \ldots, X_{ab})$ is multivariate normal (which means that any linear combination of the X_{ij} is normal distributed), we have that a random vector is comonotonic if and only if all correlations are equal to 1. The concept of comonotonicity is extensively studied in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a, b).

Assuming comonotonicity between the individual risks involved, implies for instance that increasing credit risk losses for the Banking pool would always go hand in hand with increasing operational losses within the Inurance Pool. Hence such a methodology will in general overestimate the required aggregate capital, as no diversification effect is taken into account. Otherwise stated, determining the economic capital by (7) will in general give rise to a too high capital. It is a safe strategy, but at the cost of holding too much capital.

3.2 The copula approach

In general, the individual losses X_{ij} will not be comonotonic, implying that there will exist a diversification benefit, which allows the aggregate capital, at a (1-p)-probability level, to be lower than the sum of the individual capitals, also at a (1-p)- probability level. Hence, we can take into account the degree to which the different risk types co-move or even counter-move. If we know to what degree the losses related to risk type i and business unit j tend to follow the losses related to risk type i' and business unit j', for all couple (i, j) and (i', j'), we will be able to compute the aggregate capital requirement, as the distribution of S is completely specified in this case.

One typically uses Pearson's correlation coefficient to describe the dependencies between the different risk types. It is well-known that only in a few cases, (the most important case being the one where risks are assumed to be multivariate normal), the knowledge of Pearson's correlation coefficient is sufficient to describe the full dependency structure. So, in case the risks involved are multivariate normal distributed, we can conclude that this approach used to aggregate the different risks will be justified. In general however, a more sophisticated notion for describing dependency structures will be required, namely a copula.

To state it more mathematically: we will be able to determine the (1-p)– percentile of the aggregate loss if the copula C connecting the marginal distributions $F_{ij}(x_{ij})$ of the individual losses X_{ij}, as well as these marginal distributions, are known. In this case, the multivariate distribution of the random vector $(X_{11}, X_{12}, \ldots, X_{ab})$ is given by

$$F_{X_{11}, X_{12}, \ldots, X_{ab}}(x_{11}, x_{12}, \ldots, x_{ab}) = C\left[F_{11}(x_{11}), F_{12}(x_{12}), \ldots, F_{ab}(x_{ab})\right]. \quad (9)$$

An introduction to copulas and its applications in insurance and finance is Frees & Valdez (1998). Over the last years, a whole literature has appeared on modeling dependencies by (families of) copulas.

Although many of the developed theoretical results for copulas hold for any dimension of the random vector involved, practical applicability is often restricted to random vectors of a sufficiently small dimension (typically less than 4).

Moreover, it will be extremely difficult to model the correct copula since no empirical data concerning the dependencies that exist between the different risks seem to be readily available. Indeed, we noticed that also Pearson's correlation matrix is often bassed on benchmark data.

Although the distribution of $S = \sum_{i=1}^{a} \sum_{j=1}^{b} X_{ij}$ is completely specified when the multivariate distribution $C\left[F_{11}(x_{11}), F_{12}(x_{12}), \ldots, F_{ab}(x_{ab})\right]$ is known, determining the distribution of S and/or its quantiles from (9) is, from a computational point of view, in most cases an extremely complicated task.

For all these mentioned reasons, we believe that the copula approach is not an appropriate methodology for the aggregation problem.

3.3 Individual versus collective approach

The approaches described in the previous section could be called 'individual approaches', as the basic entities are the individual losses, which are in a further stage aggregated.

Another (theoretical) approach could be called 'collective' in the sense that the loss of the whole business (the sum of all losses of all risk types and all business units) is the basic random variable of which the distribution is modeled. Using the aggregated cash flows would lead to an aggregate loss and allow us to compute aggregated economic capital.

In fact, in this collective methodology, the dependencies (this means the correlations in the multivariate normal case) are taken into account implicitly.

These 'embedded' correlations could be determined backwards by considering also the stand-alone capitals, and then look for the correlations that correspond to these marginal and aggregate quantiles.

4   The standard aggregation methodology : a reconciliation

In this section we will retrieve the standard formula many financial institutions use to compute its diversified capital and investigate under which assumptions this formula is valid.

In this individual approach, the stand-alone economic capitals are first determined per risk type and business unit. The dependencies that exist between the individual losses (for each couple $(X_{ij}, X_{i'j'})$) are described by the correlations $corr\left[X_{ij}, X_{i'j'}\right]$ [1].

In order to make the presentation somewhat simpler without losing generality of the results, we will rename the individual losses X_{ij} as $Y_1, Y_2, \ldots, Y_n$ with $n = a.b$. This will allow us to avoid double summations. The aggregate loss S can then be rewritten as

$$S = \sum_{i=1}^{a} \sum_{j=1}^{b} X_{ij} = \sum_{k=1}^{n} Y_k. \tag{10}$$

Let $\underline{\Sigma}$ be the variance-covariance matrix, which represents the variances and covariances between the different individual losses Y_i. Hence, $\underline{\Sigma}$ is an $n \times n$ matrix, with element in the k-th row and l-the column given by

$$\left(\underline{\Sigma}\right)_{kl} = cov\left[Y_k, Y_l\right] = E\left[\left(Y_k - E\left[Y_k\right]\right)\left(Y_l - E\left[Y_l\right]\right)\right]. \tag{11}$$

We also define the $n \times n$ correlation matrix $\underline{\Lambda}$, of which the elements are given by

$$\left(\underline{\Lambda}\right)_{kl} = r\left[Y_k, Y_l\right], \tag{12}$$

where $r\left[Y_k, Y_l\right]$ stands for Pearson's correlation coefficient between Y_k and Y_l :

$$r\left[Y_k, Y_l\right] = \frac{cov\left[Y_k, Y_l\right]}{\sigma_k \ \sigma_l}, \tag{13}$$

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[1] An exception is the ALM economic capital framework. In this case, a collective approach is used.

with σ_k being the standard deviation of Y_k:

$$\sigma_k = \sqrt{cov\,[Y_k, Y_k]}. \tag{14}$$

In order to compute the aggregate capital requirement, one assumes that the stand-alone losses Y_k (per risk type and business unit) are multivariate normal distributed:

Assumption: $(Y_1, Y_2, \ldots, Y_n)$ is multivariate normal distributed. $\qquad$ (15)

Notice that under this assumption, the procedure explained in Subsection 3.1 (summing the individual capitals) corresponds with choosing the correlation matrix $\underline{\Lambda}$ such that $(\underline{\Lambda})_{kl} = 1$, for all k and l.

The economic capital EC_k of individual loss Y_k is defined as

$$EC_k = F_{Y_k}^{-1}(1 - p) - \mu_k, \tag{16}$$

with

$$\mu_k = E\,[Y_k]\,,$$

see also formula (5).

The assumption of normality implies that

$$\begin{aligned} F_{Y_k}^{-1}(1 - p) &= \mu_k + \sigma_k\,\Phi^{-1}(1 - p) \\ &= \mu_k + \Phi^{-1}(1 - p)\,\sigma_k. \end{aligned} \tag{17}$$

Hence, from (16) and (17), we find

$$EC_k = \Phi^{-1}(1 - p)\,\sigma_k. \tag{18}$$

The assumption of multivariate normality of the random vector $(Y_1, Y_2, \ldots, Y_n)$ can be expressed as follows:

$$\sum_{k=1}^{n} \alpha_k Y_k \text{ is normal distributed, for any set of real numbers } \alpha_1, \alpha_2, \ldots, \alpha_n. \tag{19}$$

Notice that the mean and standard deviation of $\sum_{k=1}^{n} \alpha_k Y_k$ are given by

$$E\left[\sum_{k=1}^{n} \alpha_k Y_k\right] = \sum_{k=1}^{n} \alpha_k\,\mu_k \tag{20}$$

and

$$\sigma \left[\sum_{k=1}^{n} \alpha_k Y_k \right] = \sqrt{\underline{\alpha}^T \cdot \underline{\Sigma} \cdot \underline{\alpha}}, \tag{21}$$

respectively, where the superscript 'T' stands for the transposition operation. Thus

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{pmatrix}, \tag{22}$$

whereas

$$\underline{\alpha}^T = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}. \tag{23}$$

More in particular, the multivariate normal assumption implies that the aggregate loss $S = \sum_{k=1}^{n} Y_k$ is normal distributed with mean and standard deviation given by

$$E[S] = \sum_{k=1}^{n} \mu_k \tag{24}$$

and

$$\sigma_S = \sqrt{\underline{1}^T \cdot \underline{\Sigma} \cdot \underline{1}}, \tag{25}$$

respectively, where $\underline{1}$ is the unity vector:

$$\underline{1}^T = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}. \tag{26}$$

The aggregate economic capital EC is now defined as

$$EC = F_S^{-1}(1 - p) - E[S]. \tag{27}$$

Taking into account the previous results, we find the following expression for the aggregate economic capital in terms of the variance-covariance matrix Σ:

$$\begin{aligned} EC &= \Phi^{-1}(1 - p)\, \sigma_S \\ &= \Phi^{-1}(1 - p)\, \sqrt{\underline{1}^T \cdot \underline{\Sigma} \cdot \underline{1}}. \end{aligned} \tag{28}$$

Defining the vector $\underline{EC}$ by

$$\underline{EC} = (EC_1, EC_2, \dots, EC_n), \tag{29}$$

we find

$$\underline{1}^T \cdot \underline{\Sigma} \cdot \underline{1} = \sum_{k=1}^{n}\sum_{l=1}^{n} cov\,[Y_k, Y_l] = \sum_{k=1}^{n}\sum_{l=1}^{n} \sigma_k\,\sigma_l\,corr\,[Y_k, Y_l] \qquad (30)$$

$$= \frac{1}{[\Phi^{-1}(1-p)]^2}\sum_{k=1}^{n}\sum_{l=1}^{n} EC_k\,EC_l\,corr\,[Y_k, Y_l]$$

$$= \frac{1}{[\Phi^{-1}(1-p)]^2}\underline{EC}^T \cdot \underline{\Lambda} \cdot \underline{EC}.$$

Hence, the aggregate economic capital EC can be written in terms of the correlation matrix as

$$EC = \sqrt{\underline{EC}^T \cdot \underline{\Lambda} \cdot \underline{EC}}. \qquad (31)$$

As a special case, in addition to the multivariate normality, assume that all losses Y_k are comonotonic, then formula (31) reduces to

$$EC = \sum_{k=1}^{n} EC_k. \qquad (32)$$

To conclude, the standard aggregation methodology computea the aggregate Economic Capital of $S = \sum_{k=1}^{n} Y_k$ by (31). Notice that:

1. The random vector $(Y_1, Y_2, \ldots, Y_n)$ is assumed to be multivariate normally distributed.

2. The individual economic capitals EC_k are defined through $EC_k = F_{Y_k}^{-1}(1-p) - \mu_k$,

3. Pearson's correlation matrix $\underline{\Lambda}$ describes the correlation between the individual risks, see (13).

In fact, formula (31) shows that in the multivariate normal case, the (1-p)-percentile of the sum can be computed from the correlation matrix and the (1-p)-percentiles of the marginals involved. It follows immediately that this result holds for any percentile.

5 An alternative aggregation methodology based on the concept of comonotonicity

5.1 Introduction

The standard methodology assumes that the set of risks one faces is multivariate normally distributed. In this section we develop another methodology, for the case that this assumption is not satisfied. Indeed, the assumption of a (symmetric) normal distribution in case of credit risk for instance is hard to defend. This is because the different credit risks are subject to the same economical environment which makes these risks principlly positive dependent turning the distribution function skewed. A lognormal assumption would already mean a major improvement here. Therefore, we will describe a methodology that is appropriate in case the individual risks that are involved are either normal or lognormal distributed.

A random vector $(Y_1, Y_2, \ldots, Y_n)$ has the multivariate normal distribution if and only if every linear combination of its variates has a univariate normal distribution. Now assume that $(Y_1, Y_2, \ldots, Y_n)$ has a multivariate normal distribution, with expectations $E[Y_k] = \mu_k$, variances $var[Y_k] = \sigma_k^2$ and covariance coefficients $cov[Y_k, Y_l]$. Let Y and Λ be linear combinations of the variates Y_i. Hence,

$$Y = \sum_{i=1}^{n} \alpha_i Y_i \tag{33}$$

and

$$\Lambda = \sum_{i=1}^{n} \beta_i Y_i. \tag{34}$$

Then also (Y, Λ) has a bivariate normal distribution.
Further, if (Y, Λ) has a bivariate normal distribution, then, conditionally given $\Lambda = \lambda$, Y has a univariate normal distribution with mean and variance given by

$$E[Y \mid \Lambda = \lambda] = E[Y] + r(Y, \Lambda) \frac{\sigma_Y}{\sigma_\Lambda} (\lambda - E[\Lambda]) \tag{35}$$

and

$$Var[Y \mid \Lambda = \lambda] = \sigma_Y^2 \left(1 - r(Y, \Lambda)^2\right), \tag{36}$$

where $r(Y, \Lambda)$ is Pearson's correlation coefficient for the couple (Y, Λ).

For a normal random variable Y with expectation μ and variance σ^2, the following expressions hold for any $p \in (0, 1)$:

$$
\begin{aligned}
F_Y^{-1}(p) &= \mu + \sigma \, \Phi^{-1}(p), & (37) \\
F_{e^Y}^{-1}(p) &= e^{\mu + \sigma \, \Phi^{-1}(p)}, \\
TVaR_p[Y] &= \mu + \sigma \frac{\phi\left(\Phi^{-1}(p)\right)}{1 - p}, \\
TVaR_p\left[e^Y\right] &= e^{\mu + \sigma^2/2} \frac{\Phi\left(\sigma - \Phi^{-1}(p)\right)}{1 - p},
\end{aligned}
$$

where as before, Φ stands for the standard normal distribution and ϕ is its derivative, see e.g. Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004).

Essentially, the standard method describes a way how to determine quantiles of the sum S of the components of a multivariate normal random vector $(Y_1, Y_2, \ldots, Y_n)$.

In this section, we will consider the case where not all components are normal, but where some have a lognormal distribution. We will propose an approximate (but accurate and analytical) method to determine quantiles of such a sum.

To be more precise, consider a random variable S of the form

$$
S = \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{Y_k}, \tag{38}
$$

where $(Y_1, Y_2, ..., Y_n)$ is a multivariate normal distributed random vector. Hence, we assume that the aggregate risk is composed of dependent stand-alone risks which may be either normal or lognormal distributed.

As before, we introduce the following notations:

$$
\begin{aligned}
\mu_k &= E[Y_k], \; \mu_S = E[S], & (39) \\
\sigma_k^2 &= Var[Y_k], \; \sigma_S^2 = Var[S].
\end{aligned}
$$

We will assume that the multivariate normal distribution of $(Y_1, Y_2, ..., Y_n)$ is known, which means that the expectations and variances defined above are known, as well as the following correlations:

$$
r_{k,l} = r[Y_k, Y_l]. \tag{40}
$$

The r.v. S defined in (38) will in general be a sum of non-independent normal and lognormal r.v.'s. Its distribution function cannot be determined analytically and is in general too cumbersome to work with. Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a) derive *comonotonic upper bound and lower bound approximations* (in the convex order sense) for the distribution of S. Especially the lower bound approximation, which is given by $E[S \mid \Lambda]$ for an appropriate choice of the conditioning r.v. Λ performs extremely accurate, see for instance Vanduffel, Dhaene, Goovaerts & Kaas (2003) or Vanduffel, Hoedemakers & Dhaene (2004).

5.2   Comonotonic approximations

A central concept in the theory on comonotonic r.v.'s is the concept of convex order. A r.v. X is said to precede a r.v. Y in the convex order sense, notation $X \leq_{cx} Y$, if the means of both r.v.'q are equal and if their corresponding stop-loss premia are ordered uniformly for all retentions d, i.e. $E[(X-d)_+] \leq E[(Y-d)_+]$, for all d.

Replacing the copula describing the dependency structure of the terms in the sum (38) by the comonotonic copula yields an convex order upper bound for S. On the other hand, applying Jensen's inequality to S provides us with a lower bound.

These results are formalized in the following theorem, which can be proven using the techniques explained in Kaas, Dhaene & Goovaerts (2000).

Theorem 1 *Let the random variable S be given by (38), where we make no assumption concerning the distribution function of the random vector $(Y_1, Y_2, ..., Y_n)$. Consider the conditioning random variable Λ, given by*

$$\Lambda = \sum_{j=1}^{n} \beta_j Y_j. \tag{41}$$

Also consider random variables S^l and S^c defined by

$$S^l = \sum_{k=1}^{m} E\left[Y_k \mid \Lambda\right] + \sum_{k=m+1}^{n} E\left[e^{Y_k} \mid \Lambda\right] \tag{42}$$

and

$$S^c = \sum_{k=1}^{m} F_{Y_k}^{-1}(U) + \sum_{k=m+1}^{n} e^{F_{Y_k}^{-1}(U)}, \tag{43}$$

13

respectively. Here U is a Uniform$(0, 1)$ r.v. and Φ is the cumulative d.f. of the $N(0, 1)$ distribution. For the random variables S, S^l and S^c, the following convex order relations hold:

$$S^l \leq_{cx} S \leq_{cx} S^c. \tag{44}$$

The theorem states that (the distribution function of) S^l is a convex order lower bound for (the distribution function of) S, whereas (the distribution function of) S^c is a convex order upper bound for (the distribution function of) S.

The upper bound S^c is obtained by replacing the original copula between the marginals of the sum S by the comonotonic copula, but keeping the marginal distributions unchanged.

It is easy to see that the d.f. of the lower bound S^l corresponds with the distribution function of $E[S \mid \Lambda]$. The lower bound is obtained by changing both the copula and the marginals of the original sum. Intuitively, one can expect that an appropriate choice of the conditioning variable Λ will lead to much better approximations than the the upper bound approximations. This is because the conditioning technique introduces information about the dependency structure in the exact sum S. The approximation S^c on the other hand, only uses the marginals of the exact sum.

Notice that the result of the theorem above holds without making any assumption concerning the random vector $(Y_1, Y_2, ..., Y_n)$. The multivariate normal case is considered in the following theorem.

Theorem 2 *Let the random variable S be given by (38), where the random vector $(Y_1, Y_2, ..., Y_n)$ has a multivariate normal distribution. Then the (distribution function of the) random variables S^l and S^c defined in the previous theorem follow from*

$$S^l \stackrel{d}{=} \sum_{k=1}^{m} \left(\mu_k + r_k \sigma_k \Phi^{-1}(U) \right) + \sum_{k=m+1}^{n} e^{\mu_k + r_k \sigma_k \Phi^{-1}(U) + \frac{1}{2}\sigma_k^2(1-r_k^2)} \tag{45}$$

and

$$S^c \stackrel{d}{=} \sum_{k=1}^{m} \left(\mu_k + \sigma_k \Phi^{-1}(U) \right) + \sum_{k=m+1}^{n} e^{\mu_k + \sigma_k \Phi^{-1}(U)}, \tag{46}$$

with

$$r_k = r\left[Y_k, \Lambda\right], \tag{47}$$

14

and where U is a random variable which is uniformly distributed over the unit interval $(0, 1)$.

Proof. (a) Conditionally, given $\Lambda = \lambda$, the random variable Y_k is normally distributed with parameters

$$E[Y_k \mid \Lambda = \lambda] = \mu_k + r_k \frac{\sigma_k}{\sigma_\Lambda} (\lambda - E[\Lambda])$$

and

$$Var[Y_k \mid \Lambda = \lambda] = \sigma_k^2 \left(1 - r_k^2\right).$$

Defining the random variable U by

$$\Phi^{-1}(U) = \frac{\Lambda - E[\Lambda]}{\sigma_\Lambda},$$

we find that

$$E[Y_k \mid \Lambda] = \mu_k + r_k \sigma_k \Phi^{-1}(U)$$

and

$$
\begin{aligned}
E\left[e^{Y_k} \mid \Lambda\right] &= e^{E[Y_k \mid \Lambda] + \frac{1}{2} Var[Y_k \mid \Lambda]} \\
&= e^{\mu_k + r_k \sigma_k \Phi^{-1}(U) + \frac{1}{2} \sigma_k^2 (1 - r_k^2)}.
\end{aligned}
$$

The expression (45) follows then from the previous theorem.

(b) We have that

$$F_{Y_k}^{-1}(p) = \mu_k + \sigma_k \Phi^{-1}(p)$$

and

$$F_{e^{Y_k}}^{-1}(p) = e^{\mu_k + \sigma_k \Phi^{-1}(p)}.$$

Inserting these expressions in (43), we find (46). ∎

We have that the coefficients r_k are given by

$$r_k = \frac{1}{\sigma_k \sigma_\Lambda} \sum_{j=1}^n \beta_j \, cov[Y_k, Y_j], \tag{48}$$

with

$$\sigma_\Lambda^2 = \sum_{j=1}^n \sum_{k=1}^n \beta_j \beta_k cov[Y_k, Y_j]. \tag{49}$$

15

5.3 Determining approximations for $\mathbf{F}_S^{-1}(\mathbf{p})$ and $\mathbf{TVaR}_p[S]$

Notice that S^c is a comonotonic sum. As the quantiles and tailvars are additive for comonotonic risks, we find from (37) that for $p \in (0, 1)$, the following expressions hold:

$$F_{S^c}^{-1}(p) = \sum_{k=1}^{m} \left(\mu_k + \sigma_k \Phi^{-1}(p) \right) + \sum_{k=m+1}^{n} e^{\mu_k + \sigma_k \Phi^{-1}(p)}, \tag{50}$$

$$TVaR_p[S^c] = \sum_{k=1}^{m} \left(\mu_k + \sigma_k \frac{\phi\left(\Phi^{-1}(p)\right)}{1-p} \right) + \sum_{k=m+1}^{n} e^{\mu_k + \sigma_k^2/2} \frac{\Phi\left(\sigma_k - \Phi^{-1}(p)\right)}{1-p}. \tag{51}$$

Provided all coefficients r_i are positive, the terms in S^l are all non-decreasing functions of the same r.v. U. Hence, S^l will also be a comonotonic sum in this case. This implies that $F_{S^l}^{-1}(p)$ and $TVaR_p[S^l]$ can again be computed by summing the corresponding risk measures for the (normal and lognormal) marginals involved. Hence, assuming that all r_i are positive, we find the following expressions for the quantiles and TVaRs of S^l, for any $p \in (0, 1)$:

$$F_{S^l}^{-1}(p) = \sum_{k=1}^{m} \left(\mu_k + r_k \sigma_k \Phi^{-1}(p) \right) \tag{52}$$
$$+ \sum_{k=m+1}^{n} e^{\mu_k + r_k \sigma_k \Phi^{-1}(p) + \frac{1}{2}\sigma_k^2(1-r_k^2)},$$

$$TVaR_p[S^l] = \sum_{k=1}^{m} \left(\mu_k + r_k \sigma_k \frac{\phi\left(\Phi^{-1}(p)\right)}{1-p} \right) \tag{53}$$
$$+ \sum_{k=m+1}^{n} e^{\mu_k + \sigma_k^2/2} \frac{\Phi\left(r_k \sigma_k\right) - \Phi^{-1}(p)}{1-p}.$$

Notice that

$$TVaR_p[S^l] \le TVaR_p[S] \le TVaR_p[S^c] \tag{54}$$

always holds. The same ordering can not be proven between the quantiles. However, extensive numerical comparisons reveal that the same ordering will hold, provided p is large enough, for instance $p = 99.97\%$.

Let us now consider the general case where not all r_i are positive. In this case the lower bound is not a sum of comonotonic random variables, making the determination of the distribution function of the lower bound more complicated. The cdf of S^l then follows from

$$
\begin{aligned}
F_{S^l}(x) &= \int_0^1 \Pr\left[\sum_{k=1}^m \left(\mu_k + r_k\sigma_k\Phi^{-1}(U)\right) + \sum_{k=m+1}^n e^{\mu_k+r_k\sigma_k\Phi^{-1}(U)+\frac{1}{2}\sigma_k^2(1-r_k^2)} \leq x \mid U = u\right] du \\
&= \int_0^1 I\left(\sum_{k=1}^m \left(\mu_k + r_k\sigma_k\Phi^{-1}(u)\right) + \sum_{k=m+1}^n e^{\mu_k+r_k\sigma_k\Phi^{-1}(u)+\frac{1}{2}\sigma_k^2(1-r_k^2)} \leq x\right) du \quad (55)
\end{aligned}
$$

5.4 Moments of S, S^c and S^l

The expected values of the random variables S, S^c and S^l are all equal :

$$
E[S] = E[S^l] = E[S^c] = \sum_{k=1}^m \mu_k + \sum_{k=m+1}^n e^{\mu_k+\frac{1}{2}\sigma_k^2}. \quad (56)
$$

In order to determine the variances of S, S^c and S^l, we introduce the following notations:

$$
\begin{aligned}
S^c &= \sum_{k=1}^m \left(\mu_k + \sigma_k\Phi^{-1}(U)\right) + \sum_{k=m+1}^n e^{\mu_k+\sigma_k\Phi^{-1}(U)} \quad (57) \\
&= \sum_{k=1}^m Y_k^c + \sum_{k=m+1}^n e^{Y_k^c},
\end{aligned}
$$

and

$$
\begin{aligned}
S^l &= \sum_{k=1}^m \left(\mu_k + r_k\sigma_k\Phi^{-1}(U)\right) + \sum_{k=m+1}^n e^{\mu_k+r_k\sigma_k\Phi^{-1}(U)+\frac{1}{2}\sigma_k^2(1-r_k^2)} \quad (58) \\
&= \sum_{k=1}^m Y_k^l + \sum_{k=m+1}^n e^{Y_k^l}.
\end{aligned}
$$

Notice that any of the random vectors $(Y_1, Y_2, ..., Y_n)$, $(Y_1^c, Y_2^c, ..., Y_n^c)$ and $(Y_1^l, Y_2^l, ..., Y_n^l)$ is multivariate normal.

17

The variances of S, S^c and S^l follow from

$$Var\left[S\right] = \sum_{k=1}^{m}\sum_{l=1}^{m} cov\left[Y_k, Y_l\right] + \sum_{k=1}^{m}\sum_{l=m+1}^{n} cov\left[Y_k, e^{Y_l}\right] \tag{59}$$
$$+ \sum_{k=m+1}^{n}\sum_{l=1}^{m} cov\left[e^{Y_k}, Y_l\right] + \sum_{k=m+1}^{n}\sum_{l=m+1}^{n} cov\left[e^{Y_k}, e^{Y_l}\right],$$

$$Var\left[S^c\right] = \sum_{k=1}^{m}\sum_{l=1}^{m} cov\left[Y_k^c, Y_l^c\right] + \sum_{k=1}^{m}\sum_{l=m+1}^{n} cov\left[Y_k^c, e^{Y_l^c}\right] \tag{60}$$
$$+ \sum_{k=m+1}^{n}\sum_{l=1}^{m} cov\left[e^{Y_k^c}, Y_l^c\right] + \sum_{k=m+1}^{n}\sum_{l=m+1}^{n} cov\left[e^{Y_k^c}, e^{Y_l^c}\right]$$

and

$$Var\left[S^l\right] = \sum_{k=1}^{m}\sum_{l=1}^{m} cov\left[Y_k^l, Y_l^l\right] + \sum_{k=1}^{m}\sum_{l=m+1}^{n} cov\left[Y_k^l, e^{Y_l^l}\right] \tag{61}$$
$$+ \sum_{k=m+1}^{n}\sum_{l=1}^{m} cov\left[e^{Y_k^l}, Y_l^l\right] + \sum_{k=m+1}^{n}\sum_{l=m+1}^{n} cov\left[e^{Y_k^l}, e^{Y_l^l}\right]$$

In order to be able to determine the different covariances in formulae (59), (60) and (61), we prove the following theorem.

Theorem 3 *(a) Let (X, Y) be bivariate normal distributed and let Z be standard normal distributed and independent of (X, Y), we have that*

$$(X, Y) \stackrel{d}{=} \left(X, \ \mu_Y + \sigma_Y \ r(X, Y)\left(\frac{X - \mu_X}{\sigma_X}\right) + Z \ \sigma_Y \ \sqrt{1 - r(X, Y)^2}\right). \tag{62}$$

(b)
$$cov\left[e^X, e^Y\right] = e^{\mu_X + \mu_Y + \frac{1}{2}\left(\sigma_X^2 + \sigma_Y^2\right)}\left(e^{cov[X,Y]} - 1\right). \tag{63}$$

(c)
$$cov\left[X, e^Y\right] = cov\left[X, Y\right] \ e^{\mu_Y + \frac{1}{2}\sigma_Y^2}. \tag{64}$$

18

Proof. (a) It is easy to verify that both random couples in (62) are bivariate normal, and also that they have the same marginal distributions and the same covariance, which proves the equality in (62).

(b) The relation (63) follows from a straightforward calculation.

(c) From (62), we find

$$
\begin{aligned}
cov\left[X, e^Y\right] &= cov\left[X - \mu_X,\ e^{\mu_Y + \sigma_Y\ r[X,Y]\left(\frac{X-\mu_X}{\sigma_X}\right) + \sigma_Y\ \sqrt{1-r[X,Y]^2}Z}\right] \\
&= E\left[(X - \mu_X)\, e^{\mu_Y + \sigma_Y\ r[X,Y]\left(\frac{X-\mu_X}{\sigma_X}\right) + \sigma_Y\ \sqrt{1-r[X,Y]^2}Z}\right] \\
&= E\left[(X - \mu_X)\, e^{\mu_Y + \sigma_Y\ r[X,Y]\left(\frac{X-\mu_X}{\sigma_X}\right)}\right]\ E\left[e^{\sigma_Y\ \sqrt{1-r[X,Y]^2}Z}\right] \\
&= \sigma_X\ e^{\mu_Y}\ E\left[Z e^{\sigma_Y\ r[X,Y]Z}\right]\ e^{\frac{1}{2}\sigma_Y^2\ (1-r[X,Y]^2)} \\
&= \frac{\sigma_X}{\sigma_Y} e^{\mu_Y}\ \frac{\partial}{\partial r\,(X,Y)} E\left[e^{\sigma_Y\ r[X,Y]\ Z}\right]\ e^{\frac{1}{2}\sigma_Y^2\ (1-r[X,Y]^2)} \\
&= cov\,(X,Y)\ e^{\mu_Y + \frac{1}{2}\sigma_Y^2},
\end{aligned}
$$

which proves (64). ■

From the results above, we find that

$$
\begin{aligned}
cov\left[Y_k^c, Y_l^c\right] &= \sigma_k \sigma_l, \\
cov\left[Y_k^l, Y_l^l\right] &= r_k r_l \sigma_k \sigma_l,
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
cov\left[Y_k, e^{Y_l}\right] &= cov\left[Y_k, Y_l\right]\ e^{\mu_l + \frac{1}{2}\sigma_l^2}, \\
cov\left[Y_k^c, e^{Y_l^c}\right] &= \sigma_k \sigma_l\ e^{\mu_l + \frac{1}{2}\sigma_l^2}, \\
cov\left[Y_k^l, e^{Y_l^l}\right] &= r_k r_l \sigma_k \sigma_l\ e^{\mu_l + \frac{1}{2}r_l^2\sigma_l^2}
\end{aligned}
\tag{66}
$$

and also

$$
\begin{aligned}
cov\left[e^{Y_k}, e^{Y_l}\right] &= e^{\mu_k + \mu_l + \frac{1}{2}\left(\sigma_k^2 + \sigma_l^2\right)}\left(e^{cov[Y_k, Y_l]} - 1\right), \\
cov\left[e^{Y_k^c}, e^{Y_l^c}\right] &= e^{\mu_k + \mu_l + \frac{1}{2}\left(\sigma_k^2 + \sigma_l^2\right)}\left(e^{\sigma_k \sigma_l} - 1\right), \\
cov\left[e^{Y_k^l}, e^{Y_l^l}\right] &= e^{\mu_k + \mu_l + \frac{1}{2}\left(r_k^2\sigma_k^2 + r_l^2\sigma_l^2\right)}\left(e^{r_k r_l \sigma_k \sigma_l} - 1\right).
\end{aligned}
\tag{67}
$$

Summarizing, we considered a multivariate normal random vector $(Y_1, Y_2, ..., Y_n)$, with given covariances $cov\,[Y_k, Y_l]$ for all couples involved.

19

The variance of the random variable S defined in (38) follows then from (59), (66) and (67).

The variance of S^c given by (57) follows from (60), (65), (66) and (67).

Finally, consider the random variable S^l given by (45), with conditioning variable Λ given by (41). The coefficients r_k follow then from (48).

The variance of S^l follows from (61), (65), (66) and (67).

5.5   On the choice of the conditioning variable Λ

Since $\mathrm{Var}[S] = \mathrm{Var}\big[S^l\big] + \mathrm{E}[Var\,[S \mid \Lambda]]$, it follows that $\mathrm{Var}\big[S^l\big] \leq \mathrm{Var}[S]$ and also that an equality can only occur in case $S^l \overset{d}{=} S$. Hence the best choice for the conditioning r.v. Λ is likely to occur when the variance of S^l is maximized. Theoretically, one could use numerical procedures to determine the optimal Λ, but this would outweigh one of the main features of the convex bounds, namely that the quantiles and conditional tail expectations (and also other actuarial quantities such as stop-loss premiums) can be easily determined analytically. Having a ready-to-use approximation that can be easily implemented is important from a practical point of view. Since maximization of $\mathrm{Var}(S^l)$ is equivalent with minimization of $\mathrm{E}(Var\,[S \mid \Lambda])$ we expect to have a good choice for Λ in case it is 'close' to S. Therefore, we propose to determine the lower bound with a conditioning variable Λ, which is a first order approximation for S.

We have that

$$
\begin{aligned}
S \;&=\; \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{Y_k} = \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{\mu_k + (Y_k - \mu_k)} \\
&\approx\; \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{\mu_k}\big[1 + (Y_k - \mu_k)\big] \\
&=\; C + \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{\mu_k}\, Y_k
\end{aligned}
\tag{68}
$$

for some appropriate constant C.

Hence, we propose to chose the conditioning variable Λ as (a linear transformation of) a first order approximation to S:

$$
\Lambda = \sum_{k=1}^{n} \beta_k Y_k.
\tag{69}
$$

with

$$\begin{aligned} \beta_k &= 1, \quad k = 1, \cdots, m, \\ \beta_k &= e^{\mu_k}, \quad k = m+1, \cdots, n. \end{aligned} \tag{70}$$

Consider now the special case that all covariances $cov\left[Y_k^l, Y_l^l\right]$ are positive. Then it follows from (48) and (70) that all correlation coefficients r_k are also positive. Hence, in this case the lower bound S^l as defined in (45) is a comonotonic sum. This implies that the quantiles and tailvar's of S^l can be determined from (52) and (53), respectively.

In the special case that all risks involved are normally distributed, the condition variable Λ defined in (69) equals S. Hence, the lower bound S^l coincides with the exact aggregate claims S. In this case, the aggregation method explained in Section 4 can be applied to obtain exact results.

5.6 Determining approximations for $\mathbf{EC}[S]$

Consider a multivariate normal random vector $(Y_1, Y_2, \ldots, Y_n)$, with given expectations $E\left[Y_k\right] = \mu_k$, variances $var\left[Y_k\right] = \sigma_k^2$ and covariance coefficients $cov\left[Y_k, Y_l\right]$.
Furthermore, assume that the first m stand-alone risks ($m \leq n$) are given by $(Y_1, Y_2, \ldots, Y_m)$, whereas the last $n - m$ stand-alone risks are given by $\left(e^{Y_{m+1}}, e^{Y_{m+2}}, \ldots, e^{Y_n}\right)$.
The economic capital EC for the aggregate loss $S = \sum_{k=1}^m Y_k + \sum_{k=m+1}^n e^{Y_k}$ is defined in (1):
$$EC = F_S^{-1}(1-p) - \mu_S. \tag{71}$$

We propose to approximate $F_S^{-1}(1-p)$ by $F_{S^l}^{-1}(1-p)$ or $F_{S^c}^{-1}(1-p)$, with S^l and S^c as defined in (57) and (58), respectively.
The coefficients r_k are defined by (48):

$$r_k = \frac{1}{\sigma_k \sigma_\Lambda} \sum_{j=1}^n \beta_j \, cov\left[Y_k, Y_j\right],$$

with σ_Λ^2 determined from (49):

$$\sigma_\Lambda^2 = \sum_{j=1}^n \sum_{k=1}^n \beta_j \beta_k cov\left[Y_k, Y_j\right].$$

and the β_j given by (70):

$$
\begin{aligned}
\beta_k &= 1, \quad k = 1, \cdots, m, \\
\beta_k &= e^{\mu_k}, \quad k = m+1, \cdots, n.
\end{aligned}
$$

Quantiles and Tailvars of the comonotonic sum S^c follow immediately from (50) and (51).

Notice that in the standard approach, both S^l and S^c will be comonotonic sums. Indeed, for S^c this is a trivial statement whereas for S^l this will be the case because all covariances $cov\,[Y_k, Y_j]$ and hence also all correlations r_k are positive.

We remark also that even if some of the $cov\,[Y_k, Y_j]$ are negative, it will still be possible to make a choice for the β_j such that S^l will be a comonotonic sum, see e.g. Vanduffel, Dhaene & Goovaerts (2004).

We propose to approximate the aggregate economic capital EC by EC^l:

$$
EC^l = F_{S^l}^{-1}(1-p) - \mu_S \tag{72}
$$

with $F_{S^l}^{-1}(1-p)$ determined by (52), provided all r_k are positive, and with μ_S determined by (56).

As a second approximation for the aggregate economic capital, we propose EC^c, which is defined by

$$
EC^c = F_{S^c}^{-1}(1-p) - \mu_S, \tag{73}
$$

with $F_{S^c}^{-1}(1-p)$ given by (50).

6 A generalisation of the standard approach: determining the economic capital by a distortion risk measure in a more general multivariate setting

6.1 Introduction

In this section, we will start from the standard methodology. We will develop now some generalizations of this methodology. These generalizations allow to

adapt the current methodology which is based on multivariate normality and a quantile-based capital requirement, to multivariate elliptical distributions and a capital requirement based on any particular distortion risk measure.

6.2 Distortion risk measures

In this section we will consider the class of distortion risk measures, introduced in the actuarial literature by Wang (1996). The quantile risk measure and TVaR belong to this class.

The expectation of a random variable X, if it exists, can be written as

$$\mathrm{E}\left[X\right] = -\int_{-\infty}^{0}\left[1 - \overline{F}_X(x)\right]\mathrm{d}x + \int_{0}^{\infty}\overline{F}_X(x)\mathrm{d}x, \tag{74}$$

where $\overline{F}_X(x) = \Pr\left[X > x\right]$.

Wang (1996) defines a family of risk measures by using the concept of distortion function as introduced in Yaari's dual theory of choice under risk. A distortion function is defined as a non-decreasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$. The distortion risk measure associated with distortion function g is denoted by $\rho_g\left[\cdot\right]$ and is defined by

$$\rho_g\left[X\right] = -\int_{-\infty}^{0}\left[1 - g\left(\overline{F}_X(x)\right)\right]\mathrm{d}x + \int_{0}^{\infty}g\left(\overline{F}_X(x)\right)\mathrm{d}x, \tag{75}$$

for any random variable X with finite mean. Note that the distortion function g is assumed to be independent of the distribution function of the random variable X. The distortion function $g(q) = q$ corresponds to $\mathrm{E}[X]$. Note that if $g(q) \geq q$ for all $q \in [0, 1]$, then $\rho_g\left[X\right] \geq \mathrm{E}[X]$. In particular this result holds in case g is a concave distortion function. Also note that $g_1(q) \leq g_2(q)$ for all $q \in [0, 1]$ implies that $\rho_{g_1}\left[X\right] \leq \rho_{g_2}\left[X\right]$.

Substituting $g\left(\overline{F}_X(x)\right)$ by $\int_0^{\overline{F}_X(x)}\mathrm{d}g(q)$ in (75) and reverting the order of the integrations, one finds that any distortion risk measure $\rho_g\left[X\right]$ can be written as

$$\rho_g\left[X\right] = \int_{0}^{1}F_X^{-1}(q)[X]\mathrm{d}g(q). \tag{76}$$

From (76), one can easily verify that the quantile $F_X^{-1}(p)$, $p \in (0, 1)$, corresponds to the distortion function

$$g(x) = I_{(x>1-p)}, \qquad 0 \leq x \leq 1. \tag{77}$$

On the other hand, $\text{TVaR}_p[X] = \frac{1}{1-p} \int_p^1 F_X^{-1}(q) \, dq, \ p \in (0,1)$, corresponds to the distortion function

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), \qquad 0 \le x \le 1. \tag{78}$$

Notice that in case X has a continuous distribution (such as the normal distribution e.g.), the Tail Value-at-Risk can also be expressed as

$$TVaR_p[X] = E\left[X \mid X > F_X^{-1}(p)\right]. \tag{79}$$

An extensive overview of distortion risk measures, and its applications in a solvency context is Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004).

6.3   Generalizing the standard methodology

Let us now assume, as in the standard methodology, that the stand-alone losses Y_k (per risk type and business unit) all belong to the same family of translation-scale invariant distributions. This means that there exists a random variable Z such that for each Y_k, we have that

$$Y_k \overset{d}{=} \mu_k + \sigma_k \, Z, \tag{80}$$

where $\overset{d}{=}$ stands for 'equality in distribution'. Furthermore, we assume that the aggregate loss S belongs to the same class of translation-scale invariant distributions:

$$S \overset{d}{=} \mu_S + \sigma_S \, Z. \tag{81}$$

An important special case is the case where $(Y_1, Y_2, \ldots, Y_n)$ has a multivariate elliptical distribution, see for instance Valdez & Dhaene (2003). Notice that the multivariate normal distribution, is one particular element of the class of multivariate elliptical distributions.

The outcome of a distortion risk measure only depends on the distribution function of the underlying random variable. Hence, from the translation invariance and the positive homogeneity property of distortion risk measures, we find for the stand-alone losses Y_k and the aggregate loss S that

$$\begin{aligned}
\rho_g[Y_k] &= \rho_g[\mu_k + \sigma_k \, Z] \\
&= \mu_k + \sigma_k \, \rho_g[Z]
\end{aligned} \tag{82}$$

and

$$\rho_g\left[S\right] = \mu_S + \sigma_S\,\rho_g\left[Z\right]. \tag{83}$$

We have that

$$\begin{aligned}
\sigma_S^2 &= \underline{1}^T \cdot \underline{\Sigma} \cdot \underline{1} \tag{84}\\
&= \sum_{k=1}^n \sum_{l=1}^n cov\left[Y_k, Y_l\right] = \sum_{k=1}^n \sum_{l=1}^n \sigma_k\ r\left[Y_k, Y_l\right]\ \sigma_l\\
&= \left(\frac{1}{\rho_g\left[Z\right]}\right)^2 \sum_{k=1}^n \sum_{l=1}^n \left(\rho_g\left[Y_k\right] - \mu_k\right)\ r\left[Y_k, Y_l\right]\ \left(\rho_g\left[Y_l\right] - \mu_l\right)\ .
\end{aligned}$$

Hence,

$$\rho_g\left[S\right] = \mu_S + \sqrt{\sum_{k=1}^n \sum_{l=1}^n \left(\rho_g\left[Y_k\right] - \mu_k\right)\ r\left[Y_k, Y_l\right]\ \left(\rho_g\left[Y_l\right] - \mu_l\right)}. \tag{85}$$

Instead of determining the economic capital EC for the aggregate loss S by (27), we now assume that it is determined by

$$EC = \rho_g\left[S\right] - \mu_S \tag{86}$$

for some distortion function g. Similarly, we define the stand-alone economic capitals EC_k by

$$EC_k = \rho_g\left[Y_k\right] - \mu_k. \tag{87}$$

Defining the vector $\underline{EC}$ as in (29), we find from (85) that the formula (31) remains to hold.

For convenience, we introduce the following vector notations:

$$\rho_g\left[\underline{Y}\right]^T = \left(\rho_g\left[Y_1\right], \rho_g\left[Y_2\right], \ldots, \rho_g\left[Y_n\right]\right) \tag{88}$$

and

$$\underline{\mu}^T = \left(\mu_1, \mu_2, \ldots, \mu_n\right). \tag{89}$$

Then formula (85) can also be written as

$$\rho_g\left[S\right] = \mu_S + \sqrt{\left(\rho_g\left[\underline{Y}\right] - \underline{\mu}\right)^T \cdot \underline{\Lambda} \cdot \left(\rho_g\left[\underline{Y}\right] - \underline{\mu}\right)}. \tag{90}$$

To conclude, we proved that formulae (31) or equivalently, formula (90), holds for general distortion risk measures, and for any multivariate random

vector $(Y_1, Y_2, \ldots, Y_n)$ such that all its stand-alone risks Y_k and also the aggregate risk S belong to the same class of translation-scale invariant distributions.

Notice that a first special case of this general result is the VaR approach (for a particular choice of the probability level) within a multivariate normal framework.

Another important special case, is the case where the economic capital is determined by a Tail-Value-at-Risk approach. In this case, the formula (90) reduces to

$$\text{TVaR}_p[S] = \mu_S + \sqrt{\left(\text{TVaR}_p[\underline{Y}] - \underline{\mu}\right)^T \cdot \underline{\Lambda} \cdot \left(\text{TVaR}_p[\underline{Y}] - \underline{\mu}\right)}. \qquad (91)$$

6.4   Summary

Assume that the stand-alone risks $(Y_1, Y_2, \ldots, Y_n)$ as well as the aggregate loss $S = Y_1 + Y_2 + \ldots + Y_n$ belong to the same family of translation-scale invariant distributions, see definitions (80) and (81), with correlation matrix $\underline{\Lambda}$ as defined in (12).

The economic capital EC for the aggregate loss S is determined by (86):

$$EC = \rho_g[S] - \mu_S,$$

for some distortion risk measure ρ_g as defined in (75). The individual stand-alone economic capitals are determined by (87):

$$EC_k = \rho_g[Y_k] - \mu_k.$$

The aggregate economic capital can then be determined by (31):

$$EC = \sqrt{\underline{EC}^T \cdot \underline{\Lambda} \cdot \underline{EC}},$$

where $\underline{EC}$ is defined by (29):

$$\underline{EC} = (EC_1, EC_2, \ldots, EC_n).$$

7 Generalizations for the alternative methodology : determining the economic capital by a distortion risk measure

7.1 Description

The economic capital EC for the aggregate loss $S = \sum_{k=1}^{m} Y_k + \sum_{k=m+1}^{n} e^{Y_k}$ can also be determined using any other distortion risk measure ρ_g. In this case, the aggregate economic capital is defined by

$$EC = \rho_g[S] - \mu_S, \tag{92}$$

for some distortion risk measure ρ_g, see (86).

In this case, we propose to approximate $\rho_g[S]$ by $\rho_g[S^l]$ or $\rho_g[S^c]$. Since S^c is a comonotonic sum, we find, using the additivity property for comonotonic risks

$$\rho_g[S^c] = \sum_{k=1}^{m} \rho_g[Y_k] + \sum_{k=m+1}^{n} \rho_g[e^{Y_k}], \tag{93}$$

see Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004).

In case also S^l is a comonotonic sum (which holds in case all $cov[Y_k, Y_j]$ are positive), we have that:

$$\rho_g[S^l] = \sum_{k=1}^{m} \left(\mu_k + r_k \sigma_k \rho_g[\Phi^{-1}(U)]\right) + \sum_{k=m+1}^{n} \rho_g[e^{\mu_k + r_k \sigma_k \Phi^{-1}(U) + \frac{1}{2}\sigma_k^2(1-r_k^2)}] \tag{94}$$

As a first approximation for the aggregate economic capital, we propose EC^l, which is defined by

$$EC^l = \rho_g[S^l] - \mu_S \tag{95}$$

As a second approximation for the aggregate capital, we propose EC^c, which is defined by

$$EC^c = \rho_g[S^c] - \mu_S \tag{96}$$

For the special case that ρ_g corresponds with the (1-p)-quantile, the approximations (95) and (96) reduce to (72) and (73), respectively.

7.2 A moments-based approximation

Finally, we notice that the upper bound and lower bound approaches can be combined further to obtain an approximation for the distribution of S which preserves the mean and the variance. Such an approach has been applied by Vyncke, Goovaerts and Dhaene (2003) to the case of Asian option pricing. In contrast to these authors who mix the cdf's of S^l and S^c such that the variance of S is preserved, we propose to mix the quantile functions of S^l and S^c. Indeed, let us consider the r.v. $S_m = zS^l + (1-z)S^c$ with $(S^l, S^c) = (F_{S^l}^{-1}(U), F_{S^c}^{-1}(U))$ and $0 < z < 1$ such that $Var(S_m) = Var(S)$. Since S_m is clearly a comonotonic sum we have that $F_S^{-1}[p] = zF_{S^l}^{-1}[p] + (1-z)F_{S^c}^{-1}$. More in general, we find that for any distortion risk measure ρ_g that $\rho_g[S]$ can now be approximated by $\rho_g[S_m] = z\rho_g[S^l] + (1-z)\rho_g[S^c]$. This alternative approach has the advantage that it is straightforward now to approximate distortion risk measures for S by using both the upper bound and lower bound approximations.

8 Conclusion

In this paper, we analysed a standard approach financial institutions use to calculate their total economic capital. This approach uses the different stand-alone economic capitals (ALM, Credits, Operational, ...) and the correlations between the different risk factors. This approach is based on the assumption that the vector of individual risks is multivariate normal, whereas the risk measure used is a given percentile.

We discussed this standard method and in addition, we proposed several generalizations.
Within the multivariate normal framework, we proved that the current method can be applied to any distortion risk measure, the TailVaR in particular. Furthermore, we showed that in this framework, it is not necessary to restrict to the multivariate normal case. A more general framework within a given class of translation-scale invariant distributions is given.

We also investigated the case where both normal and lognormal random variables are involved. In this case, we proposed an efficient algorithm that gives rise to accurate and easy computable approximations for the aggregate economic capital.

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