Pricing Rate of Return Guarantees in Regular Premium Unit Linked Insurance

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Abstract

We derive general pricing formulas for Rate of Return Guarantees in Regular Premium Unit Linked Insurance under stochastic interest rates. Our approach differs from the previous literature as we don't find a fair premium principle but price the guarantee in a generic Unit Linked contract taking the premium principle as given. Our main contribution focusses on the effect of stochastic interest rates. First, we show that the effect of stochastic interest rates can be interpreted as, what is known in the financial community as, a convexity correction. Second we link the LIBOR Market Model to our model of the economy. This allows us to find guarantee prices consistent with observed cap and swaption prices. Numerical results show the effect of this more sophisticated interest rate modelling is considerable. Popular ways of approximating Asian option values through tight bounds don't provide accurate results because of long maturities of the guarantees.

Keywords: Return Guarantee, Average Rate Option, Convexity Correction, LIBOR Market Model.

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1 Introduction

Unit Linked (UL) insurance is a form of insurance where the policyholder bears the investment risk. The premiums are invested in several investment funds which usually invest a large percentage of their money in stocks. Sometimes the policyholder is even allowed to invest directly in stocks. Rate of return guarantees in a UL context can therefore be considered as some kind of stock option. Many insurance companies have given guarantees on UL contracts in the beginning of the nineties, not realizing the risk attached to this product characteristic. With the current bearish stock markets and Fair Value calculations at the center of attention it should be realized that these options are in or at least at the money. The results in this paper can help quantify the risk attached to the guarantees and provide in the need for market values of insurance liabilities.

A single premium UL contract with maturity guarantee can be viewed as a put option on a stock (Brennan and Schwartz, 1976). Bacinello and Ortu (1994) and Nielsen and Sandmann (1995, 1996, 2002b) analyse the periodic premium contract with maturity guarantee from a fair premium principle perspective.

Our contribution to the existing literature is threefold. First we take a different approach to the insurance contract as previous literature on the subject as we let the cost and mortality deductions be exogenously given. This is the case in practice and differs from the approach by Nielsen and Sandmann (1995, 1996a, 2002b) as they derive the existence of Fair Premium principles for the guarantee within the contract. The payment of multiple premiums makes the option payoff dependent on the stock price at different time points, which leads to an analogy with Asian Options. We show that for a generic structure of the contract, which allows the invested premium to be path dependent, the structure of the option payoff remains Asian like. In our approach we take full account of all insurance aspects of the contract. We will apply option pricing techniques to the context of UL products using Change of Numeraire methods. We derive a general pricing formula for the guarantee. It turns out the guarantee can be expressed as a put option on a stochastically weighted average of the stock price at maturity.

Our second contribution is that we make the analogy of the guarantee with Asian options explicit by proving equality between prices of both contracts in a constant interest rate environment. This setup allows for stochastic (stationary) volatility however the analogy brakes down when stochastic interest rates are introduced. We discuss the cause of the differences that arise when interest rates become stochastic. This already gives us some intuition on how these contracts can best be hedged.

Third and most importantly we argue for a more general setup of the randomness of the term structure. This is done in the following steps, choosing a convexity correction approach we specify a quite general lognormal model of the economy. In this model we derive results for the Levy (1992) approximation to the price of the guarantee next to an upper and lower bound to this price extending work by Nielsen and Sandmann (2002a) and Thompson (1998). It turns out the effect of stochastic interest rates can be interpreted as a convexity correction. Then we show how our setup can be linked with the popular LIBOR Market Model. For long term options typically encountered in life insurance the convexity correction effect of stochastic interest has a high impact on the price of the option. Realizing this, we conjecture that it is of interest to use more sophisticated term structure models. We provide in the need to have a stock option pricing model which has its term structure part in accordance with the dominant term structure model in the option pricing literature, the LIBOR Market Model. Our results provide the link between the standard Black Scholes stock model and the Black pricing model for Caps (and Swaptions) by providing approximate expressions for the forward bond volatility in a LMM. Building on arguments by Brace et al. (2001) we show that forward bond prices are approximately lognormal in the LMM. It thus seems natural to use the LMM not only for interest rate derivative purposes but also in pricing stock options. Numerical pricing results using real data suggest that more general term structure models can produce non neglegible price differences when compared with single (and two) factor Hull-White models.

Finally we show using an empirical example what the impact of our more general setup is for the price of the guarantee. We also present results on the tightness of the bounds which are derived. This is of independent interest since these have not been tested before for the maturities which are encountered in life insurance. It turns out these bounds are not tight for long maturities and hence can not be used for pricing. Furthermore we show that popular numerical procedures like the Rogers and Shi (1995) lower bound, recently generalized by Nielsen and Sandmann (2002a), and the upper bound by Thompson (1998) fail to produce accurate results for long maturities at the estimated parameters.

Convexity correction or convexity adjustment is frequently used in the financial industry to value payments which are made at the wrong time point (e.g. an interest rate which is known at time T is paid at a later time S) or a different currency (e.g. a foreign interest rate is paid in domestic currency). It is shown in a review article by Pelsser (2003) that convexity correction has its basis in a change of measure associated with a change of numeraire. The advantage of using convexity corrections is that this pricing approach is product based. The determinants of the price are apparent from the respective formulas instead of hidden in the equations / simulations of some general pricing framework. In the context of calculations for large insurance portfolios an additional advantage of this product based approach resulting in an analytical pricing formula is that one can avoid the use of computer intensive methods to calculate the market value of insurance liabilities.

To arrive at our results we extensively use the Change of Numeraire techniques developed by Geman et al. (1995). They extend the original ideas of Harrison and Kreps (1978) and Harrison and Pliska (1981) and show that for any self-financing portfolio of assets, with strict positive value, there exists an equivalent measure under which asset prices normalized by this portfolio are martingales, hence this portfolio can be used as numeraire. They also show how to derive the Radon-Nikodym derivative associated with any change of numeraire.

Now the remainder of the paper is organized as follows. First section 2 describes the change of numeraire techniques and convexity correction. In section 3 we describe the financial and insurance aspects of the Regular Premium Unit Linked contract and derive our general pricing formula. In section 4 we make the analogy of the UL guarantee with an Asian option explicit and discuss some hedging issues. Section 5 derives the Levy approximation and shows the interpretation of the effect of stochastic interest rates as a convexity correction. To obtain prices consistent with the popular LMM we derive expressions for the forward bond volatility in terms of LIBOR rate volatilities in section 6. In section 7 we discuss and generalize pricing bounds to the case of rate of return guarantees. Results in a parameterized framework are given in section 8. Numerical results showing the effect of stochastic interest rates and implications of more general interest rate dynamics are given in section 9. Section 10 concludes.

2 Preliminaries: Changes of Numeraire and Convexity Correction

The idea that in a complete and arbitrage-free market the unique value of any financial claim equals the expectation of the payoff normalized by the money market account under some equivalent measure extends back to the work of Harrison and Kreps (1979) and Harrison and Pliska (1981). Since under this intended probability measure the return on all assets equals the risk free rate, the probability measure is termed the Risk Neutral measure, denoted here by $\mathbf{Q}$, and expectation with respect to this measure is called Risk Neutral expectation. In this context the normalizing asset (in these papers the money market account) is called the *Numeraire*. Geman et al. (1995) show how not only the money market account, but every strictly positive self-financing portfolio of traded assets, can be used as a numeraire. Their Change of Numeraire theorem shows how an expectation under a probability measure $\mathbf{Q}^N$ associated with numeraire N is related to an expectation under an equivalent probability measure $\mathbf{Q}^M$ associated with numeraire M. As a by-product all normalised assets are martingales under the probability measure associated with the numeraire. To be more specific their theorem states that in a complete and arbitrage-free market, for any numeraires N and M with associated measures $\mathbf{Q}^N$ and $\mathbf{Q}^M$ respectively, the following holds for the price of an asset H at time t,

$$H(t) = N(t)\, E_t^N \left(\frac{H(T)}{N(T)} \right) = M(t)\, E_t^M \left(\frac{H(T)}{M(T)} \right) \tag{1}$$

Where E_t^N, E_t^M denotes expectation conditional on the information available at time t under $\mathbf{Q}^N$ and $\mathbf{Q}^M$ respectively. Throughout the paper we will use $E^{\mathbf{X}}$ and $E_t^{\mathbf{X}}$ for expectation and conditional expectation with respect to some probability measure $\mathbf{X}$. The Radon-Nikodym derivative associated with a Change of Measure from $\mathbf{Q}^N$ to $\mathbf{Q}^M$ is given by,

$$\frac{d\mathbf{Q}^M}{d\mathbf{Q}^N} = \frac{M(T)\ /\ M(t)}{N(T)\ /\ N(t)} \tag{2}$$

Hence if the price of an asset with payoff $H(T)$, known at time T, can be calculated by taking a Risk-Neutral expectation, it can be equivalently and sometimes more conveniently calculated by changing numeraires.

Many (particular interest rate) derivatives can be characterized as *exotic* European options. This means that the price of the option is determined by the joint distribution of a few relevant interest rates at one point in time. A possible approach in the case of interest rates is to specify a full (multi-factor) model, estimate the parameters and calculate the price, possibly analytic or otherwise by numerical techniques like Monte Carlo simulation. The danger of this approach is that it can lead to very unrealistic correlation structures between the relevant rates (see Rebonato, 1998 Chapter 3 and 1999). Contrary to this approach convexity correction focuses the modelling as closely as possible on the problem at hand. Of the joint distribution of the relevant rates, the marginal distributions are almost always taken lognormal hence the joint distribution is characterized by the marginals and some correlation parameter(s). In this paper we use the idea of convexity correction to interpret our results on the pricing of Rate of Return Guarantees. The application of this idea to stock options is a novelty in the literature. We will see this leads (under weak assumptions) to a very interpretable expression of the implied volatility to use in the Levy approximation for which expressions are derived in section 5. An introductory example applying the ideas of convexity correction to the pricing of a displaced LIBOR payment, the so called LIBOR in arrears payment, can be found in Pelsser (2003). An example of how convexity correction arises in the pricing of an Arithmetic Asian Option is given in Appendix A. This will already hint at the approach we take to determine the value of the guarantee. For more on convexity correction see Pelsser (2003) and the references therein.

We have now set the stage and discussed all the ingredients we need in the remainder of the paper and turn to the pricing of rate of return guarantees in a regular premium Unit Linked framework. We start by describing the insurance contract in which the guarantee arises.

3 Contract definition and general pricing formula

The Unit Linked concept refers to the way the policy holders' premiums are invested. The net premiums (i.e. after cost and risk premium deductions) are invested based on the choice of the policyholder. Common practice is to let the policyholder choose between several selected investment funds. Some insurers even give the possibility to invest in individual stocks. It should be stressed that in this construction the policyholder bears the investment risk. This means all profits are his (100% profit sharing) but also he has to account for the losses. There is, contrary to traditional insurance, in principle no guaranteed rate of return. This type of insurance has high potential profitability because profit is based on equity investments instead of fixed income. However policy holders are in for a disappointment in times of economic downfall. This is where the Rate of Return Guarantee comes in.

Typical for a Unit Linked contract is that the reserve is not counted in money but in units of several investment funds or stocks. The reserve in money terms is the number of units times the price of each unit. This reserve is termed the *fundvalue*. A gross premium is paid at regular intervals until expiry of the insurance contract. After cost deduction for investment and administration costs and mortality risk premiums, an investment premium results. For each investment premium a number of units of each funds chosen by the policy holder is purchased at the prevailing price at the payment date. In the presence of a guarantee the value of the purchased units at expiry is compared to some fixed guaranteed amount. This amount is likely to be determined by factors like the height of the premium, guaranteed return and cost and mortality deductions but could as well be exogenous. In this paper we assume that the policyholder invests only in a single investment fund or stock, hence we are dealing with only one stock price process. This assumption is not at all restrictive and our results can be generalized in a straightforward manner to include investments in multiple investment funds or stocks[1]. After this introduction to UL insurance, let us fix some notation.

Let S_t be the price of a unit at time t (in years). This should be thought of as a stock price or stock index. Let the start of the contract be at $t_0 = 0$ and let t_i, $i = 0, ..., n-1$ be the time points at which a premium P_i is credited to the reserve[2]. With premium we mean *investment* premium, so costs and mortality charge are taken into account. Since these cost deductions may depend on the fundvalue at each time t_i (the investment premium P_i may depend on the evolution of S prior to t_i), this means P_i is in general path dependent. We don't

[1] From our later results it can be seen that investing in multiple assets means the Guarantee turns into an Asian Option on a basket of assets. The techniques used to value basket options are similar to those used in Asian option pricing, hence the problem can be solved analogously. Since the weights of each asset in the basket sum to one, it is interesting to note that considering multiple assets only changes the result in Corollary 4. Theorem 2 remains unchanged. Pricing bounds can be obtained by generalization of the results in section 7.

[2] We assume there are no possibilities to surrender. This means there is no insecurity about premium payments.

make this dependence explicit in our notation, as we will show we can remove it, under weak assumptions, later. Furthermore let $T = t_n$ be the expiry date of the contract and K be the guaranteed amount at expiry. Then at time t_i the policyholder purchases $P_i \, / \, S_{t_t}$ units and each unit has value S_T at expiry. So the fundvalue at the expiry date is $\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}$. At each payment date, t_i, prior to expiry the fundvalue is given by $\sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}}$. Since the policyholder is entitled to a minimum payment of K, conditional upon survival of the insured until time T the payoff of the contract at maturity equals,

$$\max\left(\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}, K\right) = \sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}} + \left(K - \sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}\right)^+ \tag{3}$$

From this formula we can draw our first conclusion; the value of the guarantee is represented by a put option on $\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}$, which can be interpreted as some stochastically weighted average of the unit (i.e. stock) price at expiry. Making use of the put-call parity the insurance contract can also be interpreted as a traditional endowment insurance on the amount K, with an upside potential depending on the stochastically weighted average of the stock price. The quantity $\ln\left(\frac{S_T}{S_{t_i}}\right)$ represents the logreturn of the investment fund over the period $[t_i, T]$. With a minimum guaranteed rate of return of say, R, we are likely to find insurers calculate the guaranteed amount at time T according to, $K = \sum_{i=0}^{n-1} P_i\,(R)\, e^{R(T-t_i)}$. Here off course the P_i depend on the choice of R through the cost deduction scheme, hence we write $P_i\,(R)$.

Dependent on the insurer, the contract could also have a guarantee implicit if the insured dies before the end of the contract. The convention is adopted that payments to the policyholder are made at the end of the period. If the death is in the interval $[t_{i-1}, t_i)$ and let the guaranteed amount in that case be K_i, then the payoff of the contract would equal,

$$\max\left(\sum_{j=0}^{i-1} P_i \frac{S_{t_i}}{S_{t_j}}, K_i\right) = \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}} + \left(K_i - \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}}\right)^+ \tag{4}$$

which results in a payoff of the guarantee of,

$$\left(K_i - \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}}\right)^+ \tag{5}$$

Again a put option on a particular weighted average of the stock price at termination of the contract. The payoff is similar to that of a contract with maturity at time t_i. Valuation of this payoff is analogous to that of (3). In case of a specified guaranteed rate of return, the value of K_i is also likely to be determined by an algorithm similar to the one determining K.

At this point we introduce our generic form of the investment premium. It is necessary to make some assumption on the investment premiums since some cost deductions could depend on the fundvalue and hence make the investment premium stochastic. This would make the path dependency of the option even more complicated. Our assumption makes the dependence on the fundvalue explicit and maintains the structure of the payoff. Hence the payoff remains that of a put option on a stochastically weighted average of the stock price at maturity.

Let $FV_i = \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}}$ be the fundvalue at t_i. In practice these products have the following generic form of the investment premium[3] as a function of gross premium, GP_i, fixed costs, FC_i and fundvalue related cost deduction (including mortality charges), c_i, here the GP_i's, FC_i's and c_i's are deterministic[4],

$$\begin{aligned} P_i &= GP_i - FC_i - c_i FV_i && (6) \\ &= NP_i - c_i \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}} && (7) \end{aligned}$$

and hence the payoff of the guarantee is given by,

$$\left(K - \sum_{i=0}^{n-1} \left(NP_i - c_i \sum_{j=0}^{i-1} P_j \frac{S_{t_i}}{S_{t_j}} \right) \frac{S_T}{S_{t_i}} \right)^+ \tag{8}$$

In this form the path dependency of the option seems to get out of control, however rewriting gives back the original structure,

$$\left(K - \sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}} \right)^+ \tag{9}$$

where $\widetilde{P}_i = NP_i \cdot \left(1 + \sum_{k=1}^{n-i} \prod_{j=1}^{k} (-c_{n-j})\right)$ which is deterministic. If $c_i \equiv c$ then, $\widetilde{P}_i = NP_i \left(1 + \frac{-c-(-c)^{n-i}}{1+c}\right)$. The effect that we see is that because of the way the investment premiums are determined at each time t_i we already know how much of the gross premium minus fixed costs in terms of value at time T is devoted to fundvalue related loadings. The interpretation of the guarantee is still that of a put option on a particular stochastically weighted average of the stock price.

Recently attempts have been made to include stochastic mortality rates and a market price of mortality risk in the pricing of options embedded in life insurance products, see Milevsky and Promislow (2001) and Jiang, Milevsky and Promislow (2001). We adopt this approach here and give results in terms of risk-neutral mortality probabilities[5]. Furthermore we apply what is common practice and assume independence between mortality and the financial markets. This enables us to consecutively take expectations with respect to mortality and financial risk. In case of a linear dependence on mortality this results in that we can treat mortality probabilities as known constants. We should off course distinguish between risk-neutral mortality probabilities and those used to determine investment premiums and possibly the guaranteed amount at any date. The latter probabilities are known and are part of the product. For the former only estimates can be used. They do not play a role in the product but only in the pricing formula.

Let M_x denote the time of mortality of the policy holder, where x is the age of the policyholder at the issue of the contract and $D(t,T)$ be the price at time t of a zero coupon bond with maturity date T. Then using the

[3] The function we pose can always be considered as a first order approximation of the true investment premium (as a function of the fundvalue).

[4] They can be considered to parameterize the contract together with t_i and K_i, $i = 0, 1, ..., n$

[5] More formally, we give results in terms of expected mortality where expectation is taken under the risk neutral measure. Since we also assume mortality is independent of the financial markets, for mortality, this equals expectation under the T-forward measure. If one then adopts the view that mortality risk can be diverisfied by increasing the number of policies hence assume investors are risk-neutral with respect to mortality, risk neutral mortality probabilities equal real world mortality probabilities.

aforementioned assumptions on mortality we can write for the, time t, price G_t of the guarantee at maturity[6],

$$\begin{aligned} G_t &= D(t,T)\, E_t^{\mathbf{Q}^T}\left[\left(K-\sum_{i=0}^{n-1}\widetilde{P}_i\frac{S_T}{S_{t_i}}\right)^+ I_{[M_x>T]}\right] && (10)\\ &= D(t,T)\, E_t^{\mathbf{Q}}\left[I_{[M_x>T]}\right] E_t^{\mathbf{Q}^T}\left(K-\sum_{i=0}^{n-1}\widetilde{P}_i\frac{S_T}{S_{t_i}}\right)^+ && (11)\\ &= {}_{T-t}p_{x+t} D(t,T)\, E_t^{\mathbf{Q}^T}\left(K-\sum_{i=0}^{n-1}\widetilde{P}_i\frac{S_T}{S_{t_i}}\right)^+ && (12)\end{aligned}$$

and $I_{[\cdot]}$ is an indicator function. Furthermore ${}_{T-t}p_{x+t} \equiv E_t^{\mathbf{Q}} I_{[M_z>T]}$ is the $T-t$ year risk-neutral survival probability of an $x+t$ year old and x is the age of the insured at the start of the contract. Formula (11) illustrates the effect on the pricing formula of the assumption on mortality of independence with financial markets. In the remainder of the paper we assume only a guarantee at the maturity date of the contract is given. Guarantees given upon termination of the contract at an earlier date through mortality can be priced as follows. As before let an amount K_i be guaranteed when the insured dies in the interval $[t_{i-1},t_i)$ then, again using the assumptions on mortality to arrive at our results, the total price of the guarantee in the contract is,

$$G_t^* = \sum_{i=1}^{n} {}_{t_i-t}p_{x+t}\ {}_{(t_{i+1}-t_i)}q_{(x+t_i)} E_t^{\mathbf{Q}^{t_{i+1}}}\left(K_i-\sum_{j=0}^{i-1}\widetilde{P}_i\frac{S_{t_i}}{S_{t_j}}\right)^+ + {}_{T-t}\, p_{x+t} E_t^{\mathbf{Q}^T}\left(K-\sum_{j=0}^{n-1}\widetilde{P}_i\frac{S_T}{S_{t_j}}\right)^+ \qquad (13)$$

Where ${}_{(t_{i+1}-t_i)}q_{(x+t_i)} = E_t^{\mathbf{Q}}\left[M_x \in (t_{i-1},t_i]\,|\,M_x>t_{i-1}\right]$, the risk-neutral probability of mortality in the time interval $(t_{-1},t_i]$ given that the insured has survived until time ${t_{i-1}}$[7]. The elements in the sum correspond to guarantees upon death and the lone term to the guarantee at maturity. So if not only a guarantee at maturity is given but also at each intermediate point upon death of the insured, the guarantee can be interpreted as a portfolio of put options (with different maturities) on a stochastically weighted average of the stock price.

Formulas (12) and (13) give the general price of the guarantee in a Regular Premium Unit Linked contract. The next section shows the similarity of this price with that of an Asian option. Hence using the general pricing formula (12) as a starting point, guarantee prices can be obtained by extending pricing methods for Asian options to include UL Guarantees.

4 Relationship with Asian Options

The dependency of the payoff on the stock price at different time points leads to an analogy with Asian options. An average price Asian is an option on the average of the stock price at different time points. The strong

[6] Starting from the well known risk neutral valuation formula, in which the money market account is the numeraire, we can use the Change of Numeraire Theorem to find the price of the guarantee if the zero bond with maturity T is used as numeraire. These type of numeraire changes were first introduced by El Karoui and Rochet (1989) and Jamshidian (1991).

[7] This follows since $E_t^{\mathbf{Q}}\left[I_{[M_x\in(t_{i-1},t_i]]}\right] = E_t^{\mathbf{Q}}\left[I_{M_x\in(t_{i-1},t_i]}\,|\,M_x>t_{i-1}\right] E_t^{\mathbf{Q}}\left[I_{[M_x>t_{i-1}]}\right]$. In a general model of mortality these probabilities are time inhomogeneous functions. A simple case for example in which one can already see this effect is a model in which mortality exhibits a decreasing trend. This is a simple model for the longevity effect. The one-year mortality probability of a sixty year old at this moment is not the same as it will be in 30 years. The latter will (most likely) be lower.

relationship between an Asian Option and the guarantee we consider can be summarized in the following proposition,

Proposition 1 *Assume markets are arbitrage free and complete. Also assume that stock volatility, σ_S, and the short rate, r, are constant. Consider the regular premium UL contract with $\widetilde{P}_i = S_0 \,/\, n$, and without loss of generality assume, $t_i - t_{i-1} = 1$. Then, ignoring mortality, we have equality between the prices of an average price Asian Put, with strike K, and the Rate of Return Guarantee of the UL contract with the same strike. More formally,*

$$e^{-rT} E^{\mathbf{Q}} \left(K - \frac{1}{n} \sum_{i=1}^{n} S_{t_i} \right)^{+} = e^{-rT} E^{Q} \left(K - \sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}} \right)^{+} \tag{14}$$

Proof. If we can establish equality in distribution under $\mathbf{Q}$ between $\frac{1}{n}\sum_{i=1}^{n} S_{t_i}$ and $\sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}}$ we are done. We know the stock price is given by,

$$S_t = S_0 \exp\left(\left[r - \frac{1}{2}\sigma_S^2 \right] t + \sigma_S W_t \right)$$

where W_t is a brownian motion under $\mathbf{Q}$. Consider the vectors of lognormal random variables, $A = \left[\begin{array}{ccc} \frac{S_{t_1}}{n} & \cdots & \frac{S_{t_n}}{n} \end{array} \right]'$ and $UL = \left[\begin{array}{ccc} \widetilde{P}_i \frac{S_T}{S_{t_{n-1}}} & \cdots & \widetilde{P}_i \frac{S_T}{S_{t_0}} \end{array} \right]'$. Equality in distribution of A and UL implies equality in distribution of $A \cdot \mathbf{1}$ and $UL \cdot \mathbf{1}$, where $\mathbf{1}$ is a $(n \times 1)$ vector of ones. Since these are lognormal r.v.'s we have equality in distribution if we can show the first two moments are equal under the risk neutral measure. Using $\widetilde{P}_i = S_0 \,/\, n$ and $t_i = i$, straightforward calculations give,

$$E\left(A\right)_{(i)} = E\left(UL\right)_{(i)} = \frac{S_0}{n} e^{ri}$$

where $E\left(A\right)_{(i)}$ and $E\left(UL\right)_{(i)}$denote the i$^{\text{th}}$ element of the first moment of A and UL resp. which is a vector. For the second moment we have,

$$E\left(A\,A'\right)_{(i,j)} = E\left(\left[UL\right]\,\left[UL\right]'\right)_{(i,j)} = e^{r(i+j)+\sigma_S^2\,\min(i,j)}$$

where $E\left(A\,A'\right)_{(i,j)}$ and $E\left(\left[UL\right]\,\left[UL\right]'\right)_{(i,j)}$ denote the element in the i$^{\text{th}}$ row and j$^{\text{th}}$ column of the second (non central) moment of A and UL resp. This completes the proof. ■

Proposition 1 illustrates the strong similarities between Asian Options and the UL Guarantee. It shows that in this perfect Black-Scholes world there is exactly the same randomness in the i^{th} fixing of the stock price as there is in the $(n-i)^{\text{th}}$ premium payment. We can generalize this result to allow for stationary stochastic volatility. It can however not be generalized to allow for stochastic interest rates or for any non stationary time dependence in both volatility or interest rates. Instead the randomness in interest rates is complementary in those two contracts. The Asian is sensitive for stochastic interest rates over the intervals $[0, t_i]$ $i = 1, 2, ..., n$ whereas the guarantee is sensitive over the intervals $[t_i, T]$, $i = 0, 1, ..., n-1$. We can say that within the simplified setting of proposition 1 time runs in opposite directions for the two types of options. The UL Guarantee is an Asian which starts at time T and expires at zero. In the Asian option the risk of each individual term in the summation, S_{t_i}, runs from time zero to t_i. This is mainly stock price risk, represented through the choice of (sum of the stochastic parts of the) forward stock return as conditioning variable. In our UL contract we can split the risk of each individual term, $S_T \,/\, S_{t_i}$, in interest rate risk, related to some forward bond price, from time zero to t_i and forward stock price risk from time t_i to T.

This latter observation should also provide direction in how to hedge these type of options in a 'quick and dirty' way. At the start of the contract the risk is mainly interest rate related. This risk could be hedged by using caps / floors. However as time progresses the risk becomes more and more stock related. The instruments that come to mind are (forward starting) stock options. Especially forward starting options seem to be a good hedge. It is shown in the next section that the implied volatility of these options also arise in the volatility of the fundvalue. Next we analyze the effect of stochastic interest rates on the guarantee value using the method by Levy (1992). More accurate approximation methods are considered in section 7.

5 Effect of Stochastic Interest Rates

In this section we analyze the effect of stochastic interest rates on the guarantee value. This is done within the context of the Levy (1992) approximation. This consists of approximating the distribution of $\sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}}$ by a lognormal one with the same mean and variance. We choose the Levy approximation because it is simple and allows for a nice financial economic interpretation of the effect of stochastic interest rates. Under the assumption of lognormality of the weighted sum of stock prices, the effect of stochastic interest rates on the guarantee value can be inferred from the effect of stochastic interest rates on the first two moments. For this purpose we derive the first and second moment of $\sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}}$ under the T-Forward measure. Note that this also gives us the variance of $\sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}}$ since for a lognormal variable Y we have that $\mathrm{Var}(\ln Y) = \ln \frac{EY^2}{(EY)^2}$. As the first moment doesn't show the effect explicitly, this implies the effect of stochastic interest rates is isolated in the second moment. This off course holds only under the assumption of lognormality. The Levy approximation is not very accurate for the maturities typical for UL contracts. Therefore we use our results in this section mainly for expositional purposes. However we have approximate lognormality and the effect is similar for the more accurate methods of section 7. Note that we don't restrict ourselves to formulas for the Levy approximation at the time of writing of the contract. Approximate prices for the guarantee obtained through the Levy approximation during the time of the contract are implicit in the expressions for $E_t^{\mathbf{Q}^T} \sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}}$ and $\mathrm{Var}_t^{\mathbf{Q}^T} \ln \sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}}$. Results on the first moment are presented in Theorem 2. Lemma 3 and Corollary 4 give the (partial) expressions for the second moment. Without any modeling assumptions on stock or bond price movements from the general Change of Numeraire Theorem results:

Theorem 2 *Let $t_j \leq t < t_{j+1}$, $j \in \{0, 1, 2, ..., n-1\}$. Let the assumptions of the Change of Numeraire Theorem hold. Then the conditional first moment of the fundvalue at maturity is given by,*

$$E_t^{\mathbf{Q}^T} \left[\sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}} \right] = \sum_{i=0}^{j} \widetilde{P}_i \frac{F_t^T}{S_{t_i}} + \sum_{i=j+1}^{n-1} \widetilde{P}_i \frac{D\left(t, t_i\right)}{D\left(t, T\right)} \tag{15}$$

Proof. Since from the martingale property of numeraire adjusted asset prices and the Tower Law of conditional expectation we have, for $t \le t_i$

$$\begin{aligned} E_t^{\mathbf{Q}^T} \frac{S_T}{S_{t_i}} &= E_t^{\mathbf{Q}^T} \left[\frac{1}{S_{t_i}} E_{t_i}^{\mathbf{Q}^T} \left[\frac{S_T}{D(T,T)} \right] \right] = E_t^{\mathbf{Q}^T} \left[\frac{1}{S_{t_i}} \frac{S_{t_i}}{D(t,T)} \right] \\ &= E_t^{\mathbf{Q}^T} \frac{1}{D(t_i,T)} = E_t^{\mathbf{Q}^T} \frac{D(t_i,t_i)}{D(t_i,T)} = \frac{D(t,t_i)}{D(t,T)} \end{aligned} \tag{16}$$

and for $t > t_i$,

$$\begin{aligned} E_t^{\mathbf{Q}^T} \frac{S_T}{S_{t_i}} &= \frac{1}{S_{t_i}} E_t^{\mathbf{Q}^T} \frac{S_T}{D(T,T)} = \frac{1}{S_{t_i}} F_t^T \\ &= \frac{S_t}{S_{t_i} D(t,T)} \end{aligned} \tag{17}$$

■

The interpretation of (16) is clear. The expectation conditional on the information until time t, under the T-Forward measure, of the return on the asset in the period $[t_i, T]$, where $t \le t_i$, equals the continuously compounded forward rate for that period. In a non stochastic interest rate environment we have $E_t^{\mathbf{Q}^T} \frac{S_T}{S_{t_i}} = E_t^{\mathbf{Q}} \frac{S_T}{S_{t_i}} = \exp\left(\int_{t_i}^T r_s ds\right)$. The interpretation of (17) is the following: the quantity $\frac{S_t}{S_{t_i}}$ represents the return which is already locked in at time t, the quantity $\frac{1}{D(t,T)} = \frac{D(t,t)}{D(t,T)}$ has the same interpretation as (16).

Now for the second moment we do need some assumptions on price dynamics. Because all observable volatilities in the market are those of forward quantities, like forward stock prices, forward LIBOR and Swap rates, we start modelling at this level. Furthermore we assume the volatility of forward stock and bond prices to be a deterministic function of time. In a parameterized environment this would amount to taking a lognormal stock price process and a Gaussian interest rate model. We do not assume the volatilities to be constant to be able to adapt to variation in implied volatilities. Especially since we make a connection between our lognormal model of the economy and the popular LIBOR Market Model in the next section. We assume the T-Forward stock price, F_t^T, and the T-Forward bond price, $D^T(t,S)$, to follow the dynamics,

$$dF_t^T = \sigma_F(t) F_t^T dW_t^T \tag{18}$$

$$dD^T(t,S) = \sigma_{D_S^T}(t) D^T(t,S) dW_t^{ST} \tag{19}$$

where W_t^T and W_t^{ST} (for all relevant S) are Brownian Motions under the T-Forward measure[8] and σ_F and $\sigma_{D_S^T}$ are deterministic functions of time. Correlations between those Brownian Motions are given by $dW_t^T dW_t^{ST} = \rho_{F^T D_S^T}$ and $dW_t^{ST} dW_t^{UT} = \rho_{D_S^T D_U^T}$.

Start with writing the trivial result,

$$E_t^{\mathbf{Q}^T} \left(\sum_{i=0}^{n-1} \widetilde{P}_i \frac{S_T}{S_{t_i}} \right)^2 = \sum_{i=0}^{n-1} \widetilde{P}_i^2 E_t^{\mathbf{Q}^T} \left(\frac{S_T}{S_{t_i}} \right)^2 + 2 \sum_{i=0}^{n-1} \sum_{i<j} \widetilde{P}_i \widetilde{P}_j E_t^{\mathbf{Q}^T} \frac{S_T^2}{S_{t_i} S_{t_j}} \tag{20}$$

now we have the following lemma,

[8] The martingale property of F_t^T and $D^T(t,S)$ follows from the Change of Numeraire Theorem in combination with the no-arbitrage condition.

Lemma 3 *Let again the conditions of the Change of Numeraire Theorem hold. Then for $t \leq t_i < t_j$, we have*

$$\begin{aligned} E_t^{\mathbf{Q}^T} \frac{S_T^2}{S_{t_i} S_{t_j}} &= D^T(t,t_i) D^T(t,t_j) \cdot \exp\left(\int_t^{t_i} \rho_{D_i^T D_j^T}(s) \sigma_{D_i}(s) \sigma_{D_j}(s)\, ds \right) \\ & \exp\left(\int_{t_i}^{t_j} \rho_{F^T D_j^T}(s) \sigma_{F^T}(s) \sigma_{D_j^T}(s)\, ds + \int_{t_j}^{T} \sigma_{F^T}^2(u, S_u)\, du \right) \end{aligned} \tag{21}$$

Proof. For $t \leq t_i < t_j$, using the Tower Law of conditional expectation as in theorem 2, we have

$$E_t^{\mathbf{Q}^T} \frac{S_T^2}{S_{t_i} S_{t_j}} = E_t^{\mathbf{Q}^T} \frac{1}{S_{t_i}} E_{t_i}^{\mathbf{Q}^T} \frac{1}{S_{t_j}} E_{t_j}^{\mathbf{Q}^T} S_T^2 \tag{22}$$

now from (18) we obtain for the last part of this expression,

$$E_{t_j}^{\mathbf{Q}^T} S_T^2 = E_{t_j}^{\mathbf{Q}^T} \left(F_T^T\right)^2 = \left(F_{t_j}^T\right)^2 \exp\left(\int_{t_j}^{T} \sigma_F^2(u, S_u)\, du \right)$$

Continuing we obtain,

$$\begin{aligned} E_t^{\mathbf{Q}^T} \frac{S_T^2}{S_{t_i} S_{t_j}} &= \exp\left(\int_{t_j}^{T} \sigma_F^2(u, S_u)\, du \right) E_t^{\mathbf{Q}^T} \frac{1}{S_{t_i}} E_{t_i}^{\mathbf{Q}^T} \frac{1}{S_{t_j}} \left(F_{t_j}^T\right)^2 \\ &= \exp\left(\int_{t_j}^{T} \sigma_F^2(u, S_u)\, du \right) E_t^{\mathbf{Q}^T} \frac{1}{S_{t_i}} E_{t_i}^{\mathbf{Q}^T} F_{t_j}^T D^T(t_j, t_j) \end{aligned} \tag{23}$$

Now both $F_{t_j}^T$ and $D^T(t_j, t_j)$ are martingales under the T-forward measure, using the solutions to (18) and (19) gives,

$$E_{t_i}^{\mathbf{Q}^T} F_{t_j}^T D^T(t_j, t_j) = F_{t_i}^T D^T(t_i, t_j) \exp\left(\int_{t_i}^{t_j} \rho_{F^T D_j^T}(s) \sigma_{F^T}(s) \sigma_{D_j^T}(s)\, ds \right) \tag{24}$$

Plugging this expression in (22) gives,

$$\begin{aligned} E_t^{\mathbf{Q}^T} \frac{S_T^2}{S_{t_i} S_{t_j}} &= \exp\left(\int_{t_j}^{T} \sigma_{F^T}^2(u, S_u)\, du \right) \exp\left(\int_{t_i}^{t_j} \rho_{F^T D_j^T}(s) \sigma_{F^T}(s) \sigma_{D_j^T}(s)\, ds \right) E_t^{\mathbf{Q}^T} \frac{1}{S_{t_i}} F_{t_i}^T D^T(t_i, t_j) \\ &= \exp\left(\int_{t_j}^{T} \sigma_{F^T}^2(u, S_u)\, du \right) \exp\left(\int_{t_i}^{t_j} \rho_{F^T D_j^T}(s) \sigma_{F^T}(s) \sigma_{D_j^T}(s)\, ds \right) E_t^{\mathbf{Q}^T} D^T(t_i, t_i) D^T(t_i, t_j) \end{aligned} \tag{25}$$

Again using (18) and (19), we obtain,

$$E_t^{\mathbf{Q}^T} D^T(t_i, t_i) D^T(t_i, t_j) = D^T(t, t_i) D^T(t, t_j) \exp\left(\int_t^{t_i} \rho_{D_i^T D_j^T}(s) \sigma_{D_i^T}(s) \sigma_{D_j^T}(s)\, ds \right) \tag{26}$$

This gives the desired result. ■

We can split the long expression in (21) in three parts corresponding to the three integrals. The argument of each integral is the instantaneous correlation of the ´returns´ of $S_T\,/\,S_{t_i}$ and $S_T\,/\,S_{t_j}$ at the relevant time points. The first integral ranging from zero to t_i corresponds to the correlation between the normalized bonds with maturity t_i and t_j. This can be explained by the fact that the risk in the ´return´ on the quantities $S_T\,/\,S_{t_i}$ and $S_T\,/\,S_{t_j}$ is represented by the corresponding T-forward bond processes. Second $\int_{t_i}^{t_j} \rho_{FD_j}(s)\,\sigma_F(s)\,\sigma_{D_j}(s)\,ds$, represents the covariance between the forward asset price and $\frac{D(t,t_j)}{D(t,T)}$. Since after t_i, S_{t_i} is fixed and the return risk of $S_T\,/\,S_{t_i}$ is now represented by the T-forward asset price. The risk is now twofold and measured by the quadratic covariation of the forward stock and forward bond price processes. Finally after t_j we are left with pure equity risk, i.e. both S_{t_i} and S_{t_j} are known, as $\int_{t_j}^{T} \sigma_F^2(u,S_u)\,du$ represents the implied volatility of a forward start option.

At this point the effect of stochastic interest rates becomes clear. When interest rates are deterministic there's no risk in the return on $S_T\,/\,S_{t_i}$ in the interval $[0\,,\,t_i]$ so the first and second integral in (21) should equal zero[9]. These two integrals are precisely the effect of stochastic interest rates on the volatility of the fundvalue and hence (in the lognormal approximation) on the guarantee value.

Lemma 3 gives us the expression for $E_t^{\mathbf{Q}^T}\frac{S_T^2}{S_{t_i}S_{t_j}}$ assuming $t < t_i < t_j$. Modifications of this result for e.g. $t > t_i$ are easy to obtain. This gives us:

Corollary 4 *Let $t_k < t < t_{k+1}$. Then under the assumption of deterministic volatilities for the T-forward asset price and T-forward bond prices we obtain for the second moment of $\sum_{i=1}^{n}\frac{S_T}{S_{t_i}}$,*

$$
\begin{aligned}
E_t^{\mathbf{Q}^T}\left(\sum_{i=0}^{n-1}\widetilde{P}_i\frac{S_T}{S_{t_i}}\right)^2 &= \sum_{i=0}^{k}\widetilde{P}_i^2\left[\frac{F_t^T}{S_{t_i}}\right]^2\exp\left(\int_t^T\sigma_F^2(u,S_u)\,du\right)+ \\
&\sum_{i=k+1}^{n}\widetilde{P}_i^2\left[D^T(t,t_i)\right]^2\exp\left(\int_t^{t_i}\sigma_{D_i}^2(s)\,ds+\int_{t_i}^{T}\sigma_F^2(u,S_u)\,du\right)+ \\
&2\sum_{i=0}^{k}\sum_{j=i+1}^{k}\widetilde{P}_i\widetilde{P}_j\frac{\left(F_t^T\right)^2}{S_{t_i}S_{t_j}}\exp\left(\int_t^T\sigma_F^2(u,S_u)\,du\right)+ \\
&2\sum_{i=0}^{k}\sum_{j=k+1}^{n-1}\widetilde{P}_i\widetilde{P}_j\frac{F_t^TD^T(t,t_j)}{S_{t_i}}\cdot \\
&\exp\left(\int_t^{t_j}\rho_{FD_j}(s)\,\sigma_F(s)\,\sigma_{D_j}(s)\,ds+\int_{t_j}^{T}\sigma_F^2(u,S_u)\,du\right)+ \\
&2\sum_{i=k+1}^{n-1}\sum_{j=i+1}^{n-1}\widetilde{P}_i\widetilde{P}_jD^T(t,t_i)\,D^T(t,t_j)\cdot \\
&\exp\left(\int_t^{t_i}\rho_{D_iD_j}(s)\,\sigma_{D_i}(s)\,\sigma_{D_j}(s)\,ds+\int_{t_i}^{t_j}\rho_{FD_j}(s)\,\sigma_F(s)\,\sigma_{D_j}(s)\,ds+\int_{t_j}^{T}\sigma_F^2(u,S_u)\,du\right)
\end{aligned}
\tag{27}
$$

[9] This also follows since with deterministic interest rates forward bond volatilities are zero.

The expression for the second moment consists of five summations. The first summation consists of those terms of the first summation in (20) for with $t < t_i$ the second summation consists of the terms with $t > t_i$.The third to fifth correspond to parts of the double summations in (20) with $t_i < t_j < t$, $t_i < t < t_j$ and $t < t_i < t_j$ respectively.

5.1 Convexity Correction Interpretation of Stochastic Interest Rates

Now we have isolated the effect of stochastic interest rates we will show how it is related to change of measure and convexity correction. We give an interpretation to the expectations $E_{t_i}^{\mathbf{Q}^T} F_{t_j}^T D^T(t_j, t_j)$ and $E^{\mathbf{Q}^T} \frac{D(t_i,t_i)}{D(t_i,T)} \frac{D(t_i,t_j)}{D(t_i,T)}$ (assume again that $i < j$). With regard to the first expectation we have, using the Change of Numeraire Theorem,

$$
\begin{aligned}
E_{t_i}^{\mathbf{Q}^T} F_{t_j}^T D^T(t_j, t_j) &= \frac{D(t_i, t_j)}{D(t_i, T)} E_{t_i}^{\mathbf{Q}^{t_j}} F_{t_j}^T \\
&= \frac{D(t_i, t_j)}{D(t_i, T)} \widetilde{F}_{t_i}^T \\
&= \frac{D(t_i, t_j)}{D(t_i, T)} F_{t_i}^T \cdot \text{Convexity Correction}
\end{aligned}
$$

Where $\widetilde{F}_{t_i}^T$ is the convexity corrected forward asset price. Comparing with (23) and (24) we see that the expression $\exp\left(\int_{t_i}^{t_j} \rho_{F^T D_j^T}(s)\, \sigma_{F^T}(s)\, \sigma_{D_j^T}(s)\, ds\right)$ is the convexity correction arising from taking the expectation of the T-Forward Asset Price 'under the wrong measure' (i.e. not under the measure associated with the asset which is used to normalize the stock price).

To interpret the second expectation note that $\frac{D(t_i,T)}{D(t_i,t_j)} \frac{D(0,t_j)}{D(0,T)}$ is the Radon-Nikodym derivative for a change of measure from the T-Forward measure to the t_j-Forward measure, then

$$
\begin{aligned}
E_t^{\mathbf{Q}^T} D^T(t_i, t_i)\, D^T(t_i, t_j) &= E_t^{\mathbf{Q}^T} \frac{D(t_i, t_i)}{D(t_i, T)} \frac{D(t_i, t_j)}{D(t_i, T)} \\
&= \frac{D(t, t_j)}{D(t, T)} \left[1 + E_t^{\mathbf{Q}^{t_j}} L_{t_i T}(t_i)\right] \\
&= \frac{D(t, t_j)}{D(t, T)} \left[1 + \widetilde{L}_{t_i T}(t_i)\right]
\end{aligned}
$$

Where $L_{t_j T}(t) = \frac{D(t,t_j) - D(t,T)}{D(t,T)}$, the 'forward LIBOR rate for the time period (t_i, T) and $\widetilde{L}_{t_i T}(t_i)$ is the convexity corrected forward LIBOR rate. Now because $i < j$ and hence $L_{t_i T}(t_i)$ is known at t_i, $D(t, t_j)\, E^{\mathbf{Q}^{t_j}} L_{t_i T}(t_i)$ is the value of a payment of $L_{t_i T}(t_i)$, at a later time t_j. It can be seen from (25) and (26) that the expression $\exp\left(\int_t^{t_i} \rho_{D_i^T D_j^T}(s)\, \sigma_{D_i^T}(s)\, \sigma_{D_j^T}(s)\, ds\right)$ can then be explained as the convexity correction arising from taking the expectation of the LIBOR payment 'under the wrong measure'[10]. Note that with deterministic interest rates there would not be 'a wrong measure' as then $\mathbf{Q}^T = \mathbf{Q}^{t_i} = \mathbf{Q}^{t_j} = \mathbf{Q}$ and hence there would not be any convexity correction.

Since $\sigma_F(t, S_t)$ is the instantaneous volatility of the forward asset price at time t and $\sqrt{\int_s^T \sigma_F^2(u, S_u)\, du \;/\; (T - s)}$ can be given the interpretation of the implied volatility of a forward start option with maturity date T starting

[10] This is mentioned only for interpretation, these kind of LIBOR payments are not traded in the market.

at $s > t$. This motivates the use of forward starting options in hedging these contracts. We will illustrate the effect of the convexity correction in section 9.

6 Calibration of Forward Bond Volatilities

Next we are interested in obtaining prices of guarantees which are consistent with the LIBOR Market Model (LMM). Nowadays the LMM is the dominant term structure model to price interest rate derivatives. One of its advantages is that it has the ability to fit cap and swaption prices better than e.g. short rate models. Therefor it is of interest to obtain stock option prices (like the UL guarantee under consideration in this paper) which are consistent with the LMM. To do this we link the forward bond volatilities in (19) to LIBOR rate volatilities. To rewrite forward bond volatility we use a method developed and tested by both Hull and White (2000) and Brace et al. (2001). They use it to rewrite swap rate volatilities in terms of LIBOR rate volatilities. They conclude that, although assuming both swap and LIBOR rates lognormal is mutually inconsistent, swap rates are approximately lognormal in the LMM. Basing ourselves upon their arguments we conclude that forward bond prices are also approximately lognormal in the LMM. This gives us accurate approximations to the forward bond volatilities which we should use in our lognormal model to give guarantee prices consistent with observed cap and swaption prices.

The forward LIBOR rate at time t starting from T with maturity S is defined as,

$$L_{TS}(t) = \frac{1}{\alpha_{TS}} \left(\frac{D(t,T) - D(t,S)}{D(t,S)} \right) \tag{28}$$

Where α_{TS} is the daycount fraction for the period (T, S). In the interest rate market only LIBOR rates with a specific tenor $S - T$ are traded. Let this tenor be $\Delta_L T$ and define $L_j(t) = L_{T_j^L T_{j+1}^L}$ and $\alpha_j^L = \alpha_{T_j^L T_{j+1}^L}$, where $T_j^L = j\Delta_L T$, $j = 0, 1, ..., N+1$ are the so called *reset dates*. Now an m-factor version of the LIBOR Market Model (LMM) developed independently by Miltersen, Sandmann and Sonderman (1997), Brace, Gatarek and Musiela (1997) and Jamshidian (1998) poses the following dynamics for the forward LIBOR rates L_j,

$$dL_j(t) = \sum_{q=1}^{m} \sigma_{L_j}^q(t) L_j(t) \, dW_t^{j+1,q} \tag{29}$$

Where the $W^{j+1,q}$'s are Brownian Motions (which can be assumed uncorrelated, since we can rotate factors) under $\mathbf{Q}^{T_{j+1}}$, the T_{j+1}^L-Forward measure and the $\sigma_{L_j}^q(t)$'s are deterministic functions of time. The popularity of the LMM stems from the fact that the parameters of the model (the $\sigma_{L_j}^q(t)$'s) can be chosen such that the model exactly matches observed cap prices in the market.

A full factor LMM assumes m, the number of Brownian Motions, equals N, the number of (uncertain) LIBOR rates under consideration. Hence we can construct the model such that each LIBOR rate is driven by its own Brownian Motion. Furthermore, assuming a stationary volatility and correlation structure, we have, $\sigma_{L_j}^q(t) = \sigma(T_j - t)$ and $dW_t^{j+1} dW_t^{i+1} = \rho(T_j - t, T_i - t)$. Summarizing the model becomes,

$$\begin{aligned} dL_j(t) &= \sigma(T_j - t) L_j(t) \, dW_t^{j+1} \qquad , \; j = 1, ..., N \\ dW_t^{j+1} dW_t^{k+1} &= \rho(T_j - t, T_k - t) \end{aligned} \tag{30}$$

This is the model we use to obtain numerical results later. We will now discuss how to obtain approximate forward bond volatilities in this model. Results for other parameterizations of the LMM can be obtained in a similar manner.

We assume the dates of the forward bond prices under consideration in the pricing of the guarantee coincide at least partly with the reset dates of LIBOR rates[11]. Define $\widetilde{t} := \{\{t_i\}_{i=1}^{n}\}$ and $\widetilde{T_L} := \left\{\{T_i^L\}_{i=1}^{N+1}\right\}$ then our assumption boils down to assuming $\widetilde{t} \subseteq \widetilde{T_L}$. Next define, $G_{j,i} := \left\{j : T_j^L \in [t_i, T)\right\}$. Then the relationship with bond prices is[12],

$$D^T(t, t_i) = \frac{D(t, t_i)}{D(t, T)} = \prod_{j \in G_{j,i}} \left[1 + \alpha_j^L L_j(t)\right] \tag{31}$$

Now we have,

$$\frac{1}{D^T(t, t_i)} \frac{\partial D^T(t, t_i)}{\partial L_j(t)} = \frac{\alpha_j^L}{1 + \alpha_j^L L_j(t)}$$

Applying Itô to (31), bearing in mind (29) gives[13],

$$\begin{aligned} dD^T(t, t_i) &= ...dt + \sum_{j \in G_{j,i}} \frac{1}{D^T(t, t_i)} \frac{\partial D^T(t, t_i)}{\partial L_j(t)} \sigma(T_j - t) L_j(t) D^T(t, t_i) dW_t^{j+1} \\ &= ...dt + \sum_{j \in G_{j,i}} \frac{\alpha_j^L \sigma(T_j - t) L_j(t)}{1 + \alpha_j^L L_j(t)} D^T(t, t_i) dW_t^{j+1} \end{aligned}$$

This leads to a variance rate $\sigma^2_{D_i^T}$ of $D^T(t, t_i)$ of,

$$\sigma^2_{D_i^T}(t) = \sum_{j \in G_{j,i}} \left(\frac{\alpha_j^L \sigma(T_j - t) L_j(t)}{1 + \alpha_j^L L_j(t)}\right)^2 + \sum_{j \in G_{j,i}} \sum_{k > j} \rho(T_j - t, T_k - t) \left(\frac{\alpha_j^L \sigma(T_j - t) L_j(t)}{1 + \alpha_j^L L_j(t)}\right) \left(\frac{\alpha_k^L \sigma(T_k - t) L_k(t)}{1 + \alpha_k^L L_k(t)}\right) \tag{32}$$

The approximation Hull and White suggest in their paper is to approximate (32), which is a stochastic quantity, by a constant, replacing the forward LIBOR rate by their time zero values, leading to an approximate variance rate of,

$$\begin{aligned} \widetilde{\sigma}^2_{D_i^T}(t) &= \sum_{j \in G_{j,i}} \left(\frac{\alpha_j^L \sigma(T_j - t) L_j(0)}{1 + \alpha_j^L L_j(0)}\right)^2 + \\ &\quad \sum_{j \in G_{j,i}} \sum_{k > j} \rho(T_j - t, T_k - t) \left(\frac{\alpha_j^L \sigma(T_j - t) L_j(0)}{1 + \alpha_j^L L_j(0)}\right) \left(\frac{\alpha_k^L \sigma(T_k - t) L_k(0)}{1 + \alpha_k^L L_k(0)}\right) \end{aligned} \tag{33}$$

Note that we are effectively approximating $\frac{\alpha_j^L L_j(t)}{1+\alpha_j^L L_j(t)}$ with it's zero value. This is in line with the approach of Brace et al. (2001) and Brace and Womersley (2000) who observe that $\frac{\alpha_j^L L_j(t)}{1+\alpha_j^L L_j(t)}$ is a low variance $\mathbf{Q}^{T_{j+1}}$-martingale and use this to approximate swaption volatilities in the LMM. We use it to approximate forward bond volatilities in the LMM. The above expression (33) gives the variance rate at time t. A frequent assumption in

[11] Implicit in this assumption is that we consider pricing at payment dates only.

[12] We could do the exact same thing in the context of the Swap Market Model (SMM). However the LMM seems to be preferable over the SMM in terms of out of sample pricing performance (see De Jong et al., 2001).

[13] We do not calculate the drift term, or equivalently specify the Brownian Motion, since it is irrelevant for our purposes. We are interested in the quadratic variation terms.

applications is that correlation and volatility are constant between reset dates. For example we can approximate the variance of $D^T(t, t_i)$ over the interval $[0, T_l]$ as:

$$\int_0^{T_l} \sigma^2_{D_i^T}(s)\, ds \approx \sum_{k=0}^{l-1} \alpha_k^L \widetilde{\sigma}^2_{D_i^T}(T_k)$$

which as what we use in section 9.

The literature on the subject of LMM calibration is rapidly growing. Ideally one would calibrate the model not only to caps but also to swaption prices. This, and for example the inclusion of more factors, will give a better fit to the correlation structure of the forward LIBOR rates. In general, to calibrate a LMM we must specify volatility functions for the LIBOR rates and a correlation matrix for the Brownian Motions such that the option prices are fitted by the model. Important with non-vanilla products, such as ours, is that the calibrated model fits the correlation structure of the relevant rates well. Therefore in our numerical examples we use calibration results for a multifactor model. Also important in the calibration process are the users goals. Usually traders prefer exact calibration since they want their models to replicate observed prices exactly, whereas for risk management (or reserving) purposes the user might want to protect against overfitting and use a parsimonious model and non-exact calibration for better out of sample performance. The current state of the art LMM calibration is based on semidefinite programming techniques, see Brace and Womersley (2000) and d ´Aspremont (2002).

7 Bounds on the price of the guarantee

In the literature several techniques exist to bound the price of an arithmetic Asian Option. A very accurate lower bound to the price of an Asian Option is the method of Rogers and Shi (1995), recently generalized to allow for stochastic interest rates by Nielsen and Sandmann (2002a) and applied to an Equity Linked contract in Nielsen and Sandmann (2002b). In this section we provide additional results to adapt their method to the case of guarantees in regular premium UL contracts in the model (18) and (19). As the method by Rogers and Shi only provides a tight lower bound I generalize the upper bound by Thompson (1998) to the case of the regular premium UL Guarantee at the same time allowing interest rates to be stochastic. Results on the tightness of these bounds are presented in appendix B.

7.1 Derivation of Lower Bound

The method uses a conditioning variable to derive a lower bound to the price which is extremely tight. The approach by Rogers and Shi develops according to the following steps. First choose Z to be a standard normal random variable. Then calculate $E_t^{\mathbf{Q}^T}\left\{K - \sum_{i=0}^{n-1} \frac{S_T}{S_{t_i}} \mid Z\right\}$ by summing individual expectations[14]. This

[14] Since we are in the lognormal framework in these calculations one can use standard relations for conditional expectations of normally distributed random variables.

expectation will be a convex function in Z (since sum of convex functions is itself convex and the individual expectations are convex functions). Second, using the ideas of Jamshidian (1989) split the option on a sum into a portfolio of options by solving, $E_t^{\mathbf{Q}^T}\left\{K-\sum_{i=0}^{n-1}\frac{S_T}{S_{t_i}}\mid Z\right\}=0$ for Z. For details see Nielsen and Sandmann (2002a), definition 1 and theorem 1.

In this case the second step of the approach by Nielsen and Sandmann will be exactly the same. With the results from the first step we will be able to do a Jamshidian decomposition on the conditioning variable Z. This simplifies the pricing problem from one of an option on a sum to that of a sum of options. The first step is however a bit different in this case.

To obtain a price for the guarantee we are interested in the following expectation and lower bound,

$$E_t^{\mathbf{Q}^T}\left[K-\sum_{i=0}^{n-1}\frac{S_T}{S_{t_i}}\right]^+ \geq E_t^{\mathbf{Q}^T}\left[\left[E_t^{\mathbf{Q}^T}\left\{K-\sum_{i=0}^{n-1}\frac{S_T}{S_{t_i}}\mid Z\right\}\right]^+\right] \tag{34}$$

Hence we must calculate $E_t^{\mathbf{Q}^T}\left\{\sum_{i=0}^{n-1}\frac{S_T}{S_{t_i}}\mid Z\right\}$. At this point we introduce the following shorthand notation,

$$\begin{aligned}
\int_t^T \overline{\sigma}_i(s)\,d\overline{W}_s^i &\equiv \int_t^{\max(t,t_i)} \sigma_{D_i^T}(s)\,dW_s^{iT} + \int_{\max(t,t_i)}^T \sigma_F(s)\,dW_s^T \\
E_t^{\mathbf{Q}^T}\frac{S_T}{S_{t_i}} &\equiv \overline{\mu}_i(t) = \frac{D(t,t_i)}{D(t,T)} I_{[0,t_i]}(t) + \frac{S_T\,/\,D(t,T)}{S_{t_i}} I_{[t_i\,,T]}(t)
\end{aligned}$$

This enables us to write the volatility terms we encounter in the remainder through the use of a single integral. This also implies,

$$\int_t^T \overline{\sigma}_i^2(s)\,ds \equiv \int_t^{\max(t,t_i)} \sigma_{D_i^T}^2(s)\,ds + \int_{\max(t,t_i)}^T \sigma_F^2(s)\,ds$$

Using the developed notation, with the advantage becoming clear straight away, we can immediately write,

$$\frac{S_T}{S_{t_i}} = \overline{\mu}_i(0)\exp\left(-\frac{1}{2}\int_0^T \overline{\sigma}_i^2(s)\,ds + \int_0^T \overline{\sigma}_i(s)\,d\overline{W}_s^i\right)$$

This gives, using well known relations for the conditional expectation of normal random variables[15],

$$\begin{aligned}
E_t^{\mathbf{Q}^T}\left[\frac{S_T}{S_{t_i}}\mid Z\right] &= \overline{\mu}_i(t)\exp\left(-\frac{1}{2}\int_t^T \overline{\sigma}_i^2(s)\,ds + \mu_{i\mid Z}(t)\,Z + \frac{1}{2}\sigma_{i\mid Z}^2(t)\right) &(35)\\
&= \overline{\mu}_i(t)\exp\left(\mu_{i\mid Z}(t)\;Z - \frac{1}{2}\mu_{i\mid Z}(t)^2\right) &(36)
\end{aligned}$$

where,

$$\begin{aligned}
\mu_{i\mid Z}(t) &= E_t^{\mathbf{Q}^T}\left[Z\int_{t_i}^T \sigma_F(s)\,dW_s^T\right] \\
\sigma_{i\mid Z}^2(t) &= \int_t^T \overline{\sigma}_i^2(s)\,ds - \mu_{i\mid Z}(t)^2
\end{aligned}$$

[15] $E\,X\mid Z = E\,X + \frac{\mathrm{Cov}(X\,,Z)}{\sqrt{\mathrm{Var}(Z)}}$ and $\mathrm{Var}(X\mid Z) = \mathrm{Var}(X) - \frac{[\mathrm{Cov}(X\,,Z)]^2}{\mathrm{Var}(Z)}$.

Notice that the randomness in $S_T \,/\, S_{t_i}$ over the interval $[t, t_i]$ is given by $\int_t^{\max(t,t_i)} \sigma_{D_i^T}(s)\, dW_s^{iT}$. So until the premium is paid the risk for the insurer in this product is pure interest rate risk.

7.2 Choice of conditioning variable

The approximate solution to the pricing problem (i.e. the lower bound resulting from the above sketched approach) depends on the choice of Z. A sensible choice, and indeed the one Rogers and Shi make, is that conditioning variable for which the variance of the conditional payoff is small. Nielsen and Sandmann (2002a) suggest two conditioning variables, each based on a different approximation of the variance of the conditional payoff. Since their numerical results cannot distinguish between them the one I present here is just the one which is easiest to implement. This corresponds to the second suggestion for the conditioning variable in Nielsen and Sandmann. It turns out because of the structure of the product we have to go through a bit more trouble to derive the conditional variance of the fundvalue at maturity then Nielsen and Sandmann do for the Asian. However the results are similar except for a subtle difference in the resulting conditioning variable Z. In the Asian option the risk of each individual term in the summation, S_{t_i}, runs from time zero to t_i. This is mainly stock price risk, represented through the choice of (sum of the stochastic parts of the) forward stock return as conditioning variable. In our UL contract we can split the risk of each individual term, $S_T \,/\, S_{t_i}$, in interest rate risk, related to some forward bond price, from time zero to t_i and forward stock price risk from time t_i to T. This is reflected by the choice of conditioning variable which is (the sum of the stochastic parts of) the forward bond return over the interval $[0, t_i]$ and the forward stock return over $[t_i, T]$. Summarizing we have,

$$Z = \frac{1}{\alpha_t} \sum_{i=0}^{n-1} \left\{ \int_t^T \overline{\sigma}_i(s)\, d\overline{W}_s^{iT} \right\}$$

where the relevant forward bond volatilities are taken zero if $t > t_i$ and α_t is a normalising constant . Observe that our choice of Z amounts to conditioning on the stochastic parts of the 'return' on the forward stock and bond prices. Where,

$$\alpha_t = \sum_{j=0}^{n-1} \mathrm{Cov}_t^{\mathbf{Q}^T} \left(\int_t^T \overline{\sigma}_j(s)\, d\overline{W}_s^{jT} \,;\, \int_t^T \overline{\sigma}_j(s)\, d\overline{W}_s^{jT} \right) + \sum_{j=0}^{n-1} \sum_{k=j+1}^{n-1} \mathrm{Cov}_t^{\mathbf{Q}^T} \left(\int_t^T \overline{\sigma}_k(s)\, d\overline{W}_s^{kT} \,;\, \int_t^T \overline{\sigma}_j(s)\, d\overline{W}_s^{jT} \right)$$

and, for $i \le j$

$$\mathrm{Cov}_t^{\mathbf{Q}^T} \left(\int_t^T \overline{\sigma}_i(s)\, d\overline{W}_s^{iT} \,;\, \int_t^T \overline{\sigma}_j(s)\, d\overline{W}_s^{jT} \right) = \int_t^{\max(t,t_i)} \rho_{D_i D_j} \sigma_{D_i} \sigma_{D_j} ds + \int_{\max(t,t_i)}^{\max(t,t_j)} \rho_{F D_j} \sigma_F \sigma_{D_j} ds + \int_{\max(t,t_j)}^{T} \sigma_F^2 ds$$

which is the convexity correction (first two terms) and the implied volatility of a forward starting stock option (third term) present in the second moment of the fund value. Furthermore,

$$\begin{aligned}
\mu_{i\,|\,Z}(t) &= E_t^{\mathbf{Q}^T}\left[Z\int_t^T \overline{\sigma}_i(s)\,d\overline{W}_s^{iT}\right] \\
&= \sum_{j=0}^{n-1}\int_t^{\max(t,t_i)} \rho_{D_iD_j}\sigma_{D_i}\sigma_{D_j}ds + \int_{\max(t,t_i)}^{\max(t,t_j)} \rho_{FD_j}\sigma_F\sigma_{D_j}ds + \int_{\max(t,t_j)}^{T} \sigma_F^2 ds
\end{aligned}$$

which are again the expressions for the convexity correction and the implied volatility of a forward start option in the second moment of the fund value at maturity.

It is important to note that since for reasonable correlation values (mainly for the correlation between forward stock and forward bond processes) the coefficients of Z are positive for all i, which results in **a unique solution** to the equation, $\sum_{i=0}^{n-1} E_t^{\mathbf{Q}^T}[S_T\,/\,S_{t_i}\,|\,Z]-K=0$. This simplifies the calculations of the lower bound. Thompson (1998) derives an analytical solution to the equation, $\sum_{i=0}^{n-1} E_t^{\mathbf{Q}^T}[S_{t_i}\,|\,Z]-K=0$ in the case of an Asian Option based on interchanging exponentiation and summation. This doesn't work in our case since the terms in the exponent are not small enough.

7.3 Derivation of Upper Bound

In deriving an upper bound, generalizing Thompson's approach we exploit the high correlation between, $\int_t^T \overline{\sigma}_i d\overline{W}^i$ and $\sum_{j=0}^{n-1}\int_t^T \overline{\sigma}_j d\overline{W}^j$. These two variables correspond to the 'return' on premium to be paid at time t_j and the sum of the returns on each premium, i.e. the total return. We have the covariance matrix,

$$E_t^{\mathbf{Q}^T}\begin{pmatrix}\int_t^T \overline{\sigma}_i d\overline{W}^i \\ \sum_{j=0}^{n-1}\int_t^T \overline{\sigma}_j d\overline{W}^j\end{pmatrix}\begin{pmatrix}\int_t^T \overline{\sigma}_i d\overline{W}^i & \sum_{j=0}^{n-1}\int_t^T \overline{\sigma}_j d\overline{W}^j\end{pmatrix} = \begin{pmatrix}\int_t^T \overline{\sigma}_i^2(s)\,ds & \mu_{i\,|\,Z}(t)\sqrt{\mathrm{Var}_t^{\mathbf{Q}^T}(Z)} \\ \mu_{i\,|\,Z}(t)\sqrt{\mathrm{Var}_t^{\mathbf{Q}^T}(Z)} & \mathrm{Var}_t^{\mathbf{Q}^T}(Z)\end{pmatrix} \tag{37}$$

which will be usefull in our later analysis. We continue the notation adopted in the previous subsections.

The upper bound Thompson proposed is based on the observation that,

$$E^{\mathbf{Q}^T}\left(K-\sum_{i=0}^{n-1}\frac{S_T}{S_{t_i}}\right)^+ \le \sum_{i=0}^{n-1}E^{\mathbf{Q}^T}\left(f_iK-\frac{S_T}{S_{t_i}}\right)^+ \tag{38}$$

under the condition that $\sum_{i=0}^{n-1} f_i = 1$ and f is some (stochastic) function of i. The function f is split into a deterministic part and a stochastic part based on the difference between the return on the premium paid at time i and the total return. The intuition behind this is that when the return on a premium is low compared to the total return it will have a large weight in determining the payoff of the guarantee put option, to dampen this increased weight in the payoff we let the individual option (on $S_T\,/\,S_{t_i}$) be out-of-the-money to a greater extend then the other individual options by decreasing the strike. This will help minimize the RHS of (38). Summarizing we propose to use,

$$f_i = m_i + \left(\sum_{j=0}^{n-1}\int_0^T \overline{\sigma}_j d\overline{W}^j - \int_0^T \overline{\sigma}_i d\overline{W}^i\right)$$

The first part, m_i, is deterministic. The second part represents the total return minus the individual return on a premium paid at time t_i. Notice the change in sign with the proposed function f in Thompson (1998). This is due to the fact we consider a put option whereas Thompson considers call options. Now we will minimize the RHS of (38) w.r.t. m_i under the condition $\sum_{i=0}^{n-1} m_i = 1$. This leads to the first order condition,

$$\sum_{i=0}^{n-1} \left[K\,\mathbf{P}^{\mathbf{Q}^T} \left(f_i K \geq \frac{S_T}{S_{t_i}} \right) - \lambda \right] \varepsilon_i = 0 \tag{39}$$

which becomes, if we require stationarity w.r.t. small deterministic perturbations,

$$\mathbf{P}^{\mathbf{Q}^T} \left(f_i K \geq \frac{D(0,t_i)}{D(0,T)} \exp\left(-\frac{1}{2} \int_0^T \overline{\sigma}_i^2 ds + \int_0^T \overline{\sigma}_i d\overline{W}^i \right) \right) = \lambda^* \quad , \quad \forall i \tag{40}$$

Now use $\exp(x) \approx 1 + x$ to get explicit dependence of m_i on λ, to obtain,

$$\mathbf{P}^{\mathbf{Q}^T} \left(f_i K \geq \frac{D(0,t_i)}{D(0,T)} \exp\left(-\frac{1}{2} \int_0^T \overline{\sigma}_i^2 ds \right) \left(1 + \int_0^T \overline{\sigma}_i d\overline{W}^i \right) \right) = \lambda^* \quad , \quad \forall i \tag{41}$$

hence $\mathbf{P}^{\mathbf{Q}^T}(m_i K \geq N_i)$, where,

$$N_i = D(0,t_i) \,/\, D(0,T) \exp\left(-\frac{1}{2} \int_0^T \overline{\sigma}_i^2 ds \right) \left(1 + \int_0^T \overline{\sigma}_i d\overline{W}^i \right) - K \left(\sum_{j=0}^{n-1} \int_0^T \overline{\sigma}_j d\overline{W}^j - \int_0^T \overline{\sigma}_i d\overline{W}^i \right)$$

should be independent of i. This condition yields, using the earlier results on the covariance of individual return and total return,

$$m_i = \frac{1}{K} \left(\frac{D(0,t_i)}{D(0,T)} \exp\left(-\frac{1}{2} \int_0^T \overline{\sigma}_i^2 ds \right) + \gamma \sqrt{\mathrm{Var}^{\mathbf{Q}^T}(N_i)} \right)$$

where γ is some constant to be determined by the constraint $\sum_{i=0}^{n-1} m_i = 1$ and,

$$\begin{aligned} \mathrm{Var}^{\mathbf{Q}^T}(N_i) &= c_i^2 \int_0^T \overline{\sigma}_i^2 ds + K\,\mathrm{Var}^{\mathbf{Q}^T}(Z) - 2 c_i K \mu_{i\,|\,Z} \sqrt{\mathrm{Var}^{\mathbf{Q}^T}(N_i)} \\ c_i &= \frac{D(0,t_i)}{D(0,T)} \exp\left(-\frac{1}{2} \int_0^T \overline{\sigma}_i^2 ds \right) + K \end{aligned}$$

The constraint on m_i yields for γ,

$$\gamma = \left\{ K - \sum_{i=0}^{n-1} \frac{D(0,t_i)}{D(0,T)} \exp\left(-\frac{1}{2} \int_0^T \overline{\sigma}_i^2 ds \right) \right\} \Big/ \sum_{i=0}^{n-1} \sqrt{\mathrm{Var}^{\mathbf{Q}^T}(N_i)}$$

This gives the following upper bound for the value of the guarantee (at time zero):

$$G_0 \leq D(0,T) \sum_{i=0}^{n-1} E^{\mathbf{Q}^T} \left(K \left(m_i + \sum_{j=0}^{n-1} \int_0^T \overline{\sigma}_j d\overline{W}^j - \int_0^T \overline{\sigma}_i d\overline{W}^i \right) - \frac{S_T}{S_{t_i}} \right) \tag{42}$$

Now these expectations can be evaluated numerically by first conditioning on $X_i \equiv \int_0^T \overline{\sigma}_i d\overline{W}^i$, since

$$\sum_{j=0}^{n-1} \int_0^T \overline{\sigma}_j d\overline{W}^j \mid \int_0^T \overline{\sigma}_i d\overline{W}^i = Z \mid X_i \backsim N\left(\mu_{i\,|\,Z} X_i\,;\, \mathrm{Var}\left(Z\right) - \frac{\mu_{i\,|\,Z} \mathrm{Var}\left(Z\right)}{\int_0^T \overline{\sigma}_i^2 ds}\right) = N\left(\mu_{Z\,|\,i} X_i\,;\, \sigma_{Z\,|\,i}^2\right)$$

So that the conditional expectation is fairly easy. After conditioning we are left with,

$$\int_{-\infty}^{\infty} E_{Z\,|\,X_i}^{\mathbf{Q}^T}\left(K\left(m_i + Z - X_i\right) - \frac{D\left(0,t_i\right)}{D\left(0,T\right)} \exp\left(-\frac{1}{2}\int_0^T \overline{\sigma}_i^2 ds + X_i\right)\right) \frac{1}{\sqrt{\int_0^T \overline{\sigma}_i^2 ds}} \phi\left(\frac{X_i}{\sqrt{\int_0^T \overline{\sigma}_i^2 ds}}\right) dX_i \tag{43}$$

where the inner expectation is taken over Z which is normally distributed, hence the upper bound is given by the summation of integrals,

$$\sum_{i=0}^{n-1} \int_{-\infty}^{\infty} \left\{a\left(i,X_i\right)\Phi\left(\frac{a\left(i,X_i\right)}{K\sigma_{Z\,|\,i}}\right) + K\sigma_{Z\,|\,i}\phi\left(\frac{a\left(i,X_i\right)}{K\sigma_{Z\,|\,i}}\right)\right\} \frac{1}{\sqrt{\int_0^T \overline{\sigma}_i^2 ds}} \phi\left(\frac{X_i}{\int_0^T \overline{\sigma}_i^2 ds}\right) dX_i \tag{44}$$

where $a\left(i,X_i\right) = K\left(m_i + \mu_{Z\,|\,i} X_i - X_i\right) - \frac{D(0,t_i)}{D(0,T)} \exp\left(-\frac{1}{2}\int_0^T \overline{\sigma}_i^2 ds + X_i\right)$. Since we are analyzing the discrete case Thompsons' expression for the upper bound reduces from a double integral to a sum of integrals, which have to be evaluated numerically.

8 Results in a Black-Scholes Hull-White model

In this section we derive the results stated in section 5 and 8 in the context of a combined Black-Scholes Hull-White (BSHW) model. We assume the stock price has a constant volatility and the short rate follows a Hull-White (1990) process under the risk neutral measure. Furthermore we assume the correlation between the stock and the short rate equals ρ. Essential to all of our preceeding results is the covariance $E_t^{\mathbf{Q}^T}\left[\int_t^T \overline{\sigma}_i\left(s\right) d\overline{W}_s^i \int_t^T \overline{\sigma}_j\left(s\right) d\overline{W}_s^j\right]$.

Our result in this section extends the results of Nielsen and Sandmann (1996b) for Asian options to Guarantees in Regular Premium UL insurance. But more importantly, using the results and ideas of section 5, their results can quite easily be restated in terms of forward stock and forward bond volatilities. This enables us to interpret their results in terms of convexity adjusted quantities and obtain additional insight in the determinants of the price of Asian options. See also Appendix A.

In a Black-Scholes Hull-White model all volatilities of forward prices are a deterministic function of time. So we essentially parameterize our general lognormal model in (18) and (19). The convexity correction can then be interpreted as parameterized convexity correction. We now proceed with the derivation.

It is not difficult to derive that under the stated assumptions of constant stock volatility and a correlation between stock and short rate of ρ, the T-forward stock and T-forward bond price follow the dynamics,

$$dF_t^T = \sqrt{1-\rho^2}\sigma_S F_t^T dW_{1,t}^T + \rho\sigma_S F_t^T dW_{2,t}^T + \sigma_r B\left(t,T\right) F_t^T dW_{2,t}^T \tag{45}$$

$$dD^T\left(t,S\right) = -\sigma_r\left[B\left(t,S\right) - B\left(t,T\right)\right] D^T\left(t,S\right) dW_{2,t}^T \tag{46}$$

Where $W_{1,t}^T$ and $W_{2,t}^T$ are independent Brownian Motions under the T-Forward measure and $B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$. For (45) we can write equivalently (in weak SDE solution terms):

$$dF_t^T = \sqrt{\sigma_S^2 + 2\rho\sigma_S\sigma_r B(t,T) + \sigma_r^2 B(t,T)^2}\, dZ_t^T \tag{47}$$

Where Z_t^T is a Brownian Motion under the T-Forward Measure. This makes the forward asset price volatility direct. From $\ln E^{\mathbf{Q}^T} F_t^T D^T(t,S) = \int_0^t \rho_{FD}(s)\,\sigma_{F^T}(s)\,\sigma_{D^T}(s)\,dt$, the instantaneous covariance between F_t^T and $D^T(t,S)$, can be shown to equal,

$$\rho_{FD}(t)\,\sigma_{F^T}(t)\,\sigma_{D^T}(t) = -\rho\sigma_S\sigma_r\left[B(t,S) - B(t,T)\right] + \sigma_r^2 B(t,T)^2 - \sigma_r^2 B(t,T)\,B(t,S) \tag{48}$$

Instead of inferring the volatilities of interest from market data we parameterize them according to the results above. Then if we parameterize the forward stock price volatility and forward bond price volatility in (18) and (19) according to (45) and (46) respectively, we obtain (also assuming that $i < j$),

$$\begin{aligned} E_t^{\mathbf{Q}^T}\left[\int_t^T \overline{\sigma}_i(s)\,d\overline{W}_s^i \int_t^T \overline{\sigma}_j(s)\,d\overline{W}_s^j\right] &= \left(\sigma_r^2\int_0^{t_i}\left[B(s,T) - B(s,t_i)\right]\left[B(s,T) - B(s,t_j)\right]ds\right) + \\ &\left(-\rho\sigma_S\sigma_r\int_{t_i}^{t_j} B(s,t_j) - B(s,T)\,ds + \sigma_r^2\int_{t_i}^{t_j} B(s,T)^2\,ds - \sigma_r^2\int_{t_i}^{t_j} B(s,T)\,B(s,t_j)\,ds\right. \\ &\left(\sigma_S^2(T - t_j) + 2\rho\sigma_r\sigma_S\int_{t_j}^{T} B(s,T)\,ds + \sigma_r^2\int_{t_j}^{T} B(s,T)^2\,ds\right) \end{aligned}$$

We can split this long expression in three parts corresponding to the three integrals in (21) namely first $\sigma_r^2\int_0^{t_i}\left[B(s,T) - B(s,t_i)\right]\left[B(s,T) - B(s,t_j)\right]ds$ corresponds to $\int_0^{t_i}\rho_{D_iD_j}(s)\,\sigma_{D_i}(s)\,\sigma_{D_j}(s)\,ds$, the correlation between the bonds with maturity t_i and t_j normalized by the bond with maturity T, since in a one factor model the correlation between bonds (and hence forward bond prices) equals one, this expression is direct from (46), the forward bond volatility and second $-\rho\sigma_S\sigma_r\int_{t_i}^{t_j}\left[B(s,t_j) - B(s,T)\right]ds - \sigma_r^2\int_{t_i}^{t_j} B(s,T)^2\,ds + \sigma_r^2\int_{t_i}^{t_j} B(s,T)\,B(s,t_j)\,ds$ corresponds to $\int_{t_i}^{t_j}\rho_{FD_j}(s)\,\sigma_F(s)\,\sigma_{D_j}(s)\,ds$, the covariance between the forward asset price and $\frac{D(t,t_j)}{D(t,T)}$. Finally $\sigma_S^2(T - t_j) + 2\rho\sigma_S\sigma_r\int_{t_j}^{T} B(s,T)\,ds + \sigma_r^2\int_{t_j}^{T} B(s,T)^2\,ds$ corresponds to $\int_{t_j}^{T}\sigma_F^2(u,S_u)\,du$, the implied volatility of a forward start option. The same approach can be followed for any other Gaussian interest rate model in combination with a Geometric Brownian Motion for the stock.

9 Numerical results

This section provides numerical results for both the pricing approach discussed in sections 5 and 7 as well as the one based on the Black-Scholes Hull-White (HW) model. To obtain numerical results on the price of a guarantee according to the approach in section 7 which have empirical relevance, we use the estimation results of De Jong, Driessen and Pelsser (2002) on a full-factor LMM. A motivation for the use of this model can be found in their paper. We take the S&P100 as the investment funds. We take the CBOE S&P100 implied volatility index as our forward stock volatility estimate. We estimated, using population estimates, the correlation between forward stock and bond prices using weekly S&P100 and interest rate data for the same period over which the LMM parameters were estimated, namely January 1995 to June 1999. We assumed that, $\rho_{F^T D^T(t_i)}(t) \equiv \rho(T-t, T-t_i)$, only the remaining time to maturity of the relevant zeros is taken into account. A time homogeneous correlation matrix results. To let the HW parameters, like the LMM parameters, be representative for the whole sample we calibrated the HW parameters to a set of implied 'at-the-money-forward' swaption volatilities generated by the calibrated LMM using the term structure of the latest observation in our sample. These swaption volatilities where calculated using the method described by Hull and White (2000) in their paper. We minimized the sum of squared errors of the volatility implied by HW model prices with respect to the LMM implied volatility. The following results for the parameters were obtained, $a = 0.0349$ and $\sigma_r = 0.0116$. To estimate correlation between short rate and stock price Brownian Motions we used time series data on 3 month interest rate and the S&P100. A correlation coefficient of -0.0255 resulted. For pricing we used an implied forward stock volatility equal to the CBOE implied volatility index of the S&P100 (equal to 21.01%) and the term structure corresponding to the latest observation in our sample. So we can consider the results as prices as at end of June 1999.

The following parameterization of the insurance contract is taken, $GP_i \equiv GP = 100$, $FC_i = 5 + 25\,[i \leq 3]$, $c_i \equiv c = 0.02$ and yearly premium payments. Only a guaranteed amount at maturity is considered. The prices we calculated do not take into account survival rates of the insured. These can be very different over countries, age and gender[16]. Furthermore in the light of our results in section 3, no clear-cut way is known to us to determine risk neutral survival probabilities.

We proceed as follows, first we visualize the effect of stochastic interest rates using the volatility of the fundvalue. Since the mean of the fundvalue is independent of modelparameters there's a one-to-one correspondence between the price and volatility in the lognormal approximation. Second, acknowledging the importance of stochastic interest rates, we look at differences between prices obtained using the LMM for the calculation of forward bond volatility versus prices obtained using the BSHW model of section 8. Our results show this effect is non-negligible.

Figure 1 shows the convexity correction effect on the volatility of the fundvalue at maturity in the BSHW model. The effect of stochastic interest rates on the volatility (i.e. the convexity correction) increases with maturity. We see that the volatility first decreases because of an averaging effect. The volatility is an average of the volatility of each premium payment. This volatility is highest for the first payment and lowest for the last payment (see the result in Lemma 3, stock volatility is greater than bond volatility). Then the volatility increases again due to a time effect. For maturities typical in the context of pension funds this completely cancels out the averaging effect, partly because of the effect of stochastic interest rates. This time effect is induced by the increasing number of cross correlations (at a rate equal to T^2) between premium payments at longer maturities. The sudden fall in volatility after three years can be explained by the structure of the fixed

[16] A typical 10 year survival rate of a 35 year old male is 98%, for a 50 year old male this is 93%. These numbers are based on a Dutch Mortality table over the years 1990-1995. So, for reasonable maturities and age below 50, mortality will not influence the prices shown here dramatically. However for products like these in the context of pension funds the maturity of the contracts is a lot higher and the influence of mortality increases. We can say though that in general the maturity of a contract like the one discussed here doesn't go beyond age 65, which is usually when years are starting to count in terms of survival probabilities.

cost deductions. For the first three years fixed costs are 30% of gross premium. After that fixed costs are only 5% of gross premium. This is similar to reality where insurers let policy holders pay for their acquisition costs. As the main part of the volatility is caused by forward stock volatility, the premium payments in the first years are contributing most. Because of the acquisition costs the *relative* effect of the earlier premium payments decreases and hence volatility decreases.

For a contract with a maturity of 10 years, in the BSHW model the convexity correction is 40 volatility basis points on a volatility of 14.4%. For a contract with maturity 30 (very common in the context of pension funds) this is even higher at 170 vol. bp. (1.7%) on a volatility of 19.6%. When we use the LMM, the convexity correction is 27 vol. bp for a contract with a 10 year maturity. These numbers are not surprising since convexity correction is a second order effect. It remains important however since insurers are working with large portfolios and the effect increases with maturity. To illustrate this let us consider an insurance portfolio of 250,000 policies with an average premium of \$1000 a year and a maturity guarantee of 3%. Let the maturities (of 6 to 10 years) be equally distributed over the policies. The convexity correction amounts to a difference in reserve / Fair Value of 3.2% or $\$2,600,000.-$.

A comparison of implied volatilities for LMM and Hull-White results is given in Figure 2. On the horizontal axis is the maturity of the contract. The main explanation for the differences is that due to faster decorrelation[17] between forward LIBOR rates (see correlation parameterization of LMM model), and hence forward bond prices, both forward bond covariance and the covariance between forward bond and forward stock price are lower in the LMM based model. Furthermore due to higher volatility of short term forward bonds we have underpricing of the BSHW model for short maturities and considerable overpricing for long maturities. The latter effect becomes more pronounced with increasing maturity of the contract. Table 1 shows the percentage difference of the convexity correction of the BSHW model compared with LMM based results for different contract maturities. It shows that the BSHW model underprices contracts with short maturity, illustrated by the low convexity correction compared with LMM and overprices long maturity contracts, illustrated by the high convexity correction compared with LMM..

We stress that effects of using the LMM are similar for the pricing bounds both qualitatively and quantitatively. However the advantage of the Levy approximation is that we can explain and interpret these differences, whereas using the methods of section 7 the explanation is hidden behind an involved procedure.

In table 2 we present results for guarantee prices for the BSHW model in money terms and as a percentage of discounted premiums for various contract lengths and guarantee levels, we let $K = \sum_{i=0}^{n-1} \widetilde{P}_i e^{R(T-t_i)}$. As we would expect, prices increase with a higher guaranteed rate of return. The price of the guarantee increases with maturity for guaranteed rate of 3% and 6%. For a guaranteed rate of 0% (essentially a 'no-loss' guarantee) the price decreases with maturity.

Results on the absolute and relative prices of the rate of return guarantee in the LMM framework discussed in section 7 are presented in table 3. We see that prices are lower than for the BSHW framework. This is due to reduced convexity correction effect on the volatility. For contracts with a maturity of 10 years the percentage differences are, 5.44%, 3.83% and 2.58%, for guarantee levels of 0%, 3% and 6% respectively. In line with results in table 1 the overpricing is highest for the lowest guarantee levels (which are of greater importance in practice).

[17] This decorrelation is fast in comparison since in the Hull-White setting all rates are perfectly correlated.

Maturity	Perc. diff.	Maturity	Perc. diff.
3	-58%	7	10%
4	-41%	8	21%
5	-22%	9	30%
6	-5%	10	36%

Table 1. *Percentage differences between convexity corrections based on the BSHW model and the LMM. For short maturities the BSHW model underestimates the effect of stochastic interest rates, for long maturities the effect is overestimated.*

	Guaranteed Rate		
Maturity	0%	3%	6%
5	\$13.59	\$21.75	\$33.15
	3.1%	5.0%	7.6%
10	\$21.73	\$42.71	\$78.19
	3.0%	5.8%	10.7%

Table 2. *Absolute and relative prices of rate of return guarantees for BSHW based framework of section 8 for different maturities and guaranteed rates .*

	Guaranteed Rate		
Maturity	0%	3%	6%
5	\$13.58	\$21.74	\$33.14
	3.1%	5.0%	7.7%
10	\$20.61	\$41.14	\$76.22
	2.8%	5.6%	10.4%

Table 3. *Absolute and relative prices of rate of return guarantees for LMM based framework of section 6 for different maturities and guaranteed rates .*

To find out how these prices compare with those of other rate of return guarantees we compared with single premium contracts and regular premium contracts with a guaranteed amount on every invested premium. In the latter case the guarantee is a sum of forward starting put options. The prices of these forward starting put options can be calculated using the techniques of section 5. We calculated prices using the convexity correction approach based on the LMM. Since all prices are relative to the amount invested we only present relative prices. The results are shown in table 4. We see that for both single premium contracts and regular premium contracts with a guaranteed amount on every invested premium prices are higher than for the contract analyzed in this paper. This is because the possibility of averaging out losses over the lifetime of the contract is eliminated for the former two contracts. For single premium contracts the price is higher because the total amount is subject to forward stock volatility for the whole maturity. As for regular premium contracts the invested premium is only subject to stock volatility for the remaining maturity after premium payment.

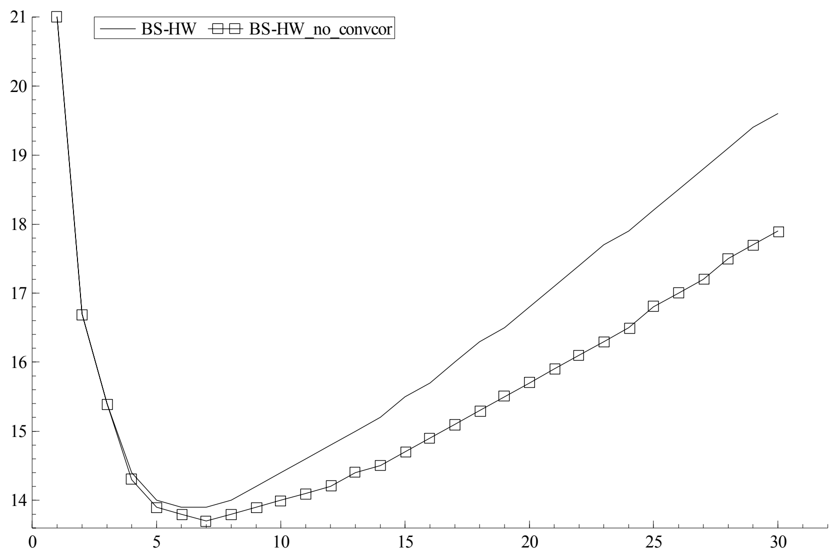

Figure 1: *Volatility for Rate of Return Guarantees calculated using the BS-HW model with and without the convexity correction effect. The normal line shows the volatility whereas the boxed line shows the volatility without the convexity correction. As can be seen from the graph, the convexity correction effect increases with maturity.*

We conclude that the convexity correction on the volatility derived in section 5 is important in the context of pricing insurance liabilities. This effect is especially important for contracts with long maturities. Furthermore it seems that a one factor model interest rate model tends to overestimate prices. We recommend guarantee pricing based on the LMM approach suggested in section 6 or some other multi-factor model.

	Guaranteed Rate		
Maturity	0%	3%	6%
5	4.6%	8.4%	14.1%
10	2.8%	7.4%	16.8%
5	4.6%	7.0%	10.4%
10	5.2%	9.9%	18.1%

Table 4. *Relative prices (as a percentage of discounted premiums) of rate of return guarantees for Single Premium contracts (row 1 and 2) and regular premium contracts with a guaranteed amount on every invested premium (row 3 and 4).*

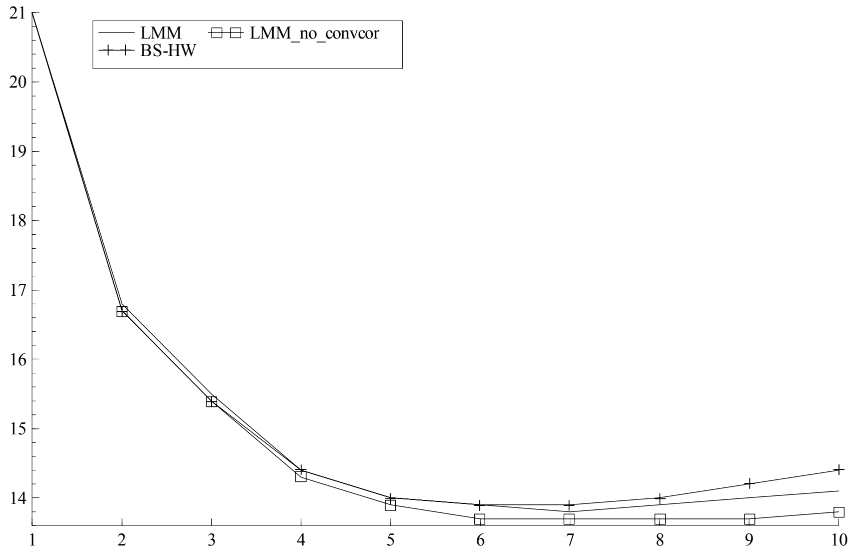

Figure 2: *Volatility for guarantees, percentage points, in regular premium contract for different maturities. The straight line represents the implied volatility with the convexity correction based on the LMM. The boxed line represents the volatility based on the LMM without the convexity correction. For completeness, the line with plusses is the volatility based on the BS-HW model.*

10 Conclusion

In recent years Unit Linked insurance has become a more prominent part of life insurance business. Hence it is of interest to be able to price guarantees in these products. Our results can be used to price and hedge guarantees without making restrictive assumptions about the stochastic processes of the underlying instruments. We have derived, using Change of Numeraire techniques, a general pricing formula for Rate of Return Guarantees in a Regular Premium Unit Linked Insurance contract. We show the guarantee is equivalent to a put option on some stochastically weighted average of the stock price at maturity. Furthermore we derive some results on and discuss the analogy of the guarantee with Asian options. The main contribution of our paper focusses on the effect of stochastic interest rates. In the context of the Levy approximation we derive general expressions for this and show it has the interpretation of a convexity correction. Then we show how we can obtain guarantee prices in accordance with the popular LIBOR Market Model. This enables one to find prices of the guarantee which are consistent with both observed stock option prices and observed cap and swaption prices. We extend earlier results on pricing bounds of Asian options to UL Guarantees and stochastic interest rates. Numerical results show non-negligible prices of guarantees. They also illustrate the importance of the convexity correction arising from stochastic interest rates. This effect grows stronger with longer maturities. We also find a one factor interest rate model overestimates prices in comparison with LMM consistent pricing. This overpricing also increases with the maturity of the contract and is highest for guarantee levels most relevant to the industry.

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Appendix A: Arithmetic Stock Price Average

In this appendix we are concerned with the expectation of the arithmetic average of the stock price assuming stochastic interest rates under the T-Forward measure. We encounter this in the calculation of the price of an Asian option. The time zero price of an arithmetic Asian option maturing at time T, strike K, with the average taken over the time points t_i, $i = 1, ..., n$, $t_n = T$ is given by,

$$V^{Asian}(t) = D(t,T)\, E_t^{\mathbf{Q}^T} \left(\sum_{i=1}^{n} S_{t_i} - K \right)^+ \tag{49}$$

If one would use the Levy approximation to calculate this price one would first be interested in $E_t^{\mathbf{Q}^T} \sum_{i=1}^{n} S_{t_i}$. This problem is not more difficult then the calculation of $E_t^{\mathbf{Q}^T} S_{t_i}$ for $i < n$. This is very similar to the problem of pricing LIBOR in arrears, but in this case we are dealing with a 'displaced' stock price (a stock price at a time point which doesn't correspond to the forward measure under which the expectation is taken). Observe that both $F_t^{t_i} \equiv \frac{S_t}{D(t,t_i)}$ and $D^{t_i}(t,T) \equiv \frac{D(t,T)}{D(t,t_i)}$ are martingales under the t_i-Forward measure. The convexity correction approach to this valuation problem is to assume both t_i-Forward stock and t_i-Forward bond prices have volatilities, $\sigma_{F^{t_i}}$ and $\sigma_{D^{t_i}}$ respectively, which are deterministic functions of time. This implies both forward stock and forward bond prices are lognormal. Also assume forward stock and forward bond prices are correlated with correlation ρ. Use the Change of Numeraire theorem, the martingale property and the assumption of lognormality, in the given order, to obtain,

$$\begin{aligned} E^T S_{t_i} &= \frac{D(0,t_i)}{D(0,T)} E^{t_i} S_{t_i} \frac{D(t_i,T)}{D(t_i,t_i)} \\ &= \frac{D(0,t_i)}{D(0,T)} \frac{S_0}{D(0,t_i)} \frac{D(0,T)}{D(0,t_i)} \exp\left(\int_0^{t_i} \rho_{FD}(s)\, \sigma_{F^{t_i}}(s)\, \sigma_{D^{t_i}}(s)\, ds \right) \\ &= F_0^{t_i} \exp\left(\int_0^{t_i} \rho_{FD}(s)\, \sigma_{F^{t_i}}(s)\, \sigma_{D^{t_i}}(s)\, ds \right) \end{aligned} \tag{50}$$

or assuming volatilities and correlation constant,

$$E^T S_{t_i} = F_0^{t_i} \exp\left(\rho \sigma_{F^{t_i}} \sigma_{D^{t_i}} t_i\right) \tag{51}$$

So the expected stock price is the forward stock price times some convexity correction. The advantages of using convexity correction techniques are clear. The determinants of the price can be seen from the formulas in an eyesight and the price can be written in terms of readily observable implied volatilities. The volatilities can be taken from implied stock option volatility and cap or swaption volatility (as we have shown in section 7). The correlation can be obtained from timeseries data.

Now if we consider a Black-Scholes Hull-White model and observing that in a model in which the short rate follows a Hull and White model the volatility of $D^{t_i}(t,T)$ equals $-\sigma_r\left[B(t,T) - B(t,t_i)\right]$ (see section 6) it follows that the expectation of the 'displaced' stock price equals,

$$E^{\mathbf{Q}^T} S_{t_i} = \frac{S_0}{D(0,t_i)} \exp\left\{ -\sigma_r^2 \int_0^{t_i} B(s,T)\, B(s,t_i)\, ds + \sigma_r^2 \int_0^{t_i} B^2(s,t_i)\, ds + \rho \sigma_S \sigma_r \int_0^{t_i} B(s,t_i) - B(s,T)\, ds \right\} \tag{52}$$

Comparing with (50) and (51) we can interpret these integrals in (52) as the quadratic covariance of $\ln\left(F_t^{t_i}\right)$ and $\ln\left(D^{t_i}(t,T)\right)$. This is completely in line with the results in Nielsen and Sandmann (1996)

Appendix B: Quality of Levy Approximation and Rogers and Shi Lower Bound

In this appendix we present some results on the quality of the Levy approximation and the lower bound approximation for option maturities typically encountered for insurance contracts. For ease of implementation we present results in the context of the BSHW model. Table B.1. presents the results of Guarantee prices obtained through Monte Carlo simulation of the Black-Scholes Hull-White model dicussed in section 8[18]. Standard errors of the MC simulation are given in parantheses. Also percentage errors of the Levy approximation

[18] We used an exact discretisation scheme for this model so there are no discretisation errors.

are presented. Since these are prices in the BSHW model prices in table B.1. should be compared with prices in table 1. The Levy approximation for the guarantee prices behaves in the same way as for Asian options. The quality deteriorates with increasing volatility and the approximation is best for at the money options. Figure 3 shows a comparison between the exact (modulo MC error) and the implied distribution of the fundvalue at maturity under the T-Forward measure for a contract with maturity 10 years and guarantee level of 3%. It can be seen from the graph that the lognormal approximation overestimates the thickness of the left tail of the distribution and hence overprices the Guarantee (which is a put option). So their is no risk of underpricing when using the Levy approximation in this case, hence results can also be interpreted as upperbounds, which might not be too tight for contracts with long maturities.

Table B.2. shows some results on the tightness of the lower bound derived in section 7. We find that in line with the results on Asian options by Nielsen and Sandmann (2002a) for short maturities the bound is very tight. However for long maturities the tightness decreases as reported in the table. Results on the tightness of the upper bound technique are not available at this time. We expect tightness to be of the same degree as that of the lower bound.

	Guaranteed Rate		
Maturity	0%	3%	6%
5	\$12.99 (0.03) 4.58%	\$21.20 (0.03) 2.57%	\$32.74 (0.04) 1.26%
10	\$19.19 (0.04) 13.24%	\$40.00 (0.07) 6.79%	\$75.93 (0.10) 2.98%
20	\$19.88 () 39.75%	\$60.10 () 17.89%	\$156.73 () 6.95%
30	\$16.13 () 70.58%	\$64.77 () 28.63%	\$218.28 () 10.05%

Table B.1. *Monte Carlo prices of the Guarantee and perc. Levy approximation errors. The errors increase with maturity (i.e. total volatility) and decrease with Guarantee level (i.e. moneyness).The results are obtained in the Black-Scholes Hull-White setting of section 8 with parameters as reported in section 9. In the simulation we used 1,000,000 paths.*

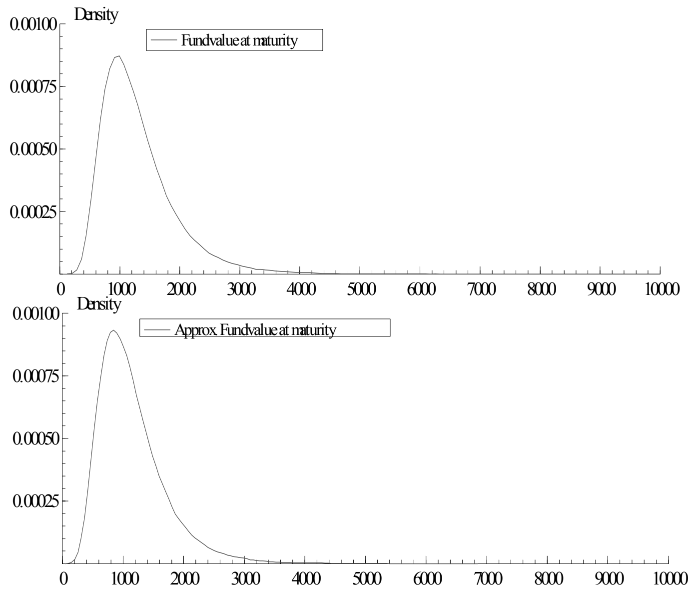

Figure 3: *Comparison of exact distribution of the Fundvalue at maturity and the lognormal distribution implied by the Levy approximation. Results are obtained in the Black-Scholes Hull-White setting of section 8 with parameters as reported in section 9. The contract considered to obtain these graphs has a maturity of 10 yrs. and guarantee level of 3%.*

	Guaranteed Rate		
Maturity	0%	3%	6%
5	\$12.94 -0.41%	\$21.14 -0.31%	\$32.65 -0.26%
10	\$19.00 -0.99%	\$39.71 -0.72%	\$75.48 -0.59%
20	\$19.47 -2.03%	\$59.03 -1.79%	\$154.36 -1.51%
30	\$15.55 -3.64%	\$62.75 -3.11%	\$212.62 -2.59%

Table B.2. *Pricing results and errors of the lower bound derived in section 7. The errors increase with maturity (i.e. total volatility) and decrease with Guarantee level (i.e. moneyness). The results are obtained in the Black-Scholes Hull-White setting of section 8 with parameters as reported in section 9. The lower bound is compared with the Monte Carlo results of table B.1.*