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AN EQUILIBRIUM CHARACTERIZATION OF THE TERM STRUCTURE

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The paper derives a general form of the term structure of interest rates. The following assumptions are made: (A.1) The instantaneous (spot) interest rate follows a diffusion process; (A.2) the price of a discount bond depends only on the spot rate over its term; and (A.3) the market is efficient. Under these assumptions, it is shown by means of an arbitrage argument that the expected rate of return on any bond in excess of the spot rate is proportional to its standard deviation. This property is then used to derive a partial differential equation for bond prices. The solution to that equation is given in the form of a stochastic integral representation. An interpretation of the bond pricing formula is provided. The model is illustrated on a specific case.

1. Introduction

Although considerable attention has been paid to equilibrium conditions in capital markets and the pricing of capital assets, few results are directly applicable to description of the interest rate structure. The most notable exceptions are the works of Roll (1970, 1971), Merton (1973, 1974), and Long (1974). This paper gives an explicit characterization of the term structure of interest rates in an efficient market. The development of the model is based on an arbitrage argument similar to that of Black and Scholes (1973) for option pricing. The model is formulated in continuous time, although some implications for discrete interest rate series are also noted.

2. Notation and assumptions

Consider a market in which investors buy and issue default free claims on a specified sum of money to be delivered at a given future date. Such claims will be called (discount) bonds. Let $P(t, s)$ denote the price at time t of a discount bond maturing at time s, $t \leqq s$, with unit maturity value,

$$P(s, s) = 1.$$

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The yield to maturity $R(t, T)$ is the internal rate of return at time t on a bond with maturity date $s = t+T$,

$$R(t, T) = -\frac{1}{T}\log P(t, t+T), \qquad T > 0. \tag{1}$$

The rates $R(t, T)$ considered as a function of T will be referred to as the term structure at time t.

The forward rate $F(t, s)$ will be defined by the equation

$$R(t, T) = \frac{1}{T}\int_t^{t+T} F(t, \tau)\,\mathrm{d}\tau. \tag{2}$$

In the form explicit for the forward rate, this equation can be written as

$$F(t, s) = \frac{\partial}{\partial s}[(s-t)R(t, s-t)]. \tag{3}$$

The forward rate can be interpreted as the marginal rate of return from committing a bond investment for an additional instant.

Define now the spot rate as the instantaneous borrowing and lending rate,

$$r(t) = R(t, 0) = \lim_{T\to 0} R(t, T). \tag{4}$$

A loan of amount W at the spot rate will thus increase in value by the increment

$$\mathrm{d}W = Wr(t)\mathrm{d}t. \tag{5}$$

This equation holds with certainty. At any time t, the current value $r(t)$ of the spot rate is the instantaneous rate of increase of the loan value. The subsequent values of the spot rate, however, are not necessarily certain. In fact, it will be assumed that $r(t)$ is a stochastic process, subject to two requirements: First, $r(t)$ is a continuous function of time, that is, it does not change value by an instantaneous jump. Second, it is assumed that $r(t)$ follows a Markov process. Under this assumption, the future development of the spot rate given its present value is independent of the past development that has led to the present level. The following assumption is thus made:

(A.1) The spot rate follows a continuous Markov process.

The Markov property implies that the spot rate process is characterized by a single state variable, namely its current value. The probability distribution of the segment $\{r(\tau), \tau \geqq t\}$ is thus completely determined by the value of $r(t)$.

Processes that are Markov and continuous are called diffusion processes. They can be described [cf. Itô (1961), Gikhman and Skorokhod (1969)] by a stochastic differential equation of the form

$$dr = f(r, t)\,dt + \rho(r, t)\,dz, \tag{6}$$

where $z(t)$ is a Wiener process with incremental variance dt. The functions $f(r, t)$, $\rho^2(r, t)$ are the instantaneous drift and variance, respectively, of the process $r(t)$.

It is natural to expect that the price of a discount bond will be determined solely by the spot interest rate over its term, or more accurately, by the current assessment of the development of the spot rate over the term of the bond. No particular form of such relationship is presumed. The second assumption will thus be stated as follows:

(A.2) The price $P(t, s)$ of a discount bond is determined by the assessment, at time t, of the segment $\{r(\tau), t \leqq \tau \leqq s\}$ of the spot rate process over the term of the bond.

It may be noted that the expectation hypothesis, the market segmentation hypothesis, and the liquidity preference hypothesis all conform to assumption (A.2), since they all postulate that

$$R(t, T) = E_t\left(\frac{1}{T}\int_t^{t+T} r(\tau)\,d\tau\right) + \bar{\pi}(t, T, r(t)),$$

with various specifications for the function $\bar{\pi}$.

Finally, it will be assumed that the following is true:

(A.3) The market is efficient; that is, there are no transactions costs, information is available to all investors simultaneously, and every investor acts rationally (prefers more wealth to less, and uses all available information).

Assumption (A.3) implies that investors have homogeneous expectations, and that no profitable riskless arbitrage is possible.

By assumption (A.1) the development of the spot rate process over an interval (t, s), $t \leqq s$, given its values prior to time t, depends only on the current value $r(t)$. Assumption (A.2) then implies that the price $P(t, s)$ is a function of $r(t)$,

$$P(t, s) = P(t, s, r(t)). \tag{7}$$

Thus, the value of the spot rate is the only state variable for the whole term structure. Expectations formed with the knowledge of the whole past development of rates of all maturities, including the present term structure, are equivalent to expectations conditional only on the present value of the spot rate.

Since there exists only one state variable, the instantaneous returns on bonds of different maturities are perfectly correlated. This means that the short bond and just one other bond completely span the whole of the term structure. It should be noted, however, that bond returns over a finite period are not correlated perfectly. Investors unwilling to revise the composition of their portfolio continuously will need a spectrum of maturities to fulfil their investment objectives.

3. The term structure equation

It follows from eqs. (6), (7) by the Itô differentiation rule [cf., for instance, Itô (1961), Kushner (1967), Åström (1970)], that the bond price satisfies a stochastic differential equation

$$\mathrm{d}P = P\mu(t,s)\,\mathrm{d}t - P\sigma(t,s)\,\mathrm{d}z, \tag{8}$$

where the parameters $\mu(t,s) = \mu(t,s,r(t))$, $\sigma(t,s) = \sigma(t,s,r(t))$ are given by

$$\mu(t,s,r) = \frac{1}{P(t,s,r)}\left[\frac{\partial}{\partial t} + f\frac{\partial}{\partial r} + \tfrac{1}{2}\rho^2\frac{\partial^2}{\partial r^2}\right]P(t,s,r), \tag{9}$$

$$\sigma(t,s,r) = -\frac{1}{P(t,s,r)}\,\rho\,\frac{\partial}{\partial r}P(t,s,r). \tag{10}$$

The functions $\mu(t,s,r)$, $\sigma^2(t,s,r)$ are the mean and variance, respectively, of the instantaneous rate of return at time t on a bond with maturity date s, given that the current spot rate is $r(t) = r$.

Now consider an investor who at time t issues an amount W_1 of a bond with maturity date s_1, and simultaneously buys an amount W_2 of a bond maturing at time s_2. The total worth $W = W_2 - W_1$ of the portfolio thus constructed changes over time according to the accumulation equation

$$\mathrm{d}W = (W_2\mu(t,s_2) - W_1\mu(t,s_1))\,\mathrm{d}t - (W_2\sigma(t,s_2) - W_1\sigma(t,s_1))\,\mathrm{d}z \tag{11}$$

[cf. Merton (1971)]. This equation follows from eq. (8) by application of the Itô rule.

Suppose that the amounts W_1, W_2 are chosen to be proportional to $\sigma(t,s_2)$, $\sigma(t,s_1)$, respectively,

$$W_1 = W\sigma(t,s_2)/(\sigma(t,s_1) - \sigma(t,s_2)),$$

$$W_2 = W\sigma(t,s_1)/(\sigma(t,s_1) - \sigma(t,s_2)).$$

Then the second term in eq. (11) disappears, and the equation takes the form

$$\mathrm{d}W = W(\mu(t, s_2)\sigma(t, s_1) - \mu(t, s_1)\sigma(t, s_2))(\sigma(t, s_1) - \sigma(t, s_2))^{-1}\,\mathrm{d}t. \quad (12)$$

The portfolio composed of such amounts of the two bonds is instantaneously riskless, since the stochastic element $\mathrm{d}z$ is not present in (12). It should therefore realize the same return as a loan at the spot rate described by eq. (5). If not, the portfolio can be bought with funds borrowed at the spot rate, or otherwise sold and the proceeds lent out, to accomplish a riskless arbitrage.

As such arbitrage opportunities are ruled out by Assumption (A.3), comparison of eqs. (5) and (12) yields

$$(\mu(t, s_2)\sigma(t, s_1) - \mu(t, s_1)\sigma(t, s_2))/(\sigma(t, s_1) - \sigma(t, s_2)) = r(t),$$

or equivalently,

$$\frac{\mu(t, s_1) - r(t)}{\sigma(t, s_1)} = \frac{\mu(t, s_2) - r(t)}{\sigma(t, s_2)}. \quad (13)$$

Since eq. (13) is valid for arbitrary maturity dates s_1, s_2, it follows that the ratio $(\mu(t, s) - r(t))/\sigma(t, s)$ is independent of s. Let $q(t, r)$ denote the common value of such ratio for a bond of any maturity date, given that the current spot rate is $r(t) = r$,

$$q(t, r) = \frac{\mu(t, s, r) - r}{\sigma(t, s, r)}, \qquad s \geqq t. \quad (14)$$

The quantity $q(t, r)$ can be called the market price of risk, as it specifies the increase in expected instantaneous rate of return on a bond per an additional unit of risk.

Eq. (14) will now be used to derive an equation for the price of a discount bond. Writing (14) as

$$\mu(t, s, r) - r = q(t, r)\sigma(t, s, r),$$

and substituting for μ, σ from eqs. (9), (10) yields, after rearrangement,

$$\frac{\partial P}{\partial t} + (f + \rho q)\frac{\partial P}{\partial r} + \tfrac{1}{2}\rho^2 \frac{\partial^2 P}{\partial r^2} - rP = 0, \qquad t \leqq s. \quad (15)$$

Eq. (15) is the basic equation for pricing of discount bonds in a market characterized by Assumptions (A.1), (A.2), (A.3). It will be called the term structure equation.

The term structure equation is a partial differential equation for $P(t, s, r)$. Once the character of the spot rate process $r(t)$ is described and the market price of risk $q(t, r)$ specified, the bond prices are obtained by solving (15) subject to the boundary condition

$$P(s, s, r) = 1. \tag{16}$$

The term structure $R(t, T)$ of interest rates is then readily evaluated from the equation

$$R(t, T) = -\frac{1}{T}\log P(t, t+T, r(t)). \tag{17}$$

4. Stochastic representation of the bond price

Solutions of partial differential equations of the parabolic or elliptic type, such as eq. (15), can be represented in an integral form in terms of an underlying stochastic process [cf. Friedman (1975)]. Such representation for the bond price as a solution to the term structure equation (15) and its boundary condition is as follows:

$$\begin{aligned} P(t, s) = E_t \exp\Big(&-\int_t^s r(\tau)\,\mathrm{d}\tau - \tfrac{1}{2}\int_t^s q^2(\tau, r(\tau))\,\mathrm{d}\tau \\ &+\int_t^s q(\tau, r(\tau))\,\mathrm{d}z(\tau)\Big), \qquad t \leqq s. \end{aligned} \tag{18}$$

To prove (18), define

$$V(u) = \exp\left(-\int_t^u r(\tau)\,\mathrm{d}\tau - \tfrac{1}{2}\int_t^u q^2(\tau, r(\tau))\,\mathrm{d}\tau + \int_t^u q(\tau, r(\tau))\,\mathrm{d}z(\tau)\right),$$

and apply Itô's differential rule to the process $P(u, s)V(u)$. Then

$$\begin{aligned} \mathrm{d}(PV) &= V\mathrm{d}P + P\mathrm{d}V + \mathrm{d}P\mathrm{d}V \\ &= V\left(\frac{\partial P}{\partial t} + f\frac{\partial P}{\partial r} + \tfrac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}u + V\frac{\partial P}{\partial r}\rho\,\mathrm{d}z + PV(-r - \tfrac{1}{2}q^2)\,\mathrm{d}u \\ &\quad + PVq\,\mathrm{d}z + \tfrac{1}{2}PVq^2\,\mathrm{d}u + V\frac{\partial P}{\partial r}\rho q\,\mathrm{d}u \\ &= V\left(\frac{\partial P}{\partial t} + (f+\rho q)\frac{\partial P}{\partial r} + \tfrac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2} - rP\right)\mathrm{d}u + PVq\,\mathrm{d}z + V\frac{\partial P}{\partial r}\rho\,\mathrm{d}z \\ &= PVq\,\mathrm{d}z + V\frac{\partial P}{\partial r}\rho\,\mathrm{d}z, \end{aligned}$$

by virtue of eq. (15). Integrating from t to s and taking expectation yields

$$E_t(P(s,s)V(s)-P(t,s)V(t)) = 0,$$

and eq. (18) follows.

In the special case when the expected instantaneous rates of return on bonds of all maturities are the same,

$$\mu(t,s) = r(t), \qquad s \geqq t,$$

(this corresponds to $q = 0$), the bond price is given by

$$P(t,s) = E_t \exp\left(-\int_t^s r(\tau)\,\mathrm{d}\tau\right). \tag{19}$$

Eq. (18) can be given an interpretation in economic terms. Construct a portfolio consisting of the long bond (bond whose maturity approaches infinity) and lending or borrowing at the spot rate, with proportions $\lambda(t)$, $1-\lambda(t)$, respectively, where

$$\lambda(t) = (\mu(t,\infty)-r(t))/\sigma^2(t,\infty).$$

The price $Q(t)$ of such portfolio follows the equation

$$\mathrm{d}Q = \lambda Q(\mu(t,\infty)\,\mathrm{d}t-\sigma(t,\infty)\,\mathrm{d}z)+(1-\lambda)Qr\,\mathrm{d}t.$$

This equation can be integrated by evaluating the differential of $\log Q$ and noting that $\lambda(t)\sigma(t,\infty) = q(t, r(t))$. This yields

$$\begin{aligned}\mathrm{d}(\log Q) &= \lambda\mu(t,\infty)\,\mathrm{d}t-\lambda\sigma(t,\infty)\,\mathrm{d}z+(1-\lambda)r\,\mathrm{d}t-\tfrac{1}{2}\lambda^2\sigma^2(t,\infty)\,\mathrm{d}t \\ &= r\,\mathrm{d}t+\tfrac{1}{2}q^2\,\mathrm{d}t-q\,\mathrm{d}z,\end{aligned}$$

and consequently

$$\frac{Q(t)}{Q(s)} = \exp\left(-\int_t^s r(\tau)\,\mathrm{d}\tau-\tfrac{1}{2}\int_t^s q^2(\tau, r(\tau))\,\mathrm{d}\tau+\int_t^s q(\tau, r(\tau))\,\mathrm{d}z(\tau)\right).$$

Thus, eq. (18) can be written in the form

$$P(t,s) = E_t Q(t)/Q(s), \qquad t \leqq s. \tag{20}$$

This means that a bond of any maturity is priced in such a way that the same portion of a certain well-defined combination of the long bond and the riskless asset (the portfolio Q) can be bought now for the amount of the bond price as is expected to be bought at the maturity date for the maturity value.

Equivalently, eq. (20) states that the price of any bond measured in units of the value of such portfolio Q follows a martingale,

$$\frac{P(t,s)}{Q(t)} = E_t \frac{P(\tau,s)}{Q(\tau)}, \qquad t \leqq \tau \leqq s.$$

Thus, if the present bond price is a certain fraction of the value of the portfolio Q, then the future value of the bond is expected to stay the same fraction of the value of that portfolio.

In empirical testing of the model, as well as for applications of the results, it is necessary to know the parameters f, ρ of the spot rate process, and the market price of risk q. The former two quantities can be obtained by statistical analysis of the (observable) process $r(t)$. Although the market price of risk can be estimated from the defining eq. (14), it is desirable to have a more direct means of observing q empirically. The following equality can be employed:

$$\left.\frac{\partial R}{\partial T}\right|_{T=0} = \tfrac{1}{2}(f(t,r(t)) + \rho(t,r(t))\cdot q(t,r(t))). \tag{21}$$

Once the parameters f, ρ are known, q could thus be determined from the slope at the origin of the yield curves. Eq. (21) can be proven by taking the second derivative with respect to s of (18) (Itô's differentiation rule is needed), and putting $s = t$. This yields

$$\left.\frac{\partial^2 P}{\partial s^2}\right|_{s=t} = r^2(t) - f(t,r(t)) - \rho(t,r(t))\cdot q(t,r(t)). \tag{22}$$

But from (1),

$$\left.\frac{\partial^2 P}{\partial s^2}\right|_{s=t} = r^2(t) - 2\left.\frac{\partial R}{\partial T}\right|_{T=0}. \tag{23}$$

By comparison of (22), (23), eq. (21) follows.

5. A specific case

To illustrate the general model, the term structure of interest rates will now be obtained explicitly in the situation characterized by the following assumptions: First, that the market price of risk $q(t, r)$ is a constant,

$$q(t, r) = q,$$

independent of the calendar time and of the level of the spot rate. Second, that the spot rate $r(t)$ follows the so-called Ornstein–Uhlenbeck process,

$$\mathrm{d}r = \alpha(\gamma - r)\,\mathrm{d}t + \rho\,\mathrm{d}z, \tag{24}$$

with $\alpha > 0$, corresponding to the choice $f(t, r) = \alpha(\gamma - r)$, $\rho(t, r) = \rho$ in eq. (6). This description of the spot rate process has been proposed by Merton (1971).

The Ornstein–Uhlenbeck process with $\alpha > 0$ is sometimes called the elastic random walk. It is a Markov process with normally distributed increments. In contrast to the random walk (the Wiener process), which is an unstable process and after a long time will diverge to infinite values, the Ornstein–Uhlenbeck process possesses a stationary distribution. The instantaneous drift $\alpha(\gamma - r)$ represents a force that keeps pulling the process towards its long-term mean γ with magnitude proportional to the deviation of the process from the mean. The stochastic element, which has a constant instantaneous variance ρ^2, causes the process to fluctuate around the level γ in an erratic, but continuous, fashion. The conditional expectation and variance of the process given the current level are

$$E_t r(s) = \gamma + (r(t) - \gamma)\mathrm{e}^{-\alpha(s-t)}, \qquad t \leqq s, \tag{25}$$

$$\mathrm{Var}_t r(s) = \frac{\rho^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha(s-t)}), \qquad t \leqq s, \tag{26}$$

respectively.

It is not claimed that the process given by eq. (24) represents the best description of the spot rate behavior. In the absence of empirical results on the character of the spot rate process, this specification serves only as an example.

Under such assumptions, the solution of the term structure equation (15) subject to (16) [or alternatively, the representation (18)] is

$$P(t, s, r) = \exp\left[\frac{1}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)})(R(\infty) - r) - (s-t)R(\infty)\right.$$
$$\left. - \frac{\rho^2}{4\alpha^3}(1 - \mathrm{e}^{-\alpha(s-t)})^2\right], \qquad t \leqq s, \tag{27}$$

where

$$R(\infty) = \gamma + \rho q/\alpha - \tfrac{1}{2}\rho^2/\alpha^2. \tag{28}$$

The mean $\mu(t, s)$ and standard deviation $\sigma(t, s)$ of the instantaneous rate of return of a bond maturing at time s is, from eqs. (9), (10),

$$\mu(t, s) = r(t) + \frac{\rho q}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}),$$

$$\sigma(t, s) = \frac{\rho}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}),$$

with $t \leqq s$. It is seen that the longer the term of the bond, the higher the variance of the instantaneous rate of return, with the expected return in excess of the spot rate being proportional to the standard deviation. For a very long bond (i.e., $s \to \infty$) the mean and standard deviation approach the limits

$$\mu(\infty) = r(t) + \rho q/\alpha,$$

$$\sigma(\infty) = \rho/\alpha.$$

The term structure of interest rates is then calculated from eqs. (17) and (22). It takes the form

$$R(t, T) = R(\infty) + (r(t) - R(\infty))\frac{1}{\alpha T}(1 - \mathrm{e}^{-\alpha T})$$

$$+ \frac{\rho^2}{4\alpha^3 T}(1 - \mathrm{e}^{-\alpha T})^2, \qquad T \geqq 0, \tag{29}$$

Note that the yield on a very long bond, as $T \to \infty$, is $R(\infty)$, thus explaining the notation (28).

The yield curves given by (29) start at the current level $r(t)$ of the spot rate for $T = 0$, and approach a common asymptote $R(\infty)$ as $T \to \infty$. For values of $r(t)$ smaller or equal to

$$R(\infty) - \tfrac{1}{4}\rho^2/\alpha^2,$$

the yield curve is monotonically increasing. For values of $r(t)$ larger than that but below

$$R(\infty) + \tfrac{1}{2}\rho^2/\alpha^2,$$

it is a humped curve. When $r(t)$ is equal to or exceeds this last value, the yield curves are monotonically decreasing.

Eq. (29), together with the spot rate process (24), fully characterizes the behavior of interest rates under the specific assumptions of this section. It provides both the relationship, at a given time t, among rates of different maturities, and the behavior of interest rates, as well as bond prices, over time. The relationship between the rates $R(t, T_1)$, $R(t, T_2)$ of two arbitrary maturities can be determined by eliminating $r(t)$ from eq. (29) written for $T = T_1$, $T = T_2$. Moreover, (29) describes the development of the rate $R(t, T)$ of a given maturity over time. Since $r(t)$ is normally distributed by virtue of the properties of the Ornstein–Uhlenbeck process, and $R(t, T)$ is a linear function of $r(t)$, it follows that $R(t, T)$ is also normally distributed. The mean and variance of $R(\tau, T)$ given $R(t, T)$, $t < \tau$, are obtained from (29) by use of (25), (26). The calculations are elementary and will not be done here. It will only be noted that eqs. (24), (29) imply that the discrete rate series,

$$R_n = R(nT, T), \qquad n = 0, 1, 2, \dots,$$

follows a first-order linear normal autoregressive process of the form

$$R_n = c + a(R_{n-1} - c) + \varepsilon_n, \tag{30}$$

with independent residuals ε_n [cf. Nelson (1972)]. The process (30) is the discrete elastic random walk, fluctuating around its mean c. The parameters c, a, and $s^2 = E\varepsilon_n^2$ could be expressed in terms of γ, α, ρ, q. In particular, the constant a, which characterizes the degree to which the next term in the series $\{R_n\}$ is tied to the current value, is given by $a = \mathrm{e}^{-\alpha T}$.

Also, eq. (29) can be used to ascertain the behavior of bond prices. The price $P(\tau, s)$ given its current value $P(t, s)$, $t \leqq s$, is lognormally distributed, with parameters of the distribution calculated using eqs. (1), (25), (26), and (29).

The difference between the forward rates and expected spot rates, considered as a function of the term is usually referred to as the liquidity premium [although, as Nelson (1972) argues, a more appropriate name would be the term premium]. Using eqs. (3) and (25), the liquidity premium implied by the term structure (29) is given by

$$\begin{aligned}\pi(T) &= F(t, t+T) - E_t r(t+T) \\ &= \left(R(\infty) - \gamma + \tfrac{1}{2}\frac{\rho^2}{\alpha^2}\mathrm{e}^{-\alpha T}\right)(1 - \mathrm{e}^{-\alpha T}), \qquad T \geqq 0. \end{aligned} \tag{31}$$

The liquidity premium (31) is a smooth function of the term T. It is similar in form to the shape of the curves used by McCulloch (1975) in fitting observed estimates of liquidity premia. Its values for $T = 0$ and $T = \infty$ are $\pi(0) = 0$, $\pi(\infty) = R(\infty) - \gamma$, respectively, the latter being the difference between the yield on the very long bond and the long-term mean of the spot rate. If $q \geqq \rho/\alpha$, $\pi(T)$ is a monotonically increasing function of T. For $0 < q < \rho/\alpha$, it has a humped shape, with maximum of $q^2/2$ occurring at

$$T = \frac{1}{\alpha} \log \left(\frac{\rho/\alpha}{\rho/\alpha - q} \right).$$

If the market price of risk $q \leqq 0$, then $\pi(T)$ is a monotonically decreasing function.

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