B.A.J. 1, V, 777–964 (1995)

MORE ON A STOCHASTIC ASSET MODEL FOR ACTUARIAL USE

BY A. D. WILKIE, M.A., F.F.A., F.I.A.

[Presented to the Institute of Actuaries, 24 April 1995]

ABSTRACT

In this paper the 'Wilkie investment model' is discussed, updated and extended. The original model covered price inflation, share dividends, share dividend yields (and hence share prices) and long-term interest rates, and was based on data for the United Kingdom from 1919 to 1982, taken at annual intervals. The additional aspects now covered include: the extension of the data period to 1994 (with omission of the period from 1919 to 1923); the inclusion of models for a wages (earnings) index, short-term interest rates, property rentals and yields (and hence property prices) and yields on index-linked stock; consideration of data for observations more frequently than yearly, in particular monthly data; extension of the U.K. model to certain other countries; introduction of a model for currency exchange rates; extension of certain aspects of the model to a larger number of other countries; and consideration of more elaborate forms of time-series modelling, in particular cointegrated models and ARCH models.

KEYWORDS

Stochastic Asset Models; Wilkie Model; Time-Series Models; Price Inflation; Wages; Share Dividend Yields, Dividends and Prices; Long-Term and Short-Term Interest Rates; Property Rentals, Yields and Prices; Index-Linked Yields; Exchange Rates; Cointegrated Time Series; ARCH Models

1. INTRODUCTION

1.1 *History of the Model*

1.1.1 What has become known to some as 'the Wilkie Investment Model' had its origin in work done while I was a member of the Maturity Guarantees Working Party, whose Report was presented to the Institute and to the Faculty in 1980. In that Report an index of United Kingdom share prices was decomposed into share dividends and share dividend yields, which were analysed with separate time-series models.

1.1.2 A more comprehensive model, including both price inflation and long-term interest rates, was presented to the Faculty in 1984 (Wilkie, 1986a). This model was described in further papers, one of which was presented to the then Institute of Actuaries Students' Society (Wilkie, 1986b, 1987).

1.1.3 An updated model, with extensions to other countries, the United States of America and France, and including a model for currency exchange rates, was presented to the 24th International Congress of Actuaries in Montréal (Wilkie, 1992). Earlier that year Geoghegan *et al.* (1992) reported to the Institute on their investigations on the Wilkie model.

1.1.4 Others have used similar methodologies to investigate investment series in other countries. Carter (1991) and Hua (1994) have looked at Australian data, with mixed success. Metz & Ort (1993) considered certain aspects of Swiss data. In a number of papers (Pentikäinen *et al.*, 1989; Bonsdorff *et al.*, 1994; Pukkilla *et al.*, 1994) and also in Daykin, Pentikäinen & Pesonen (1994) a Finnish group have put forward certain alternative models. Most recently Thomson (1994) has developed a similar, but interestingly different, model for South Africa.

1.2 *Outline of the Paper*

1.2.1 The original model covered price inflation, share dividends, share dividend yields (and hence share prices) and long-term interest rates. It was based on the data for 1919 to 1982, taken at annual intervals. This paper covers a number of additional aspects:

— extension of the data period to 1994, with omission of the period from 1919 to 1923;
— inclusion of models for a wages (earnings) index, short-term interest rates, property rentals and yields (and hence property prices) and index-linked stock;
— consideration of data for observations more frequently than yearly, in particular monthly data;
— extension of the U.K. model to certain other countries;
— introduction of a model for currency exchange rates;
— extension of certain aspects of the model to a larger number of other countries; and
— consideration of more elaborate forms of time-series modelling, in particular cointegrated models and ARCH models.

1.2.2 Applications of the model are not discussed. Many of these are well known, and have been described in a number of papers, for example the Report of the Faculty Solvency Working Party (1986), Ross (1989), Purchase *et al.* (1989), Daykin & Hey (1990, 1991), Macdonald (1991, 1993, 1994, 1995), Hardy (1993, 1994) and others. In my earliest papers on the subject the possible applications of the model received emphasis at the expense, perhaps, of statistical justification.

1.2.3 The emphasis in this paper is on the statistical analysis of the data, but, in order not to disturb the flow of discussion too greatly, most of the theoretical material is included in appendices. My intention is to give readers a feel for how certain statistical models behave, and therefore whether they might reflect satisfactorily the long-term features of the investment series to which they might be applied.

1.2.4 I discuss first a model for price indices (consumer prices or retail prices), then for a wages index, followed by share prices (dividends and dividend yields), interest rates (long-term and short-term), property (rental and yield), index-linked stocks and currency exchange rates. In the earlier sections the statistical discussion is rather fuller; it is not necessary to repeat much of the discussion in the later sections. I discuss all aspects of each series in the relevant section. Some forecasting results are shown in Section 11, and some suggestions for further research in Section 12.

1.3 *Purpose of the Paper*

1.3.1 One of the scrutineers appointed by the Institute to review the paper asked what the purpose of this long paper was. [At this point I should like to thank the scrutineers for their efforts in studying such a long paper, and for their helpful suggestions.] I replied that there were at least four reasons for producing it:

— because I found the topic interesting, and I hoped that others would too;
— because others might find the content useful for applications, especially the new series considered and the extensions to other countries;
— so that users of the models might feel more confident that they were reasonably justified (or not as the case may be) once the models had been exposed to discussion and criticism; and
— so that actuaries who were interested in time-series modelling might acquire a feel for some aspects of the subject, such as the difference between these sorts of models and pure random walk models.

1.3.2 It will be seen that this paper is more like a progress report than the last word on the subject. The more aspects that are investigated the more alternatives worthy of investigation open up, and in ¶12.2 I list some points that I think deserve further consideration. However, I believe that enough has been done to make it worth exposing the results obtained so far to the profession for criticism and discussion.

1.3.3 It is also my intention to show how the type of model I discuss, which is intended as a long-term model, is consistent with the short-term models favoured by many financial economists. In effect, the short-term properties of both types of model are the same, but the long-term properties are different, and, in my view, the long-term properties of the models I describe mean that they should be preferred.

1.4 *Statistics and Economics*

1.4.1 There is sometimes a conflict between the perceived properties of the data series, as derived from the application of conventional statistical methodology, and the properties that would be considered desirable or necessary, either from general economic or investment principles, or from plain common sense. This is partly a matter of whether one considers the given data series in isolation, or whether one brings in additional information, especially about the long-term behaviour of other similar series. For example, the evidence in Homer (1969) shows that interest rates, or at least real interest rates, must be modelled as statistically stationary series, whatever the apparent short-term properties of the data. One could use explicit Bayesian statistical methods for the analysis, but I consider it sufficient to use one's external knowledge informally in making sensible choices of model.

1.4.2 There is also a conflict between a model that may have good short-term forecasting properties and one that adequately describes the long-term variability of the series under consideration. For the former, minimum short-term errors are the ideal; for the latter an appropriate variance structure is desirable.

1.4.3 An example of these conflicts is found in a recent paper presented to the Staple Inn Actuarial Society by Huber (1995), who makes some trenchant criticisms of my model, which are not, however, supported by his analysis. Although he has shown that statistical models could be devised that would give better one-step-ahead forecasts for the last twelve years than mine does, he has not considered the long-term properties of his models, which are, in my view, quite unrealistic. It is unfortunate when application of a restricted statistical methodology fails to take into account the total picture for the data under consideration.

2. Consumer Price Indices

2.1 *The Original Model*

2.1.1 The original model for the U.K. Retail Prices Index (RPI) (what in other countries would usually be called a consumer price index), based on annual data from June 1919 to June 1982, where $Q(t)$ is the value of a retail price index at time t, is:

$$Q(t) = Q(t-1).\exp\{I(t)\}$$

so that $I(t) = \ln Q(t) - \ln Q(t-1)$ is the rate (strictly force) of inflation over the year $(t-1,t)$:

$$I(t) = QMU + QA.(I(t-1) - QMU) + QE(t)$$

$$QE(t) = QSD.QZ(t)$$

$$QZ(t) \sim \text{iid } N(0,1)$$

that is $QZ(t)$ is a series of independent, identically distributed unit normal variates, i.e. they have zero mean and unit standard deviation.

2.1.2 The suggested parameters, based on the experience from 1919 to 1982, were: $QMU = 0.05$, $QA = 0.6$, $QSD = 0.05$.

2.1.3 This model stated that the difference in the logarithms of the RPI each year could be modelled as a first order autoregressive series. In the time-series literature this would be denoted as an AR(1) model; this is a statistically stationary series (i.e. in the long run the mean and variance are constant), of a type which is also described as an I(0) series. The sum of the annual forces of inflation gives the logarithm of the RPI, which can be described as an ARIMA(1,1,0) model, and is an example of an I(1) series. Some properties of these models are described in Appendix A.

2.1.4 Since AR(1) models seem to apply widely to the investment series I discuss, it is convenient to abbreviate the notation and to write, for example:

$$I(t) \sim \text{AR1}(QMU,\ QA,\ QSD)$$

or, inserting numerical values, to write:

$$I(t) \sim \text{AR1}(0.05,\ 0.6,\ 0.05).$$

2.1.5 The model can be described in words: each year the force of inflation is equal to its mean rate (0.05), plus 60% of last year's deviation from the mean, plus a random innovation which has zero mean and a standard deviation of 0.05.

2.2 *The Experience from 1982 to 1994*

2.2.1 It is of interest to see how this model has fared since 1982. One can investigate the experience in two ways: by looking at the residuals, the difference between the 'forecast' and the actual values year by year, the observed *QE*s, or their 'standardised' versions, the *QZ*s; and by looking at the cumulative result, the logarithm of the RPI, and comparing it with the values that would have been forecast in 1982.

2.2.2 According to the model, the residuals, the *QE*s, are distributed $N(0, QSD^2)$, i.e. normally with zero mean and variance QSD^2; it is convenient first to divide each *QE* by *QSD* to give *QZ*s; these are assumed to be distributed N(0,1). The sum of n such *QZ*s is distributed $N(0, n)$, and the sum of the squares of n such *QZ*s is distributed as χ^2_n.

2.2.3 Originally I used values for June in each year, and I continue with these values. Values of the RPI up to June 1994 are now available; this gives us 12 new values. Table 2.1 shows, for each year, the observed value $I(t)$, the expected value conditional on the relevant information up to year $(t-1)$, $E[I(t)\,|\,\mathcal{F}_{t-1}]$, the observed residual $QE(t) = I(t) - E[I(t)\,|\,\mathcal{F}_{t-1}]$, and the standardised residual $QZ(t) = QE(t)/QSD$. The notation $\mathcal{F}_t$ just means the 'facts' at time t, and I use a modified version of this notation later.

2.2.4 We can compare the sum of the 12 values of *QZ*, which is -0.98, with the expected value, zero, and the standard deviation $\sqrt{12} = 3.46$. It is well within one standard deviation away from its expected value. We can also compare the sum of the 12 values of QZ^2, which is 2.53, with a χ^2_{12} distribution; the probability of a value of χ^2 as great or greater is 0.998, which suggests that this value of χ^2 is unexpectedly low. None of the (absolute) values of *QZ* exceeds 1.0. Perhaps the value of *QSD* is too high.

2.2.5 We can now consider the forecast values of $\ln Q(t)$, conditional on the information as at 1982. It is easier to work with the change in the logarithm, i.e. $QF(t) = \ln Q(t) - \ln Q(1982)$, which is just the cumulative sum of the values of I given in Table 2.1. Using the formulae for the expected values and variances of the forecast log changes, which are set out in Appendix E.2, we get the results

Table 2.1. Comparison of actual and expected values of $I(t)$, 1983-94

Year	$I(t)$	$E[I(t) \mid \mathcal{F}_{t-1}]$	$QE(t)$	$QZ(t)$
1982	0.0877			
1983	0.0359	0.0726	−0.0367	−0.73
1984	0.0501	0.0415	0.0086	0.17
1985	0.0673	0.0501	0.0172	0.34
1986	0.0247	0.0604	−0.0357	−0.71
1987	0.0411	0.0348	0.0063	0.13
1988	0.0451	0.0447	0.0004	0.01
1989	0.0793	0.0471	0.0323	0.65
1990	0.0934	0.0676	0.0258	0.52
1991	0.0568	0.0761	−0.0193	−0.39
1992	0.0380	0.0541	−0.0160	−0.32
1993	0.0121	0.0428	−0.0307	−0.61
1994	0.0259	0.0273	−0.0014	−0.03
Total			−0.0492	−0.98
ΣQZ^2				2.53

Table 2.2. Comparison of actual and expected values of $QF(t)$, 1983-94, all conditional on $\mathcal{F}_{1982}$

Year	$QF(t)$	$E[QF(t)]$	Deviation	Standard deviation	Standardised deviation
1983	0.0359	0.0726	−0.0367	0.05	−0.73
1984	0.0860	0.1362	−0.0502	0.0943	−0.53
1985	0.1533	0.1943	−0.0410	0.1360	−0.30
1986	0.1780	0.2492	−0.0712	0.1742	−0.41
1987	0.2191	0.3021	−0.0830	0.2089	−0.40
1988	0.2642	0.3539	−0.0897	0.2405	−0.37
1989	0.3435	0.4049	−0.0614	0.2694	−0.23
1990	0.4369	0.4555	−0.0186	0.2961	−0.06
1991	0.4937	0.5059	−0.0122	0.3210	−0.04
1992	0.5317	0.5561	−0.0244	0.3442	−0.07
1993	0.5439	0.6063	−0.0624	0.3660	−0.17
1994	0.5698	0.6564	−0.0866	0.3867	−0.22

shown in Table 2.2. This shows the value of $QF(t)$ for each year, its expected value conditional on the relevant information up to 1982, $E[QF(t) \mid \mathcal{F}_{1982}]$, the observed deviation $QF(t) - E[QF(t) \mid \mathcal{F}_{1982}]$, the standard deviation of $QF(t) \mid \mathcal{F}_{1982}$, and the standardised residual, the observed deviation divided by the corresponding standard deviation.

2.2.6 The successive values of $\ln Q(t)$ are not independent, and the results represent only one experience for 12 years, not 12 independent experiences for 1, 2, ..., 12 years. The values of $\ln Q(t)$ are each well within one standard deviation of their forecast values. Again, one might think that the 'expanding funnel of doubt' has been too wide over this period. Perhaps the value of QSD was too high; perhaps the innovations do not have a constant variance (see Section 2.8); perhaps they are not normally distributed (see Section 2.9); a different covariance structure (i.e. a different value of QA) would lead to a lower variance for $\ln Q$, but this seems implausible in view of the evidence in Section 2.3.

2.2.7 Although I discuss the deviations between the observed and the forecast values, I do not believe that the model should be judged on its short-term forecasting performance alone, or even at all. There are many alternative models that could possibly give better short-term forecasts, especially those that take into account known exogenous variables, such as current government policy and the current state of the economy; but these factors are not readily forecastable for the future. The purpose of my model is to provide a realistic variance and covariance structure for many years ahead, to quantify the expanding funnel of doubt, and it should be judged on whether it does this satisfactorily for the purposes that actuaries might wish to use it.

2.3 *Updating and Rebasing to 1923-94*

2.3.1 I now consider refitting the parameters of the model, including the data up to 1994. However, it is desirable to reconsider the starting date. Originally I had started in June 1919, in part because this gave the longest available series for a share price index. It may be noted that what is now called the BZW Index gives share prices from December 1918 and share dividend yields from December 1919. However, as I wrote in my recent paper, 'The Risk Premium on Ordinary Shares' (Wilkie, 1995):

> "(December 1919), however, is not a particularly satisfactory date at which to start. It was only a little over a year after the end of the First World War. Retail prices over the two years 1919 and 1920 rose very sharply, and over the following three fell even more sharply. Share dividends paid in 1920 were almost double those paid in 1919; but share prices did not increase correspondingly. Companies were presumably now free of any restraints imposed by the war, and were able to increase dividends considerably; but shareholders recognised these as exceptional, and it seems as if they did not expect them to continue. Over the next three years, 1920-23, dividends fell back, but possibly by less than people expected, so share prices rose, and the very high dividend yields recorded at the end of 1920 (9.5%) also fell back.

"It is therefore reasonable to start the investigation at the end of 1923, when share dividend yields were 6.43%, Consols yields were 4.5%, and over the previous year retail prices had dropped by 1.7%, share dividends had grown by 12.4%, and share prices had risen by 5.09%. This date gives a suitable starting position, after the disturbances of the post-war period had been passed."

2.3.2 I now prefer to start the annual series in June 1923, when conditions were becoming reasonably stable, and to start the monthly investigations, described in Section 2.5, in December 1923. I have also used a revised series, by using for 1920 to 1936 the values of the Cost of Living Index published by the Central Statistical Office instead of the Board of Trade wholesale price indices. The values of the two series do not seem to be very different, but they produce rather different estimates of the parameters of the model.

2.3.3 To show the difference made by the different series and by different observation periods, I show, in Table 2.3, the parameter estimates and their standard errors (in parentheses) from fitting an AR(1) model to the original data from 1919 to 1982, and to the revised data for 1919-82, 1919-94, 1923-82 and 1923-94.

Table 2.3. Estimates of parameters and standard errors of AR(1) models for inflation over different periods

	Original data	Revised data			
Period	1919-82	1919-82	1919-94	1923-82	1923-94
QMU	0.0364	0.0374	0.0382	0.0489	0.0473
	(0.0169)	(0.0156)	(0.0132)	(0.0145)	(0.0120)
QA	0.5977	0.5057	0.5025	0.5863	0.5773
	(0.0976)	(0.1061)	(0.0975)	(0.0882)	(0.0798)
QSD	0.0543	0.0618	0.0574	0.0457	0.0427
	(0.0048)	(0.0055)	(0.0047)	(0.0042)	(0.0036)

See Appendix C.1 for an explanation of the method of estimating the parameters.

2.3.4 One can see that the estimates of *QA* are rather sensitive to the data and the period used. This is because of the severe falls in prices in 1919-21; even small changes to these extreme values affect the estimates of the parameters. Omitting the data for these years produces more stable estimates, in which the value of *QA* for periods starting in 1923 happens to be similar to the value found in the original investigation. Adding the data for 1982 to 1994 makes little difference to the parameter estimates whether one starts in 1919 or in 1923.

2.3.5 The estimated value of *QMU* is higher for periods starting in 1923 than for periods starting in 1919, because the years of extreme falls of price have been omitted; the added 12 years from 1982 to 1994 make little difference. I suggested using a value of *QMU* of 0.05, higher than that observed for 1919-82, because I felt that that was unduly influenced by the experience of the early years. Note, however, that the standard errors of the estimates of the mean are all relatively high. Plus or minus two standard errors for the 1923-94 period would give a range of about 0.025 to 0.07, or 2½% to 7%.

2.3.6 The estimates of *QSD* are lower when the early years are omitted, and lower also when the later years are added, because of the low variation of these recent years.

2.3.7 One can round off the values found for 1923-94 to give:

$$QMU = 0.047;\ QA = 0.58;\ QSD = 0.0425.$$

2.3.8 An important feature of the way I believe this model should be used is that those using it should form their own opinions about the choice of appropriate mean values. Actuaries, investment managers and others often have strong views about the likely mean rate of inflation in future, and also the likely mean rate of real dividend increase, mean share dividend yield, etc. These estimates of mean values to be used in future can depend on much more than simply an analysis of the historic values of any past period, and can take into account the actuary's view of likely future political and economic developments. It is not the part of my model to preempt the actuary's judgement. However, the model does allow a variance and correlation structure to be built around whatever mean or median values are chosen.

2.3.9 Extensive diagnostic testing of the model was carried out. The parameters for all periods are seen to be significantly different from zero. The residuals, the observed values of *QE*, are calculated, first with the exact parameter estimates, and then with my rounded ones for 1923-94. The autocorrelation coefficients of the residuals show nothing unusual, nor does the Wald-Wolfowitz test of runs of the same sign (the same as Steven's sign test). However, the skewness and kurtosis coefficients, based on the third and fourth moments of the residuals, are rather large: $\sqrt{b_1} = 1.13$, demonstrating substantial positive skewness; and $b_2 = 5.11$, implying quite heavy 'tails' in the distribution. These are even larger when the data for 1919-23 are included.

2.3.10 Individual large deviations can be picked out. Values of the residual errors greater than twice the standard deviation occur for 1940 (3.64 times the standard deviation), 1975 (2.83 times) and 1980 (2.54 times). There seems to be evidence that the residuals should not be taken as normally distributed, and this is discussed further in Sections 2.8 and 2.9.

2.3.11 A composite test of the skewness and kurtosis coefficients has been devised by Jarque & Bera (see Appendix C.3.8), and this also shows significant non-normality. The test statistic is 28.71, which should be compared with a χ^2_2 variate. The probability that such a result would occur at random is negligible.

2.4 *Previous Centuries*

2.4.1 I have available data for a consumer price index of sorts for a very long run of annual values, starting in 1264. For the data sources see Appendix F.1.

The indices for years prior to 1870 were not, of course, calculated contemporaneously, but have been constructed more recently by economic historians. They, therefore, do not have the same reliability as the modern Retail Prices Index, but, nevertheless, I believe that they have some validity, especially when the annual changes are considered.

2.4.2 The series from 1264 to 1994 $Q(t)$, is graphed with a vertical logarithmic scale in Figure 2.1, and the annual differences in the logarithms $I(t)$, are graphed in Figure 2.2. One can observe that from 1264 until about the middle of the 16th Century the price index oscillated about a fairly constant mean level, but there were considerable annual fluctuations, and considerable movements away from the overall mean level. Around 1540 a rise commenced, which peaked about 1650. Thereafter there were larger upwards and downwards drifts, but still with substantial annual changes in the rate of inflation. From 1933 the movement has been almost entirely upwards, and generally the rates of inflation have been more stable, as well as being almost uniformly positive.

2.4.3 I investigated part of these data in an earlier paper on 'Indexing Long Term Contracts' (Wilkie, 1981). I found that no simple autoregressive or moving average model readily represented the inflation series over these earlier centuries, and the same is true with the additional data now available.

2.4.4 I have subdivided the period before 1914 into three sub-periods, two of which overlap, which are: 1264-1540; 1540-1650; and 1600-1914. I have also considered the whole period, 1264-1914. Augmented Dickey-Fuller (ADF) tests for unit roots (see Appendix A.5.4) show that it is reasonable to take ln Q as an

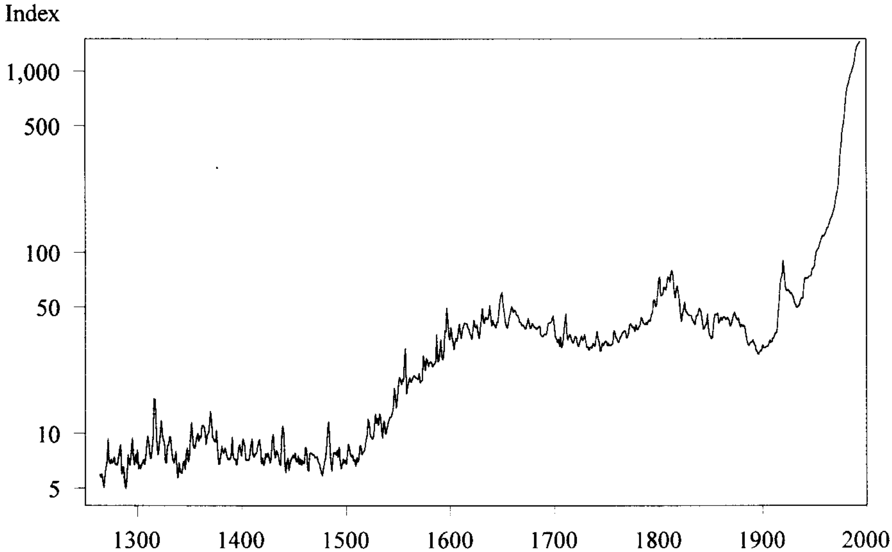

Figure 2.1. Consumer price index, 1264-1994

integrated I(1) series in each sub-period, when enough additional difference terms are included. When the differences of the logarithms, $I(t)$, are investigated, it is found that in each case there is significant negative autocorrelation two years and three years apart.

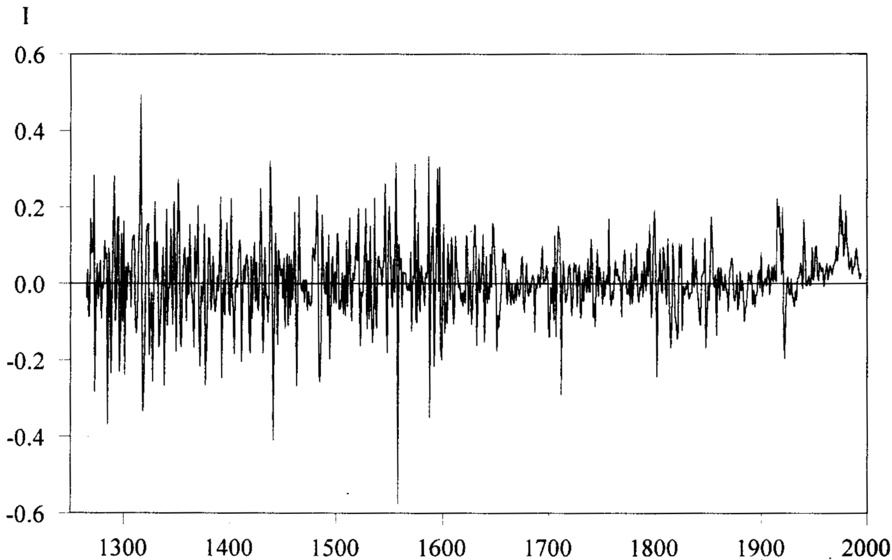

Figure 2.2. Annual force of inflation, 1264-1994

2.4.5 Fitting AR(p) models to each of these periods, where $p = 3$ or 5 as indicated, gives the results shown in Table 2.4. A higher order autoregressive model is not justified, and for 1540-1650 and 1600-1914 $p = 3$ is sufficient. It is interesting that most of the autoregressive coefficients are negative. However, tests of the normality of the residuals in each case fail significantly.

2.5 *Observations at Monthly Intervals*

2.5.1 Values of the Retail Prices Index, or its predecessor, the Cost-of-Living Index, are available from August 1914 at monthly intervals. Since these are available, it is worth investigating what can be done statistically with this series. For this purpose I have limited my observation period to the values of the RPI from December 1923 to June 1994, inclusive. This gives 847 monthly values of Q, from which I can derive 846 monthly values of the monthly force of inflation (note that I am not taking the annual inflation over the preceding 12 months, but the increase in each month).

2.5.2 I first calculated the autocorrelation coefficients of these monthly values up to lag 120. A graph of these is shown in Figure 2.3. Note that there are only three negative values in the first seven years, at lags 42, 78 and 79. All the others are positive. Note also that there are 'spikes' at lags 12, 24, 36, etc., i.e. at annual values, and that this continues right through to 10 years out. Note further that the correlation coefficients do not seem to decline very quickly, almost as if the series had 'long memory' or was 'fractionally integrated' (see, for example, Brockwell & Davis, 1991, Chapter 13; or Granger & Teräsvirta, 1993, Chapter 5, for explanations of these; and see Maddocks *et al.*, 1991; or Craighead, 1994, for examples of investigations using these ideas). The investigations I shall describe show, however, that this assumption of long memory would be unjustified.

Table 2.4. Results of fitting AR(p) models to four sub-periods

Period	1264-1540	1540-1650	1600-1914	1264-1914
Mean of $I(t)$	0.0023	0.0157	0.0002	0.0028
S.d. of $I(t)$	0.1262	0.1270	0.0711	0.1065
Order of AR(p)	5	3	3	5
Coefficients:				
a_1	−0.0394	−0.2459	0.1186	−0.0436
a_2	−0.3795	−0.3575	−0.1933	−0.3213
a_3	−0.2414	−0.2144	−0.1249	−0.2139
a_4	−0.1626	-	-	−0.1072
a_5	−0.1388	-	-	−0.1029
Residual s.d.	0.1149	0.1164	0.0683	0.0995
Skewness $\sqrt{\beta_1}$	0.22	0.19	−0.16	0.19
Kurtosis β_2	4.21	6.06	3.63	5.67
Jarque-Bera χ^2	18.91	43.89	6.52	195.39
$p(\chi^2)$	0.000	0.000	0.038	0.000

2.5.3 It is now useful to introduce a little notation, which I shall generalise in terms of an integrated series $x(t)$, observed at monthly intervals $t = 1, 2, ..., N$. One can take differences of this series at intervals of m months. There are m possible starting points. For example, if $m = 2$ we can take differences at intervals of two months, starting either with month 1 or month 2, e.g. we can take the first interval as January and February, the second as March and April, etc.; or we can ignore January and start with the first interval as February and March, the second April and May, etc. I shall denote differences taken at intervals of m months, starting with month h as $y_{m/h}(u)$, $u = 1, 2, 3, ...$ where:

$$y_{m/h}(1) = x(h+m) - x(h)$$

$$y_{m/h}(2) = x(h+2m) - x(h+m), \text{ etc.}$$

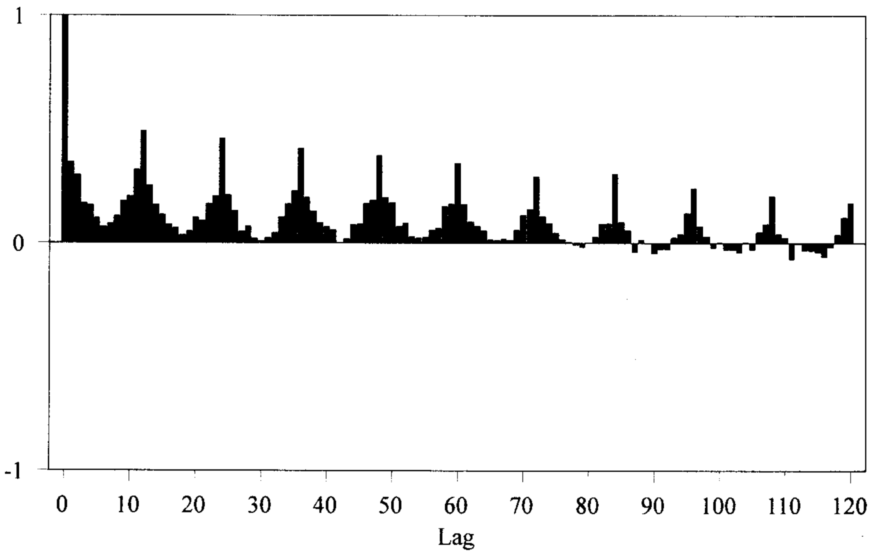

Figure 2.3. Autocorrelation function of monthly inflation, December 1923-June 1994

2.5.4 Since different numbers of months may be omitted at the beginning and the end of each series, the number of observations for each value of h is not necessarily the same. There are roughly N/h observations for each series, but there may be one or more fewer than this.

2.5.5 The initial series differenced at monthly intervals is denoted $y_{1/1}$. For the inflation series I shall denote the monthly differences by I, so that the original monthly inflation values are denoted by $I_{1/1}$.

2.5.6 We can now consider the series $I_{2/1}$ and $I_{2/2}$, i.e. the two series constructed by taking price changes over each pair of months. The first term of one series is the change in (the logarithm of) the price index from December 1923 to February 1924, and of the other series the change from January 1924 to March 1924.

2.5.7 The autocorrelation coefficients for the first 60 pairs of months are shown in Figures 2.4 and 2.5. Again 'spikes' can be seen at lags 6, 12, 18, etc., corresponding to annual intervals. Again the autocorrelation coefficients do not decline slowly, and are positive for several years.

2.5.8 Similar calculations have been carried out for differencing intervals of 3, 4 and 6 months, and similar results are obtained. In each case there is a moderately high autocorrelation at lag 1, and generally a larger one at the lag corresponding to annual intervals. As the differencing period lengthens, both the first and the annual correlation coefficients increase. This is shown in Table 2.5.

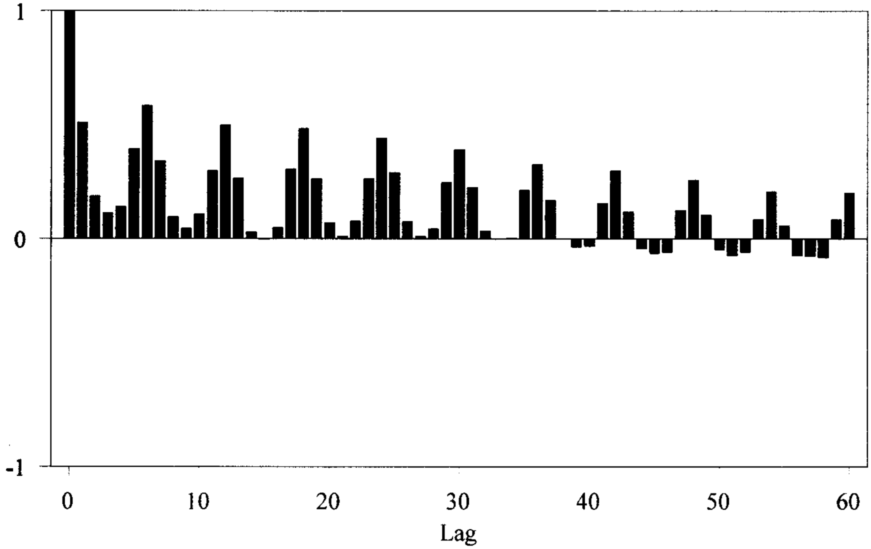

Figure 2.4. Autocorrelation function, inflation series 2/1, December 1923-June 1994

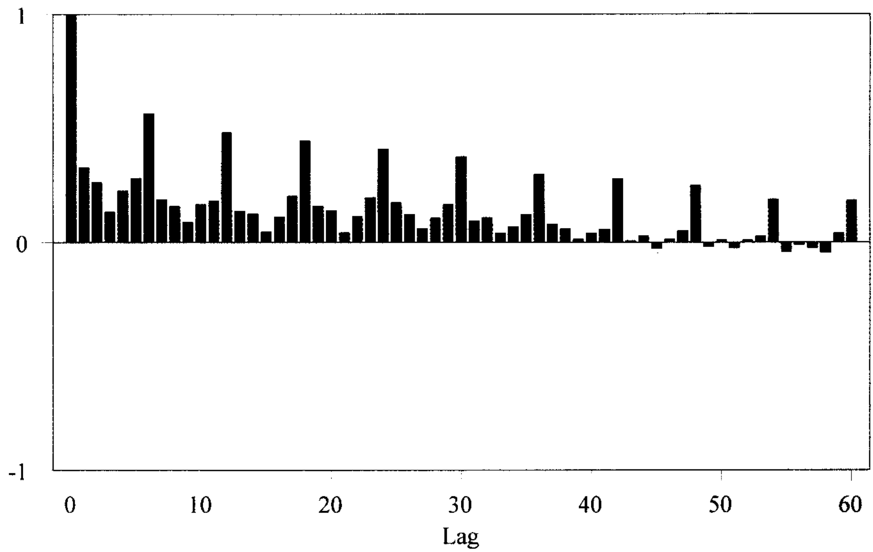

Figure 2.5. Autocorrelation function, inflation series 2/2, December 1923-June 1994

Table 2.5. Statistics and autocorrelation coefficients for various *m*/*h* series for RPI, December 1923 to June 1994

m/*h*	Number of values of $I_{m/h}$	12/*m* times mean value of $I_{m/h}$	$\sqrt{(12/m)}$ times standard deviation of $I_{m/h}$	r_1	$r_{12/m}$
1/1	846	0.0442	0.0299	0.3537	0.4913
2/1	423	0.0442	0.0342	0.5091	0.5833
2/2	422	0.0443	0.0354	0.3280	0.5651
3/1	282	0.0442	0.0400	0.3575	0.5995
3/2	281	0.0443	0.0370	0.3972	0.5831
3/3	281	0.0442	0.0389	0.3961	0.6052
4/1	211	0.0442	0.0432	0.2913	0.6407
4/2	211	0.0443	0.0421	0.3346	0.6218
4/3	211	0.0441	0.0410	0.3744	0.6183
4/4	210	0.0442	0.0395	0.4162	0.5803
6/1	141	0.0442	0.0509	0.1189	0.6771
6/2	140	0.0442	0.0439	0.4685	0.6068
6/3	140	0.0441	0.0430	0.5485	0.5824
6/4	140	0.0442	0.0421	0.6343	0.5572
6/5	140	0.0448	0.0435	0.4807	0.6401
6/6	140	0.0450	0.0484	0.2124	0.6788
					Standard deviation after fitting regression
12/1 (Dec)	70	0.0442	0.0525	0.7213	0.0364
12/2	70	0.0442	0.0524	0.7239	0.0362
12/3	70	0.0441	0.0524	0.7276	0.0356
12/4	70	0.0442	0.0532	0.7067	0.0376
12/5	70	0.0448	0.0537	0.6802	0.0394
12/6	70	0.0450	0.0542	0.6567	0.0410
12/7	70	0.0452	0.0550	0.6186	0.0434
12/8	69	0.0454	0.0539	0.6326	0.0419
12/9	69	0.0453	0.0545	0.6449	0.0417
12/10	69	0.0453	0.0543	0.6506	0.0413
12/11	69	0.0450	0.0525	0.7222	0.0361
12/12	69	0.0446	0.0527	0.7227	0.0357

2.5.9 When we reach annual differencing the pattern suddenly 'gels', and an autocorrelation function corresponding to an AR(1) model emerges. After lag 1 the autocorrelation coefficients die away exponentially in each case. There is no evidence of long memory beyond one year, except as transmitted through the first annual autocorrelation.

2.5.10 Experiments using differencing intervals which are not fractions of a year, e.g. 10, 11, 13 and 14 months, show similar results, with a single first order autocorrelation coefficient and subsequent values declining exponentially. However, the evidence from the initial monthly differencing shows clearly that there is an annual cycle, and not, for example, an 11-month or a 13-month cycle.

2.5.11 It is not surprising, nowadays, that autocorrelation in the inflation rate over a year should be observed. The 'headline' inflation rate quoted each month is always the annual rate for the preceding 12 months. Many enterprises review, and often alter, their prices at annual intervals, and the Government introduces an annual budget (though the position of this in the year has changed recently). There therefore seems good justification in continuing to model inflation on an annual basis, rather than attempting a more complex model based on more frequent intervals.

2.5.12 Carter (1991), studying Australian data, seems to have fallen into a trap. In order to increase the number of observations available, he observed quarterly inflation data (in Australia the Consumer Prices Index is published only at quarterly intervals), and found, not surprisingly, that my annual model did not fit very well.

2.5.13 Note that the line for 12/1 shows differencing at annual intervals using the December values, and so on, so that the line 12/7 corresponds to the June values I have used in Section 2.3. It can be seen that the estimate there of *QMU* is not identical with the mean value of $I_{m/h}$ and that *QA* is not identical with r_1. This is because the method of estimation of the parameters is not identical with the method of calculation of the values in Table 2.5, the values at the start of the series being treated slightly differently. The value of *QSD* should be compared with the value in the last column, which, for the 12/*h* series, shows the standard deviation after taking account of the regression using the value of r_1 shown as the parameter.

2.5.14 It is interesting to note the tidy cyclicality of the first autocorrelation coefficients for the 12/*h* series. The value of r_1 is at a maximum for series 12/3, that for February, and at a minimum for series 12/7, that for June, and the standard deviation of the residuals is correspondingly reversed. Thus inflation over the year February to February appears rather more predictable than over the year June to June. Whether this will continue now that the annual budget has been moved to late November or early December will be interesting to discover.

2.6 *Data for Selected Other Countries for Other Periods*

2.6.1 I have been able to obtain data on consumer price indices for a number of other countries for reasonably long periods. I have also obtained data for a larger number of countries for a uniform, but shorter, period, and these are discussed in Section 2.7. The periods and the results, giving broadly rounded values, are shown in Table 2.6.

Table 2.6. Fitted parameters for CPI model for selected other countries

	U.K.	U.S.A.	France	Canada	Sweden	Finland	Norway
Period	1923-94	1926-89	1951-89	1923-93	1923-93	1950-93	1930-93
QMU	0.047	0.03	0.06	0.034	0.046	0.06	0.047
QA	0.58	0.65	0.55	0.64	0.56	0.46	0.46
QSD	0.0425	0.035	0.04	0.032	0.034	0.039	0.039

2.6.2 The figures for the U.S.A. and for France have been shown in my paper for the Montréal Congress (Wilkie, 1992); those for Canada have been published in Wilkie (1994b): those for Sweden, Finland and Norway have not previously been published.

2.6.3 Values of the parameters are reasonably similar for each country. The most variable is the mean rate of inflation *QMU*, which ranges from 0.03 to 0.06, or roughly 3% to 6% inflation. This partly depends on the different periods chosen, since inflation in all countries has been rather higher in recent decades than before the Second World War, except that, because of the exceptionally high inflation in France during and just after that war, I chose to start my investigations for that country in 1951.

2.6.4 In order to complete a model for several countries, it is necessary to know the simultaneous crosscorrelations of the residuals, and to investigate whether there are any significant lagged crosscorrelations. While, in principle, it should be easy to do this, the way I have investigated the series makes it complicated to produce the answers readily, and I limit myself to quoting simultaneous correlation coefficients for the U.K., U.S.A. and France from my Montréal paper. These are not calculated over identical periods. They are:

U.K. *v* U.S.A.: 0.19; U.K. *v* France: 0.47; U.S.A. *v* France: 0.29.

Simultaneous correlation coefficients over a consistent, but shorter, period, and for more countries, are discussed in Section 2.7.

2.7 *Data for Several Other Countries for 1969-94*

2.7.1 In this Section I update the report given in my paper to the 4th AFIR Colloquium (Wilkie, 1994a), when I took the data up to May 1993. On this occasion I go up to June 1994.

2.7.2 Values of the Consumer Price Index (CPI) for 23 countries (whose names are listed in the tables) are available, in general at monthly intervals from January 1969 to June 1994. Details of the data sources are given in Appendix

F.8. Since monthly values are available, it is possible to investigate them in the same way as I have for the U.K. over a longer period, in Section 2.5, differencing at different numbers of months, i.e. the various *m/h* series. In general similar results are found for each country, but it would be laborious to quote them in full.

2.7.3 Table 2.7 shows summary results from fitting annual models to the data for each of the 23 countries. It shows, for each country, certain values of estimates of *QMU*, the mean value of the $I_{12/h}$ series, of *QA*, the autoregressive parameter, estimated by the first autocorrelation coefficient, and of *QSD*, the standard deviation of the residuals after fitting an AR(1) model with this parameter. For each of these three parameters it shows the lowest value observed for any of the 12 annual series, one corresponding to each month, a mean value (explained further below) and the highest value for any of the 12 series. In the case of *QA* and *QSD*, the mean values shown are the means of the values for the 12 separate series. For *QMU* the overall mean is shown, which is 12 times the mean of the total monthly series from January 1969 to June 1994. For almost every country this overall mean is lower than the mean of the twelve separate series, because different months are included at the beginning and end of the different series, and inflation was generally low in these end months.

2.7.4 There is quite a range of values of *QMU*, from just below 4% for Germany and Switzerland to over 14% for Greece and Portugal. The values of *QA* lie mostly between 0.5 and 0.8, though Greece, Norway and Portugal show values for individual months below 0.5, and all the values for Sweden are below this level.

2.7.5 The range of values of *QMU*, *QA* and *QSD* for the yearly series, for any one country, gives an indication of the variability that can be found in such an investigation when the number of observations is not large. The range is typically of the same order as one standard error of the parameter estimates.

2.7.6 Most of the countries have a range of values of *QSD* that includes some part of the interval 0.02 to 0.03. The values for Austria, Germany and the Netherlands are wholly below this range, and the values for Greece, Japan, New Zealand, Portugal and the U.K. are wholly above, with the lowest value for Portugal being 0.0448. There is some connection between high values of *QA* and low values of *QSD*, and vice versa, but this is not uniform.

2.7.7 There is some tendency for countries with low mean inflation rates to have low standard deviations of the original observations, with low values of *QSD*, and vice versa. This suggests, perhaps, that a certain type of stable economic environment may lead to inflation rates that are both low and stable, whereas it may be difficult to control inflation to be both high and stable. However, the experience of previous centuries in the U.K. suggests that inflation with a low mean and a high standard deviation is also possible. In the past, however, money was gold, and was not based on paper or, as nowadays, computer records, and the responsibility of governments was, perhaps, restricted to maintaining the gold value of the coinage. In these conditions a low mean

Table 2.7.
Analysis of Consumer Price Index for 23 countries from 1/1969 to 6/1994; parameters of AR(1) model for 12 series with 24 or 25 yearly steps

	QMU Low - overall mean - high	*QA* Low - mean - high	*QSD* Low - mean - high
Australia	0.0754 - 0.0743 - 0.0778	0.605 - 0.641 - 0.685	0.0240 - 0.0261 - 0.0290
Austria	0.0452 - 0.0448 - 0.0464	0.665 - 0.731 - 0.765	0.0130 - 0.0142 - 0.0162
Belgium	0.0518 - 0.0515 - 0.0530	0.649 - 0.725 - 0.780	0.0184 - 0.0206 - 0.0238
Canada	0.0586 - 0.0585 - 0.0610	0.679 - 0.704 - 0.735	0.0186 - 0.0200 - 0.0210
Denmark	0.0661 - 0.0656 - 0.0679	0.502 - 0.658 - 0.775	0.0209 - 0.0264 - 0.0331
Finland	0.0738 - 0.0731 - 0.0771	0.643 - 0.711 - 0.783	0.0227 - 0.0270 - 0.0311
France	0.0672 - 0.0667 - 0.0692	0.704 - 0.804 - 0.831	0.0185 - 0.0204 - 0.0262
Germany	0.0372 - 0.0369 - 0.0380	0.724 - 0.751 - 0.783	0.0113 - 0.0125 - 0.0134
Greece	0.1448 - 0.1444 - 0.1503	0.308 - 0.431 - 0.559	0.0391 - 0.0482 - 0.0537
Ireland	0.0881 - 0.0873 - 0.0906	0.745 - 0.780 - 0.826	0.0290 - 0.0342 - 0.0388
Italy	0.0981 - 0.0970 - 0.1007	0.587 - 0.716 - 0.812	0.0262 - 0.0336 - 0.0428
Japan	0.0484 - 0.0483 - 0.0495	0.596 - 0.659 - 0.724	0.0308 - 0.0348 - 0.0381
Luxembourg	0.0494 - 0.0488 - 0.0509	0.673 - 0.738 - 0.772	0.0173 - 0.0192 - 0.0226
Netherlands	0.0437 - 0.0438 - 0.0451	0.782 - 0.817 - 0.830	0.0159 - 0.0165 - 0.0181
New Zealand	0.0925 - 0.0914 - 0.0959	0.503 - 0.567 - 0.638	0.0356 - 0.0393 - 0.0433
Norway	0.0692 - 0.0683 - 0.0722	0.258 - 0.539 - 0.671	0.0197 - 0.0233 - 0.0293
Portugal	0.1458 - 0.1441 - 0.1513	0.371 - 0.479 - 0.646	0.0448 - 0.0568 - 0.0632
South Africa	0.1139 - 0.1137 - 0.1172	0.565 - 0.616 - 0.660	0.0179 - 0.0190 - 0.0206
Spain	0.1027 - 0.1011 - 0.1058	0.705 - 0.782 - 0.831	0.0232 - 0.0277 - 0.0346
Sweden	0.0743 - 0.0733 - 0.0768	0.226 - 0.369 - 0.428	0.0236 - 0.0260 - 0.0299
Switzerland	0.0395 - 0.0390 - 0.0412	0.553 - 0.589 - 0.642	0.0176 - 0.0196 - 0.0213
U.K.	0.0845 - 0.0838 - 0.0873	0.548 - 0.633 - 0.698	0.0343 - 0.0388 - 0.0445
U.S.A.	0.0550 - 0.0549 - 0.0561	0.636 - 0.673 - 0.700	0.0204 - 0.0221 - 0.0243

Table 2.8.
Analysis of Consumer Price Index for 23 countries from 1/1969 to 6/1994; 12 series with 24 or 25 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Aus	Ost	Bel	Can	Den	Fin	Fra	Ger	Gre	Ire	Ita
Australia	**1.0**	.15	.52	.42	.38	.52	.51	.15	.30	.47	**.60**
Austria (Ost)	.12	**1.0**	**.73**	.42	.57	**.60**	**.70**	.59	.38	**.69**	.55
Belgium	.42	**.61**	**1.0**	.51	.48	**.65**	**.82**	.59	.29	**.75**	**.63**
Canada	.54	.25	**.60**	**1.0**	.52	.50	**.64**	.46	.35	.47	.51
Denmark	.51	.48	.46	.50	**1.0**	.43	**.79**	.41	.39	.44	**.66**
Finland	.59	.40	.57	.53	.35	**1.0**	.55	.56	−.01	**.68**	**.79**
France	.40	**.66**	**.77**	.59	**.69**	.47	**1.0**	**.61**	.44	**.70**	**.70**
Germany	.16	.41	.30	.47	.25	.48	.38	**1.0**	−.02	**.64**	.54
Greece	.34	−.02	−.06	.23	.13	.29	.01	.28	**1.0**	.24	.30
Ireland	.57	**.66**	**.60**	.46	**.68**	.54	**.84**	.42	.19	**1.0**	**.67**
Italy	.52	**.63**	**.62**	.32	**.80**	.40	**.77**	.16	.14	**.78**	**1.0**
Japan	.55	**.63**	.53	.43	**.72**	.56	**.68**	.29	.36	**.67**	**.73**
Luxembourg	.50	.45	**.79**	.36	.22	.51	.54	.39	.02	.52	.55
Netherlands	.24	**.61**	**.63**	.46	.33	.57	**.66**	.55	−.06	**.67**	.43
New Zealand	.52	−.01	.11	.52	.33	.18	.03	.03	.28	.27	.22
Norway	.43	.21	.33	.48	.34	.41	.31	−.16	.25	.35	.39
Portugal	.46	.13	.32	.17	.49	.22	.52	−.06	.13	.48	.48
South Africa	.29	−.01	.19	.34	.15	.04	.19	−.02	.24	.26	.27
Spain	.50	.08	.38	.53	**.71**	.41	.35	.10	.12	.30	.44
Sweden	.15	.23	.55	.49	.39	.37	.51	.11	.18	.29	.49
Switzerland	.35	.41	.34	.56	.35	.44	.33	**.82**	.52	.35	.23
U.K.	.47	.26	.51	.54	.13	.57	.44	.35	.22	.53	.26
U.S.A.	.36	.42	.47	**.62**	.47	.42	**.71**	.39	.42	.53	.53

Table 2.8 (continued).
Analysis of Consumer Price Index for 23 countries from 1/1969 to 6/1994; 12 series with 24 or 25 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Jap	Lux	Net	N.Z.	Nor	Por	S.A.	Spa	Swe	Swi	U.K.	U.S.A.
Australia	.54	.33	.11	.54	.32	.30	.35	.42	.25	.27	.57	.40
Austria (Ost)	**.61**	**.63**	**.60**	.12	.32	.39	.04	.33	.31	.39	.48	.52
Belgium	**.65**	**.77**	.54	.25	.19	.53	.24	.37	.26	**.61**	.58	.53
Canada	.47	.36	.16	.36	.51	.25	.37	.32	.49	.55	.35	**.61**
Denmark	**.78**	.29	.33	.10	.32	.46	.08	.56	.36	.34	.50	**.67**
Finland	.46	**.77**	.54	.33	.58	.16	.15	.21	.48	.43	**.69**	.45
France	**.76**	.56	.45	.37	.26	.57	.28	.47	.32	.54	**.63**	**.83**
Germany	.44	**.65**	**.70**	.18	.15	.08	.21	.37	.18	**.65**	.51	.54
Greece	.56	.08	−.14	.13	.29	.25	.35	.13	.17	.08	.07	.42
Ireland	.46	**.69**	**.69**	.35	.30	.32	.42	.37	.18	.47	**.77**	**.63**
Italy	**.73**	**.70**	.41	.21	.47	.32	.20	.39	.54	.39	**.74**	**.66**
Japan	**1.0**	.42	.30	.11	.17	.40	.27	.47	.24	.41	.50	**.69**
Luxembourg	.29	**1.0**	**.62**	.11	.34	.30	.09	.26	.43	.49	.53	.35
Netherlands	.44	.47	**1.0**	.19	.09	.07	.09	.34	.20	.55	.58	.30
New Zealand	.12	.04	−.01	**1.0**	.45	.08	.43	.22	.25	.12	.47	.39
Norway	.30	.17	.12	.43	**1.0**	−.11	.09	.15	.55	−.05	.26	.20
Portugal	.44	.28	.20	−.02	.05	**1.0**	.18	.49	.27	.31	.42	.38
South Africa	.19	.05	.29	.46	.31	−.08	**1.0**	.11	−.11	.32	.34	.37
Spain	.48	.14	.16	.35	.31	.41	−.02	**1.0**	.26	.22	.52	.31
Sweden	.30	.35	.37	.17	.43	.23	.27	.50	**1.0**	.23	.42	.29
Switzerland	.54	.32	.42	.17	−.04	.06	.17	.19	.11	**1.0**	.40	.38
U.K.	.24	.51	.47	.42	.13	.20	.17	.18	.37	.25	**1.0**	**.68**
U.S.A.	**.64**	.31	.30	.19	.21	.32	.16	.27	.46	.46	.56	**1.0**

inflation rate may have been the effective target, regardless of annual fluctuations in prices; but I do not expect that these targets were so explicitly expressed by the Chancellors of the Exchequer of the time.

2.7.8 The skewness and kurtosis coefficients of the values, both of the original observations and of the residuals of each of the 276 series, were calculated, along with the Jarque-Bera χ_2^2 statistic. Generally these indicated no marked departure from normality, but certain countries, in particular Belgium, Finland, Greece and Japan, showed particularly high values of this statistic.

2.7.9 The next step is to calculate the correlation coefficients between the residuals of the different series, both simultaneously and with appropriate lags. Thus, considering the annual series for month h, we calculate the residuals for each country after fitting an AR(1) model, and calculate correlation coefficients between these residuals for each pair of countries, both simultaneously and with various lags. There is, thus, a set of arrays of correlation coefficients for each month.

2.7.10 Simultaneous correlation coefficients for the annual June series and the annual December series are shown in Table 2.8, with the December values in the lower triangle and the June values in the upper triangle. Values of 0.60 or greater are shown in bold type.

2.7.11 One can pick out clusters of countries with high correlations: Germany and Switzerland are one pair, Belgium and Luxembourg another, and Belgium, Denmark, France, Ireland, Italy, Japan and Netherlands form a third.

2.7.12 Lagged correlation coefficients are not shown. For lag 1 they are much smaller than the simultaneous correlation coefficients; but it is clear that they are not symmetrically distributed around zero, which they should be if there were, in fact, no lagged crosscorrelations. If two series are independent and normally distributed, the sample correlation coefficient calculated from a sample of size n is normally distributed with mean zero and variance $1/n$. In this case we have 25 or 26 observations at lag 0, and fewer for higher lags. One standard error is therefore approximately 0.2. A single correlation coefficient as large as 0.4 is not significantly different from zero at a 5% probability level.

2.7.13 The average of all 529 correlation coefficients at lag 1 for June is 0.17, and for December is 0.16; similar values are found for the other months. The standard deviation of the 529 values is between 0.19 and 0.22 for all months, which is about the same as $1/\sqrt{n}$.

2.7.14 For lag 1, individual high values appear for December, with the first country lagging one year behind the second: Belgium after Finland 0.62; Belgium after Japan 0.64; Belgium after Switzerland 0.69; Canada after Japan 0.62; Netherlands after U.S.A. 0.63; Norway after New Zealand 0.63; U.K. after Japan 0.69; U.K. after U.S.A. 0.61. While one might find some rationale for certain of these relationships, I think it would be difficult to explain, for example, why the Consumer Price Index in the U.S.A. (December series) follows that of Greece with a two-year lag and a correlation coefficient of -0.51. It seems more reasonable to assume a general world-wide cross-correlation at lag 1 of modest extent, rather than a series of specific effects.

2.8 *ARCH Modelling*

2.8.1 I have assumed, so far, that the values of the parameters themselves are stationary, except for the obvious break sometime during the first half of this century, as compared with previous centuries. There are many possible ways of modelling time series with changing parameters, but it is not practicable to carry out stochastic simulations unless one has a precise model of the way in which the parameters might change.

2.8.2 One comparatively simple method, however, is to allow for a non-stationary standard deviation, through what are known as autoregressive conditional heteroscedastic (ARCH) models. These are described more fully in Appendix D.1. The sort of model that is comparatively easy to investigate and to implement is:

$$QSD(t)^2 = QSA + QSB.(I(t-1) - QSC)^2$$

where QSA, QSB and QSC are parameters to be determined. This model states that the variance each year depends on the square of the deviation of last year's observation of the force of inflation $I(t)$, from some middle value QSC, which might or might not equal the mean value QMU. The value of QSA must be positive; this is necessary because, without it, there is the chance of the second term being exactly zero, the variance reducing to zero, and inflation moving deterministically thereafter. QSB must not be negative. If it is zero, then the model has a constant variance of QSA, the same as I have used so far.

2.8.3 This model reflects the notion that, if the rate of inflation over any year has been unusually high, then the uncertainty about the rate of inflation in the following year is increased. It might be again high or it might be much lower. The same applies if the rate of inflation is unusually low, and this is made effective through the squared term. If, on the other hand, the rate of inflation happens to have been near the value of QSC, then the standard deviation in the succeeding year is smaller.

2.8.4 One way of testing for this sort of ARCH model is to fit a conventional linear model, and then to consider the squares of the residuals, the QEs and the squares of the observations, or possibly the squares of the deviations of the observations from the means $(I(t) - QMU)^2$, and then to look at the autocorrelation coefficients and crosscorrelation coefficients of the squared values. Another possible indicator is to use the expected observations, or their deviations from the mean, in this case given by:

$$IH(t) = QA.(I(t - 1) - QMU).$$

2.8.5 One expects the residuals to be simultaneously correlated with the observations, because the residual forms part of each observation; the squares are likely also to show simultaneous correlation. However, if the model has been appropriately fitted, then one expects the residuals and the expected values to have no correlation, so correlation in the squared values is of interest.

2.8.6 For convenience, I shall refer to the relevant series as *I*-squared, *IH*-squared and *QE*-squared, though the first two reflect squared deviations from the mean.

2.8.7 The *QE*-squared series shows no significant autocorrelations; indeed the autocorrelation coefficients are conspicuously low. The crosscorrelation coefficient between *QE*-squared and $I(t-1)$-squared is 0.086, also quite small. The simultaneous cross-correlation coefficient between *QE*-squared and *IH*-squared is 0.032, even smaller. There is, therefore, no evidence to suggest that an ARCH model of this type would be useful. However, the original residuals are distinctly fat-tailed (see ¶¶2.3.6 to 2.3.8), and one feature of ARCH models is that they can reproduce fat-tailed residuals.

2.8.8 Undeterred, therefore, I fitted the first order ARCH model described above to the yearly data for 1923 to 1994, first allowing *QSC* to be chosen freely (model (ii)) and then fixing *QSC* to equal *QMU* (model (iii)), with the results shown in Table 2.9, which also shows, denoted as model (i), results from the original model, but with the new parameterisation, i.e. *QSA* replaces QSD^2.

Table 2.9. Estimates of parameters of AR(1) ARCH models for inflation, 1923-94

	(i) Original model *QSB* = 0		(ii) ARCH model *QSC* free		(iii) ARCH model *QSC* = *QMU*	
	Parameter estimate	Standard error	Parameter estimate	Standard error	Parameter estimate	Standard error
QMU	0.0473	0.0120	0.0443	0.0113	0.0404	0.0108
QA	0.5773	0.0805	0.6217	0.1303	0.6179	0.1292
QSA	0.001825 $=0.0427^2$	0.000304	0.000662 $=0.0257^2$	0.000228	0.000656 $=0.0256^2$	0.000224
QSB	-	-	0.5490	0.2171	0.5524	0.2147
QSC	-	-	0.0389	0.0068	0.0404	-
Log likelihood	0.0		+4.45		+4.36	
Jarque-Bera χ_2^2	28.71		6.85		5.76	
$p(\chi^2)$	0.000		0.033		0.056	
Long-run standard deviation *QSD*	0.0427		0.0627		0.0626	

2.8.9 All the parameters for the ARCH models are more than 2.5 times their standard deviations away from zero, and the log likelihood for model (ii) compared with model (i) is improved by 4.45. Twice this figure should be compared with a χ_2^2 variate, since two extra parameters (*QSB* and *QSC*) have been added. The improvement is distinctly worthwhile ($p = 0.012$). For model (iii) the reduction in log likelihood compared with model (ii) is only 0.09, which suggests that model (iii) is almost as satisfactory as model (ii) and has one fewer free parameter.

2.8.10 The Jarque-Bera statistic is considerably reduced in both the ARCH models, from 28.71 (p = 0.000) in model (i) to 6.85 (p = 0.033) in model (ii) and 5.76 (p = 0.056) in model (iii); the kurtosis coefficient b_2 is now 3.89 (model (ii)) and 3.74 (model (iii)) instead of 5.11 (model (i)); the skewness coefficient $\sqrt{b_1}$ remains relatively large at 0.61 and 0.59 respectively. The number of extreme errors is also reduced, though high values remain for 1940 (3.44 times the standard deviation in model (ii) and 3.40 times in model (iii)), 1948 (2.03 and 2.02 times) and 1951 (2.15 and 2.18 times); but one must expect values a little greater than 2.0 times the standard deviation from time to time.

2.8.11 For model (ii) the value of the standard deviation $QSD(t)$ ranges from 0.176 in 1923, following the large fall in inflation in 1922, and 0.145 in 1976, at the high end, to less than 0.026, in six different years; the smallest value possible is $\sqrt{QSA}$ = 0.0257. The average value of the standard deviations is 0.0453, and the square root of the average value of the variance, which is a better measure, is 0.0530. Very similar values are found for model (iii).

2.8.12 An ARCH model of this type is easy to use for simulation, as noted in Appendix B to Geoghegan *et al.* (1992), but it produces larger variances in the long run. The long-run or unconditional variance of QE is given by:

$$QSD^2 = \{QSA + QSB(QMU - QSC)^2\}/\{1 - QSB/(1 - QA^2)\}$$

and the value of this is also shown in the table, assuming that QA = 0.58. Graphs of simulated forecasts for ARCH model (iii) are shown in Figure 2.8 and numerical values are given in Table 11.3 in Section 11.5, in which certain features of the simulations are also discussed.

2.8.13 Suitably rounded parameters for ARCH model (iii) would be:

$$QMU = 0.04; \; QA = 0.62; \; QSA = 0.0256^2; \; QSB = 0.55; \; QSC = 0.04.$$

2.8.14 Taylor (1986) and Mills (1994) have also investigated using ARCH models for certain economic time series.

2.9 *Other Distributions for the Residuals*

2.9.1 When I started developing this paper I thought that it would be useful to investigate alternative distributions for the residuals of the models for many of the series, other than the normal, because so many of the distributions were fat-tailed. Many of the series of residuals for monthly series are indeed 'leptokurtic', but the annual series, after taking into account any necessary ARCH effects, as for the inflation series, are not particularly non-normal, so I have not investigated this.

2.9.2 There are two ways of approaching the distributions of the residuals: one is to assume a non-normal distribution for short-term changes, but one with finite variance, so that over a longer period the cumulative residuals approach normality because of the Central Limit Theorem; the other is to assume stably distributed residuals (known as α-stable, Lévy-stable or stable Paretian), which have infinite variance (if $\alpha < 2$, so that the distribution is not normal), but have the feature that the sum of such a set of residuals is also distributed α-stable, with the same characteristic parameter α. Not only is the variance of α-stable variates infinite, the mean of a log stable variate is also infinite, which would have certain interesting implications.

2.9.3 Possible distributions for the first approach include the *t*-distribution (see Praetz, 1972), gamma (see ¶2.10.4), Pearson Type IV, and no doubt others, such as can be found in the compendious work by Johnson & Kotz (1970) [N.L. Johnson is a Fellow of the Institute]. For a good introduction to stable distributions, in French, see Walter (1990), which has a good list of references, mostly in English, and for fuller, but not comprehensive, texts see Samorodnitsky & Taqqu (1994) or Janicki & Weron (1994); stable distributions are interesting, and have many nice features, but they are not easy to manipulate.

2.10 *Other Models*

2.10.1 Other authors have investigated or proposed models for inflation on the lines that I have suggested. The usual econometric approach is much more short term, and takes account of many exogenous variables, such as the government's current monetary and fiscal policy. I am not aware of any modelling that has been done by econometricians on similar lines to mine, though I have not investigated the literature fully.

2.10.2 Metz & Ort (1993) use my model and other ARIMA models to investigate the Swiss consumer price index from 1925 to 1990, and also for a shorter period, 1940 to 1990. Using an ARIMA(1,1,0) model, in effect the same as mine, they obtain parameters $QMU = 0.0237$, $QA = 0.6760$, $QSD = 0.0282$ for the first period and $QMU = 0.0345$, $QA = 0.6035$ and $QSD = 0.0238$ for the second period. These are consistent with my model, and not too different from the figures quoted for Switzerland in Table 2.7 for a yet shorter period. Metz & Ort tried several higher order ARIMA models, but concluded that, on balance, the ARIMA(1,1,0) model was the most satisfactory.

2.10.3 Deaves (1993) studied Canadian inflation data using quarterly rates of inflation, and produced results quite similar to what I would do for m/h series with $m = 3$. However, he did not investigate different frequencies of sampling, as I have done.

2.10.4 Daykin, Pentikäinen & Pesonen (1994) (DPP) describe work done in Finland on modelling inflation. They propose (in Chapter 7) essentially the same autoregressive model as I have, except that they use the annual rate of inflation ($Q(t)/Q(t-1) - 1$) instead of the difference between the logarithms, i.e. an *i*-type rate rather than a δ-type rate. While this makes little numerical difference when the values are small, I prefer to use δ-type rates because of their convenient additive properties, i.e. if each $I(t)$ is normally distributed, then their sum, the force of inflation for several years, is also normally distributed, and the resulting price index is lognormally distributed.

2.10.5 Pentikäinen *et al.* (1989, see also DPP, Chapter 7) suggest that a skew distribution for the residuals is more appropriate than a normal one, and propose a three-parameter gamma function, i.e. a gamma distribution with the origin shifted away from zero. They do not suggest parameters for this distribution, and there seems to me to be the same problem about applying this distribution to their model, in that, while the sum of gamma distributions remains gamma, it is not clear what distribution is created by the product of gamma distributed variables.

2.10.6 Clarkson (1991) suggests various non-linearities in my model, to reflect the upward skewness of the residuals. Using a modification of my notation he suggests:

$$I(t) = QMU + QA.(I(t-1) - QMU) + QB.\text{Trend}_+(I(t)) + QE(t) + QP(t).QF(t)$$

where:

— QB is a fixed parameter;
— $\text{Trend}_+(I(t))$ is the current trend in $I(t)$, if it is positive; the trend can be estimated either by least squares regression on recent values, or by an exponentially weighted moving average of past values;
— $QP(t)$ is a Bernoulli variable that takes the value 1 with probability p, but only if the values of $QP(t-k)$, for $k = 1$, 2 and 3, have been zero, and otherwise takes the value zero; and
— $QF(t)$ is another random variate whose distribution is to be chosen.

2.10.7 While Clarkson explains the rationale of all his terms, the model seems to me to be unnecessarily overparameterised. I have not investigated the statistical properties of such a model, e.g. the means, variances and covariances of $I(t+k)$, given all the facts at time t, nor have I investigated statistical methods of parameter estimation; nor, in his published paper, has Clarkson. 'Mixture distributions', such as are created by the extra Bernoulli term, are nice for simulation, but there are problems in the estimation of the parameters (see Everitt & Hand, 1981).

2.11 *Forecasting*

2.11.1 Having chosen a model to represent the stochastic development of the Retail Prices Index, or any other variable, one wishes to use it for producing 'forecasts' of the future. In my earlier papers I did this wholly by simulation, which is necessary for some of the series I considered. This method remains useful for complicated functions of the variables, but since the model for ln Q is purely linear, it is possible to provide forecast means and variances by manipulation of the parameters, as shown in Appendix E.2.

2.11.2 The word 'forecast' may be misleading. What one is calculating are the moments of the probability distributions of future values of ln Q, according to the chosen stochastic model. A good forecast, perhaps like a weather forecast, would take into account all sorts of other exogenous variables, and would not go very far ahead. The purpose of my model is for longer-term simulation, and it is the properties, particularly of the variance, the 'expanding funnel of doubt' that are of as much interest as the means. 'Projections' might be a better word than forecasts.

2.11.3 In Figure 2.6, I show a set of ten simulations of $Q(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1950, all on a logarithmic scale. One can get an impression of the shape of the expanding funnel of doubt from these.

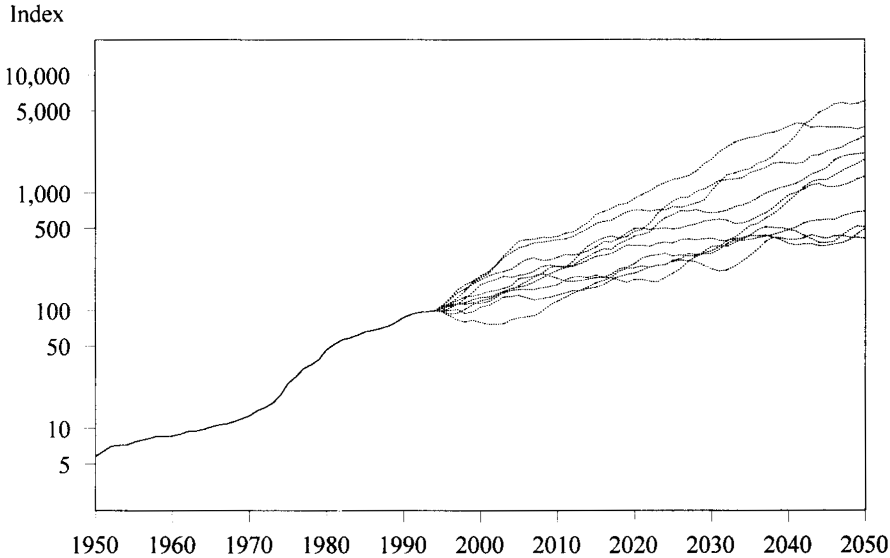

Figure 2.6. Retail prices index, 1950-94, and simulations, 1994-2050

2.11.4 In Figure 2.7, I show the forecast median of $Q(t)$, starting with the conditions in June 1994, also on a logarithmic scale, along with two sets of confidence intervals. The wider pair shows the mean plus and minus two standard deviations, using the formulae in Appendix E.2. The inner pair shows what the two standard deviation confidence interval would be for a random walk model for ln Q, with the same one-year standard deviation. The standard deviation is proportional to the square root of $(t - 1994)$. This shows how much the autoregressive nature of the model increases the uncertainty about the future.

2.11.5 In Figure 2.8, I show a further set of ten simulations of $Q(t)$ at annual intervals from June 1994 to 2050, this time using ARCH model (iii) of Section 2.8. One can see how the simulations fluctuate more than with the homoscedastic (non-ARCH) model.

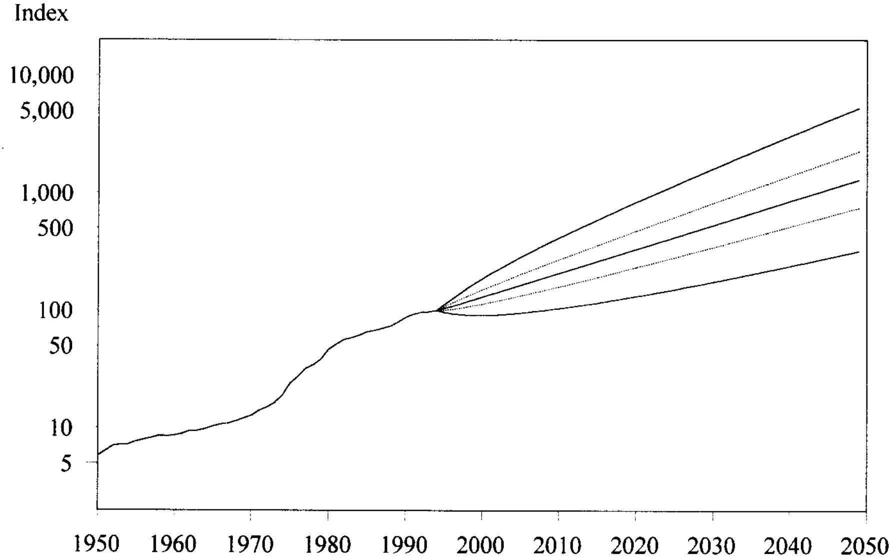

Figure 2.7. Retail prices index, 1950-94, and forecast medians and confidence intervals for AR(1) model and for a random walk model, 1994-2050

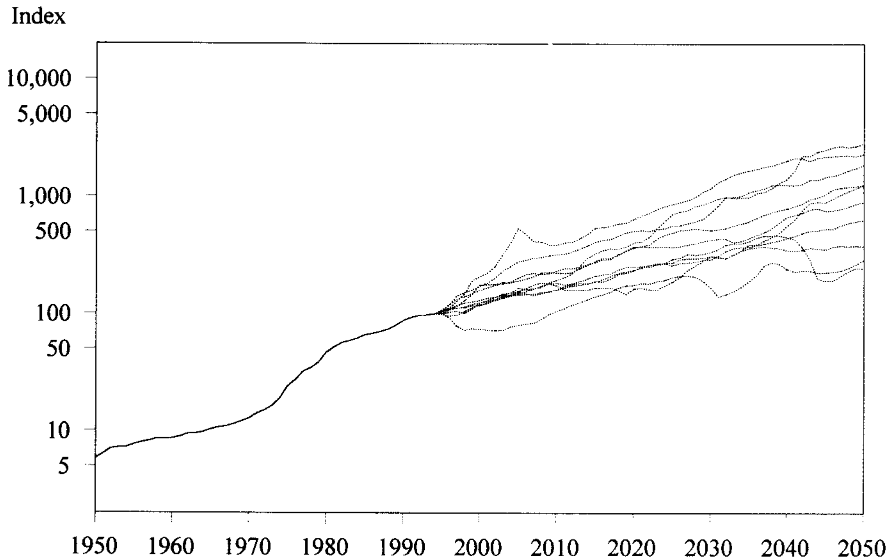

Figure 2.8. Retail prices, 1950-94, and simulations, 1994-2050, using ARCH model

3. WAGES INDEX

3.1 *Wages Data*

3.1.1 A wages index of sorts is available at annual intervals from 1809, and at monthly intervals from January 1920. The data sources are given in Appendix F.2. I describe it throughout as a 'wages' index, although in recent years it has been an index of earnings. However, I wish to avoid confusion between the earnings of individuals and the earnings of companies, so I refer to the former always as wages.

3.1.2 The wages index and the prices index I have used are both plotted in Figure 3.1, with a vertical logarithmic scale. A real wages index, calculated as the wages index divided by the prices index, is plotted, also on a vertical logarithmic scale, in Figure 3.2. It is interesting to see that during the 19th Century there was a gentle upwards drift in wages and a downwards drift in prices. This was the way in which improved productivity was transferred to consumers. However, when inflation became the norm, as in the latter half of this century, both the wages index and the prices index moved upwards broadly together.

3.1.3 It is not surprising that real wages fell during the First World War. However, it is a surprise to me to see that real wages appear to have fallen between 1900 and 1914, the Edwardian period, which, in retrospect, appears to have been a prosperous one. I do not know whether there is some peculiarity about the indices I have had to use for this period. For this reason I have omitted some of these years in certain of the calculations.

3.1.4 The real wages index shows a general upwards drift around an apparent trend of about 1.4% a year. The question of whether wages can be modelled better as random fluctuations about a deterministic trend, or as a random walk with an upwards drift, is investigated in Section 3.3.

3.2 *A Univariate Model for Wages*

3.2.1 It is appropriate to identify a suitable univariate model for wages. I have used annual values, from June 1923 to June 1994, to correspond with my investigation of the prices index. A similar model fits, namely that the first differences of the logarithms of the wages index can be modelled by an AR(1) model, where $W(t)$ is the value of the wages index at time t:

$$W(t) = W(t-1).\exp\{J(t)\}$$

so that $J(t) = \ln W(t) - \ln W(t-1)$ is the rate (strictly force) of wage inflation over the year $(t-1,t)$. Then:

$$J(t) \sim \text{AR1}(WMU,\ WA,\ WSD).$$

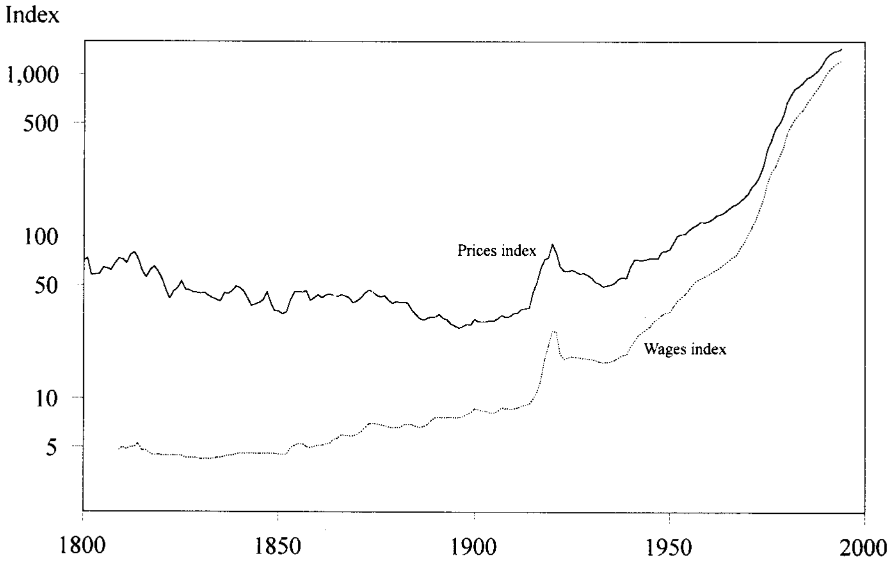

Figure 3.1. Wages index and prices index, 1809-1994

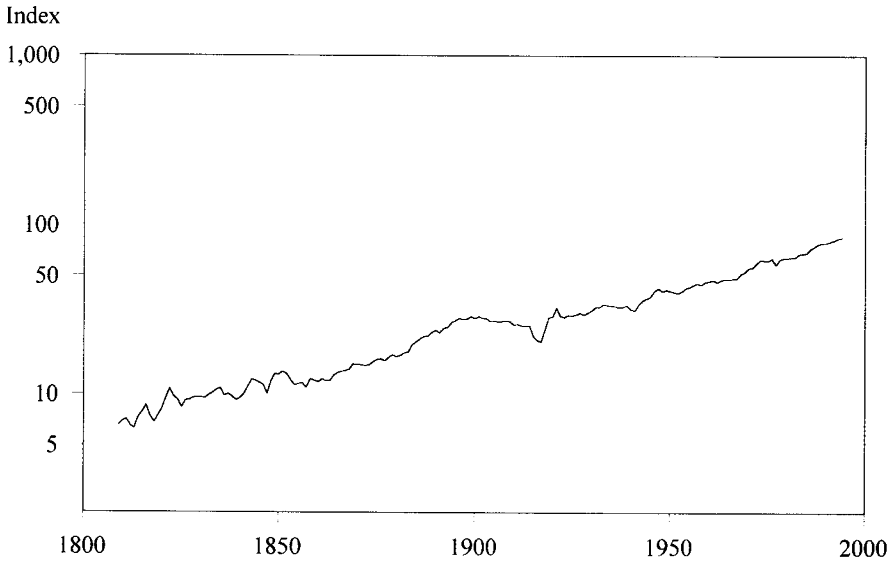

Figure 3.2. Index of real wages, 1809-1994

3.2.2 Possible parameters, based on the experience from 1923 to 1994, would be: WMU = 0.064, WA = 0.72, WSD = 0.033. However, it is unreasonable not to investigate also the connection between prices and wages.

3.3 *Are Wages and Prices Cointegrated?*

3.3.1 The concept of cointegrated time series is discussed in Appendix B.3. An example discussed later is the connection between share prices and share dividends. It is found that the logarithms of share prices and of share dividends are indeed cointegrated, so it is reasonable to model the difference in the logarithms, the logarithm of dividend yield, as a stationary series.

3.3.2 For prices and wages the equivalent is real wages. This clearly has an upwards trend or drift, which must be taken into account. The logarithms of prices and of wages, $\ln Q$ and $\ln W$, are clearly I(1) series, and ADF unit root tests (see Appendix A.5.4) for the whole period and for various subperiods confirm this for wages, as has been discussed already for prices. The question then is whether some linear combination of $\ln Q$ and $\ln W$ forms a stationary series. The obvious relationship is that the logarithm of real wages, i.e. the difference between $\ln W$ and $\ln Q$, is stationary and thus is an I(0) series; or, alternatively, if there is no cointegration, then $\ln W/Q$ is also an integrated I(1) series.

3.3.3 The difference between these models is very important. If there is cointegration, then one can model real wages as following a deterministic trend, with random fluctuations about that trend, in such a way that, if real wages rise too far above the trend, there is a tendency for them to fall back, and, if they fall too far below the trend, there is a tendency for them to rise. If, on the other hand, there is no cointegration, then wages are subject to short-term influences from current and past values of prices, and possibly influence current and future values of prices, but real wages do not move back to any particular trend.

3.3.4 There is some economic rationale for both of these models. If wages move ahead because labour, in the economic sense, has obtained a larger share of national income, then it would not be surprising if, at some later date, this movement were reversed. One could argue that this strengthening of labour happened during the 1970s and was reversed during the 1980s. On the other hand, if national income is increased because of some piece of good fortune, such as North Sea oil or technological improvements, this may provide additional returns to both labour and capital, and might well not be reversed. Contrariwise, if national income falls because of some external misfortune, such a fall might also not readily be reversed.

3.3.5 I have tested for cointegration, using the Johansen method included in the *MICROFIT* computer package, for three periods: 1809-1994; 1809-1900; and 1923-94. I omitted the period from 1900 to 1923 when subdividing the long period, for the reasons discussed in ¶3.1.3.

3.3.6 There are two tests in the Johansen method. For the long period from 1809 to 1994, both tests show that $\ln Q$ and $\ln W$ are most unlikely to be cointegrated. This is also clearly true for the recent period 1923-94. However, for the earlier period 1809-1900, the relevant test statistics both lie between the 90% and 95% critical values, suggesting that cointegration is plausible. However, the 'cointegrating vector', i.e. the relationship between ln Q and ln W for this period, seems implausible:

$$\ln Q + 0.4322 \ln W \sim I(0)$$

whereas the sort of relationship one would expect is $\ln Q - \ln W \sim I(0)$.

3.3.7 Another approach is to test ln W/Q, the logarithm of real wages, for unit roots. For the whole period and for each of the subperiods the unit root test suggests, fairly clearly, that ln W/Q has a unit root, so it is an I(1) series. It therefore seems reasonable to investigate the relationship between wages and prices without taking cointegration into account. Thus, the models I investigate take into account short-term dependence between wages and prices, but include no long-term relationship.

3.4 *A Transfer Function Model*

3.4.1 It would be possible to construct a model in which both wages and prices depended on previous values of themselves and of the other, and this is discussed in Section 3.5. In the first place, however, I use a transfer function model, in which the model for prices that has been found in Section 2 is left unchanged, and a model for wages is constructed in which wages depend on prices, but not vice versa. There may be good economic arguments against such a model, but there are considerable practical advantages, not least that those who wish to use a price index as part of an integrated investment model do not necessarily wish to be forced to consider also a wages model as part of the package. A model for wages is useful for many purposes, but one would like to be able to omit it if desired.

3.4.2 I first investigate the data from 1923 to 1994, at annual intervals using June values of both series. This gives 71 values of the change in the wages index. It is clear that the increase in wages, which I have defined above as $J(t)$, depends on the change in inflation in the same year $I(t)$, giving us a model like:

$$J(t) = WW1.I(t) + \text{other terms}$$

where 'other terms' have still to be decided.

3.4.3 Investigations show that $J(t)$ also depends on inflation in the preceding year $I(t-1)$, giving us a model:

$$J(t) = WW1.I(t) + WW2.I(t-1) + \text{other terms.}$$

However, unless we put suitable conditions on $WW1$ and $WW2$ we shall not get 'unit gain' from prices to wages, i.e. an unexpected change in prices will not produce a corresponding change in wages in the long run, so that real wages are significantly influenced by the level of inflation. This, at least in the first place, seems to me to be an undesirable feature, and it is preferable to constrain these coefficients appropriately. For a transfer function model the requirement is that they sum to unity, i.e. $WW1 + WW2 = 1$ (see McLeod, 1982), thus giving the model:

$$J(t) = WW1.I(t) + (1-WW1).I(t-1) + \text{other terms.}$$

3.4.4 Further investigations show that, after fitting a suitable value for the transfer function coefficient $WW1$, the remaining part might be modelled either as an AR(0) or as an AR(1) series, so that the 'other terms' are modelled as $WN(t)$, where:

$$WN(t) \sim \text{AR1}(WMU, WA, WSD)$$

and WA might be zero.

3.4.5 This gives us, so far, a total of six models, and Table 3.1 shows parameters for these six: (i) with $WW1$ only, (ii) with $WW1$ and WA; (iii) and (iv) with $WW1$ and $WW2$ unconstrained, (iii) without WA and (iv) with WA; and (v) and (vi) with $WW2 = 1 - WW1$, (v) without WA and (vi) with WA. Values of the parameter estimates (with the standard errors of these estimates in parentheses) are shown, along with the difference in the log likelihood function between each model and model (i), and the Jarque-Bera statistic for testing the normality of the residuals. See Appendix C.1 for an explanation of the method of estimating the parameters.

3.4.6 Adding WA in model (ii) to the first model (i), is worthwhile, since the parameter estimate is almost three standard errors away from zero, and the improvement in log likelihood is well over 2; but WA does little to improve the fit of the other models, which, however, are all clearly better than their counterparts with $WW2 = 0$. Allowing $WW2$ to be estimated, rather than fixing it equal to $1 - WW1$, gives an improvement in log likelihood of 3.06, although the difference in the estimates of $WW2$ is only about one standard error.

3.4.7 The value of the Jarque-Bera statistic is comfortably low for models (iii) and (iv), but rather on the high side for models (v) and (vi). Even so, for model (iii) there are high residuals in 1946 (2.30 times the standard deviation), 1968 (2.31 times), 1973 (2.23 times) and 1977 (−3.22 times); but the skewness is almost zero.

3.4.8 Model (v) shows a significant cross correlation of −0.24 between $WE(t)$ and $QE(t-1)$, demonstrating that the connection with inflation has been made too strong. Model (iii) avoids this feature, but the long-run gain of wages on prices is less than unity at 0.87, i.e. a rise in prices of 1% results, in due course, in a rise of only 0.87% in wages, so that in periods of high inflation wages do not keep up, and never catch up in this respect. Wages, however, have a positive expected growth of 0.0214 regardless of inflation, so the long-run mean growth of wages is higher than that of prices, at 0.0623, calculated from:

$$(WW1 + WW2).QMU + WMU.$$

Table 3.1. Parameter estimates for models for ln W, 1923-94

Model	(i)	(ii)	(iii)	(iv)	(v)	(vi)
	$WW2 = 0$		$WW2$ included		$WW2 = 1 - WW1$	
	$WA = 0$	WA included	$WA = 0$	WA included	$WA = 0$	WA included
$WW1$	0.7948 (0.0562)	0.6387 (0.0931)	0.6021 (0.0645)	0.5824 (0.0643)	0.6878 (0.0572)	0.6871 (0.0554)
$WW2$	-	-	0.2671 (0.0577)	0.2467 (0.0587)	0.3122	0.3129
WMU	0.0238 (0.0040)	0.0318 (0.0063)	0.0214 (0.0035)	0.0235 (0.0043)	0.0159 (0.0029)	0.0161 (0.0032)
WA	-	0.3036 (0.1134)	-	0.1489 (0.0944)	-	0.0908 (0.0946)
WSD	0.0266 (0.0022)	0.0255 (0.0021)	0.0233 (0.0020)	0.0229 (0.0019)	0.0244 (0.0020)	0.0242 (0.0020)
Log likelihood	0.0	+2.94	+9.40	+10.63	+6.34	+6.80
Jarque-Bera χ^2_2	0.44	7.61	1.29	1.82	10.17	9.42
$p(\chi^2)$	0.80	0.022	0.52	0.40	0.0062	0.0090

3.4.9 On balance model (iii) looks the best, provided that one does not feel committed to unit gain; but if one does, model (v) might be preferred. Suitably rounded parameter values are, for model (iii):

$$WW1 = 0.60;\ WW2 = 0.27;\ WMU = 0.021;\ WSD = 0.0233$$

or, for model (v):

$$WW1 = 0.69;\ WW2 = 0.31;\ WMU = 0.016;\ WSD = 0.0244.$$

3.4.10 One can describe these models in words: wages each year increase by 60% (69%) of the current year's price inflation, plus 27% (31%) of the previous year's price inflation, plus a further 2.1% (1.6%) increase, plus a residual with a standard deviation of a bit less than 2½%.

3.4.11 Both models, however, show significant crosscorrelations between $QE(t)$ and $WE(t-1)$, suggesting that we could model inflation rather better by taking into account the previous year's change in wages. This is entirely plausible, but it cannot be incorporated in the transfer model I have been using so far. Instead, we need a simultaneous model, which is described in Section 3.5.

3.5 *A Vector Autoregressive Model*

3.5.1 So far I have made changes in the wages index depend on price inflation, but not vice versa. A way in which influence both ways can be represented is to use a vector autoregressive (VAR) model. A first order VAR model would be:

$$I'(t) = A_{11}.I'(t-1) + A_{12}.J'(t-1) + QE^*(t)$$

$$J'(t) = A_{21}.I'(t-1) + A_{22}.J'(t-1) + WE^*(t)$$

where $I'(t) = I(t) - QMU$, $J'(t) = J(t) - WMU$, and $QE^*(t)$ and $WE^*(t)$ are correlated, with correlation coefficient ρ. Alternatively we can replace $WE^*(t)$ by $B.QE^*(t) + WE(t)$, where $QE^*(t)$ and $WE(t)$ are independent, and $B = \rho.WSD/QSD$, or we can replace $QE^*(t)$ in a similar way.

3.5.2 Noting that:

$$QE^*(t) = I'(t) - A_{11}.I'(t-1) - A_{12}.J'(t-1)$$

we can rearrange $J'(t)$ to get:

$$J'(t) = B.I'(t) + [A_{21} - B.A_{11}].I'(t-1) + [A_{22} - B.A_{12}].J'(t-1) + WE(t)$$

which starts out the same as model (iv) in Section 3.4, with:

$$WW1 = B \text{ and } WW2 = A_{21} - B.A_{11}$$

but model (iv) uses $WN(t-1)$ in the place of $J'(t-1)$. These are related by:

$$WN(t-1) = J(t-1) - WW1.I(t-1) - WW2.I(t-2)$$

so we need a VAR(2) model to reproduce any of the models with *WA* terms included.

3.5.3 Parameter estimates for a number of VAR models are shown in Table 3.2, both for the full model (i) and for a number of simpler models, including: (ii), omitting A_{12}, so that it is, in principle, the same as the full transfer model (iv) of Table 3.1; (iii) omitting A_{12} and A_{21}, so that each series depends only on its own lagged value, as well as on the current value of the other through the simultaneous correlation, which cannot be omitted with this type of model; (iv), omitting A_{12} and A_{22}, so that lagged dependence on wages is omitted for both series; and (v) omitting A_{11} and A_{21}, so that lagged dependence on prices is omitted for both series.

3.5.4 The parameters have been estimated using the NAG routine G13DCF, which makes different assumptions about the starting values than I do, so that the parameters are not exactly the same as would be derived from Table 3.1.

3.5.5 For such a model the 'ultimate response matrix', G, which shows the ultimate response of ln Q and ln W to a 'spike' of unity in $I(t)$ and in $J(t)$, is given by:

$$G = (I - A)^{-1}$$

where A is the matrix of coefficients, and I is the identity matrix (see Appendix E.3). The terms of the ultimate response matrices for all the models are also shown in Table 3.2.

Table 3.2. Parameter estimates for VAR models for ln Q and ln W, 1923-94

Model	(i) Full	(ii) $A_{12} = 0$	(iii) $A_{12} = A_{21} = 0$	(iv) $A_{12} = A_{22} = 0$	(v) $A_{11} = A_{21} = 0$
A_{11}	0.1817	0.6569	0.4521	0.6508	-
	(0.1705)	(0.0927)	(0.0861)	(0.0926)	
A_{12}	0.5927	-	-	-	0.7627
	(0.1822)				(0.1341)
QMU	0.0359	0.0380	0.0395	0.0386	0.0205
	(0.0180)	(0.0147)	(0.0097)	(0.0145)	(0.0303)
A_{21}	0.2315	0.5148	-	0.6686	-
	(0.1417)	(0.1089)		(0.0759)	
A_{22}	0.5618	0.1997	0.5532	-	0.7770
	(0.1548)	(0.1051)	(0.0778)		(0.1397)
WMU	0.0509	0.0538	0.0526	0.0544	0.0344
	(0.0179)	(0.0136)	(0.0098)	(0.0131)	(0.0317)
QSD	0.0408	0.0439	0.0452	0.0438	0.0418
	(0.0034)	(0.0037)	(0.0038)	(0.0037)	(0.0035)
WSD	0.0335	0.0345	0.0371	0.0362	0.0358
	(0.0028)	(0.0029)	(0.0031)	(0.0030)	(0.0030)
ρ	0.7139	0.7365	0.6955	0.7508	0.7318
Log likelihood	+0.0	−4.78	−16.11	−6.36	−4.23
Q Jarque-Bera χ_2^2	46.51	20.68	24.02	20.85	48.76
$p(\chi^2)$	0.0000	0.0000	0.0000	0.0000	0.0000
W Jarque-Bera χ_2^2	21.71	13.80	26.45	7.89	19.09
$p(\chi^2)$	0.0000	0.0010	0.0000	0.019	0.0000
Ultimate response matrix:					
G_{11}	1.9793	2.9144	1.9438	2.6920	1.0
G_{12}	1.0455	1.8749	0.0	1.7240	0.0
G_{21}	2.6776	0.0	0.0	0.0	3.4203
G_{22}	3.6965	1.2495	2.5219	1.0	4.4843

3.5.6 The most complicated model (i) is justified for its log likelihood value, all the others showing significantly worse values. However, the values of A_{11} and A_{21}, which measure the dependence of the two series on inflation in the previous year, are less than twice their standard errors, and the model setting these to zero, (v), is the best of the simpler models.

Table 3.3.
Parameter estimates for transfer models for ln W, 1809-1900 and 1809-1994

	1809-1900			1809-1994		
Model	(i)	(iii)	(iv)	(i)	(iii)	(iv)
	$WW2 = 0$ $WA = 0$	$WW2$ included	$WW2$ and WA included	$WW2 = 0$ $WA = 0$	$WW2$ included	$WW2$ and WA included
$WW1$	0.1639 (0.0361)	0.1486 (0.0365)	0.1474 (0.0367)	0.5704 (0.0433)	0.4465 (0.0444)	0.3940 (0.0428)
$WW2$	-	0.0664 (0.0363)	0.0646 (0.0368)	-	0.2754 (0.0443)	0.2260 (0.0425)
WMU	0.0079 (0.0024)	0.0083 (0.0024)	0.0083 (0.0025)	0.0205 (0.0032)	0.0179 (0.0029)	0.0192 (0.0045)
WA	-	-	0.0358 (0.1077)	-	-	0.4023 (0.0733)
WSD	0.0231 (0.0017)	0.0227 (0.0017)	0.0227 (0.0017)	0.0425 (0.0032)	0.0386 (0.0029)	0.0358 (0.0027)
Log likelihood	+0.0	+1.64	+1.70	+0.0	+17.79	+14.02
Jarque-Bera χ^2_2	11.26	7.22	8.77	285.14	231.49	483.56
$p(\chi^2)$	0.0036	0.027	0.012	0.0000	0.0000	0.0000

3.5.7 The ultimate response matrix for model (i) shows values of G_{11} and G_{21} of about 2.0 and 2.7, indicating that a 'spike' of 1% in prices ultimately results in an additional 2% rise in prices and a 2.7% rise in wages. The values of G_{12} and G_{22} are about 1.0 and 3.7, indicating that a spike of 1% in wages ultimately results in about a 1% rise in prices and a 3.7% rise in wages. The ultimate response matrices for the other models must be interpreted similarly, but it is not easy to see the rationale of the numbers.

3.5.8 The condition for equal responses to a spike in prices is that the two coefficients in the formula for J sum to unity, i.e. $A_{21} + A_{22} = 1$, and the condition for equal response to a spike in wages is that the two coefficients in the formula for I sum to unity, i.e. $A_{11} + A_{12} = 1$. This looks intuitively the wrong way round.

3.5.9 I do not know whether econometricians have studied these sorts of series in this way, but I have not found any in the time-series literature on investment models, which is perhaps not surprising.

3.6 *Earlier Periods*

3.6.1 Fitting the same sorts of transfer function model to the earlier period 1809-1900 shows less connection between prices and wages. Results for certain of the models investigated above are shown in Table 3.3, both for this period and for the whole period 1809-1994. For 1809-1900 there is a weak, but significant, relation between changes in wages and simultaneous changes in prices, but little connection with last year's changes, either of prices or of wages, as indicated by the not significant values of $WW2$ and WA.

Table 3.4.
Statistics and autocorrelation coefficients for various *m/h* series for wages index, December 1923 to June 1994

m/h	Number of values of $I_{m/h}$	$12/m$ times mean value of $I_{m/h}$	$\surd(12/m)$ times standard deviation of $I_{m/h}$	r_1	$r_{12/m}$
1/1	846	0.0603	0.0324	0.0432	0.5629
					Standard deviation after fitting regression
12/1 (Dec)	70	0.0606	0.0510	0.7841	0.0315
12/2	70	0.0604	0.0519	0.6965	0.0372
12/3	70	0.0605	0.0513	0.7109	0.0360
12/4	70	0.0608	0.0494	0.7919	0.0299
12/5	70	0.0603	0.0501	0.7715	0.0316
12/6	70	0.0605	0.0500	0.7681	0.0317
12/7	70	0.0603	0.0498	0.7618	0.0318
12/8	69	0.0608	0.0503	0.7784	0.0311
12/9	69	0.0606	0.0512	0.7720	0.0320
12/10	69	0.0606	0.0517	0.7239	0.0352
12/11	69	0.0607	0.0504	0.7642	0.0318
12/12	69	0.0609	0.0503	0.7776	0.0310

3.6.2 Fitting the models to the whole period 1809-1994, as also shown in Table 3.3, shows very significant values for all the parameters, and a great improvement in the log likelihood for the more elaborate models. The residual standard deviation is larger than for the later period, and the Jarque-Bera statistic indicates that there are some very high residuals, notably in 1918 and 1922, the latter year's residual being bigger than -5 standard deviations for any model.

3.6.3 It is welcome to find any model that will even plausibly fit a long period including both the 19th and 20th centuries; but the models described in Sections 3.4 and 3.5 show lower residual standard deviations, so must be preferred for current use.

3.7 *Observations at Monthly Intervals*

Values of the wages index are available at monthly intervals from January 1920, but I have analysed them only over the period from December 1923 to June 1994, as for the prices index. Table 3.4 shows figures comparable with those in Table 2.2, but only for the monthly 1/1 series and the annual 12/*h* series, showing the annualised mean, the annualised standard deviation, the first autocorrelation coefficient, and the autocorrelation coefficient for an annual frequency. The results for $m = 2, 3, 4$, and 6 are similar. One can conclude that, as for prices, wages should be analysed, as I have done, with an annual frequency. I have not fitted the sort of bivariate models described in Sections 3.4 and 3.5 to the 11 other annual series, for each month other than June. I have no reason to suspect that very different answers would be obtained.

3.8 *Data for Other Countries*

3.8.1 I have not investigated a wages index fully, on the lines described above, for any other country. I have fitted one of these models to data for Finland, but only over the period from 1964 to 1992, using transfer function model (vi) from Table 3.1. Parameters for Finland are shown in Table 3.5.

Table 3.5. Parameters for models for U.K. and Finland

	U.K.	Finland
	1923-94	1964-92
$WW1 = 1 - WW2$	0.69	1.0
WMU	0.0159	0.0225
WA	0.09	0.35
WSD	0.024	0.026

3.8.2 The model can be fitted to the data for Finland with $WW1 = 1.0$, and $WW2 = 0$, implying simultaneous, but not lagged, connections. The values of *WMU* and *WSD* are not far away from those for the U.K.

3.8.3 Substantial quantities of data for wages indices of various kinds are available for a large number of OECD countries, but I have had no occasion to analyse these. There is some research waiting to be done here.

3.9 *Forecasting*

3.9.1 I have forecast the wages model on the same lines as the inflation model in Section 2.11, using model (iii) from Table 3.1. In Figure 3.3, I show a set of ten simulations of $W(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1950, all on a logarithmic scale. As for retail prices, one can get an impression of the shape of the expanding funnel of doubt from these.

3.9.2 In Figure 3.4, I show the forecast median of $W(t)$, starting with the conditions in June 1994, also on a logarithmic scale, along with two sets of confidence intervals. The wider pair shows the mean plus and minus two standard deviations, using the formulae in Appendix E.3. As for the inflation model, the inner pair shows what the two standard deviation confidence interval would be for a random walk model for $\ln W$, with the same one-year standard deviation. The standard deviation is proportional to the square root of $(t - 1994)$. This again shows how much the autoregressive nature of the model increases the uncertainty about the future.

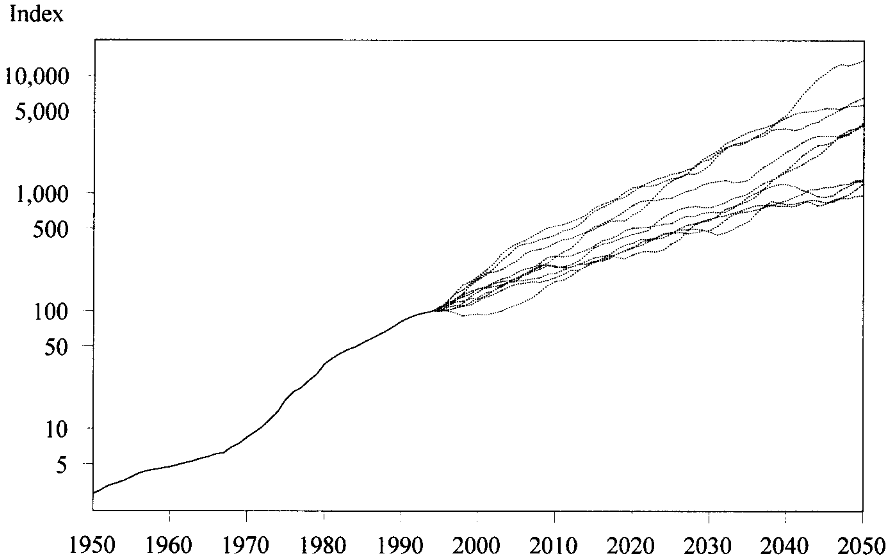

Figure 3.3. Wages index, 1950-94, and simulations, 1994-2050

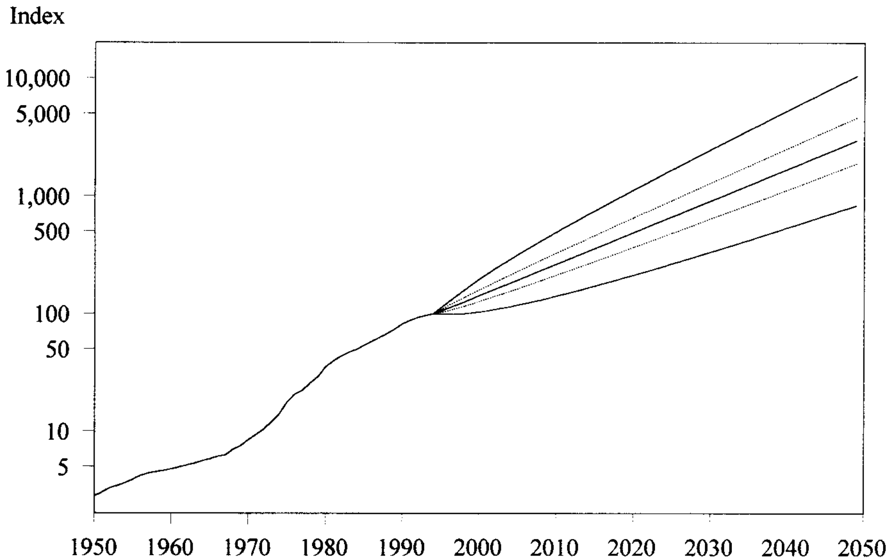

Figure 3.4. Wages index, 1950-94, and forecast medians and confidence intervals for transfer function model and for a random walk model, 1994-2050

4. Share Dividend Yields

4.1 *The Original Model for Share Dividend Yields*

4.1.1 The original model for share dividend yields, based on annual data from June 1919 to June 1982, where $Y(t)$ is the dividend yield on ordinary shares at time t, is:

$$Y(t) = \exp\{YW.I(t) + \ln YMU + YN(t)\}$$

or

$$\ln Y(t) = YW.I(t) + \ln YMU + YN(t)$$

with:

$$YN(t) = YA.YN(t-1) + YE(t)$$

$$YE(t) = YSD.YZ(t)$$

$$YZ(t) \sim \text{iid } N(0,1).$$

The last three lines can be reexpressed, noting that YN is AR(1), as:

$$YN(t) \sim AR1(0, YA, YSD).$$

4.1.2 The suggested parameter values were:

$$YW = 1.35;\ YA = 0.6;\ YMU = 4.0\%;\ YSD = 0.175.$$

4.1.3 The model can be described in words: at any date the logarithm of the dividend yield is equal to its mean value (ln 4.0%), plus 60% of its deviation a year ago from the mean (excluding the following adjustment), plus an additional influence from inflation equal to 1.35 times the force of inflation in the previous year, plus a random innovation which has zero mean and a standard deviation of 0.175.

4.2 *The Experience from 1982 to 1994*

4.2.1 It is of interest to see how this model has fared since 1982, and this is investigated in the same two ways as the inflation series is in Section 2.2. According to the model, the residuals, the YEs, are distributed $N(0, YSD^2)$, and the standardised residuals, the YZs, are distributed $N(0,1)$. The sum of n such YZs is distributed $N(0, n)$, and the sum of the squares of n such YZs is distributed as χ^2_n.

4.2.2 Table 4.1 shows, for each year, the observed value of the dividend yield on the FTSEA All-Share Index $Y(t)$, shown as a percentage, the logarithm of the yield (treated now as a fraction), the expected value of the logarithm conditional on the relevant information for Y up to year $(t-1)$, and also the observed value of $Q(t)$ $E[\ln Y(t) \mid \mathcal{F}_{t-1}+Q]$, the observed residual:

$$YE(t) = \ln Y(t) - E[\ln Y(t) \mid \mathcal{F}_{t-1}+Q]$$

Table 4.1. Comparison of actual and expected values of ln $Y(t)$, 1983-94

Year	$Y(t)$%	ln $Y(t)$	E[ln $Y(t)$]	$YE(t)$	$YZ(t)$
1982	6.09	−2.7985			
1983	4.56	−3.0878	−2.9892	−0.0986	−0.56
1984	4.87	−3.0221	−3.1017	0.0796	0.45
1985	4.80	−3.0366	−3.0505	0.0140	0.08
1986	3.86	−3.2545	−3.1307	−0.1238	−0.71
1987	3.04	−3.4933	−3.2047	−0.2886	−1.65
1988	4.18	−3.1749	−3.3560	0.1811	1.03
1989	4.31	−3.1442	−3.1219	−0.0223	−0.13
1990	4.72	−3.0534	−3.1122	0.0589	0.34
1991	5.06	−2.9838	−3.1186	0.1348	0.77
1992	4.86	−3.0241	−3.0725	0.0483	0.28
1993	3.88	−3.2493	−3.1165	−0.1329	−0.76
1994	4.04	−3.2089	−3.2120	0.0031	0.02
Total				−0.1465	−0.84
ΣYZ^2					6.20

and the standardised residual $YZ(t) = YE(t)/YSD$, where $YSD = 0.175$. The notation $\mathscr{F}_{t-1}+Q$ now means all relevant facts at time $(t-1)$ plus the value of $Q(t)$.

4.2.3 We can compare the sum of the 12 values of YZ, which is −0.84, with the expected value, zero, and the standard deviation $\sqrt{12} = 3.46$. It is well within one standard deviation away from its expected value. We can also compare the sum of the 12 values of YZ^2, which is 6.20, with a χ^2_{12} distribution; the probability of a value of χ^2 as great or greater is 0.906, which is highish, but not unreasonable. Two of the (absolute) values of YZ exceed 1.0.

4.2.4 We can now consider the forecast values of ln $Y(t)$, conditional on the information as at 1982. Using the formulae for the expected values and variances of the forecast logarithms, which are set out in Appendix E.4, we get the results shown in Table 4.2. This shows the value of ln Y(t) for each year, its expected value conditional on the relevant information up to 1982 E[ln $Y(t) \mid \mathscr{F}_{1982}$], the observed deviation ln $Y(t)$ − E[ln $Y(t) \mid \mathscr{F}_{1982}$], the standard deviation of ln $Y(t) \mid \mathscr{F}_{1982}$, and the standardised residual, the observed deviation divided by the corresponding standard deviation.

4.2.5 Remembering that the 12 observations are not independent, we can note that the observed value of ln $Y(t)$ is within one standard deviation of the expected value for all but one of the 12 years. Again it looks as if a lower value of the standard deviation YSD, might be indicated. Observe that the standard deviation of ln $Y(1983) \mid \mathscr{F}_{1982}$ is 0.1876, which is greater than the value of YSD (0.175); this is because, given the position in 1982, the value of $Q(1983)$ is also uncertain, and there is a contribution from QSD to be allowed for.

Table 4.2. Comparison of actual and expected values of ln $Y(t)$, 1983-94, all conditional on $\mathscr{F}_{1982}$

Year	ln $Y(t)$	E[ln $Y(t)$]	Deviation	Standard deviation	Standardised deviation
1982	−2.7985				
1983	−3.0878	−2.9397	−0.1482	0.1876	−0.79
1984	−3.0221	−3.0243	0.0022	0.2187	0.01
1985	−3.0366	−3.0752	0.0386	0.2289	0.17
1986	−3.2545	−3.1056	−0.1489	0.2325	−0.64
1987	−3.4933	−3.1239	−0.3694	0.2337	−1.58
1988	−3.1749	−3.1349	−0.0400	0.2342	−0.17
1989	−3.1442	−3.1415	−0.0027	0.2344	−0.01
1990	−3.0534	−3.1454	0.0920	0.2344	0.39
1991	−2.9838	−3.1478	0.1640	0.2345	0.70
1992	−3.0241	−3.1492	0.1251	0.2345	0.53
1993	−3.2493	−3.1501	−0.0992	0.2345	−0.42
1994	−3.2089	−3.1506	−0.0583	0.2345	−0.25

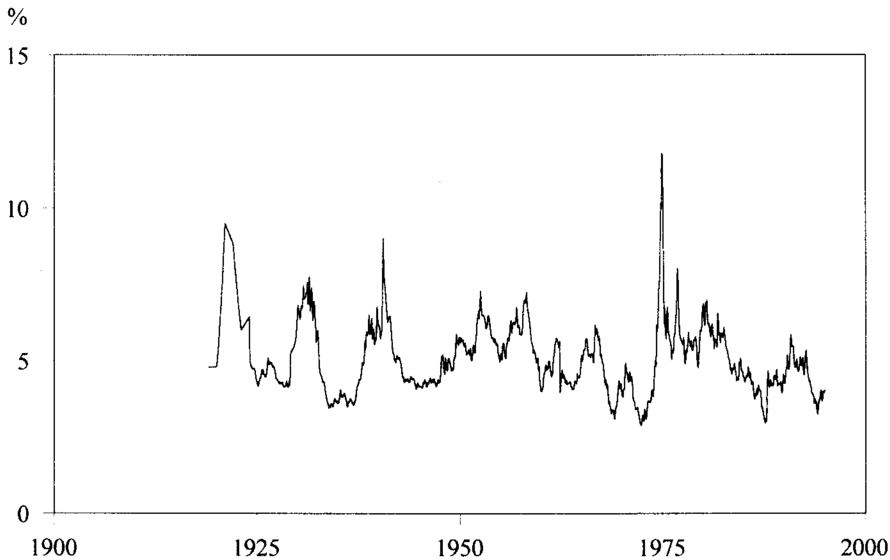

Figure 4.1. Dividend yield, monthly, 1919-94

4.3 *Updating and Rebasing to 1923-94*

4.3.1 I now consider refitting the parameters of the model, starting in June 1923 and including the data up to June 1994. This gives 72 values of the dividend yield. Values of the dividend yield for monthly values from 1919 to 1994 are shown in Figure 4.1. Possible models include: (i) the original model including *YW*, and (ii) with *YW* set to zero; it is clear that *YMU*, *YA* and *YSD* are essential features of the model. It had been suggested to me that *YW* was not a necessary feature of the model, and it seems worth investigating this. Table 4.3 shows the estimated parameters for these two models, with standard errors of the estimates in parentheses. Also shown is the difference in the log likelihood as compared with that of model (i). See Appendix C.1 for an explanation of the method of estimating the parameters.

4.3.2 Omitting *YW* worsens the log likelihood function by 6.79; since an improvement in this function of about 2.0 is sufficient to justify an additional parameter, it can be seen that the improvement is significant (see Appendix C.2.5 for the rationale of this). In addition the parameter estimate for *YW* is over three standard errors away from zero, so is clearly significant. The correlation coefficient of the residuals from model (ii) with the residuals from the fitted inflation series is 0.42, indicating an inadequate fit. Further, the distribution of the residuals in model (ii) is quite fat-tailed; b_2 = 5.19, and the Jarque-Bera statistic is 21.97, giving $p(\chi_2^2) = 0.00002$.

4.3.3 A scatter diagram of ln $Y(t)$ against $I(t)$ is shown in Figure 4.2. Although only a few values lie outside the general mass in the middle, they are

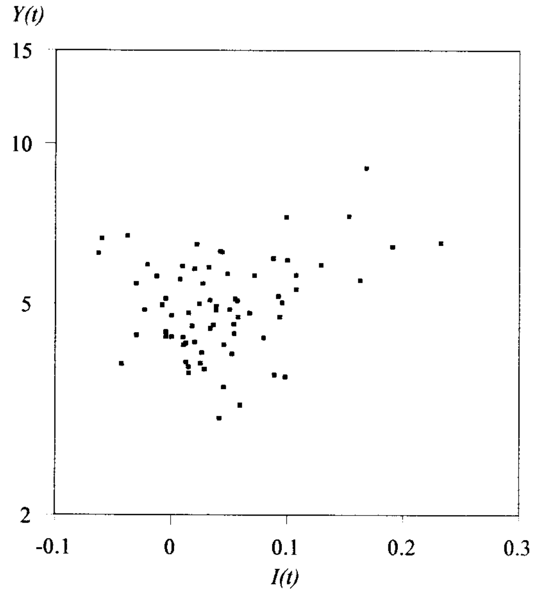

Figure 4.2. Scatter diagram of ln $Y(t)$ versus $I(t)$, 1923-94

sufficient to justify the proposition that high inflation, when it occurs, leads to a fall in share prices and hence to high dividend yields.

4.3.4 Comparing the parameters of model (i) with those in the original model, we see that *YW* is bigger and *YMU* correspondingly smaller. *YA* and *YSD* are also smaller than before. However, all the new parameter estimates are within, or not much above, one standard deviation away from the original estimates, so there is no strong evidence of a change in the parameters of the model. Nevertheless, it may be preferable to use the new parameters rather than the old ones. Suitably rounded values might be:

$$YW = 1.8;\ YA = 0.55;\ YMU = 3.75\%;\ YSD = 0.155.$$

4.3.5 The median value of ln *Y* is given by: *YMU*.exp(*YW*.*QMU*), and using $QMU = 0.047$ we get Med[ln *Y*] = 4.08%. The effect of the *YW* term is to increase the median yield by about 9%.

4.3.6 Diagnostic tests for model (i) show that the residuals appear to be independent; the autocorrelation function has no high values and the runs test is satisfactory. The crosscorrelations with the residuals of the inflation model are also satisfactory. The residuals appear to be normally distributed. The skewness and kurtosis coefficients are not far from their expected values ($\sqrt{b_1} = 0.22$ and $b_2 = 3.09$). The Jarque-Bera statistic is 0.63, giving $p(\chi_2^2) = 0.730$. There are high observed residuals in 1933 (−2.16 times the standard deviation) and 1974 (3.16 times). This seems a satisfactory model.

Table 4.3. Parameter estimates for model for ln *Y*, 1923-94

Model	(i) original model including *YW*		(ii) *YW* = 0	
Parameter	Estimate	Standard error	Estimate	Standard error
YW	1.7940	0.5862	-	-
YA	0.5492	0.1013	0.6764	0.0868
YMU %	3.77 %	0.18 %	4.09 %	0.25 %
YSD	0.1552	0.0129	0.1705	0.0142
Log likelihood	0.00		−6.79	
Jarque-Bera χ_2^2	0.63		21.97	
$p(\chi^2)$	0.73		0.00002	

4.4 *Observations at Monthly Intervals*

4.4.1 Values of the dividend yield are available from January 1923 at monthly intervals (with a little interpolation over parts of 1929 and 1930 when only quarterly values are available). It is worth investigating them. The values are shown in Figure 4.1, and the autocorrelation function is shown in Figure 4.3. It is immediately clear that an AR(p) model is appropriate, but it is not yet clear what the value of p should be. However, the first four partial autocorrelation coefficients (not shown) are significantly different from zero, so it might be suspected that an AR(4) model might be suitable. Initially I omit the relationship with the inflation rate I.

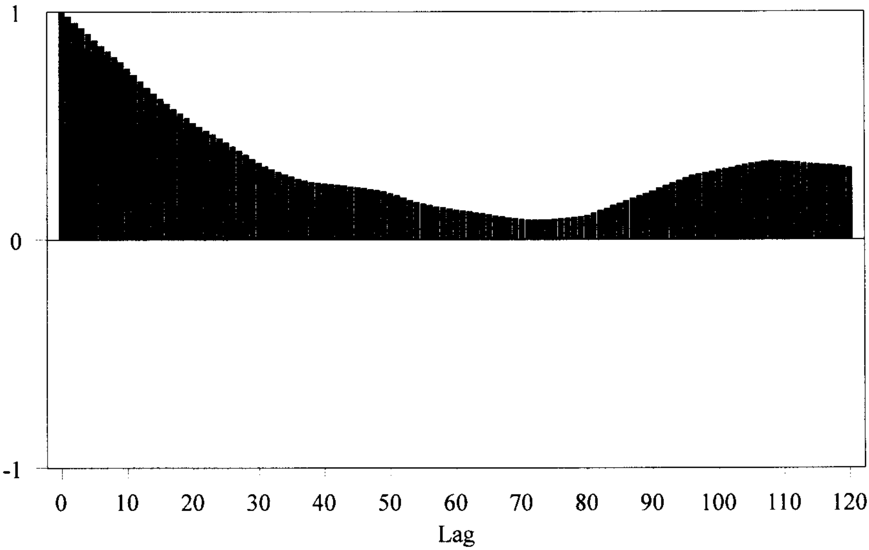

Figure 4.3. Autocorrelation function of dividend yields, monthly, 1923 - 1994

4.4.2 The results of fitting AR(p) models for p = 1 to 4 are shown in Table 4.4. The model assumed is expressed in terms of $YN(t) = \ln Y(t) - m$:

$$YN(t) = \Sigma_{i=1,p} a_i . YN(t-i) + YE(t)$$

where t is measured in months, and $YE(t) \sim N(0, s^2)$. In each case the value of the mean m, is 1.4061 = ln 4.08%. The values of a_i, $i = 1, ..., p$ and of s are given in the table. Along with the values of a_i are shown the standard errors of the parameter estimates in parentheses. Also shown is $L(p) - L(p = 1)$, i.e. the difference between the log likelihood for the optimal set of parameters for the particular value of p and the log likelihood for $p = 1$. Then the roots of the polynomial:

$$z^p - \Sigma_{i=1,p} a_i . z^{p-i} = 0$$

Table 4.4. Results of fitting AR(p) models to monthly values of ln $Y(t)$, December 1923 to June 1994

p	1	2	3	4
a_1	0.9777 (0.0072)	1.0906 (0.0342)	1.0985 (0.0344)	1.1043 (0.0343)
a_2	-	−0.1155 (0.0342)	−0.1911 (0.0507)	−0.2069 (0.0510)
a_3	-	-	0.0694 (0.0344)	0.1606 (0.0510)
a_4	-	-	-	−0.0831 (0.0344)
s	0.0509	0.0505	0.0504	0.0502
Log likelihood	0.0	+5.7	+7.7	+10.7
g_1	0.9777	0.9718	0.9756	0.9708
g_2	-	0.1188	0.2667 exp(1.3382i)	0.4487 exp(1.2397i)
g_3	-	-	0.2667 exp(−1.3382i)	0.4487 exp(−1.2397i)
g_4	-	-	-	0.4251

denoted g_i, $i = 1, ..., p$, which determine the properties of the forecast values of $Y(t)$, are shown. Where these roots are complex they are expressed in the form $r \exp(i\theta)$.

4.4.3 It can be seen that, as the value of p is increased, the log likelihood improves by more than 2 for each parameter added, and also the value of each added parameter is more than twice its standard error, and sometimes much more than twice. Nevertheless, the higher order models make almost no difference to the residual sum of squares. There is a difference between statistical significance and numerical importance.

4.4.4 The values of g_i control the development of the forecast means and variances of $Y(t)$. Each forecast mean, conditional on $\mathscr{F}_0$, i.e. the facts as at $t = 0$, is of the form:

$$\ln Y(t) = \ln Y(0) + \Sigma_{i=1,p} A_i.g_i^t$$

where the values of the A_i depend on the values of the g_is and on the values included in $\mathscr{F}_0$. Table 4.5 shows projected means and standard deviations of $YN(t) = \ln Y(t) - \ln Y(0)$, for all four AR($p$) models, based on:

$$YN(0) = 1;\ YN(-1) = YN(-2) = YN(-3) = 0$$

that is, it shows the response to a 'spike' of unity at $t = 0$. Such a response function is a helpful aid to understanding how any particular time-series model reacts.

Table 4.5.
Response of AR(p) models for monthly yields to a 'spike' of 1 at $t = 0$

p	1		2		3		4	
t	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1	0.9777	0.0509	1.0906	0.0506	1.0985	0.0505	1.1043	0.0504
2	0.9559	0.0712	1.0739	0.0749	1.0156	0.0750	1.0126	0.0750
3	0.9346	0.0863	1.0453	0.0925	0.9752	0.0909	1.0504	0.0907
4	0.9137	0.0985	1.0160	0.1066	0.9534	0.1034	1.0447	0.1050
5	0.8933	0.1089	0.9874	0.1183	0.9315	0.1141	1.0073	0.1175
6	0.8734	0.1181	0.9595	0.1285	0.9087	0.1234	0.9808	0.1280
7	0.8539	0.1262	0.9325	0.1373	0.8864	0.1316	0.9552	0.1372
8	0.8349	0.1334	0.9062	0.1452	0.8647	0.1391	0.9270	0.1454
9	0.8162	0.1400	0.8806	0.1523	0.8436	0.1458	0.8999	0.1527
10	0.7980	0.1461	0.8558	0.1587	0.8230	0.1519	0.8739	0.1593
11	0.7802	0.1516	0.8316	0.1645	0.8029	0.1574	0.8484	0.1652
12	0.7628	0.1567	0.8082	0.1698	0.7832	0.1626	0.8236	0.1707
24	0.5819	0.1971	0.5733	0.2092	0.5820	0.2024	0.5773	0.2106
36	0.4439	0.2172	0.4066	0.2264	0.4324	0.2213	0.4046	0.2277
48	0.3386	0.2281	0.2884	0.2347	0.3213	0.2311	0.2836	0.2356
60	0.2583	0.2342	0.2046	0.2387	0.2388	0.2363	0.1988	0.2394

4.4.5 From Table 4.5, one can see the steady exponential decay of the response in the AR(1) model, which, for the other models, is more erratic initially, settling down to a very similar pattern after a few months. The higher order terms affect the effective starting position, expressed in A_1. The standard deviation for each model increases as t increases, with slightly larger values for the higher order models than for the AR(1) one. In each case the value of the mean for $t = 12$ is not so far away from g_1^{12}.

4.4.6 Although the autocorrelation coefficient for the AR(1) monthly series is very significantly different from zero, it is only 0.0223 (= 1 − 0.9777) away from unity, which is only 3.1 times the estimated standard error of 0.0072. The same is true for the largest values of g_i for the higher order series. The estimated standard error is proportional to the reciprocal of the square root of the number of observations, and one can calculate that, if there had been only 354 monthly observations instead of 847, the t-ratio would have been just 2.0. An investigator who had only 30 years of monthly data (which might seem quite a long series) would thus have been tempted to assume that the series had a unit root, and was non-stationary. In fact, the distribution of the sample autocorrelation coefficient is not close to normal, which is why the ADF test needs to be used.

4.4.7 Since, in the short run, the dividend on a share index changes only very little, most of the change in share prices comes from the change in the yield, so this analysis of the yield transfers almost directly to the price index, and many investigators have concluded that share prices are close to a pure random walk, without relating them to dividends. What seems true for monthly observations may seem all the more true for daily observations, for which the first autocorrelation coefficient, assuming a corresponding AR(1) model for these too, would be about 0.9992 (= $0.9777^{1/30}$), even closer to unity. Thus, my annual autoregressive model is quite consistent with an apparent random walk for short-term share price movements.

4.4.8 As the differencing interval is reduced, one approaches a continuous model. Just as a Wiener process or Brownian motion is the continuous equivalent of a random walk, so the Ornstein-Uhlenbeck process is the continuous equivalent of an AR(1) model (see Appendix A.7). This may have some useful theoretical applications.

4.5 *Different Differencing Intervals*

4.5.1 Another approach to the analysis of monthly data is to look at it with different differencing intervals. I use the notation $Y_{m/h}$ to denote the series starting with the hth observation and picking every mth observation thereafter. Thus $Y_{1/1}$ is the full monthly series starting in December 1923, and $Y_{12/12}$ is the annual series starting in November 1924 and finishing in November 1993. Summary statistics are shown in Table 4.6, including the first autocorrelation coefficient for the series r_1, the annualised equivalent ${r_1}^{12/m}$, the standard deviation of the residuals after fitting an AR(1) model to the series with the value of r_1 as the autoregressive parameter s, and the annualised standard deviation s_y, calculated from the formula (see Appendix A.6.2):

$$s_y^2 = s^2.(1 - {r_1}^{24/m})/(1 - r_1^2).$$

4.5.2 One can see that the annualised autocorrelation coefficient and standard deviation are reasonably similar all the way down the table. I have not shown the mean values. The overall mean is ln 4.08%, as noted above. The means for the various series shown range from ln 4.00% (for series 12/5, yearly, April) to ln 4.13% (series 12/8, yearly, July), quite a small range. The values of r_1 for the yearly series range from 0.6498 (series 12/10, September) to 0.7534 (series 12/3, February). The annualised values of the more frequent series are generally higher, ranging from 0.7026 (series 6/6) to 0.7629 (series 1/1, monthly). This gives an indication of the range of parameter estimates that might be obtained by using different months for the annual series.

4.5.3 The residual standard deviation for the annual series ranges from 0.1479 (series 12/3, February) to 0.1998 (series 12/12, November). There is a tendency for the lower standard deviations to be associated with high autocorrelation

Table 4.6.
Statistics and autocorrelation coefficients for various m/h series for ln Y

m/h	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	847	0.9777	0.7627	0.0509	0.1566
2/1	424	0.9486	0.7284	0.0766	0.1658
2/2	423	0.9532	0.7499	0.0735	0.1607
3/1	283	0.9261	0.7355	0.0924	0.1660
3/2	282	0.9269	0.7381	0.0902	0.1622
3/3	282	0.9287	0.7440	0.0898	0.1619
4/1	212	0.9024	0.7349	0.1051	0.1654
4/2	212	0.9072	0.7465	0.1015	0.1605
4/3	212	0.8947	0.7163	0.1074	0.1678
4/4	211	0.9006	0.7304	0.1064	0.1672
6/1	142	0.8408	0.7069	0.1357	0.1773
6/2	141	0.8631	0.7450	0.1209	0.1597
6/3	141	0.8595	0.7387	0.1204	0.1588
6/4	141	0.8497	0.7219	0.1266	0.1661
6/5	141	0.8430	0.7106	0.1293	0.1692
6/6	141	0.8382	0.7025	0.1356	0.1769
12/1 (Dec)	71	0.6690	0.6690	0.1975	0.1975
12/2	71	0.7354	0.7354	0.1620	0.1620
12/3	71	0.7534	0.7534	0.1479	0.1479
12/4	71	0.7393	0.7393	0.1513	0.1513
12/5	71	0.7210	0.7210	0.1519	0.1519
12/6	71	0.7058	0.7058	0.1621	0.1621
12/7	71	0.6819	0.6819	0.1724	0.1724
12/8	70	0.7079	0.7079	0.1719	0.1719
12/9	70	0.6595	0.6595	0.1865	0.1865
12/10	70	0.6498	0.6498	0.1947	0.1947
12/11	70	0.6674	0.6674	0.1941	0.1941
12/12	70	0.6651	0.6651	0.1998	0.1998

coefficients, which is not surprising. The high values of Y at the end of 1974 contribute to the high standard deviations for the November and December series. The annualised standard deviations for the more frequent series have a narrower range, from 0.1567 (series 1/1, monthly) to 0.1773 (series 6/1).

4.5.4 The kurtosis coefficients for almost all the series are high, indicating that both the original values and the residuals after the AR(1) regression are fatter-tailed than normal.

4.5.5 The same analysis has been carried out on the values of $\ln Y(t)$ after deducting $YW.I(t)$ from each, where the value of YW is taken as 1.8, and $I(t)$ is the force of inflation over the preceding 12 months. The results are shown in Table 4.7 for the monthly 1/1, the two $2/h$ series and the yearly $12/h$ series.

4.5.6 The values of r_1 and hence of $r_1^{12/m}$ are all reduced as compared with those shown in Table 4.6. The values of the standard deviation of the basic residuals for the more frequent series are increased, but the equivalent annual standard deviations, and the standard deviations for the yearly series are reduced. The best value of YW for each series would presumably be different, and I have not investigated what value might be best overall; it is sufficient to demonstrate the effect with one value.

4.5.7 The kurtosis coefficients of the residuals of the series are reduced considerably, and the Jarque-Bera probabilities for most of the series are well within a 5% confidence limit. Exceptions are the yearly series for October, November, December and January, which are all affected by particularly by the events around the end of 1974.

4.5.8 Monthly observations of the dividend yield on the FTA All-Share Index and its constituent sector indices have been investigated by Toutounchi (1984), with similar results.

4.6 *Cointegration*

4.6.1 I have so far taken it for granted that the dividend yield series is stationary, and therefore that the relationship between share prices and share dividends is stationary. This, however, strictly requires investigation. In particular, it is an informative example of cointegration (see Appendix B.3) between two series, those for the logarithms of share price and share dividend, $\ln P$ and $\ln D$.

4.6.2 First, I should justify the use of logarithms for considering share prices and share dividends. Both of these are based on indices (as are the retail prices index and the wages index). Any economic index of this type is defined only up to a multiplicative constant; one has to choose an arbitrary radix, and this is often recorded in the form, e.g. 'January 1980=100'. Absolute changes have no importance, as compared with proportionate changes. Taking logarithms is the obvious way to reflect, in the differences, proportionate changes. One could work, instead, in terms of percentage changes, but these are less convenient, since, for example, they are not additive across periods, whereas log changes are.

Table 4.7.
Statistics and autocorrelation coefficients for m/h series for ($\ln Y - 1.8I$)

m/h	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	847	0.9657	0.6580	0.0526	0.1526
2/1	424	0.9233	0.6196	0.0779	0.1592
2/2	423	0.9284	0.6403	0.0758	0.1566
12/1 (Dec)	71	0.5393	0.5393	0.1897	0.1897
12/2	71	0.6188	0.6188	0.1602	0.1602
12/3	71	0.6577	0.6577	0.1441	0.1441
12/4	71	0.6276	0.6276	0.1436	0.1436
12/5	71	0.6202	0.6202	0.1441	0.1441
12/6	71	0.6156	0.6156	0.1517	0.1517
12/7	71	0.5832	0.5832	0.1564	0.1564
12/8	70	0.6164	0.6164	0.1584	0.1584
12/9	70	0.5087	0.5087	0.1836	0.1836
12/10	70	0.4769	0.4769	0.1926	0.1926
12/11	70	0.5085	0.5085	0.1889	0.1889
12/12	70	0.5269	0.5269	0.1922	0.1922

Further, the range of log changes is unrestricted in both directions, whereas percentage changes have, in principle, a lower limit of −100%, which, fortunately, indices usually do not reach. The prices of individual shares, however, do sometimes reach zero, and this can be awkward if logarithms are used.

4.6.3 The dividend yield $Y(t)$ is defined as $D(t)/P(t)$, and its logarithm $\ln Y(t)$, is therefore equal to $\ln D(t) - \ln P(t)$. Investigating $\ln Y$ instead of Y ensures that the value of Y remains positive. There is no particular reason to use the dividend yield rather than its reciprocal, the P/D ratio, analogous to the P/E ratio. Taking logarithms means that the logarithm of the P/D ratio is just the negative of the logarithm of the yield. Further, using $\ln Y$ allows $\ln P$ to be derived linearly from other variables, which makes calculation of its forecast means and variances practicable (see Appendix E.4.9).

4.6.4 Before testing whether $\ln D$ and $\ln P$ are cointegrated, it is necessary to check whether they are integrated I(1) series. Figure 4.4 shows P and 22 times D, at monthly intervals, on the same graph, on a vertical logarithmic scale. Both graphs have the appearance of non-stationary series. Further, the way they move close to one another over the whole period suggests that they are cointegrated. ADF unit root tests (see Appendix A.5.4), applied to the monthly series for both $\ln D$ and $\ln P$, show clearly that, over the period shown, it is entirely reasonable to assume that they each have a unit root, and so are I(1) series. The ADF tests applied to the annual series are also strongly significant.

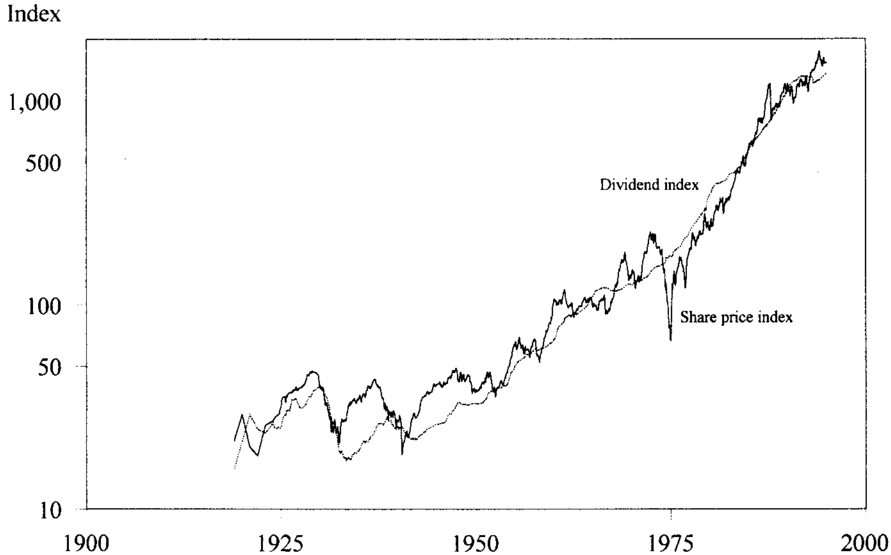

Figure 4.4. Share price index and 22 × dividend index, 1919-94

4.6.5 Strictly, it is not possible to test whether a series has a unit root, but only whether it probably has not. Any series that looks like an I(1) series could be generated by an stationary model, with a root close to, but less than, unity. The longer the sample, the closer the root to unity would need to be to remain plausible. However, there are also economic reasons for imagining that there is no tendency for retail prices, or any other series that reflects money values, including wages, share dividends and share prices, to have any fixed natural level, to which it has any tendency to return.

4.6.6 I have then used the Johansen cointegration tests provided in *MICROFIT* to test for cointegration of $\ln D$ and $\ln P$, using both the monthly series and the June annual series. The results for the monthly series are clear: the test statistics are very much higher than the 5% significance level. For the annual series the position is more marginal, but still just significant. The 'best' cointegration vector, for the monthly series, assuming 6 monthly lags in the model, is:

$$\ln D - 1.1466 \ln P \sim I(0)$$

but the 'natural' vector to use:

$$\ln D - 1.0 \ln P \sim I(0)$$

is significantly different from this ($\chi_1^2 = 5.85$, $p = 0.016$). For the annual series the best cointegration vector is:

$$\ln D - 1.0307 \ln P \sim \mathrm{I}(0)$$

and the 'natural' vector is not significantly different from this ($\chi_1^2 = 0.41$, $p = 0.524$).

4.6.7 It is therefore quite appropriate to analyse ln Y as a stationary I(0) series, as has been done. ADF unit root tests on ln Y confirm this.

4.7 *Data for Selected Other Countries for Other Periods*

4.7.1 I have data for share dividend yields for certain other countries for longer periods, the same as those discussed for inflation in Section 2.6. Data for a larger number of countries for a shorter period are discussed in Section 4.8. The longer periods and the results, giving broadly rounded values, are shown in Table 4.8.

Table 4.8. Fitted parameters for dividend yield model for selected other countries

	U.K.	U.S.A.	France	Canada	Sweden	Finland
Period	1923-94	1926-89	1951-89	1923-93	1923-93	1950-93
YW	1.8	0.5	1.64	1.17	0.71	1.3
YA	0.55	0.7	0.88	0.7	0.835	0.8
YMU%	3.75%	4.3%	2.4%	3.75%	3.45%	4.0%
YSD	0.16	0.21	0.165	0.19	0.18	0.23
Med[*Y*%]	4.08%	4.36%	2.65%	3.90%	3.56%	4.32%

4.7.2 The dividend yield for France is net of tax, and the low value of *YMU* reflects this. It should be grossed up appropriately to be comparable with the other countries. Values of the parameters are not too different for each country. The most variable is the influence of inflation *YW*, but it is noticeable that this is positive for all the countries considered. The median value of *Y* is calculated as *YMU*.exp(*YW*.*QMU*).

4.7.3 As explained in ¶2.6.4, I have not calculated the simultaneous crosscorrelations of the residuals for the series all these countries, so I quote only the simultaneous correlation coefficients for the U.K., U.S.A. and France from my Montréal paper. These are not calculated over identical periods. They are:

U.K. *v* U.S.A.: 0.29; U.K. *v* France: 0.34; U.S.A. *v* France: 0.40.

Simultaneous correlation coefficients over a consistent, but shorter, period, and for more countries, are discussed in Section 4.8.

4.8 *Data for Other Countries for 1970-94.*

4.8.1 I have been able to obtain data on share dividend yields and share dividends for a number of other countries. They start in January 1970, and I have taken the investigation up to June 1994. For data sources see Appendix F.9; in recent years I have used the *Financial Times*-Actuaries World Indices.

4.8.2 I have analysed the monthly series and 12 yearly series, one for each month, and the results are summarised in Table 4.9. I have also investigated series with different differencing intervals, like the different *m*/*h* series for the U.K.

4.8.3 Since an AR(1) model of the yield can apply to the monthly series as well as to the yearly series, I show certain figures for the monthly series as well as for the yearly ones. I have not taken account of the influence of inflation, the *YW.I*(*t*) terms.

4.8.4 In the column headed *YMU*% three values are shown the lowest and highest values of *YMU* for any of the 12 yearly series and the overall mean value for the monthly series. These are not the mean values of *YMU*, but the exponentials of the means of the logarithms of *Y*.

4.8.5 In the next column two *YA* values derived from the monthly series appear, the first being the first autocorrelation coefficient for the monthly series itself, and the second being the 12th power of this value, i.e. the equivalent annual autocorrelation coefficient. This can be compared with the three values in the next column, which are derived from the yearly series, and show the lowest and highest values of *YA*, the first autocorrelation coefficient for the yearly series, and its mean value for the 12 series.

4.8.6 Values of the standard deviation, labelled *YSD*, are shown similarly. For the monthly series the standard deviation of the monthly residuals is shown and also the equivalent yearly standard deviation calculated from:

$$YSD^2_{yearly} = YSD^2_{monthly}.(1 - YA^{24})/(1 - YA^2).$$

For the yearly series the lowest and highest values of the standard deviation for any month and the mean of the 12 months are shown.

4.8.7 One can see quite a range in the values of *YMU*, from below 2% for Austria and Japan to above 5% for Belgium, France, the Netherlands and Spain.

4.8.8 The monthly values for *YA* all appear close to unity, but range from 0.901 (Switzerland) to 0.989 (Spain). When converted into annual values these show a range from 0.285 to 0.879. Generally the annualised value is comparable with the values calculated from the yearly series, but there are a few exceptions. Note that low values of *YA* mean rather rapid reversion to the mean, and high values rather slow reversion. France, Japan and Spain show high values, i.e. the value of *Y* spends long periods away from the mean, whereas Austria, Hong Kong, Italy and Switzerland show low values, i.e. return to the mean more readily.

4.8.9 The monthly standard deviations show a considerable range, from below

Table 4.9. Analysis of share dividend yield for 18 countries from 1/1970 to 6/1994; parameters of AR(1) model for monthly series with 294 values and for 12 yearly series with 24 or 25 values

	YMU % low - overall mean - high	*YA* monthly series monthly - yearly	*YA* yearly series low - mean - high	*YSD* monthly series monthly - yearly	*YSD* yearly series low - mean - high
Australia	4.61 - 4.72 - 4.97	0.938 - 0.464	0.300 - 0.412 - 0.543	0.0833 - 0.2128	0.1786 - 0.2142 - 0.2646
Austria	1.68 - 1.73 - 1.77	0.951 - 0.547	0.166 - 0.229 - 0.365	0.0786 - 0.2128	0.2506 - 0.2640 - 0.2742
Belgium	5.24 - 5.46 - 5.66	0.978 - 0.762	0.638 - 0.725 - 0.798	0.0530 - 0.1631	0.1498 - 0.1816 - 0.2089
Canada	3.67 - 3.74 - 3.86	0.966 - 0.657	0.434 - 0.538 - 0.643	0.0512 - 0.1486	0.1425 - 0.1713 - 0.2010
Denmark	2.07 - 2.14 - 2.19	0.980 - 0.782	0.578 - 0.639 - 0.698	0.0790 - 0.2457	0.2751 - 0.3052 - 0.3543
France	5.26 - 5.46 - 5.62	0.987 - 0.860	0.752 - 0.831 - 0.864	0.0708 - 0.2294	0.2324 - 0.2540 - 0.2993
Germany	2.67 - 2.80 - 2.89	0.954 - 0.570	0.527 - 0.639 - 0.718	0.0792 - 0.2177	0.1687 - 0.1993 - 0.2409
Hong Kong	3.51 - 3.66 - 3.91	0.938 - 0.466	0.202 - 0.307 - 0.448	0.1187 - 0.3038	0.2469 - 0.3289 - 0.3936
Italy	2.31 - 2.39 - 2.51	0.930 - 0.418	0.124 - 0.303 - 0.452	0.1086 - 0.2683	0.2567 - 0.2859 - 0.3081
Japan	1.38 - 1.43 - 1.46	0.989 - 0.872	0.810 - 0.843 - 0.875	0.0612 - 0.1994	0.1853 - 0.2140 - 0.2443
Netherlands	5.13 - 5.27 - 5.51	0.967 - 0.668	0.500 - 0.604 - 0.703	0.0538 - 0.1569	0.1474 - 0.1734 - 0.1919
Norway	2.47 - 2.53 - 2.63	0.973 - 0.722	0.479 - 0.584 - 0.687	0.0866 - 0.2606	0.2742 - 0.3102 - 0.3561
Singapore	3.88 - 4.15 - 4.42	0.973 - 0.716	0.547 - 0.652 - 0.709	0.1068 - 0.3203	0.3131 - 0.3493 - 0.4329
Spain	5.11 - 5.27 - 5.59	0.989 - 0.879	0.806 - 0.855 - 0.894	0.0676 - 0.2209	0.2156 - 0.2582 - 0.3184
Sweden	3.25 - 3.31 - 3.45	0.978 - 0.770	0.638 - 0.705 - 0.773	0.0716 - 0.2213	0.2087 - 0.2505 - 0.2979
Switzerland	2.24 - 2.36 - 2.45	0.901 - 0.285	0.114 - 0.381 - 0.611	0.0737 - 0.1625	0.1052 - 0.1572 - 0.2119
U.K.	4.60 - 4.79 - 5.01	0.958 - 0.597	0.340 - 0.485 - 0.617	0.0653 - 0.1825	0.1481 - 0.2031 - 0.2554
U.S.A.	3.89 - 3.97 - 4.06	0.978 - 0.765	0.598 - 0.733 - 0.822	0.0495 - 0.1527	0.1347 - 0.1619 - 0.2045

0.05 for the U.S.A. to above 0.11 for Hong Kong. The ranking is not preserved when these are converted to the yearly equivalents, because these depend also on the values of the monthly *YA*, but Canada, the U.S.A., the Netherlands, Belgium and Switzerland remain with low values, while Hong Kong and Singapore have high values. In general the annualised monthly values correspond with those from the yearly series, but in some cases the equivalent yearly *YSD* is outside the range of the values for the separate yearly series.

4.8.10 For the monthly series the Jarque-Bera statistics are all significantly high. For the annual series they are almost all not significantly high. The U.K. is the only country to show more than one significantly high value for an annual series, which it does for five separate months.

4.8.11 Simultaneous and lagged crosscorrelation coefficients have been calculated, both for the monthly series and for the 12 different annual series. The simultaneous correlation coefficients for the monthly series are shown in Table 4.10, and those for the June (upper triangle) and December (lower triangle) yearly series are shown in Table 4.11. Values of 0.40 or greater in the monthly table and of 0.60 or greater in the yearly table are shown in bold. The values of the correlation coefficients for the yearly series are considerably higher than those for the monthly series. If the series were generated from pure monthly AR(1) models with simultaneous, but not lagged, correlations, and the series had the same value of *YA*, then the correlation coefficients for the yearly series would be the same as for the monthly series.

4.8.12 The monthly correlation coefficients for lag 1 are small, but the average of the 324 coefficients is 0.0758, significantly different from zero; one can compare this with the theoretical standard deviation of $1/\sqrt{n} = 0.0556$ and the actual standard deviation of the 324 coefficients of 0.0627. Although a few lagged correlation coefficients exceed 0.2 (therefore more than 3.4 standard errors away from zero), there seems no particular consistency or logic in their pattern. For the yearly series the lagged correlation coefficients are generally small, consistent with there being no connection between movements in the yields between different countries at intervals of a year or more.

4.9 *ARCH Models*

4.9.1 With the $YW.I(t)$ term included, the residuals of the yield model for the U.K. for 1923 to 1994 are not conspicuously fat-tailed, and there are no extreme values. This is partly because I have used the June series, thus avoiding the extremes towards the end of 1974. It is, nevertheless, of interest to investigate possible ARCH effects, as I have done for inflation in Section 2.8.

4.9.2 I use series that I denote *Y*-squared, which is calculated as $(\ln Y(t) - \ln YMU)^2$, *YH*-squared, which is the square of the expected value of each $(\ln Y(t) - YMU)$, and *YE*-squared, the squares of the residuals. *YE*-squared shows no significant autocorrelation, no lagged correlation with *Y*-squared and no simultaneous correlation with *YH*-squared.

Table 4.10.
Analysis of share dividend yield for 18 countries from 1/1970 to 6/1994; monthly series with 294 values; correlation coefficients of residuals

	Aus	Ost	Bel	Can	Den	Fra	Ger	H.K.	Ita
Australia	**1.0**								
Austria (Ost)	.06	**1.0**							
Belgium	.27	.19	**1.0**						
Canada	**.50**	.05	.38	**1.0**					
Denmark	.07	.18	.22	.13	**1.0**				
France	.31	.17	**.53**	**.46**	.25	**1.0**			
Germany	.21	.22	.35	.30	.18	.36	**1.0**		
Hong Kong	**.41**	.16	.30	.35	.25	.25	.22	**1.0**	
Italy	.16	.14	.23	.20	.13	.21	.21	.20	**1.0**
Japan	.15	.06	.23	.25	.15	.26	.21	.25	.19
Netherlands	.39	.18	**.49**	**.54**	.29	**.41**	**.40**	**.43**	.22
Norway	.34	.06	**.41**	.39	.17	.35	.24	.25	.22
Singapore	.35	−.02	.28	.37	.08	.20	.21	**.47**	.10
Spain	.24	.18	.34	.33	.12	.24	.28	.29	.26
Sweden	.35	.08	.36	.34	.13	.28	.22	.20	.12
Switzerland	.21	.09	**.47**	.38	.25	**.41**	.37	.32	.15
U.K.	**.42**	.12	**.47**	**.58**	.18	**.45**	.22	**.41**	.16
U.S.A.	**.43**	.06	.35	**.65**	.24	.39	.31	.36	.19

Table 4.10 (continued).
Analysis of share dividend index for 18 countries from 1/1970 to 6/1994; monthly series with 294 values; correlation coefficients of residuals.

	Jap	Net	Nor	Sing	Spa	Swe	Swi	U.K.	U.S.A.
Japan	**1.0**								
Netherlands	.34	**1.0**							
Norway	.16	**.48**	**1.0**						
Singapore	.25	.36	.28	**1.0**					
Spain	.29	**.40**	.27	.25	**1.0**				
Sweden	.26	.35	.35	.28	.34	**1.0**			
Switzerland	.27	**.46**	.29	.27	.30	.34	**1.0**		
U.K.	.29	**.57**	.39	**.48**	.35	**.41**	**.41**	**1.0**	
U.S.A.	.27	**.49**	.39	.35	.29	.26	.36	**.50**	**1.0**

Table 4.11.
Analysis of share dividend yield for 18 countries from 1/1970 to 6/1994; 12 yearly series with 24 or 25 values; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Aus	Ost	Bel	Can	Den	Fra	Ger	H.K.	Ita
Australia	**1.0**	−.03	.41	**.78**	.10	.27	.34	**.60**	.21
Austria (Ost)	.16	**1.0**	.19	.07	.41	.25	.48	.07	.24
Belgium	.58	.40	**1.0**	.48	.37	**.75**	.36	.13	.18
Canada	**.84**	.27	**.62**	**1.0**	.22	.41	.54	.31	.19
Denmark	.41	.46	**.60**	.53	**1.0**	.45	.58	−.09	.12
France	.54	.41	**.73**	**.72**	**.70**	**1.0**	.56	−.17	−.04
Germany	.49	**.65**	**.65**	**.62**	**.78**	**.84**	**1.0**	.07	−.01
Hong Kong	**.69**	.35	.59	.48	.27	.19	.26	**1.0**	.29
Italy	.21	.45	.22	.14	.13	.12	.14	.41	**1.0**
Japan	.59	.24	.59	.56	.39	.59	.58	.37	.22
Netherlands	**.72**	.46	**.74**	**.77**	**.72**	**.73**	**.79**	.50	−.01
Norway	.27	.23	.58	.43	.34	.40	.25	.29	.24
Singapore	**.80**	.06	**.60**	**.76**	.46	.46	.42	.54	.11
Spain	.45	.41	**.69**	.33	.29	.36	.41	.49	.34
Sweden	.52	.34	**.60**	.59	**.75**	**.67**	**.74**	.17	.21
Switzerland	**.77**	.59	**.80**	**.81**	.55	**.73**	**.74**	**.65**	.31
U.K.	**.74**	.21	**.70**	**.78**	.48	.54	.51	**.62**	−.06
U.S.A.	**.71**	.12	.54	**.79**	.40	.55	.50	.46	−.02

4.9.3 Perhaps large values of inflation or of the inflation residuals affect the standard deviation for *Y*. It is, therefore, worth comparing *YE*-squared with *I*-squared, *IH*-squared and *QE*-squared, all defined in Section 2.8. There are no large crosscorrelations, either simultaneously or lagged, except where lagged in the 'wrong direction'. For example, the correlation coefficient between *YE*-squared and $I(t+1)$-squared is 0.43, which is clearly significant, being almost four times its standard error (0.118). The correlation coefficient between *YE*-squared and $QE(t+1)$-squared is smaller, at 0.23, just less than twice its standard error. This implies that the standard deviation for the inflation model is increased when the share dividend yield is either unusually high or unusually low. The 'cascade' structure of my model does not allow this to be reflected, without considerable alteration, and I record these observations in the hope that others might investigate this further.

Table 4.11 (continued).
Analysis of share dividend index for 18 countries from 1/1970 to 6/1994; 12 yearly series with 24 or 25 values: correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Jap	Net	Nor	Sing	Spa	Swe	Swi	U.K.	U.S.A.
Australia	.57	**.67**	.28	**.73**	.44	.27	**.60**	**.69**	.59
Austria (Ost)	.17	.16	.01	−.25	.28	.16	.28	−.01	.17
Belgium	.57	.57	.57	.43	.47	.49	**.76**	**.75**	**.69**
Canada	**.69**	**.82**	.44	**.73**	.54	.56	**.68**	**.73**	**.79**
Denmark	.32	.45	.18	.19	.23	**.69**	.32	.34	.33
France	.43	**.61**	.53	.25	.21	.58	**.67**	.54	.51
Germany	.47	**.68**	.08	.31	.26	**.68**	.56	.50	.50
Hong Kong	.24	.11	−.02	.44	.34	−.06	.27	.37	.35
Italy	.45	.11	.28	.29	**.62**	.23	.11	.16	.11
Japan	**1.0**	.57	.31	**.63**	**.74**	.55	**.63**	**.69**	.48
Netherlands	.57	**1.0**	.48	.58	.32	**.67**	**.71**	**.76**	**.77**
Norway	.05	.43	**1.0**	.42	.26	.25	.38	.35	.46
Singapore	**.61**	**.74**	.36	**1.0**	.42	.43	.58	**.68**	.55
Spain	.56	.53	.35	.46	**1.0**	.47	.41	.44	.46
Sweden	.48	**.64**	.19	.55	.42	**1.0**	.55	.55	.54
Switzerland	.53	**.80**	.43	**.62**	.52	**.61**	**1.0**	**.80**	**.74**
U.K.	.50	**.75**	.23	**.67**	.43	.57	**.76**	**1.0**	**.82**
U.S.A.	.47	**.69**	.32	**.63**	.31	.42	**.66**	**.67**	**1.0**

4.9.4 I have, therefore, not attempted to fit an ARCH model to share dividend yields.

4.10 *Forecasting*

4.10.1 In Figure 4.5, I show a set of ten simulations of $Y(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1950, on a linear scale. One can see how erratic these are from year to year, but they do not fan out into an expanding funnel of doubt; the shape is more like a tunnel.

4.10.2 In Figure 4.6, I show the forecast mean of $Y(t)$, starting with the conditions in June 1994, also on a linear scale, along with confidence intervals for the mean plus and minus two standard deviations, using the formulae in Appendix E.4. One can see that the confidence interval soon becomes constant.

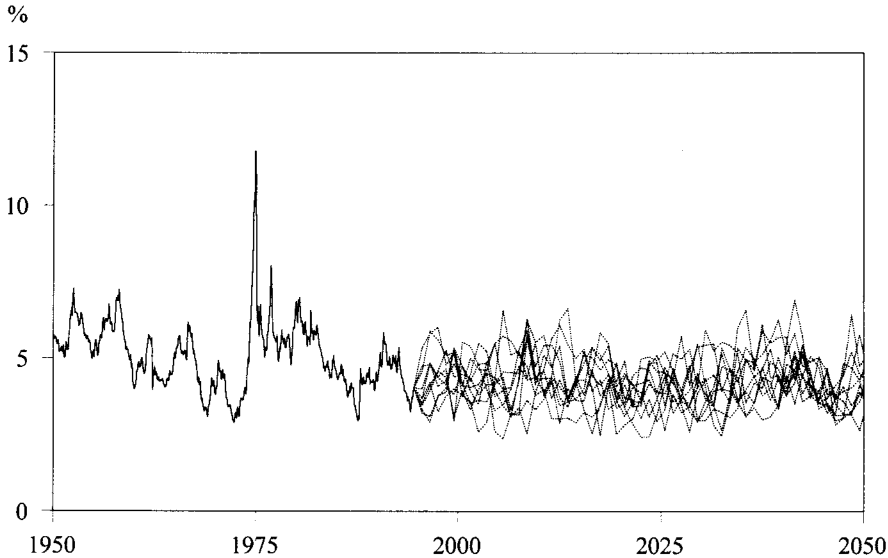

Figure 4.5. Dividend yield, 1950-94, and simulations, 1994-2050

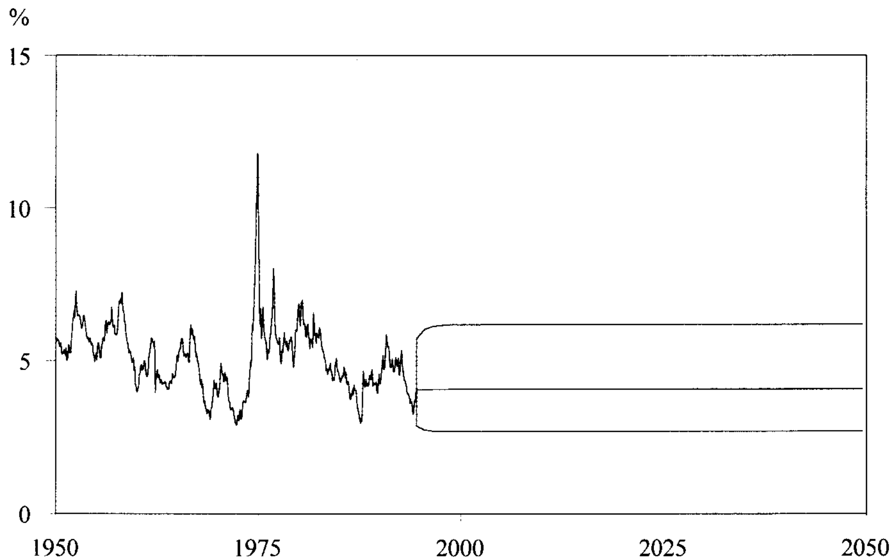

Figure 4.6. Dividend yield, 1950-94, and forecast medians and confidence intervals, 1994-2050

5. SHARE DIVIDENDS AND SHARE PRICES

5.1 *The Original Model for Share Dividends*

5.1.1 The original model for share dividends, where $D(t)$ is the value of a dividend index on ordinary shares at time t, is:

$$D(t) = D(t-1).\exp\{DW.DM(t) + DX.I(t) + DMU$$

$$+ DY.YE(t-1) + DB.DE(t-1) + DE(t)\}$$

$$DM(t) = DD.I(t) + (1-DD).DM(t-1)$$

$$DE(t) = DSD.DZ(t)$$

$$DZ(t) \sim \text{iid N}(0,1).$$

It is convenient also to denote the annual change in the logarithm as:

$$K(t) = \ln D(t) - \ln D(t-1)$$

and to identify the effect of inflation as:

$$DI(t) = DW.DM(t) + DX.I(t)$$

so that:

$$K(t) = DI(t) + DMU + DY.YE(t-1) + DB.DE(t-1) + DE(t).$$

5.1.2 I constrained DX to equal $1 - DW$, so that a change in ln Q ultimately produced the same change in ln D; the transfer function had 'unit gain'. Suggested parameters were:

$$DW = 0.8;\ DD = 0.2;\ DMU = 0.0;\ DY = -0.2;\ DB = 0.375;\ DSD = 0.075.$$

5.1.3 Hence, we have a model for $P(t)$, the value of a price index of ordinary shares at time t:

$$P(t) = D(t)/Y(t)$$

or

$$\ln P(t) = \ln D(t) - \ln Y(t).$$

5.1.4 The model can be described in words: in each year the change in the logarithm of the dividend index is equal to a function of current and past values of inflation, plus a mean real dividend growth (which is taken as zero), plus an influence from last year's dividend yield innovation, plus an influence from last year's dividend innovation, plus a random innovation which has zero mean and a standard deviation of 0.075.

5.1.5 It seems economically necessary that share dividends should respond to inflation, and plausible that it should be with unit gain, though there is also a case for arguing that dividends 'in real terms' do better in times of stable prices than in periods of high and uncertain inflation. However, it is difficult to test this proposition with the series available; a much longer series would really be necessary.

5.1.6 The rationale of the term $DY.YE(t-1)$, involving the unexpected change in the yield in the previous year, is that investment analysts can forecast changes in dividends in the forthcoming year quite well, so share prices react in advance to changes in dividends. This is partly because final dividends are declared by companies well after the end of the year to which they relate, so the trading conditions of the year are already known. Using the terminology of the time-series literature, we could say that changes in the dividend yield 'Granger-caused' changes in dividends (see Hamilton, 1994, Chapter 11).

5.1.7 The term $DB.DE(t-1)$, involving last year's dividend innovation, was the only term included in the original MGWP model (1980), and can be justified if boards of directors of companies pay out only part of any additional earnings in dividend in one year, with a further part in the following year. This term makes the basic model for 'real dividends' into an MA(1) model.

5.2 The Experience from 1982 to 1994

5.2.1 It is again of interest to see how this model has fared since 1982, and this is investigated in the same two ways as the previous series. According to the model, the residuals, the DEs, are distributed $N(0, DSD^2)$, and the standardised residuals, the DZs, are distributed $N(0,1)$. The sum of n such DZs is distributed $N(0, n)$, and the sum of the squares of n such DZs is distributed as χ^2_n.

5.2.2 Table 5.1 shows, for each year, the observed value of the change in the logarithm of the dividend index $K(t)$, the expected value of $K(t)$ conditional on the relevant information for D up to year $(t-1)$, and also the observed value of $Q(t)$ (the observed value of $Y(t)$ is not relevant) $E[K(t)\,|\,\mathcal{F}_{t-1}+Q]$, the observed residual $DE(t) = K(t) - E[K(t)\,|\,\mathcal{F}_{t-1}+Q]$, and the standardised residual $DZ(t) = DE(t)/DSD$, where $DSD = 0.075$.

5.2.3 We can compare the sum of the 12 values of DZ, which is 2.41, with the expected value, zero, and the standard deviation $\sqrt{12} = 3.46$. It is still within one standard deviation away from its expected value. We can also compare the sum of the 12 values of DZ^2, which is 6.99, with a χ^2_{12} distribution; the probability of a value of χ^2 as great or greater is 0.858, which is highish, but not unreasonable. Three of the (absolute) values of DZ exceed 1.0.

5.2.4 We can now consider the forecast values of $\ln D(t)$, conditional on the information as at 1982; again it is convenient to work with the change since

Table 5.1. Comparison of actual and expected values of $K(t)$, 1983-94

Year	$K(t)$	$E[K(t)]$	$DE(t)$	$DZ(t)$
1982	0.0801			
1983	0.0625	0.0577	0.0048	0.06
1984	0.1267	0.1038	0.0229	0.31
1985	0.1852	0.0747	0.1105	1.47
1986	0.0966	0.1024	−0.0058	−0.08
1987	0.1074	0.0845	0.0229	0.31
1988	0.1383	0.1254	0.0128	0.17
1989	0.1652	0.0372	0.1279	1.71
1990	0.1521	0.1283	0.0238	0.32
1991	0.0609	0.0633	−0.0024	−0.03
1992	0.0063	0.0297	−0.0234	−0.31
1993	−0.0620	0.0259	−0.0879	−1.17
1994	0.0618	0.0365	0.0254	0.34
Total			0.2316	3.09
ΣDZ^2				6.99

Table 5.2. Comparison of actual and expected values of $DF(t)$, 1983-94, all conditional on $\mathcal{F}_{1982}$

Year	$DF(t)$	$E[DF(t)]$	Deviation	Standard deviation	Standardised deviation
1983	0.0625	0.0709	−0.0084	0.0771	−0.11
1984	0.1892	0.1628	0.0265	0.1380	0.19
1985	0.3744	0.2470	0.1275	0.1830	0.70
1986	0.4711	0.3248	0.1462	0.2223	0.66
1987	0.5784	0.3973	0.1811	0.2588	0.70
1988	0.7167	0.4655	0.2512	0.2932	0.86
1989	0.8819	0.5302	0.3517	0.3262	1.08
1990	1.0340	0.5920	0.4420	0.3578	1.24
1991	1.0949	0.6514	0.4435	0.3881	1.14
1992	1.1012	0.7090	0.3922	0.4174	0.94
1993	1.0392	0.7651	0.2741	0.4455	0.62
1994	1.1011	0.8200	0.2811	0.4720	0.59

1982, $DF(t) = \ln D(t) - \ln D(1982)$. Using the formulae for the expected values and variances of the forecast logarithms, which are set out in Appendix E.4.7, we get the results shown in Table 5.2. This shows the value of $DF(t)$, its expected value conditional on the relevant information up to 1982 $E[DF(t) | \mathcal{F}_{1982}]$, the observed deviation $DF(t) - E[DF(t) | \mathcal{F}_{1982}]$, the standard deviation of $DF(t) | \mathcal{F}_{1982}$, and the standardised residual the observed deviation divided by the corresponding standard deviation.

5.2.5 Dividends since 1982 have risen by more than the then mean forecast, but well within a reasonable funnel of doubt, having moved outside one standard deviation away from the mean for three years, 1989-91, and that by only a small amount.

5.3 *Updating and Rebasing to 1923-94*

5.3.1 I have refitted the parameters of the model, starting in June 1923 and including the data up to June 1994. This gives 71 values of the change in the dividend index. A graph of the index is shown in Figure 4.3, and of the dividend index 'in real terms' in Figure 5.1. Possible models include the original one, with terms in *DY* and *DB*, as well as a model including *DMU*. Parameter estimates for several models are shown in Table 5.3. In models (i) to (iv) $DX = 1 - DW$; in model (v) $DW = DX = 0$, i.e. there is no influence of inflation on dividend growth. See Appendix C.1 for an explanation of the method of estimating the parameters.

5.3.2 It can be seen that both *DB* and *DY* are useful parameters. In model (i) their estimates are more than four times the standard errors. If either is omitted the log likelihood is worsened considerably, and, in addition, the residuals become conspicuously non-normal. Further, if *DB* is omitted, the first autocorrelation coefficient of the residuals becomes large, and if *DY* is omitted, the crosscorrelation of the residuals with *YE* at lag 1 (i.e. $DE(t)$ with $YE(t-1)$) becomes large. The presence of both *DY* and *DB* in the model is fully justifiable.

5.3.3 The value of the smoothing parameter *DD* is not much more than one standard error away from zero, and it therefore could be thought that it should be zero. Model (v) investigates what happens if we omit the influence of inflation by setting both *DW* and *DX* to zero. The log likelihood is worsened substantially, and, in addition, the crosscorrelation between the residuals *DE* and the residuals from inflation *QE* is large. In any case, a model without some direct transfer from retail prices to dividends, probably with unit gain, would be economically quite unrealistic. However, the large standard errors show that the speed with which inflation feeds through to dividends is rather uncertain, and many other parameter values would do about as well.

5.3.4 Finally (not shown) I have tried the model with $DW = 0.8$, $DX = 0.2$, and $DD = 0.2$, i.e. these parameters as in the original model. The log likelihood is worsened by only 0.62 and the estimates of the other (free) parameters are almost exactly as in model (i).

5.3.5 Diagnostic tests of the residuals for model (i) show no remaining autocorrelation. The residuals have too many runs of the same sign, with 47 instead of the expected 37, a *t*-ratio of -2.37; this indicates too frequent an alternation of sign. The first autocorrelation coefficient is negative, but small

Table 5.3. Parameter estimates for model for ln D, 1923-94

Model	(i) Original	(ii) $DY = 0$	(iii) $DB = 0$	(iv) $DB = DY = 0$	(v) $DW = DX = 0$
DW	0.5793	0.6192	0.7670	0.7311	-
	(0.2157)	(0.2408)	(0.2447)	(0.2678)	
DD	0.1344	0.1217	0.1613	0.1345	-
	(0.0800)	(0.0853)	(0.0602)	(0.0656)	
DMU	0.0157	0.0154	0.0145	0.0143	0.0572
	(0.0124)	(0.0133)	(0.0089)	(0.0101)	(0.0155)
DY	−0.1761	-	−0.2536	-	−0.1510
	(0.0439)		(0.0574)		(0.0571)
DB	0.5733	0.5156	-	-	0.5309
	(0.1295)	(0.0995)			(0.1222)
DSD	0.0671	0.0748	0.0753	0.0849	0.0867
	(0.0056)	(0.0063)	(0.0063)	(0.0071)	(0.0073)
Log likelihood	0.0	−7.76	−8.29	−16.90	−18.43
Jarque-Bera χ_2^2	8.16	15.87	28.35	56.52	10.02
$p(\chi^2)$	0.017	0.0004	0.0	0.0	0.0067

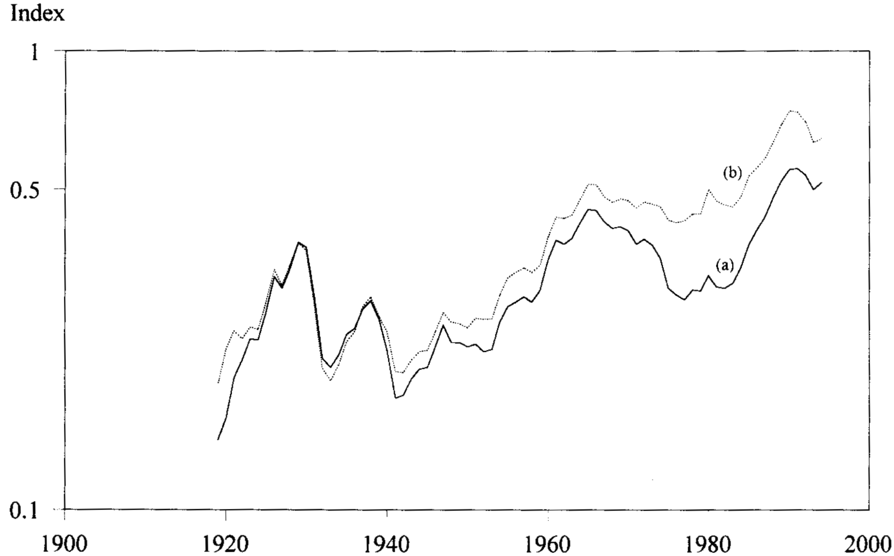

Figure 5.1. Share dividend index 'in real terms', (a) relative to price index, and (b) relative to lagged price index, 1919-94

(−0.02), and far from significant. There is no significant crosscorrelation at low lags with the residuals of the inflation, wages and dividend yield series, though there is moderate positive correlation with the residuals of the yield series at lags

2 and 3, i.e. between $DE(t)$ and $YE(t+2)$ (+0.25) and $YE(t+3)$ (+0.26); I can see no economic rationale for this, and the coefficients are only a little more than two standard errors away from zero, so I have not pursued them.

5.3.6 The residuals are not conspicuously non-normal, but the skewness coefficient $\sqrt{b_1} = -0.62$, and the kurtosis coefficient $b_2 = 4.08$, are both rather outside two standard errors away from zero and 3.0 respectively, the Jarque-Bera statistic is 8.16, with $p(\chi_2^2) = 0.0169$, so there is some evidence of fat-tailedness. The large residuals are all in the earlier years: 2.2 times the standard deviation in 1925, 2.1 times in 1928, −3.3 times in 1931, −2.6 times in 1932 and −2.3 times in 1941. The drop in gross dividend yield in 1993, caused by the change in the rate of Advance Corporation Tax, produced a residual of only −1.25 standard deviations.

5.3.7 The estimated value of *DMU*, the mean rate of growth of real dividends, is not much more than one standard error away from zero. It can, therefore, be argued that it could be set to zero, as in the original model. The current evidence does not strongly support moving away from that value. However, the real rate of growth of dividends is an important element in the total return on shares, and it is the sort of parameter about which individual actuaries may well have their own views, so it would be unreasonable to pre-empt their judgement by omitting it from the model.

5.3.8 Convenient rounded values, using model (i) are:

$$DW=0.58;\ DD=0.13;\ DMU=0.016;\ DY=-0.175;\ DB=0.57;\ DSD=0.07.$$

5.3.9 The course of real dividends deserves some further consideration. Figure 5.1 shows real dividends:

(a) as the dividend index divided by the retail price index (D/P); and also

(b) as the dividend index divided by the 'lagged' price index, allowing for the terms:

$$DI(t) = DW.DM(t) + (1 - DW).I(t)$$

using the parameters, $DW = 0.58$ and $DD = 0.13$.

5.3.10 One can see how the course of real dividends calculated on basis (b) is smoother than that calculated on basis (a), at least in more recent years. Real dividends appear to have zigzagged up and down over the years. The 1980s showed a conspicuous rise, as did the 1940s and 1950s, whereas the 1930s and 1970s showed falls. These, however, are the sorts of pattern that can be expected with what is essentially a random walk, modified only by the MA(1) term, which has influence only over a short time.

5.3.11 One can also see how difficult it is to estimate the mean rate of real dividend growth. By starting in 1923 instead of 1919 we omit the sharp rise at the beginning of the graph; but the peak in 1990 was not very much higher than the previous peaks in 1965 and 1929. On basis (b), which shows the higher rise, the growth over the 61 years from 1929 to 1990 was 1.09% p.a. If the calculations had been made in 1984, the rate of growth form 1965 would have been negative, and from 1929 only 0.4% p.a. This uncertainty is reflected in the standard errors of the estimates of *DMU*. Using model (i), a 95% confidence interval for *DMU* could be from about −0.9% to about + 4.0%.

5.4 Cointegration of Share Prices, Dividends, Retail Prices and Wages

5.4.1 I have made the dividend index depend on the retail price index through the transfer function described in ¶5.1.1. This allows real dividends to be an integrated I(1) series. However, it is possible that dividends and retail prices are cointegrated; since dividends and share prices are cointegrated, both may be cointegrated with retail prices; alternatively, one or both might be cointegrated with wages.

5.4.2 Using just the annual series from 1923 to 1994, I find that dividends and retail prices could be cointegrated, with a 'best' cointegration vector:

$$\ln D - 1.2936 \ln Q \sim I(0)$$

but the natural cointegration vector would be:

$$\ln D - 1.0 \ln Q \sim I(0)$$

which is significantly different from the best vector ($\chi_1^2 = 10.84$, $p = 0.001$); further, the best vector implies that dividends have almost a 30% gain over prices, so that they do much better when inflation is high than when it is low. Is this plausible?

5.4.3 Using the monthly series from December 1923 to June 1994, I also find evidence for cointegration. This time the best vector is:

$$\ln D - 1.4630 \ln Q \sim I(0)$$

so dividends show a 46% gain over prices. The natural vector is again significantly different ($\chi_1^2 = 15.64$, $p = 0.000$).

5.4.4 I also tested the real dividend series, calculated both on basis (a) and on basis (b), for unit roots. There is strong evidence for a unit root for both series, both at monthly and at yearly intervals. This suggests that, on balance, it is not necessary to model *D* as cointegrated with *Q*.

5.4.5 I also tested the connection between dividends and wages. Using the annual series from 1923 to 1994, I find strong evidence of cointegration between dividends and wages, with a best cointegration vector:

$$\ln D - 1.0370 \ln W \sim I(0)$$

not significantly different from:

$$\ln D - 1.0 \ln W \sim I(0)$$

so that wages and dividends remain close. However, both are connected with retail prices.

5.4.6 If I include prices, using the annual data again, the tests show one significant cointegrating vector:

$$\ln Q - 1.4457 \ln W + 0.6293 \ln D \sim I(0)$$

but I find it difficult to see a suitable interpretation of this.

5.4.7 Including also share prices gives one, but only one, possible cointegration vector:

$$\ln Q - 1.3339 \ln W - 0.3861 \ln D + 0.9658 \ln P \sim I(0)$$

but again I find it difficult to interpret this. Perhaps others can follow up this line of investigation.

5.5 *Observations at Monthly Intervals*

Values of the dividend index are available from January 1923 at monthly intervals. Although these are available, there was little more to be learned from them. Monthly changes in dividends are not closely sensitive to changes in inflation.

5.6 *Data for Selected Other Countries for Other Periods*

5.6.1 I show in Table 5.4 the estimated parameters for the dividend model for the countries whose dividend yields were considered in Section 4.7.

Table 5.4. Fitted parameters for dividend model for selected other countries

	U.K.	U.S.A.	France	Canada	Sweden	Finland
Period	1923-94	1926-89	1951-89	1923-93	1923-93	1950-93
DW	0.58	1.0	1.0	0.19	1.0	0.4
DD	0.13	0.38	0.2	0.26	0.6	0.8
DMU	0.016	0.0155	0.0	0.001	0.0075	−0.01
DY	−0.175	−0.35	0.0	−0.11	−0.2	−0.65
DB	0.57	0.5	0.7	0.58	0.5	0.5
DSD	0.07	0.09	0.085	0.07	0.096	0.16

5.6.2 Values of the parameters are reasonably similar for each country. Over the periods investigated the simultaneous and lagged correlation coefficients between the residuals for U.K. and U.S.A., U.K. and France, and U.S.A. and France were not significantly different from zero. I have not calculated the crosscorrelation coefficients for the other countries.

Table 5.5. Analysis of real dividend index for 16 countries from 1/1970 to 6/1994; 12 series with 24 or 25 yearly steps; parameters of random walk model

	DMU low - overall mean - high	*DSD* monthly series monthly - yearly	*DSD* yearly series low - mean - high
Australia	−0.0202 - −0.0159 - −0.0123	0.0493 - 0.1708	0.1332 - 0.1498 - 0.1686
Austria	−0.0424 - −0.0334 - −0.0314	0.0481 - 0.1668	0.1386 - 0.1591 - 0.1811
Belgium	−0.0143 - −0.0065 - −0.0044	0.0265 - 0.0918	0.0666 - 0.0721 - 0.0794
Canada	−0.0223 - −0.0170 - −0.0165	0.0216 - 0.0750	0.0905 - 0.1005 - 0.1134
Denmark	−0.0612 - −0.0482 - −0.0480	0.0543 - 0.1880	0.0921 - 0.1198 - 0.1885
France	−0.0189 - −0.0124 - −0.0108	0.0358 - 0.1241	0.1176 - 0.1355 - 0.1522
Germany	−0.0131 - −0.0124 - −0.0079	0.0576 - 0.1996	0.0945 - 0.1186 - 0.1578
Italy	−0.0522 - −0.0533 - −0.0462	0.0855 - 0.2963	0.2222 - 0.2592 - 0.3097
Japan	−0.0257 - −0.0246 - −0.0220	0.0319 - 0.1105	0.0555 - 0.0724 - 0.0969
Netherlands	+0.0047 - +0.0070 - +0.0081	0.0211 - 0.0732	0.0663 - 0.0727 - 0.0797
Norway	−0.0144 - +0.0076 - +0.0069	0.0431 - 0.1492	0.1254 - 0.1450 - 0.1701
Spain	−0.0543 - −0.0489 - −0.0481	0.0287 - 0.0994	0.0621 - 0.0729 - 0.0848
Sweden	+0.0095 - +0.0169 - +0.0158	0.0347 - 0.1204	0.0839 - 0.1039 - 0.1263
Switzerland	−0.0096 - +0.0046 - +0.0022	0.0555 - 0.1924	0.0718 - 0.0935 - 0.1366
U.K.	+0.0088 - +0.0105 - +0.0108	0.0119 - 0.0412	0.0630 - 0.0644 - 0.0651
U.S.A.	−0.0054 - −0.0049 - −0.0025	0.0250 - 0.0867	0.0512 - 0.0614 - 0.0745

Table 5.6.
Analysis of real dividend index for 16 countries from 1/1970 to 6/1994; 12 series with 23 or 24 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Aus	Ost	Bel	Can	Den	Fra	Ger	Ita
Australia	**1.0**	.04	**.68**	**.69**	.24	.34	.42	.23
Austria (Ost)	.04	**1.0**	.16	.03	.31	.28	.39	.19
Belgium	**.62**	.10	**1.0**	**.62**	.22	.09	.49	.12
Canada	**.67**	.04	.26	**1.0**	.33	.46	.26	−.19
Denmark	.05	.27	.18	.30	**1.0**	.27	.15	−.08
France	.33	.17	.14	.51	.52	**1.0**	.34	.02
Germany	.27	.47	.22	.01	.29	.29	**1.0**	.26
Italy	.02	.16	−.01	−.10	.08	.23	.10	**1.0**
Japan	.21	.19	−.18	.21	.02	.46	.21	.50
Netherlands	.45	.55	.27	.25	.03	.10	.52	.36
Norway	.54	−.05	.40	.20	.13	.34	.44	.08
Spain	−.14	.23	−.10	−.26	−.13	.07	.21	.09
Sweden	.35	.40	.30	.21	.40	.09	.38	.17
Switzerland	.20	.10	−.03	.06	.02	.09	.50	−.01
U.K.	.30	.46	.27	.31	.26	.15	.48	.26
U.S.A.	.22	−.03	−.04	.27	−.04	.29	.15	.08

5.7 *Data for Other Countries for 1970-94*

5.7.1 I have analysed share dividend indices for the same countries and period (January 1970 to June 1994) as the share dividend yields in Section 4.7. I have restricted myself to analysing indices of real dividends, i.e. the index constructed by dividing the dividend index by the Consumer Price Index for the relevant country. I have analysed these both on a monthly basis and on a yearly basis.

5.7.2 The autocorrelation coefficients for the monthly series are generally small, with the exception of the second autocorrelation coefficient, which, for a number of countries, is relatively large and negative (−0.4 or so). However, it is possible that this is an artefact of the way in which the dividend indices had to be constructed (see Appendix F.9), and I have not investigated this further.

5.7.3 For the monthly series, the Jarque-Bera χ_2^2 statistic is, in each case, very high, and this may also be the result of the way the indices have had to be constructed. It seems inappropriate to fit any sort of autoregressive model to the monthly series, and I have therefore calculated residuals assuming a pure random walk for real dividends. The simultaneous cross correlations are small, but with

Table 5.6 (continued).
Analysis of real dividend index for 16 countries from 1/1970 to 6/1994; 12 series with 23 or 24 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Jap	Net	Nor	Sp	Swe	Swi	U.K.	U.S.A.
Australia	.01	.55	.56	.01	.44	.28	.35	.28
Austria (Ost)	.26	.16	−.35	.08	.35	.21	.44	.17
Belgium	.02	.42	.39	.07	.32	.34	.31	.33
Canada	.02	.31	.44	.24	.19	.30	.31	.29
Denmark	−.04	−.00	.09	.28	.49	.15	.26	.03
France	.30	.35	.15	.34	.12	−.00	.33	.07
Germany	.28	**.65**	.27	.06	.47	.13	.36	.10
Italy	.32	**.62**	.01	−.31	.39	.02	.45	.02
Japan	**1.0**	.58	−.04	.31	.21	.23	.47	.14
Netherlands	.43	**1.0**	.21	.05	.37	.30	**.63**	.21
Norway	−.07	.17	**1.0**	−.13	.19	.30	−.03	.09
Spain	−.04	.06	−.03	**1.0**	.04	.02	−.03	.02
Sweden	.38	.42	.10	.02	**1.0**	−.02	.40	−.03
Switzerland	.13	.32	.24	.11	.21	**1.0**	.45	.53
U.K.	.37	**.67**	−.15	−.12	.46	.41	**1.0**	.37
U.S.A.	.52	.27	−.08	−.19	−.04	.29	.25	**1.0**

a small positive bias, and the lagged correlation coefficients are not significantly different from zero.

5.7.4 Results for the yearly series are not vitiated by the rounding problems that upset the monthly differences, and the results can be taken as reasonably reliable. However, the U.K. is the only country which shows consistently a high first autocorrelation coefficient (which justified fitting an MA(1) model to the data), though France, Japan, Netherlands and Sweden show autocorrelation coefficients in excess of 0.4 for a number of the annual series. By contrast, several countries show consistent or frequent negative correlations, as large as −0.44 for one month for Denmark. This contrasts with the results for longer periods shown in Section 5.6.

5.7.5 The annual series, which contain only 23 or 24 annual difference, are too short to allow us to see how dividends respond to inflation in the lagged way that I have developed for longer series, and again I have restricted the analysis to the assumption of a random walk for real dividends.

5.7.6 Table 5.5 shows certain statistics for the real dividend index for 16 countries. The values in the column *DMU* show the lowest and highest mean real dividend growth for any of the 12 annual series, and the overall growth for the whole period, on an annual basis. It can be noted that only five of the countries show any positive values of *DMU*, and only three (Netherlands, Sweden and the U.K.) do not show some negative values. At the bottom end are Italy (−0.0535 overall mean), Spain (−0.0490) and Denmark (−0.0483). At the top end come Sweden (+0.0169) and the U.K. (+0.0106). Even Austria and Japan, which show remarkably low average dividend yields, have shown substantially negative real dividend growth, −0.0336 and −0.0247 respectively, i.e. roughly −3.4% and −2.5% p.a.

5.7.7 The column headed *DSD* monthly shows the monthly standard deviation and the equivalent annualised standard deviation, which is $\sqrt{12} = 3.464$ times this. Then come three values of the standard deviation *DSD* from the yearly series, the lowest and highest for any month and the mean value. It is interesting that the U.K. has amongst the lowest standard deviation of real dividends, along with the U.S.A., while most countries have substantially higher standard deviations; that for Italy is not less than 0.2 and exceeds 0.3 for one month.

5.7.8 Correlation coefficients between the residuals (which in this case are just the annual changes) have been calculated, and are shown in Table 5.6, for the June series (upper triangle) and December series (lower triangle). Note that these correlation coefficients are affected by the correlation coefficients between annual changes in the price index used to calculate real dividends. In general, the values are lower than for some of the other variables considered, but still far from zero. Small groups of connected countries can be observed: Australia, Belgium and Canada form one such group, though I cannot imagine any reason to connect these; the Netherlands and the U.K. form another, and these can reasonably be connected through the existence of large multinational companies, Royal Dutch Shell and Unilever, whose dividends enter the indices for both countries: however, this can account for only a moderate effect.

5.8 *ARCH Models*

5.8.1 I investigated the dividend model for ARCH effects, as described previously for inflation in Section 2.8 and dividend yields in Section 4.9. I define three extra terms, *K*-squared, equal to $(K(t) - DMU)^2$, *KH*-squared, the square of the expected value of $(K(t) - DMU)$, and *DE*-squared, the square of the residuals.

5.8.2 *DE*-squared shows a first correlation coefficient of 0.23, almost twice the standard error of 0.118. However, the crosscorrelation with *K*-squared, lagged, is small, and the simultaneous crosscorrelation with *KH*-squared is also small.

5.8.3 When *DE*-squared is compared with simultaneous and previous values for the inflation and dividend yield series, there is only one crosscorrelation of interest, that between *DE*-squared and $QE(-1)$-squared, which is 0.25. The corresponding correlation coefficient between $DE(t)$ and $QE(t-1)$ is −0.14. The implication is that a large change in inflation, in either direction, creates a larger standard deviation for the dividend residual.

5.8.4 However, since the Jarque-Bera statistic for dividends was not excessively high, and all the large residuals occurred before 1942, I have not investigated further an ARCH model for dividends.

5.9 *Company Earnings and P/E Ratios*

5.9.1 Just as the prices of many individual shares are nowadays assessed in terms of price/earnings (P/E) ratios, so it would be reasonable to investigate the share price index as equal to the corresponding earnings index times an aggregate P/E ratio. I did not do this previously because P/E ratios on an index were available only from 1962, and then only on the FTA 500-Share Index. A P/E ratio on the All-Share Index started only in early 1993, and it includes financial companies for which P/E ratios have a different relevance than they do for industrial companies. However, changes in such a P/E ratio may be relevant.

5.9.2 Associated with the P/E ratio is the payout ratio, the ratio of dividends to earnings. If dividends are increased without a corresponding increase in earnings, so that the payout ratio is increased. This may not be sustainable, and share prices may not fully reflect the increase in dividends; and vice versa. Payout ratios may therefore be a useful short-term predictor of share price movements. The payout ratio itself must be represented by a stationary time series model, since it can hardly move without limit either up or down.

5.9.3 There are now more than 30 years of data on the FTA 500-Share Index (now the Non-Financials Index), and some analysis could be done. Rather awkwardly, there are two definitions of earnings available, assuming tax is paid on the basis of the actual dividend payout and on a nil payout basis. The published P/E is based on one of these, and the earnings yield on the other.

5.9.4 Lack of time has prevented me from being able to investigate this further at this time. However, the P/E ratio is plotted in Figure 5.2, and the payout (D/E) ratio in Figure 5.3. There are apparent discontinuities in 1965 and 1969 because of changes in the taxation system, but there are large jumps at other times too. However, both series appear to be stationary.

5.10 *Alternative Models for Share Prices*

5.10.1 The popular model for share price movements among financial economists is the random walk model, which is consistent with markets being 'efficient', share prices being unpredictable from their past history, and thus share prices being 'martingales' in a probability sense. There are good economic arguments for this assumption, and it is clear that my model, which presumes that, at times when dividend yields are exceptionally low, share prices are 'too high', and vice versa, is inconsistent with an efficient market. If enough investors were to believe in the short-term predictions of my model, then share prices would not fluctuate so far away from an average dividend yield.

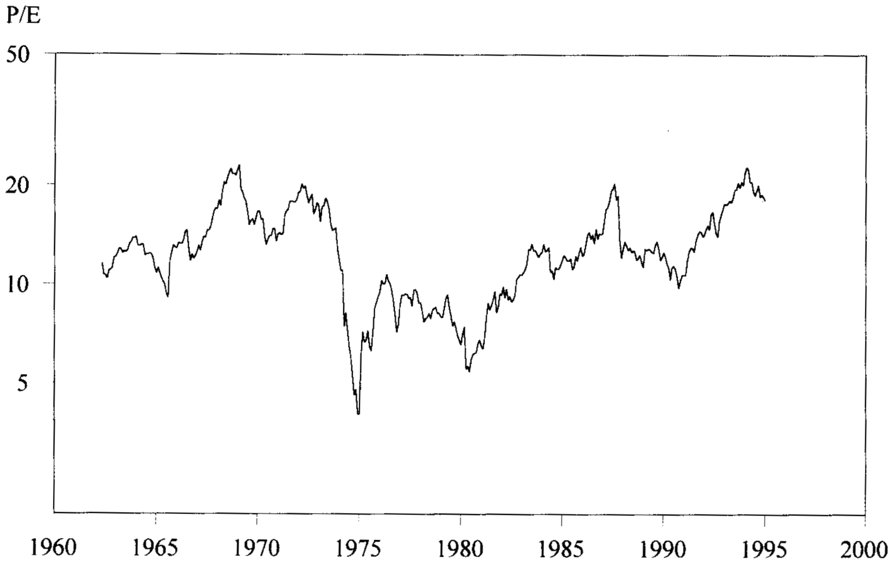

Figure 5.2. P/E ratio on FTA 500-Share Index, 1962-94

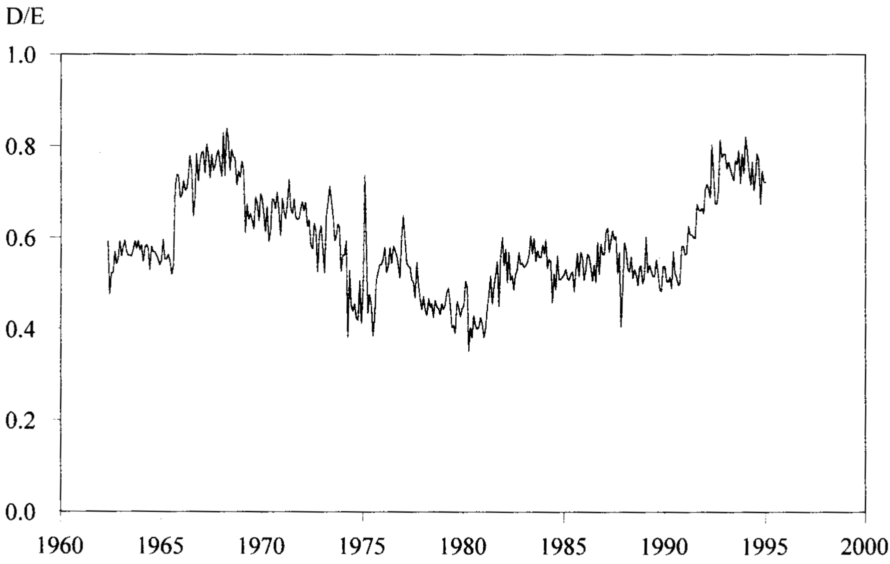

Figure 5.3. Payout ratio (D/E) on FTA 500-Share Index, 1962-94

5.10.2 It is clear from Section 4.5 that the short-term behaviour of share prices according to my model is very like that of a random walk. It is in the long term that they differ. It is not surprising that those who have investigated share prices at daily intervals over a year or two, or at weekly or monthly intervals over even as long as ten years, do not find that share dividend yields are stationary. The same problem arises for exchange rates, as discussed in ¶10.3.4. The observation period must be long enough, if a realistic long-term model is to be observed.

5.10.3 I am inclined to believe that it would be beneficial for the economy of any country for share prices to be more stable and to trade on more constant dividend yields or P/E ratios, but it is not my purpose in this paper to argue this case. It is sufficient to observe that the facts seem to be against the pure random walk model, since share dividend yields, in many countries, seem clearly to be modelled best by a stationary time series model.

5.10.4 The extensive work done by Ibbotson & Sinquefield (1989) in the U.S.A. appears to assume a random walk model for total returns on all the series they investigate. Although there is reference to some autocorrelation in their inflation series, no details are given.

5.10.5 However, some notice has been taken of dividend yields in the classical financial economics journals. Fama & French (1988) show that changes in share prices over a given period in the U.S.A. are partially predictable from the level of dividend yield at the start of the period. In Wilkie (1993) I showed the same for the U.K.; the maximum correlation coefficient between share price change over a period and dividend yield at the start of the period is 0.69 for a period of 76 months. Fama & French stopped at 48 months, and so failed to see that the effect they had noticed was actually much stronger when the period was lengthened.

5.10.6 In a recent talk, so far unpublished, Professor Michael Brennan described investigations into the movements of share dividend yields, and long- and short-term interest rates in the U.S.A. The results seemed quite consistent with my model for dividend yields.

5.10.7 A Finnish group has done work on share prices on the same lines as I have (see Pentikäinen *et al.*, 1989). Their model is described by Daykin, Pentikäinen & Pesonen (1994, Chapter 8). It seems, at first sight, rather different from mine, because it is put in the form of several factors multiplied together, rather than in logarithmic additive form, but it can be reexpressed in my style as:

$$\ln P(t) = \ln Q(t) + PMU.t + PD(t)$$

where $PD(t)$ is a stationary, I(0) series with zero mean, modelled as an AR(2) series:

$$PD(t) = PA1.PD(t-1) + PA2.PD(t-2) + PE(t)$$

with $PE(t) \sim$ IID N(0, PSD^2). This is a simple cointegrated model connecting ln P and ln Q.

5.10.8 The Finnish model develops share prices first, and makes dividends depend on both retail prices and share prices, putting:

$$\ln D(t) = \ln Q(t) + DMU.t + DD(t)$$

where:

$$DD(t) = DB.PD(t-1) + (1-DB).DD(t) + DE(t)$$

with $DE(t) \sim$ IID N(0, DSD^2). This, too, is a cointegrated model connecting ln D and ln Q, and, in some respects, like mine. Although I do not think it produces a satisfactory model for the yield, or rather its logarithm (ln D − ln P), it is worth investigating this style of model further. Whether ln Q, ln D and possibly also ln W, i.e. retail prices, dividends and wages, are cointegrated or not is an open question.

5.11 *Forecasting*

5.11.1 In Figure 5.4, I show a set of ten simulations of $D(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1950, all on a logarithmic scale. One can get, as for previous indices, an impression of the shape of the expanding funnel of doubt from these.

5.11.2 In Figure 5.5, I show the forecast median of $D(t)$, starting with the conditions in June 1994, also on a logarithmic scale, along with two sets of confidence intervals. The wider pair shows the mean plus and minus two standard deviations, using the formulae in Appendix E.4. The inner pair shows what the two standard deviation confidence interval would be for a random walk model for ln D, with the same one-year standard deviation. The standard deviation is proportional to the square root of (t − 1994). This shows how much the autoregressive nature of the model increases the uncertainty about the future.

5.11.3 In Figure 5.6, I show a set of ten simulations of $P(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1950, all on a logarithmic scale. One can get, as for previous indices, an impression of the shape of the expanding funnel of doubt from these.

5.11.4 In Figure 5.7, I show the forecast median of $P(t)$, starting with the conditions in June 1994, also on a logarithmic scale, along with two sets of confidence intervals. The inner pair shows the mean plus and minus two standard deviations, using the formulae in Appendix E.4. This time the outer pair shows what the two standard deviation confidence interval would be for a random walk model for ln P, with the same one-year standard deviation. The standard deviation is proportional to the square root of (t − 1994). This shows how the autoregressive nature of the model for dividend yields, in this case, decreases the uncertainty about the future.

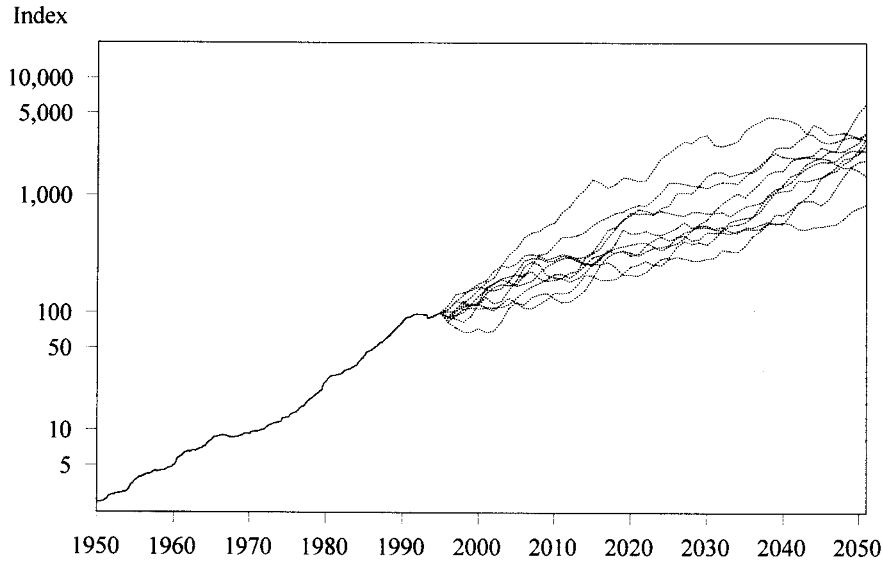

Figure 5.4. Dividend index, 1950-94, and simulations, 1994-2050

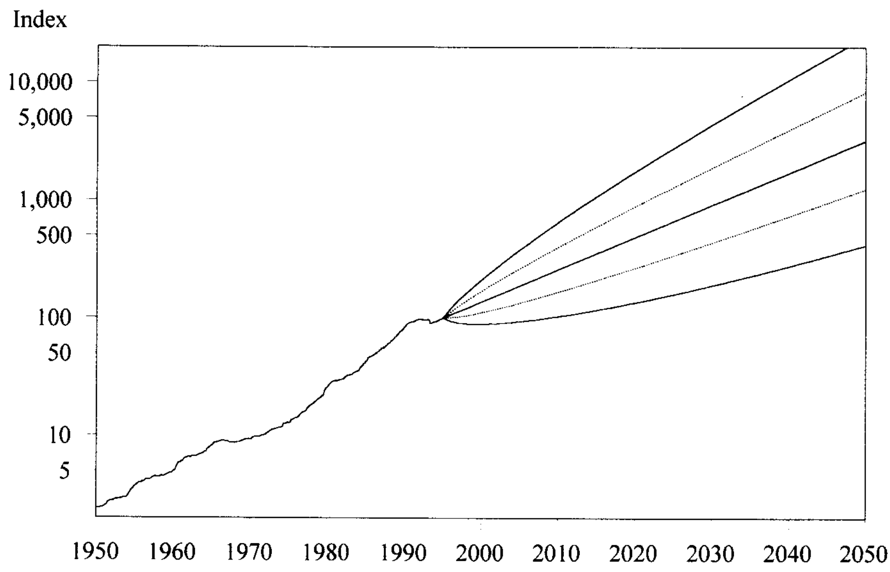

Figure 5.5. Dividend index, 1950-94, and forecast medians and confidence intervals, 1994-2050

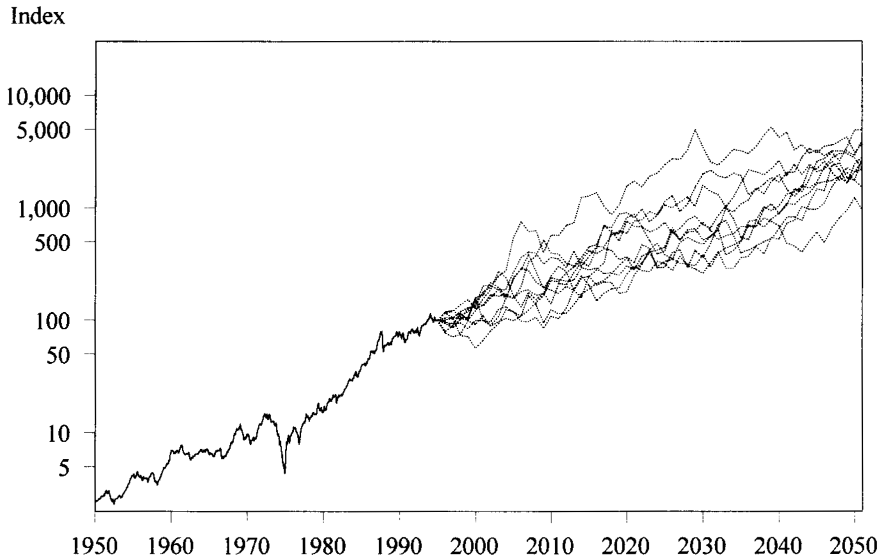

Figure 5.6. Share price index, 1950-94, and simulations, 1994-2050

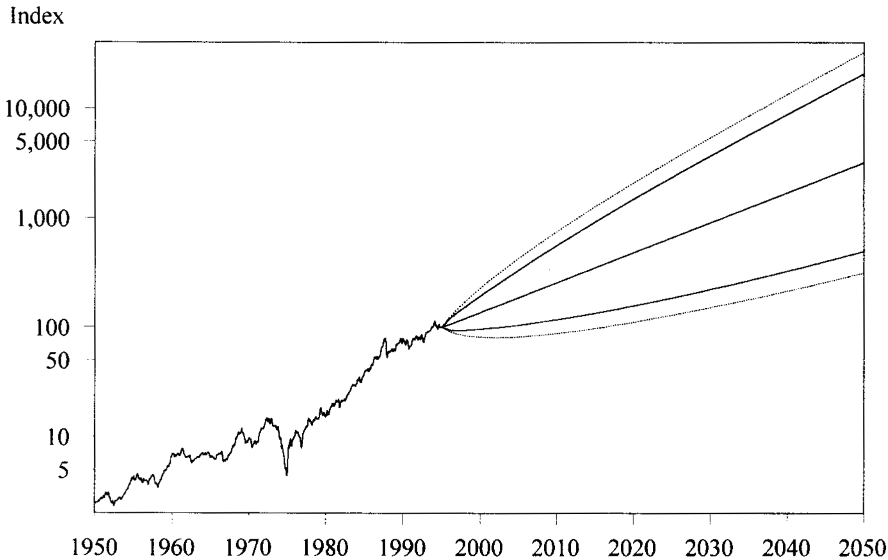

Figure 5.7. Share price index, 1950-94, and forecast medians and confidence intervals, 1994-2050

6. LONG-TERM INTEREST RATES

6.1 *Original Model for Long-Term Interest Rates*

6.1.1 The original model for the yield on 'consols', i.e. a perpetual fixed-interest stock, based on annual data from June 1919 to June 1982, where $C(t)$ is the yield on consols at time t, is:

$$C(t) = CW.CM(t) + CMU.\exp\{CN(t)\}$$

$$CM(t) = CD.I(t) + (1-CD).CM(t-1)$$

$$CN(t) = CA1.CN(t-1) + CA2.CN(t-2) + CA3.CN(t-3) + CY.YE(t) + CE(t)$$

$$CE(t) = CSD.CZ(t)$$

$$CZ(t) \sim \text{iid N}(0,1).$$

6.1.2 It is convenient to denote the 'real' part of the consols yield as:

$$CR(t) = C(t) - CW.CM(t)$$

so that:

$$\ln CR(t) = \ln CMU + CN(t).$$

The suggested parameters, based on the experience from 1919 to 1982, were: $CW = 1.0$; $CD = 0.045$; $CMU = 3.5\%$; $CA1 = 1.20$; $CA2 = -0.48$; $CA3 = 0.20$; $CY = 0.06$; $CSD = 0.14$. This model, with $CW = 1$, fully takes into account the 'Fisher effect' (see Fisher, 1907, 1930), in which the nominal yield on bonds reflects both expected inflation over the life of the bond and a 'real' rate of interest.

6.1.3 The model for the logarithm of $CR(t)$ is essentially an AR(3) model, but originally I suggested that a possible simplification was to use an AR(1) model instead, omitting the terms in $CA2$ and $CA3$, and with the remaining parameter $CA1 = 0.91$. At the time I also suggested that the CY term might also be omitted by putting $CY = 0.0$, but I have found that this term is best retained.

6.1.4 The model can be described in words: at any date the consols yield can be decomposed into two parts, an allowance for expected future inflation, and a real yield; the allowance for future inflation is based on an exponentially weighted moving average of past inflation; the logarithm of the real yield is equal to its mean value (ln 3.5%), plus 91% of its deviation a year ago from the mean, plus (if the full AR(3) model is used) influences from the values two years ago and three years ago, plus an additional influence from the current dividend yield innovation, plus a random innovation which has zero mean and a standard deviation of 0.14.

6.2 The Experience from 1982 to 1994

6.2.1 It is again of interest to see how this model has fared since 1982, and this is investigated in the same two ways as before. According to the model, the residuals, the *CE*s, are distributed N(0, CSD^2), and the standardised residuals, the *CZ*s, are distributed N(0,1). The sum of n such *CZ*s is distributed N(0, n), and the sum of the squares of n such *CZ*s is distributed as χ^2_n.

6.2.2 Table 6.1 shows, for each year, the observed value of the long-term fixed-interest yield (now represented by the yield on the FTA BGS Irredemables Index) $C(t)$, shown as a percentage (note that in the formulae I treat C as a fraction), the logarithm of the real yield, the expected value of the logarithm conditional on the relevant information for C up to year $(t-1)$, and also the observed values of $Q(t)$ and $Y(t)$ E[ln $CN(t)\,|\,\mathcal{F}_{t-1}+Q+Y$], the observed residual $CE(t) = CN(t) - \text{E}[CN(t)\,|\,\mathcal{F}_{t-1}+Q+Y]$, and the standardised residual $CZ(t) = CE(t)/CSD$, where $CSD = 0.14$. The notation $\mathcal{F}_{t-1}+Q+Y$ now means all relevant facts at time $(t-1)$ plus the values of $Q(t)$ and $Y(t)$.

Table 6.1. Comparison of actual and expected values of $CN(t)$, 1983-94

Year	$C(t)$%	$CN(t)$	E[$CN(t)$]	$CE(t)$	$CZ(t)$
1982	12.46				
1983	9.74	−0.0854	0.4254	−0.5108	−3.65
1984	10.44	0.1289	−0.2393	0.3682	2.63
1985	10.07	0.0280	0.2974	−0.2694	−1.92
1986	8.91	−0.2897	−0.0528	−0.2370	−1.69
1987	8.89	−0.2604	−0.3527	0.0923	0.66
1988	9.49	−0.0369	−0.1569	0.1201	0.86
1989	9.37	−0.0985	0.0215	−0.1200	−0.86
1990	10.63	0.2036	−0.1491	0.3526	2.52
1991	10.33	0.1385	0.2923	−0.1538	−1.10
1992	9.15	−0.1700	0.0517	−0.2217	−1.58
1993	8.24	−0.4393	−0.2378	−0.1965	−1.40
1994	8.54	−0.2524	−0.4116	0.1592	1.14
Total				−0.6167	−4.40
ΣCZ^2					42.02

6.2.3 We can compare the sum of the 12 values of *CZ*, which is −4.40, with the expected value, zero, and the standard deviation $\sqrt{12} = 3.46$. It is between one and two standard deviations away from its expected value. We can also compare the sum of the 12 values of CZ^2, which is 42.02, with a χ^2_{12} distribution; the probability of a value of χ^2 as great or greater is 0.00003, which is exceptionally low. Only three of the (absolute) values of *CZ* are less than 1.0

and three exceed 2.0. It now looks as if the value of *CSD* is too low, or as if the model should be modified in some other way.

6.2.4 By comparing the numbers in Table 6.1 with those in Table 2.1, we can see that long-term interest rates dropped in 1983, the (logarithm of the) presumed real rate dropping by more than three standard deviations, when inflation also reduced ($I(1983) = 0.0359$, whereas $I(1982) = 0.0877$). When inflation rose again the following year, $\ln CR(t)$ rose by more than two standard deviations. When inflation rose again in 1990, $\ln CR(t)$ again rose by more than two standard deviations. It is possible that expectations of future inflation now respond more quickly to current inflation than they did previously, as modelled by the fairly low value of *CD* of 0.045.

6.2.5 We can also consider forecasts made on the reduced basis referred to in ¶6.1.2. These are not shown in full. The sum of the 12 values of *CZ* is -4.70, which should be compared with its standard deviation $\sqrt{12} = 3.46$. It is again between one and two standard deviations away from zero. The sum of CZ^2 is 28.37, rather smaller than previously, but still improbably high ($p = 0.005$).

6.2.6 We can also consider the forecast values of $C(t)$, conditional on the information as at 1982. Using the formulae for the expected values and variances of the forecast rates, which are set out in Appendix E.5, we get the results shown in Table 6.2. Note that $C(t)$ is distributed neither normally nor lognormally. The results are shown as if it were distributed normally. They show $C(t)$ for each year, its expected value conditional on the relevant information up to 1982 $E[C(t) \mid \mathcal{F}_{1982}]$, the observed deviation $C(t) - E[C(t) \mid \mathcal{F}_{1982}]$, the standard deviation of $C(t) \mid \mathcal{F}_{1982}$, and the standardised residual, the observed deviation divided by the corresponding standard deviation.

Table 6.2. Comparison of actual and expected values of $C(t)$, 1983-94, all conditional on $\mathcal{F}_{1982}$

Year	$C(t)$ %	$E[C(t)$ %]	Deviation	Standard deviation	Standardised deviation
1982	12.46				
1983	9.74	12.1323	−2.3923	0.7999	−2.99
1984	10.44	11.8980	−1.4580	1.2314	−1.18
1985	10.07	11.7461	−1.6761	1.4605	−1.15
1986	8.91	11.5969	−2.6869	1.6074	−1.67
1987	8.89	11.4322	−2.5422	1.7217	−1.48
1988	9.49	11.2676	−1.7776	1.8166	−0.98
1989	9.37	11.1134	−1.7434	1.8950	−0.92
1990	10.63	10.9701	−0.3401	1.9592	−0.17
1991	10.33	10.8359	−0.5059	2.0120	−0.25
1992	9.15	10.7098	−1.5599	2.0558	−0.76
1993	8.24	10.5920	−2.3520	2.0925	−1.12
1994	8.54	10.4819	−1.9419	2.1236	−0.91

6.2.7 One can see that the value of $C(t)$ jumps well outside a two standard deviation range in 1983, but thereafter remains within such a range, partly because of the reversal in 1984. The same calculations have been carried out using the reduced basis described in ¶6.1.2, and the results are similar.

6.3 Updating and Rebasing to 1923-94

6.3.1 I have refitted the parameters of the model, starting in June 1923 and including the data up to June 1994. This gives 72 values of the consols yield. In the first place I fix CW = 1.0 and CD = 0.045, and include various of the other parameters. I start with a minimal model (i), setting $CA2$, $CA3$ and CY all to zero. In model (ii) $CA2$ is included, and in model (iii) both $CA2$ and $CA3$. Model (iv) is like model (i), but includes CY. See Appendix C.1 for an explanation of the method of estimating the parameters.

6.3.2 One can see that the effect of including additional autoregressive parameters, $CA2$, or both it and $CA3$, is small. In model (ii) the estimate of $CA2$ is less than twice its standard error, and the improvement in the log likelihood is small. In model (iii) the estimate of $CA3$ is just over twice its standard error, and the improvement in the log likelihood, as compared with model (i), is 2.76. Twice this value should be compared with χ_2^2; p = 0.063, so the pair of parameters are of marginal significance. It is more consistent with a model observed at more frequent intervals to omit these parameters, and I have found (see Section 6.6) that the extra autoregressive parameters have no significance in other countries. I therefore prefer to omit them.

6.3.3 In model (iv) the log likelihood is improved by 2.59, enough to justify one extra parameter, and the estimate of CY is more than twice its standard error. It is also plausible that the residuals of the consols yield and share dividend yield should be correlated. There therefore seems good justification for including this term.

6.3.4 When I fitted the model originally, I noticed that there was a very large residual value in 1974, and I included a dummy variable which took the value 1 in 1974 and zero elsewhere, what Box & Jenkins call an 'intervention variable'. Its effect was to screen out this very large residual, since its inclusion could affect the estimation of the other parameters unduly. In model (v) I test the effect of including $CB1$, the coefficient of this dummy variable, and in model (vi) I include a second dummy variable for 1983, which also shows a very large residual. It is clear that both these parameters are significant individually, each being more than twice its standard error; further, the improvement in log likelihood is large. However, the estimates of the other parameters in the model are not unduly distorted. The large residuals have more effect on the autoregressive parameters. The standard deviation CSD is reduced, but it is not fair to assume that this lower standard deviation can apply in future. In spite of the large residuals in 1974 (2.37 times the standard deviation) and 1983 (−3.05 times), model (iv) does not show an excessively high Jarque-Bera statistic, and it seems to me to be the most suitable.

Table 6.3. Parameter estimates for model for C, 1923-94

Model	(i) omitting $CA2$, $CA3$ and CY	(ii) including $CA2$	(iii) including $CA2$ and $CA3$	(iv) including CY	(v) including $CB1$ for 1974	(vi) including $CB1$ for 1974 $CB2$ for 1983
CW = 1.0 and CD = 0.045 throughout						
CMU %	3.09 % (0.92 %)	3.07 % (0.80 %)	3.21 % (0.99 %)	3.05 % (0.65 %)	2.86 % (0.52 %)	3.09 % (0.65 %)
$CA1$	0.9234 (0.0445)	1.0550 (0.1139)	1.0930 (0.1145)	0.8974 (0.0442)	0.8854 (0.0423)	0.9068 (0.0396)
$CA2$	-	−0.1417 (0.1137)	−0.3928 (0.1656)	-	-	-
$CA3$	-	-	0.2285 (0.1110)	-	-	-
CY	-	-	-	0.3371 (0.1436)	0.2004 (0.1435)	0.1668 (0.1333)
$CB1$	-	-	-	-	0.5230 (0.1948)	0.5149 (0.1789)
$CB2$	-	-	-	-	-	−0.5810 (0.1645)
CSD	0.1921 (0.0160)	0.1901 (0.0158)	0.1848 (0.0154)	0.1853 (0.0154)	0.1764 (0.0147)	0.1631 (0.0136)
Log likelihood	+0.0	+0.73	+2.76	+2.59	+6.12	+11.75
Jarque-Bera χ_2^2	6.81	5.00	3.63	4.88	9.30	3.92
$p(\chi^2)$	0.033	0.082	0.16	0.087	0.010	0.14

6.3.5 Diagnostic tests of the residuals of model (iv) show large residuals in 1939 (−2.28 times the standard deviation), 1940 (−2.41 times), 1974 (2.37 times), and 1983 (−3.05 times), the last two already noted. The autocorrelation function of the residuals is satisfactory, but there are marginally significant crosscorrelation coefficients with $QE(t-1)$, $YE(t-1)$ and $YE(t+1)$. I have not investigated models which could eliminate these.

6.3.6 It would be reasonable to investigate alternative values of CD or CW; another option would be to allow the effect of inflation to be omitted before some date and to include it, perhaps gradually, after that date. This could represent better the way investors may actually have thought: before a certain time, inflation was not considered to be permanent and the nominal rate was taken as the real one; an appreciation of the difference only affected the market slowly. I have experimented with various time-varying models of this kind, but none seems to be conspicuously better for future simulation than the model suggested with the parameters as in model (iv).

6.3.7 Suitably rounded parameters for this model might be:

$CW = 1.0$; $CD = 0.045$; $CMU = 3.05\%$; $CA1 = 0.9$; $CY = 0.34$; $CSD = 0.185$.

The standard deviation is larger than in the original model, and is, in effect, further increased by the much larger value of CY; but this correctly reflects the more varied experience of the last 12 years. The total variance of $CN(t)$, allowing for the term $CY.YE(t)$, can be denoted CSD^*, which can be calculated from:

$$CSD^{*2} = CY^2.YSD^2 + CSD^2$$

the value of which, using the rounded parameters, with $YSD = 0.155$, is 0.1924^2.

6.4 *Previous Centuries*

6.4.1 I have available data for consols yields at annual intervals from 1756. The values are plotted in Figure 6.1, along with graphs of Bank Rate, and of the yield on index-linked stocks. Strictly what I have for consols are annual averages of the yields, and such values are not perfect for analysis. Spurious autocorrelation coefficients can be induced into certain types of model with such data, as was shown by Working (1960). However, the autocorrelation coefficients in this case are of a size not to be significantly upset by this feature. A scatter diagram of ln $C(t)$ versus ln $C(t-1)$ is shown in Figure 6.2.

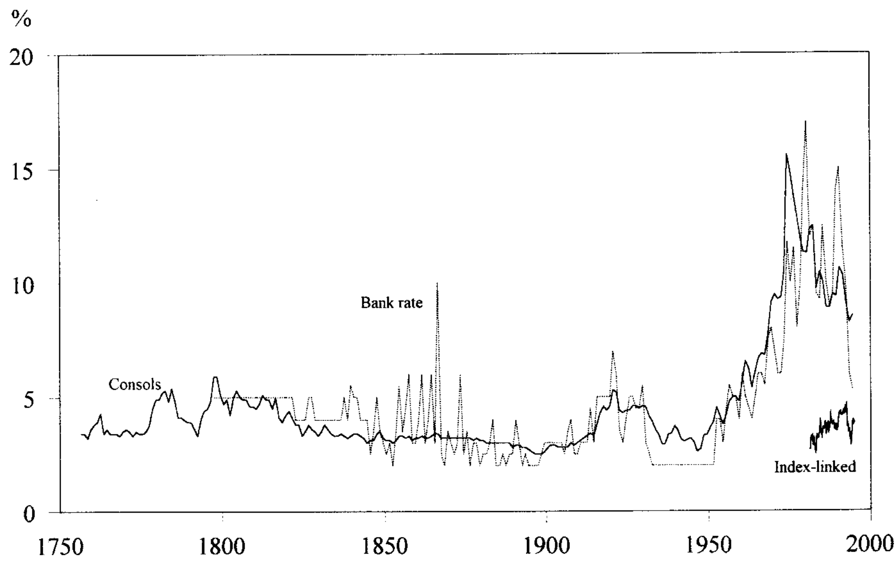

Figure 6.1. Yield on consols, 1756-1994, Bank Rate, 1797-1994, and yield on index-linked stocks, 1981-94

6.4.2 I chose in Wilkie (1986a) to model these yields from 1756 to 1956, with 201 values. It was not necessary to take into account the effect of inflation, and nominal yields were modelled directly. An AR(1) model for ln $C(t)$ is all that is necessary, with parameters and standard errors as shown in Table 6.4.

Table 6.4. Parameter estimates for model for ln C, 1756-1956

	Parameter estimate	Standard error
CMU	3.52	0.19 %
CA	0.9402	0.0260
CSD	0.0689	0.0034

6.4.3 The value of *CA* is quite similar to that of *CA*1 in model (iv), and the value of *CMU* is not far distant. The value of *CSD* is much smaller; this may be partly caused by the annual averaging, which would reduce fluctuations as compared with using values at a specific date on each year.

6.4.4 The Jarque-Bera statistic is on the high side, at 11.19 (p = 0.0037). There are 12 standardised residuals outside the range (−2, +2) including two with absolute value greater than 3 (1763, −3.24 and 1797, +3.26). On the whole the model seems quite reasonable.

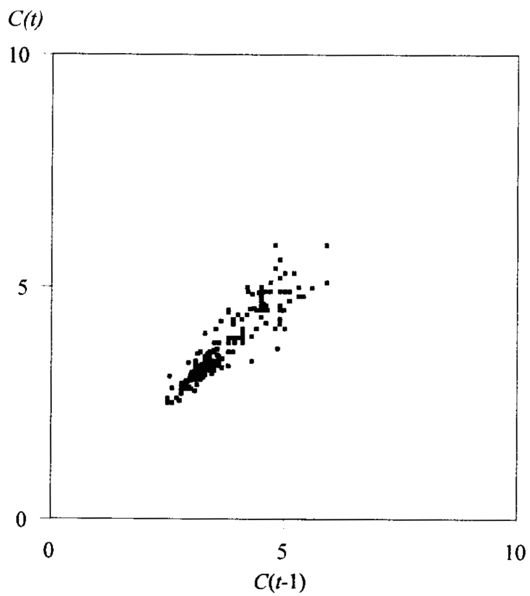

Figure 6.2. Scatter diagram of $C(t)$ versus $C(t-1)$, yearly, 1756-1956

6.5 *Observations at Monthly Intervals*

6.5.1 Values of the consols yield are available for the period since December 1923 at monthly intervals. Since these are available, it is worth investigating them. I have constructed *m/h* series, as for share dividend yields. The results are shown in Table 6.5, for m = 1, 2 and 12.

Table 6.5. Statistics and autocorrelation coefficients for various *m/h* series for ln *C*, December 1923 to June 1994

m/h	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	847	0.9975	0.9699	0.0322	0.1102
2/1	424	0.9954	0.9726	0.0431	0.1043
2/2	423	0.9945	0.9674	0.0473	0.1143
12/1 (Dec)	71	0.9695	0.9695	0.1203	0.1203
12/2	71	0.9725	0.9725	0.1118	0.1118
12/3	71	0.9762	0.9762	0.1004	0.1004
12/4	71	0.9743	0.9743	0.1023	0.1023
12/5	71	0.9726	0.9726	0.1062	0.1062
12/6	71	0.9758	0.9758	0.0945	0.0945
12/7	71	0.9741	0.9741	0.1003	0.1003
12/8	70	0.9758	0.9758	0.0994	0.0994
12/9	70	0.9737	0.9737	0.1077	0.1077
12/10	70	0.9655	0.9655	0.1236	0.1236
12/11	70	0.9657	0.9657	0.1253	0.1253
12/12	70	0.9724	0.9724	0.1126	0.1126

6.5.2 The overall mean for the monthly values is ln 5.80%; for the different yearly series the mean ranges from ln 5.75% (12/2, January) to ln 5.87% (12/7, June). An AR(1) model for all the series fits reasonably well, as can be seen from the fact that the annualised values $r_1^{12/m}$ are similar for all values of m, ranging from 0.9655 (12/10) to 0.9770 (6/2). The kurtosis coefficients b_2 of the residuals, and hence the Jarque-Bera statistics, are generally very high.

6.5.3 It is also of interest to analyse the estimated real yield, calculated by deducting an exponentially weighted moving average of past inflation from the value of $C(t)$ to get $CR(t)$, before taking logarithms. For this purpose I have used a smoothing formula for each month t:

$$CM(\text{month } t) = CD.(\ln Q(t) - \ln Q(t-12)) + (1 - CD).CM(t-1)$$

where $CD = 0.00383 = 0.045^{1/12}$ and the smoothing is applied to the annual force of inflation for the preceding twelve months. This gives smoothed values similar to those in the annual model with CD = 0.045. The resulting m/h series for ln $CR(t)$ have then been analysed, and the results are shown in Table 6.6 also for m = 1, 2 and 12.

Table 6.6. Statistics and autocorrelation coefficients for various m/h series for ln CR, December 1923 to June 1994

m/h	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	847	0.9868	0.8523	0.0843	0.2721
2/1	424	0.9696	0.8307	0.1284	0.2919
2/2	423	0.9698	0.8319	0.1261	0.2869
12/1 (Dec)	71	0.7364	0.7364	0.3381	0.3381
12/2	71	0.7695	0.7695	0.3031	0.3031
12/3	71	0.8796	0.8796	0.2148	0.2148
12/4	71	0.8922	0.8922	0.2151	0.2151
12/5	71	0.8959	0.8959	0.2204	0.2204
12/6	71	0.9223	0.9223	0.1936	0.1936
12/7	71	0.9156	0.9156	0.2018	0.2018
12/8	70	0.9064	0.9064	0.2016	0.2016
12/9	70	0.8663	0.8663	0.2382	0.2382
12/10	70	0.8349	0.8349	0.2644	0.2644
12/11	70	0.8468	0.8468	0.2656	0.2656
12/12	70	0.8510	0.8510	0.2542	0.2542

6.5.4 The mean of the monthly data is ln 2.98%, and the means of the yearly series range from ln 2.91% (12/2, January) to ln 3.05% (12/8, July). For the monthly series an AR model with p in the range 3 to 5 might be suitable. I have not tried to fit one. There are no spikes in the autocorrelation function at annual frequencies, i.e. 12, 24, etc., so there is no annual seasonality. For all the less frequent series, from m = 2 onwards, an AR(1) model could well be suitable, though, for certain of the series for m = 6 or of the yearly series for m = 12, it looks as if an AR(2) model might be worth investigating. There is quite a wide range in the values of $r_1^{12/m}$, from 0.7364 (12/1, December) to 0.9223 (12/6, May). The Jarque-Bera statistic is high for all the series except 12/4, 12/5, 12/6, 12/7 and 12/8, which include the June series. The results are broadly consistent with the fuller AR(1) model developed in Section 6.3, but there is clearly uncertainty about some of the parameter values.

6.6 Data for Selected Other Countries for Other Periods

6.6.1 Results from fitting the same model to the data for the consols yield, or other indicator of long-term interest rates, for the same countries as discussed previously, are shown in Table 6.7. The median value of *C* is given by *CMU* + 100 *CW.QMU.*

Table 6.7. Fitted parameters for long-term bond model for selected countries

Period	U.K. 1923-94	U.S.A. 1926-89	France 1951-89	Canada 1923-93	Sweden 1923-93	Finland 1950-93
CW	1.0	1.0	1.0	1.0	1.0	1.0
CD	0.045	0.058	0.2	0.04	0.018	0.05
CA	0.90	0.96	0.90	0.95	0.98	0.24
CMU%	3.05%	2.65%	2.5%	3.7%	3.35%	4.0%
CY	0.34	0.07	1.0	0.1	0.25	0.1
CSD	0.185	0.21	0.3	0.185	0.15	0.33
Med[*C*%]	7.75%	5.65%	8.5%	7.1%	7.95%	10.0%

6.6.2 The estimated values of the parameters are reasonably similar for each country, except Finland, though the value of *CSD* for France is noticeably high. The *CA* parameter for Finland is low, at 0.24, and the value of *CSD* is also high. An alternative way of modelling the Finnish series is to omit the connection with inflation entirely, and put: *CMU* = 9.5%, *CA* = 0.15, *CY* = 0, and *CSD* = 0.15. It is as if those that influenced interest rates in Finland paid no attention to the 'Fisher effect', and nominal interest rates fluctuated almost independently from year to year around a mean of 9.5%. However, the series for Finland starts much later than those for the other countries.

6.6.3 As explained in ¶2.6.4 I have not calculated the simultaneous crosscorrelations of the residuals for the series all these countries, so I quote only the simultaneous correlation coefficients for the U.K., U.S.A. and France from my Montréal paper. These are not calculated over identical periods. They are:

U.K. *v* U.S.A.: 0.29; U.K. *v* France: 0.32; U.S.A. *v* France: 0.40.

These are quite similar to the crosscorrelations for share dividend yields (see ¶4.7.3).

6.6.4 I do not have available records of interest rates for a large number of other countries. A problem in some countries is that the terms of the longest bonds in the market are quite short compared with those in the U.K. Short-term interest rates of some kind should, however, be obtainable.

6.7 *ARCH Models*

6.7.1 I have investigated possible ARCH effects for the U.K. model, as for the series for inflation, share dividend yields and share dividends. I define three new terms: *CN*-squared, the squares of the values of *CN*(*t*); *CH*-squared, the square of the expected values of *CN*(*t*), and *CE*-squared.

6.7.2 *CE*-squared shows no significant autocorrelation. It also shows no significant crosscorrelation with lagged values of *CN*-squared, or with simultaneous or lagged values of *CH*-squared; nor are there any particularly large crosscorrelations with any of the other squared series, the largest being between *CE*-squared and *YH*(+1)-squared, 0.31 and between *CE*-squared and *YE*-squared, 0.24. I have not developed an ARCH model for consols yields. There is no immediate evidence for it, and the residuals of the model shown in Table 6.3 are not particularly non-normal.

6.8 *Forecasting*

6.8.1 In Figure 6.3, I show a set of ten simulations of *C*(*t*) at annual intervals from June 1994 to 2050, along with the past record since 1950, on a linear scale. One can see that these are less erratic from year to year than the dividend yields shown in Figure 4.5, but like them they do not fan out into an expanding funnel of doubt.

6.8.2 In Figure 6.4, I show the forecast median of *C*(*t*), starting with the conditions in June 1994, also on a linear scale, along with confidence intervals for the mean plus and minus two standard deviations, using the formulae in Appendix E.5, and assuming that the consols yield is normally distributed. One can see that the confidence interval soon becomes constant; the funnel of doubt does not go on expanding.

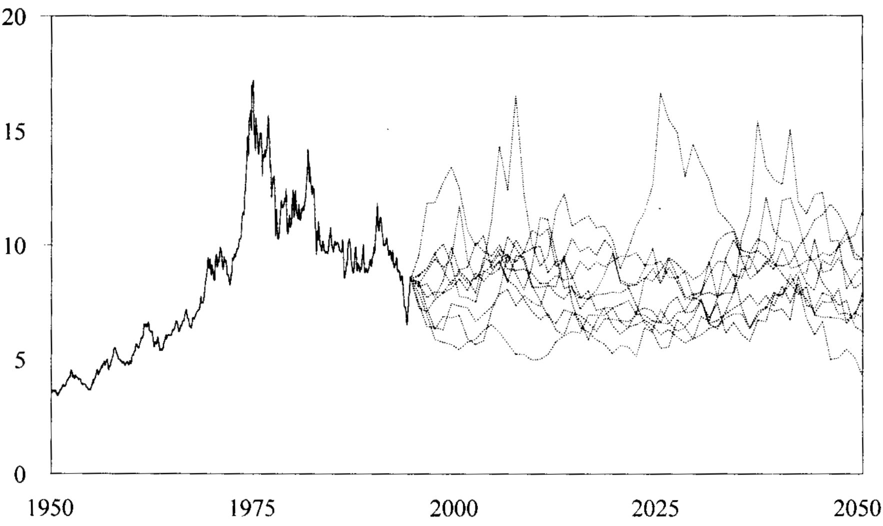

Figure 6.3. Consols yield, 1950-94, and simulations, 1994-2050

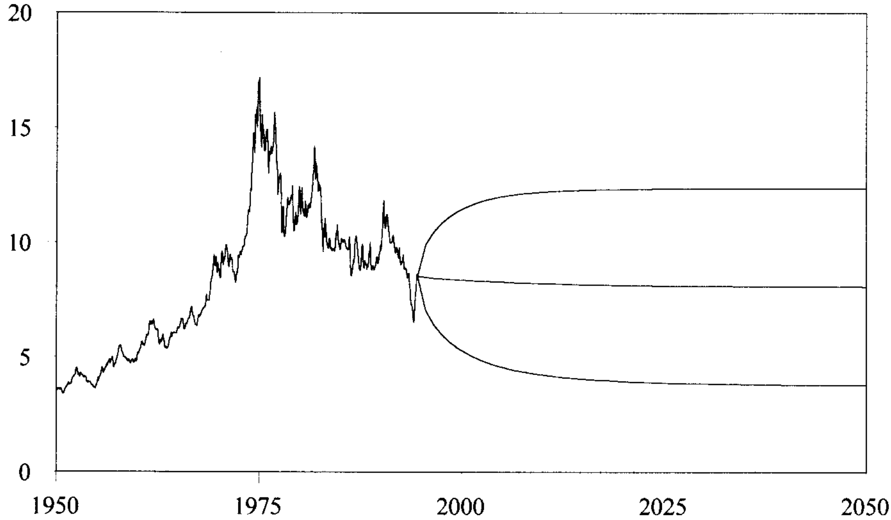

Figure 6.4. Consols yield, 1950-94, and forecast medians and confidence intervals, 1994-2050

7. Short-Term Interest Rates

7.1 *A Possible Model*

7.1.1 I have gathered a series of short-term interest rates drawn successively from Bank Rate, Minimum Lending Rate and bank base rates, in effect daily from 1 January 1797. It is practicable to obtain daily records because these rates have remained fixed for periods of varying length, and the dates of change have been recorded. Such a measure of short-term interest rates does not give a good picture of daily fluctuations in current interest rates, as would a series of LIBOR, local authority seven-day money, Treasury Bill rates or CD rates, but none of these indicators are readily available for long periods into the past. I have created a monthly series by picking the value of the Bank Rate series at the end of each month, and a yearly series likewise. The values are plotted in Figure 6.1, along with the yields on consols.

7.1.2 The first consideration is what sort of model to choose. Short-term interest rates are clearly connected with long-term ones, as shown in Figure 6.1. One approach would be to model the spread:

$$C(t) - B(t)$$

where $B(t)$ is the value of Bank Rate at time t; another would be to model the difference between the logarithms:

$$\ln C(t) - \ln B(t) = -\ln (B(t)/C(t))$$

i.e. the logarithm of the ratio of the rates; yet others would be to use the estimated real rate $CR(t)$ in either of these formulae. I have chosen the middle one of these, on the grounds that a constant proportionate spread is more plausible than a constant simple spread over a time when long-term rates have varied from 2½% to 17%. A spread of 3% is not implausible when the overall level of rates is high, but unlikely when the overall level is low. The spread is plotted in Figure 7.1 and the log ratio in Figure 7.2.

7.1.3 Inspection of the data shows that the obvious starting model is an AR(1) model for the log ratio. We define the short-term rate of interest at time t as $B(t)$, and put:

$$B(t) = C(t).\exp\{-BD(t)\}$$

where:

$$BD(t) = BMU + BA.(BD(t-1) - BMU) + BE(t)$$

$$BE(t) = BSD.BZ(t)$$

$$BZ(t) \sim \text{iid } N(0,1)$$

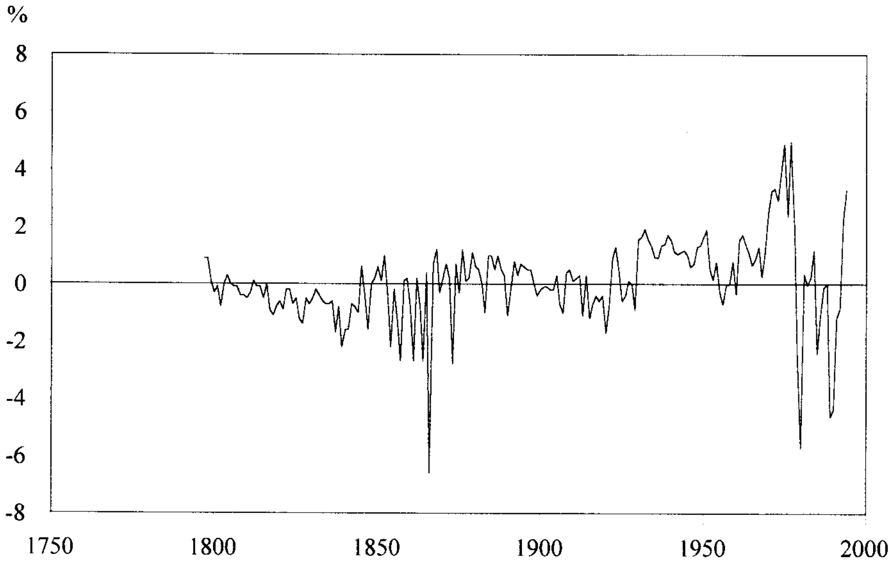

Figure 7.1. Spread $C(t) - B(t)$, yearly, 1797-1994

or

$$BD(t) \sim \text{AR1}(BMU, BA, BSD).$$

We can equally put:

$$\ln B(t) = \ln C(t) - BMU - BN(t)$$

with:

$$BN(t) \sim \text{AR1}(0, BA, BSD).$$

Note that BD has a minus sign in front of it, because short-term yields are, on average, lower than long-term ones.

7.1.4 A possible elaboration is to put:

$$BN(t) = BA.BN(t-1) + BC.CE(t) + BE(t)$$

where $BC.CE(t)$ allows an extra influence from the residuals of the consols yield. It turns out that this can be taken as zero in the U.K., but it is useful for other countries.

7.1.5 Parameter estimates for these two models are shown in Table 7.1 for 1923-94, along with estimates for model (i) for 1797-1994.

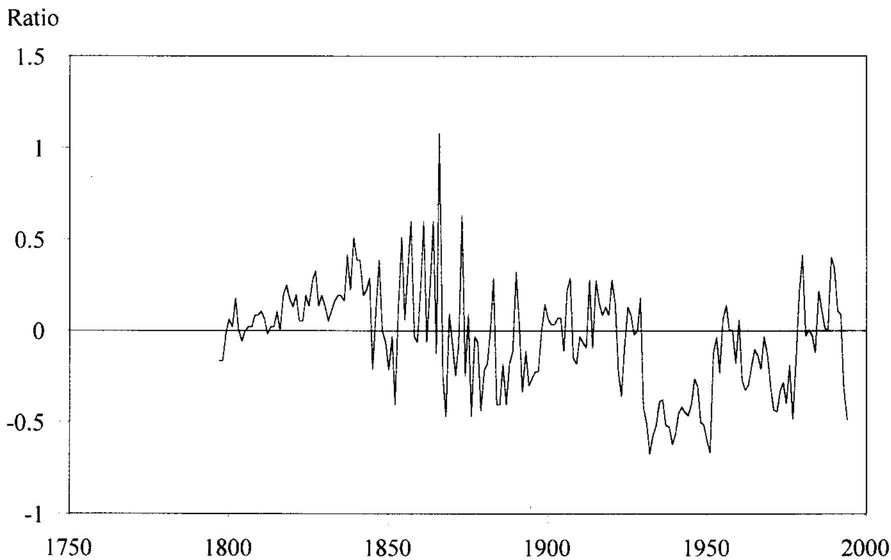

Figure 7.2. Log ratio ln $(B(t)/C(t))$, yearly, 1797-1994

Table 7.1. Parameter estimates for model for ln (*B*/*C*), 1923-94 and 1797-1994

Model	1923-94 (i) $BC = 0$	1923-94 (ii) including BC	1797-1994 (i) $BC = 0$
BMU	0.2273	0.2176	0.0419
	(0.0797)	(0.0809)	(0.0370)
BA	0.7420	0.7434	0.5391
	(0.0823)	(0.0827)	(0.0606)
BC	-	−0.0257	-
		(0.1155)	
BSD	0.1808	0.1808	0.2397
	(0.0151)	(0.0151)	(0.0121)
Log likelihood	+0.0	+0.02	-
Jarque-Bera χ^2_2	1.57	1.59	59.56
$p(\chi^2)$	0.45	0.45	0.0000

7.1.6 For 1923-94 the improvement given by including *BC* is trivial. The residuals for model (i) are satisfactory, showing no significant autocorrelations, and no crosscorrelations with any of the previously derived sets of residuals. The Jarque-Bera statistic is not high, although individual large values of the residuals were observed in 1930 (−2.74 times the standard deviation), 1952 (2.35 times), 1979 (2.23 times), 1985 (2.00 times) and 1988 (2.53 times).

7.1.7 Suitably rounded values of the parameters are:

$$BMU = 0.23;\ BA = 0.74;\ BSD = 0.18.$$

These are not identical with those quoted in Appendix D of my paper on 'The Risk Premium on Ordinary Shares' (Wilkie, 1995), which were based on an earlier investigation ending in 1992.

7.2 *Previous Centuries*

7.2.1 I also fitted the simple AR(1) model (i) to the whole period for which data were available, from 1797-1994. The results are also shown in Table 7.1.

7.2.2 The model fits plausibly, though with different parameter values, *BMU* and *BA* being much reduced, and *BSD* increased. There is significant autocorrelation of the residuals at lag 3, and also some lagged crosscorrelations with the residuals of the price and the wage series fitted over the same period. More investigation could be done here.

7.3 *Monthly Data*

7.3.1 From the monthly data for December 1923 to June 1994, I also constructed various monthly series in the same way as previously, and I record the same statistics for various *m*/*h* series, for $m = 1$, 2 and 12, first for the values of ln $B(t)$, results for which are shown in Table 7.2, and then for the log ratio $\ln(B(t)/C(t))$, results for which are shown in Table 7.3.

7.3.2 For all the series in both tables an AR(1) is a reasonable fit. The values of the yearly parameters are similar for all the series in each table, and similar in Table 7.3 to those found in Section 7.1. The Jarque-Bera statistic is very high for all the series for the log ratio, and for all except the yearly series of ln $B(t)$.

Table 7.2. Statistics and autocorrelation coefficients for various m/h series for ln B, December 1923 to June 1994

m/h	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	847	0.9938	0.9285	0.0747	0.2502
2/1	424	0.9855	0.9161	0.1144	0.2704
2/2	423	0.9877	0.9282	0.1057	0.2511
12/1 (Dec)	71	0.9109	0.9109	0.2844	0.2844
12/2	71	0.9086	0.9086	0.2876	0.2876
12/3	71	0.9188	0.9188	0.2706	0.2706
12/4	71	0.9312	0.9312	0.2414	0.2414
12/5	71	0.9393	0.9393	0.2244	0.2244
12/6	71	0.9382	0.9382	0.2292	0.2292
12/7	71	0.9470	0.9470	0.2155	0.2155
12/8	70	0.9331	0.9331	0.2445	0.2445
12/9	70	0.9156	0.9156	0.2698	0.2698
12/10	70	0.8971	0.8971	0.3005	0.3005
12/11	70	0.8974	0.8974	0.3047	0.3047
12/12	70	0.9087	0.9087	0.2902	0.2902

7.4 *Data for Selected Other Countries for Other Periods*

Results from fitting the same model to the data for a short-term yield, for the U.K., Canada and Sweden, are shown in Table 7.4. For Canada the value of BC was strongly significant, so it is included in the model. The median value of B is given by Med[C].exp($-BMU$), and this too is shown in the table.

7.5 *Other Models*

7.5.1 Ong (1994) extends my model by fitting a number of transfer function models to a series of Treasury Bill yields, using annual data from 1955 to 1993. He models the spread, $(C-B)$, rather than the log ratio, and he finds that an AR(1) model with zero mean, autoregressive parameter 0.43, and standard deviation 0.02, i.e. 2% in the interest rate, fits the data satisfactorily.

Table 7.3. Statistics and autocorrelation coefficients for various *m/h* series for ln (*B/C*), December 1923 to June 1994

m/h	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	847	0.9634	0.6394	0.0715	0.2051
2/1	424	0.9203	0.6075	0.1046	0.2123
2/2	423	0.9330	0.6597	0.0959	0.2002
12/1 (Dec)	71	0.6230	0.6230	0.2084	0.2084
12/2	71	0.6227	0.6227	0.2170	0.2170
12/3	71	0.6599	0.6599	0.2113	0.2113
12/4	71	0.7056	0.7056	0.1897	0.1897
12/5	71	0.7130	0.7130	0.1836	0.1836
12/6	71	0.6930	0.6930	0.1904	0.1904
12/7	71	0.7398	0.7398	0.1791	0.1791
12/8	70	0.6695	0.6695	0.1983	0.1983
12/9	70	0.6010	0.6010	0.2072	0.2072
12/10	70	0.6108	0.6108	0.2159	0.2159
12/11	70	0.5983	0.5983	0.2258	0.2258
12/12	70	0.6187	0.6187	0.2127	0.2127

Table 7.4. Fitted parameters for short-term bond model for selected countries

	U.K.	Canada	Sweden
Period	1923-94	1923-93	1923-93
BMU	0.23	0.26	0.10
BA	0.74	0.38	0.48
BC	0.0	0.73	0.0
BSD	0.18	0.21	0.13
Med[*B*%]	6.73%	5.47%	7.19%

7.5.2 A complete model for interest rates requires a yield curve to join the short-term and long-term rates. There are different functions that can be modelled for a yield curve, implied forward rates, zero-coupon rates, redemption yields or par yields. A parametric form of yield curve could be chosen and the stochastic behaviour of the parameters investigated. Unfortunately, it is difficult to recreate yield curves for the past; the data for stock prices would be available, but the work involved in extracting them and fitting a suitable functional form would be considerable. In addition, one might want to construct a 'yield curve' for expected inflation, which is not necessarily the same for all terms.

7.5.3 Tilley (1990) and Tilley & Mueller (1991) have investigated nominal redemption yields in the U.S.A., at four-week intervals, from December 1981 to August 1989, and find that a four-parameter cubic formula is sufficient to describe a yield curve. They investigate the stochastic properties of their parameters. However, they have not taken account of inflationary expectations, and the period they cover is relatively short.

7.5.4 Many alternative yield curve models have been proposed in the academic literature. There is a useful review in Sharp (1988) and further references in Ingersoll (1987), Pedersen, Shiu & Thorlacius (1989) and Duffie (1992). Unfortunately, to my mind, they are usually based on an assumption about how yield curves ought to behave rather than being based on how they actually do behave. Some do take inflationary expectations into account, but none satisfactorily combines nominal yields, real yields on index-linked stock (which are available in quantity only in the U.K.) and inflationary expectations, all in a single model.

7.6 *Forecasting*

In Figure 7.5, I show a set of ten simulations of $B(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1950, all on a linear scale. One can see how erratic these are from year to year, but they do not fan out into an expanding funnel of doubt. I do not show results of the theoretical calculations of the means and confidence intervals.

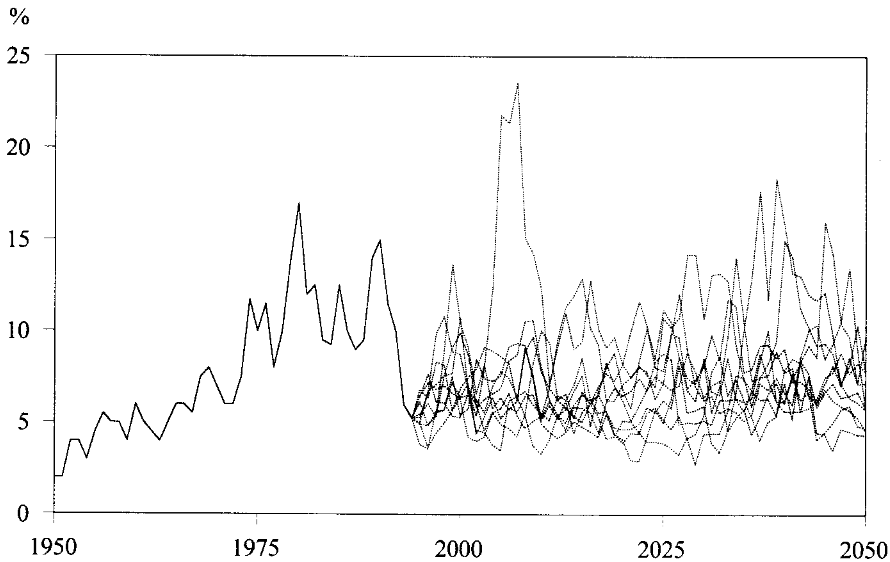

Figure 7.3. Base rate, 1950-94, and simulations, 1994-2050

8. PROPERTY

8.1 *Problems of a Property Index*

8.1.1 It is likely to be useful for institutional investors to have a model for property investment (real estate). However, there are well-known problems about constructing any index to represent the price and income performance of a property portfolio. No two properties are identical, and none are traded in fractional parts, as companies are through their shares. Actual transactions in property are infrequent, unlike the shares of quoted companies, and one has to rely substantially on valuers' estimates of capital values. There is no easily definable universe of properties, whereas for shares one can define a universe as all shares traded on a particular exchange, or as a readily definable subset of such shares.

8.1.2 There are two different rental values associated with each property: one is the 'rack rental', the estimated rental that might be obtained if the property were being newly let on modern lease conditions, and the other is the actual income that is receivable, which depends on the terms of the current lease. The usual conditions on which leases have been granted have changed over the years. Prior to the Second World War leases were often for 99 years on fixed terms, making property investment akin to fixed-interest investment, with a rather far distant reversion. Many such leases are still in force, with another 40 or so years to run. After the Second World War the customary conditions changed, first to 21-year fixed-term leases, then to leases with rents revisable upwards only, typically every five years or so. Currently there is a resistance to upwards only rent reviews, and it is possible that a new style of lease may come into general use.

8.1.3 The estimated rental value and the current income bear some analogies to the earnings and dividends of companies, though the analogy cannot be pushed too far. They lead to two different calculations of yield, which in principle coincide for a new lease, but may diverge thereafter.

8.1.4 A further problem for the sort of investigation I have tried to carry out is that almost no property indices go back very far. I have used the Jones Lang Wootton Indices, mainly because they go back to 1967, and give both the price index and a yield index based on actual income, even though the indices for the early years were calculated in arrears and not contemporaneously. They are available yearly for June 1967 to June 1977 and quarterly thereafter. I have used the yearly June values, and therefore have a series from 1967 to 1994, with 28 values. The property price index and 20 times the property income index are plotted in Figure 8.1. They track closely together from about 1971 to 1984, but diverge before and after this period. The property yield is plotted in Figure 8.2.

8.2 *A Model for Property Yields*

8.2.1 It is reasonable to compare a model for property with a model for shares. Income is analogous to dividends, and prices and yields perform the

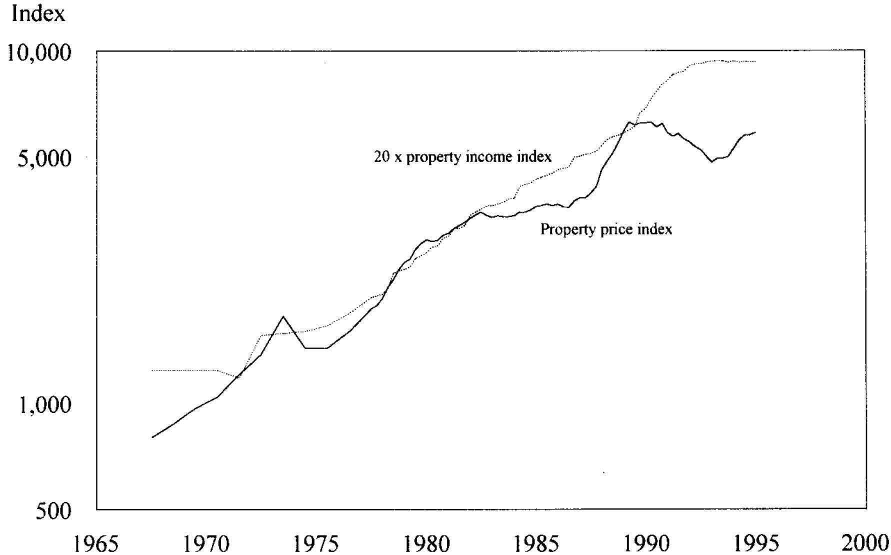

Figure 8.1. Values of property price index and 20 × property income index, yearly, 1967-77, and quarterly, 1977-94

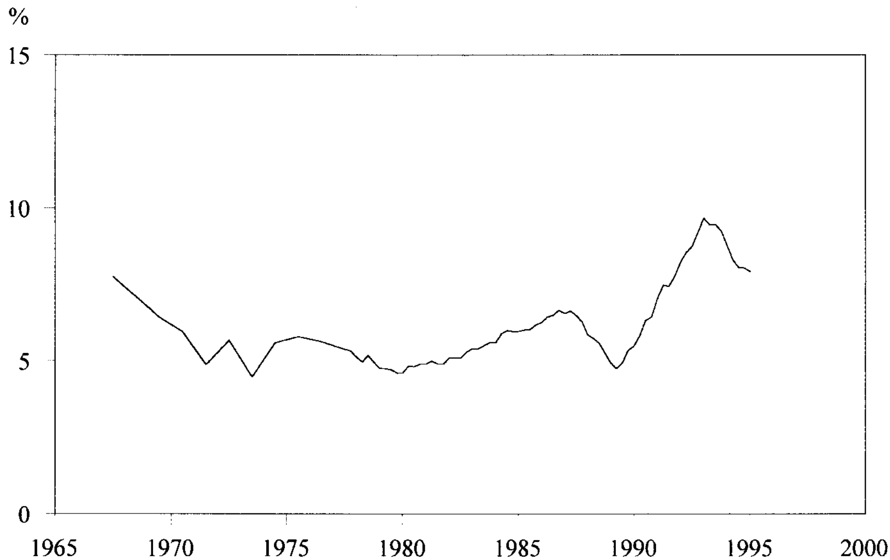

Figure 8.2. Values of property yield, yearly, 1967-77, and quarterly, 1977-94

same function in each case. I therefore start by postulating a model for property income which is similar to that for dividends, and a model for property yields which is similar to that for share dividend yields.

8.2.2 I define the property income, assumed received at time t, as $E(t)$, and the property yield at time t as $Z(t)$. The model for the property yield, $Z(t)$, is:

$$Z(t) = \exp\{ZW.I(t) + ZN(t)\}$$

or

$$\ln Z(t) = ZW.I(t) + ZN(t)$$

with:

$$ZN(t) \sim \text{AR1}(\ln ZMU, ZA, ZSD).$$

It turns out that the ZW term is not necessary, leaving the simple AR(1) model for $\ln Z$:

$$\ln Z(t) \sim \text{AR1}(\ln ZMU, ZA, ZSD).$$

8.2.3 Parameter values that I had suggested informally and which were quoted in Daykin & Hey (1990) were:

$$ZA = 0.6;\ ZMU = 5.0\%;\ ZSD = 0.075$$

but this was based on hunch and not on any data.

8.2.4 Parameter estimates for this model are shown in Table 8.1.

Table 8.1. Parameter estimates for model for ln Z, 1967-94

	Parameter estimate	Standard error
ZMU %	7.41 %	1.03 %
ZA	0.9115	0.1007
ZSD	0.1177	0.0157

The mean looks high; but this correctly reflects the observed experience. Suitable rounded values are:

$$ZMU = 7.4\%;\ ZA = 0.91;\ ZSD = 0.12.$$

8.2.5 Diagnostic tests of the residuals show no remaining autocorrelation, and no significant crosscorrelations. It would not be surprising to find that property yields moved in the same direction as share yields, i.e. their residuals were correlated, and also in the same direction as fixed-interest yields. The correlation coefficient between ZE and YE is 0.29, not significant, but at least interesting. The simultaneous correlation coefficients between ZE and CE and between ZE and BE are small and negative, but there are larger lagged crosscorrelation coefficients, between $ZE(t)$ and $CE(t+1)$, -0.31, between ZE and $BE(t+1)$, -0.26, and between $ZE(t)$ and $BE(t-1)$, $+0.28$. I can see no economic rationale for these, and they are not statistically significant at a 5% level, so I have not investigated them further.

8.2.6 There is only one residual greater than two standard deviations away from zero, that for 1973 (-2.22). The skewness is near zero ($\sqrt{b_1} = 0.18$), and the kurtosis coefficient b_2, is less than 3, at 2.40. The Jarque-Bera statistic is 0.42 ($p = 0.81$).

8.2.7 The series for property yields is a short one, with only 28 yearly values, but there is no reason, so far, to assume anything other than an AR(1) model, with normal residuals.

8.3 *A Model for Property Income*

8.3.1 A possible model for property income, $E(t)$, similar to that for share dividends is:

$$E(t) = E(t-1).\exp\{EW.EM(t) + EX.I(t) + EMU + EE(t)\}$$

where:

$$EM(t) = ED.I(t) + (1-ED).EM(t-1)$$

$$EE(t) = ESD.EZ(t)$$

$$EZ(t) \sim \text{iid } N(0,1).$$

As for share dividends, it is convenient to denote the annual change in the logarithm as:

$$EK(t) = \ln E(t) - \ln E(t-1)$$

and to identify the effect of inflation as:

$$EI(t) = EW.EM(t) + EX.I(t)$$

so that:

$$EK(t) = EI(t) + EMU + EE(t).$$

8.3.2 Possibly other terms to represent crosscorrelations need to be added. I find a simultaneous correlation between EE and ZE, the residuals for property yield, so I add a term to make the final model:

$$E(t) = E(t-1).\exp\{EW.EM(t)+ EX.I(t) + EMU + EE(t)\}$$

I constrained EX to equal $1 - EW$, so that a change in $\ln Q$ ultimately produced the same change in $\ln E$; the transfer function had 'unit gain'.

8.3.3 The parameters I suggested to Daykin & Hey informally were:

$$EW = 1.0;\ ED = 0.1;\ EMU = -0.01;\ ESD = 0.05$$

but, like the yield model, these were not based on any analysis of data.

8.3.4 Parameter estimates for this model are shown in Table 8.2. I find that, if EW is unconstrained, the optimum value is greater than unity, making EX negative, so I constrain EW to equal 1.0; this means a loss in log likelihood of only 0.30. Of the two models shown, (i) excludes EBZ and (ii) includes EBZ.

Table 8.2. Parameter estimates for model for ln E, 1968-94

Model	(i) $EBZ = 0$	(ii) EBZ included
EW	1.0	1.0
ED	0.1121	0.1289
	(0.0663)	(0.0689)
EMU	0.0006	0.0032
	(0.0152)	(0.0132)
EBZ	-	0.2363
		(0.0974)
ESD	0.0661	0.0599
	(0.0090)	(0.0082)
Log likelihood	+0.0	+2.67
Jarque-Bera χ^2_2	35.70	16.56
$p(\chi^2)$	0.0000	0.0003

8.3.5 Diagnostic tests of the residuals for both models show no remaining autocorrelation. There were large values of the residuals in 1972 (3.51 times the standard deviation for model (i) and 3.36 times for model (ii)) and, for model (i), in 1990 (2.38 times).

8.3.6 Possible rounded values of the parameters are:

$$EW = 1.0;\ ED = 0.13;\ EMU = 0.003;\ EBZ = 0.24;\ ESD = 0.06.$$

8.3.7 Following from the models for property yield and property income, we have a model for a property price index, denoted by $A(t)$:

$$A(t) = E(t)/Z(t)$$

or

$$\ln A(t) = \ln E(t) - \ln Z(t).$$

8.4 *Forecasting*

8.4.1 In Figure 8.3, I show a set of ten simulations of $Z(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1967, all on a logarithmic scale. These are again a typical stationary series.

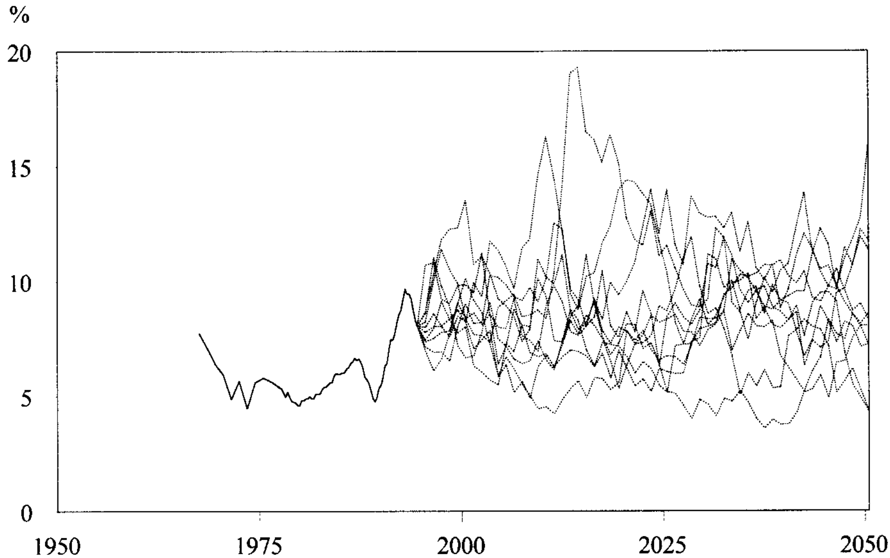

Figure 8.3. Property yield, 1967-94, and simulations, 1994-2050

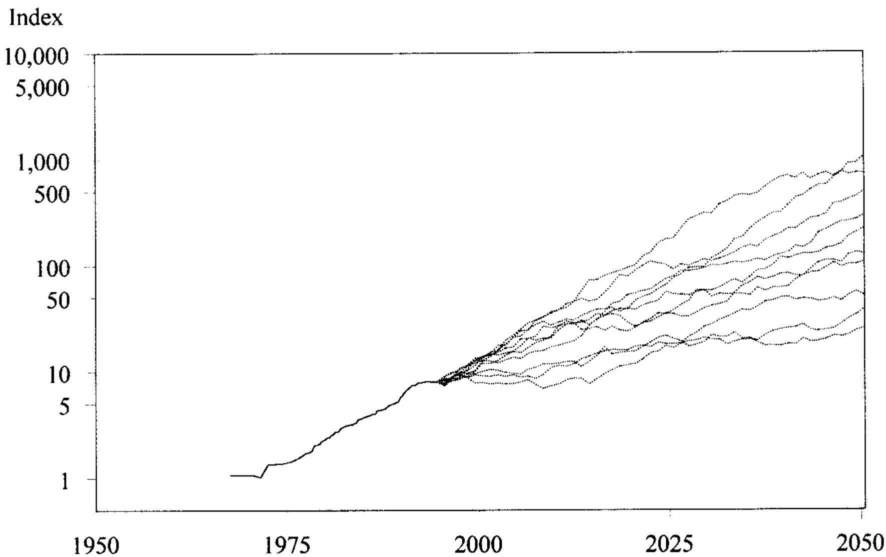

Figure 8.4. Property income index, 1967-94, and simulations, 1994-2050

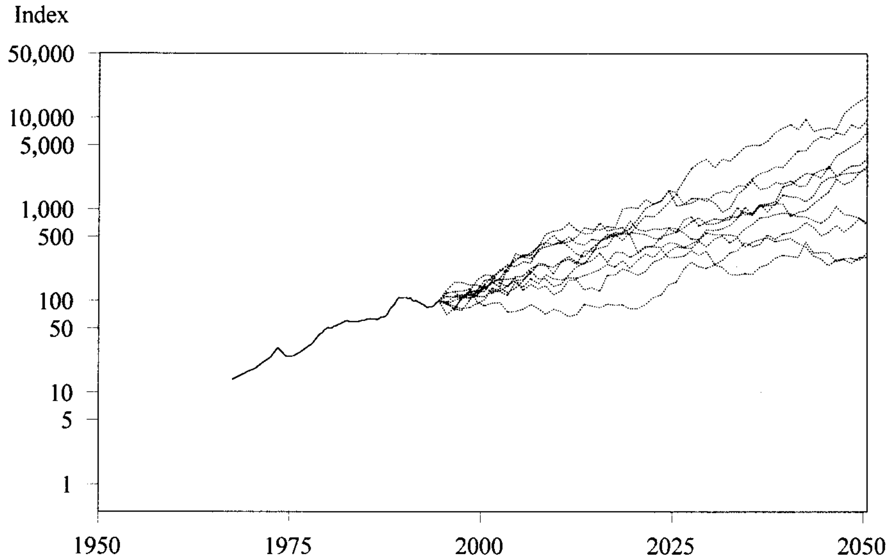

Figure 8.5. Property price index, 1967-94, and simulations, 1994-2050

8.4.2 In Figure 8.4, I show a set of ten simulations of $E(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1967, all on a logarithmic scale. These fan out in the way one can now expect for an integrated series. I have not calculated the theoretical means and confidence intervals.

8.4.3 In Figure 8.5, I show a set of ten simulations of $A(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1967, all on a logarithmic scale.

9. INDEX-LINKED STOCKS

9.1 *Index-Linked Government Stocks*

9.1.1 Index-linked government stocks have been issued and on the market since 1981. When I first developed a stochastic asset model they were too recent for any history to be useable, but more than 13 years of data are now available, and one can begin to get some idea of their stochastic behaviour.

9.1.2 However, there is a comparison available. Before inflation became recognised as endemic, fixed-interest government stocks were treated as 'gilt-edged', and it was, I believe, assumed by investors that they would provide a yield that was in effect 'real', as well as nominal. It is therefore reasonable to look back at previous centuries to see what a suitable model might be. I have done this for the consols model, using the data from 1756 to 1956, and the results are shown in Section 6.4.

9.1.3 I have used the values of the FTA index-linked real yields, assuming 5% inflation, and from 1986 using stocks with terms '5 and over', and I have taken the monthly values from May 1981 to October 1994, so as to give the maximum number of months at the time of writing. For the immediate investigations I have used the June values, from 1981 to 1994 inclusive, giving 14 values of the yield. The monthly values of the yields are plotted in Figure 6.1 and again in Figure 9.1 with a more convenient scale.

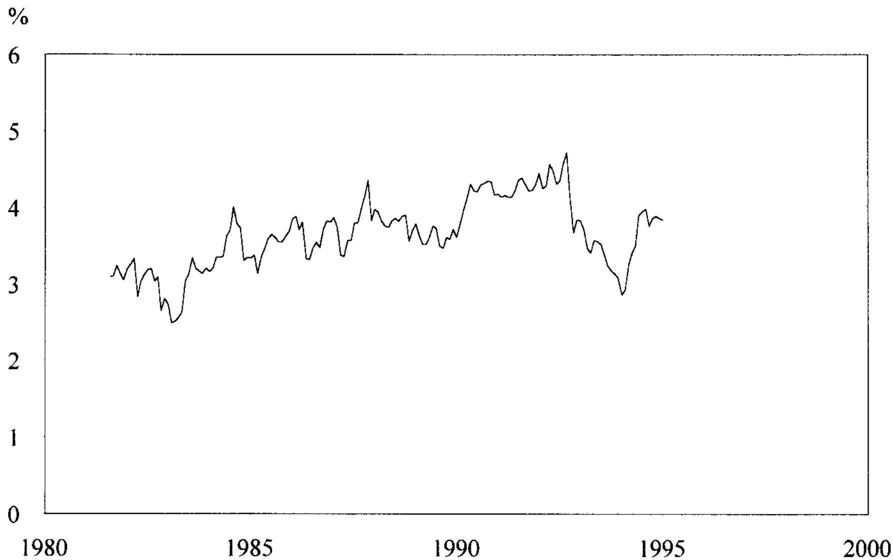

Figure 9.1. Real yield on index-linked stocks, monthly, 1981-94

9.2 *A Possible Model for Index-Linked Yields*

9.2.1 A first possible model for the real yield on index-linked stocks at time t, denoted $R(t)$, is an AR(1) model:

$$\ln R(t) = \ln RMU + RA.(\ln R(t-1) - \ln RMU) + RE(t)$$

$$RE(t) = RSD.RZ(t)$$

$$RZ(t) \sim \text{iid } N(0,1)$$

or

$$\ln R(t) \sim AR1(\ln RMU, RA, RSD).$$

9.2.2 Investigations with this model show, however, that the residuals are correlated both with the residuals from the consols yield model and with the residuals from the property yield model, so a fuller model is:

$$\ln R(t) = \ln RMU + RA.(\ln R(t-1) - \ln RMU) + RBC.CE(t) + RBZ.ZE(t) + RE(t).$$

Parameter estimates for models on these lines, (i) excluding both *RBC* and *RBZ*, (ii) including *RBC*, and (iii) including *RBZ*, are shown in Table 9.1. It would be plausible for there to be correlation also with share dividend yields, but the observed correlation coefficient is small.

Table 9.1. Parameter estimates for model for ln *R*, 1981-94

Model	(i) *RBC* = *RBZ* = 0	(ii) including *RBC*	(iii) including *RBC* and *RBZ*
RMU %	3.86 %	4.03 %	3.96 %
	(0.17)	(0.17)	(0.14)
RA	0.4936	0.5686	0.5326
	(0.1609)	(0.1076)	(0.0953)
RBC	-	0.2234	0.2419
		(0.0598)	(0.0548)
RBZ	-	-	0.2110
			(0.1160)
RSD	0.0731	0.0518	0.0468
	(0.0143)	(0.0102)	(0.0092)
Log likelihood	+0.0	+4.81	+6.23
Jarque-Bera χ^2_2	0.94	0.28	2.39
$p(\chi^2)$	0.62	0.86	0.30

9.2.3 The improvement in log likelihood makes the extra term in *RBC* clearly worth while, and the parameter estimate is well over two standard deviations away from zero. The extra improvement for *RBZ* is more marginal; the log likelihood improves only by 1.42, and the parameter estimate is only 1.8 times its standard error. There is therefore justification for rejecting this extra term. The Jarque-Bera statistics are all low, and there are no large residuals; but with only 13 values of the residual series this is not too surprising.

9.2.4 Possible rounded parameter values are:

$$RMU = 4.0\%, \; RA = 0.55, \; RBC = 0.22, \; RSD = 0.05.$$

9.2.5 The total standard deviation *RSD** is given by:

$$RSD^{*2} = RBC^2.CSD^2 + RSD^2 = 0.0645^2$$

using $CSD = 0.185$.

9.3 *Monthly Data*

9.3.1 Monthly data for the real yield on index-linked stocks are available from May 1981 to October 1994, giving 162 months in all. Table 9.2 shows statistics for the *m*/*h* series for $m = 1$, 2 and 12, constructed by picking observations at intervals of every *m* months starting in month *h*.

9.3.2 The overall mean value of the monthly values of ln *R* is ln 3.60%. The range of means of the yearly series is from ln 3.50% (12/10, February) to ln 3.73% (12/3, July). An AR(1) model fits each series quite well, Although the kurtosis coefficient b_2 of the residuals is highish, the Jarque-Bera statistic is generally satisfactory.

9.3.3 The yearly autocorrelation coefficient, estimating the value of *RA* in the model formula, is very variable, and often lower than in Table 9.1. The June series is 12/2, which shows a relatively high value of 0.4723. The yearly values of the standard deviation, which estimate the value of *RSD*, are generally higher than that shown in Table 9.1, model (i) (0.0731), but the value for the June series is similar, at 0.0759. The differences show the effect of different methods of estimating these parameters.

Table 9.2. Statistics and autocorrelation coefficients for various *m*/*h* series for ln *R*, May 1981 to October 1994

m/*h*	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	162	0.9165	0.3512	0.0476	0.1114
2/1	81	0.8367	0.3431	0.0634	0.1088
2/2	81	0.8290	0.3247	0.0682	0.1154
12/1 (May)	14	0.5268	0.5268	0.0756	0.0756
12/2	14	0.4723	0.4723	0.0759	0.0759
12/3	14	0.2986	0.2986	0.1008	0.1008
12/4	14	0.3173	0.3173	0.1168	0.1168
12/5	14	0.3717	0.3717	0.0997	0.0997
12/6	14	0.4689	0.4689	0.1195	0.1195
12/7	13	0.5992	0.5992	0.0888	0.0888
12/8	13	0.4785	0.4785	0.1212	0.1212
12/9	13	0.4450	0.4450	0.1293	0.1293
12/10	13	0.3520	0.3520	0.1289	0.1289
12/11	13	0.4792	0.4792	0.1195	0.1195
12/12	13	0.5679	0.5679	0.1066	0.1066

9.3.4 Another statistic that can be investigated is the difference between the long-term fixed-interest yield and the real yield on index-linked stocks, what one can describe as the 'implied inflation' as assessed by 'the market'. I have calculated this allowing correctly for the fact that the yields on the FTA BGS Indices are convertible half-yearly, whereas the usual rate of inflation rate is convertible yearly. I have then converted the annual rate to a continuous δ-type rate, to make it comparable with the force of inflation $I(t)$ and the smoothed inflation in the consols model $CI(t)$. I denote the difference as $RI(t)$, defined as:

$$RI(t) = \ln \{(1 + C(t)/2)^2/(1 + R(t)/2)^2\}.$$

9.3.5 Values of $RI(t)$, $CI(t)$ and $I(t)$ are shown in Figure 9.2. One can see that the rather slow smoothing in CI does not reflect the apparently faster response of the market to actual inflation, as shown by RI.

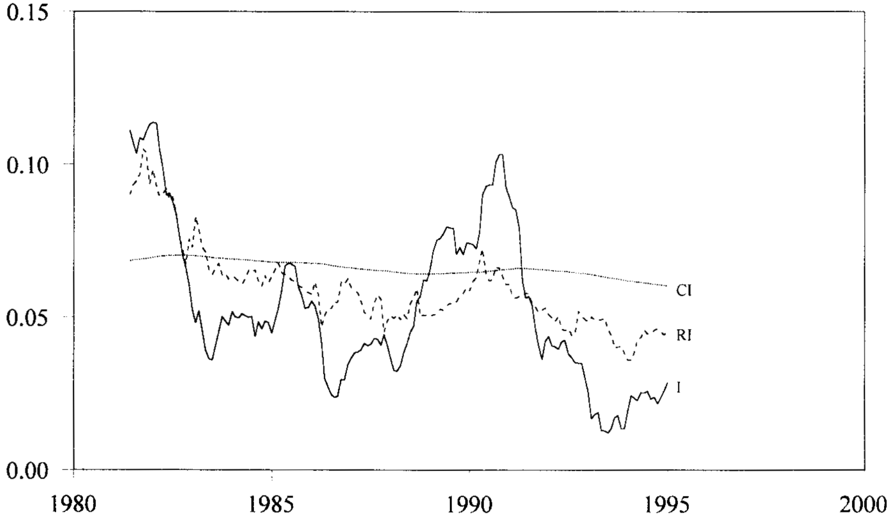

Figure 9.2. Values of $I(t)$, $RI(t)$ and $CI(t)$, monthly, 1981-94

9.3.6 Statistics for the m/h series for RI for m = 1, 2 and 12, are shown in Table 9.3. The overall monthly mean value is 0.0637, and the means of the yearly series range from 0.0623 (12/11, March) to 0.0644 (12/3, July). An AR(1) model fits well, though the Jarque-Bera statistic is high for the monthly series and for many of the series for m = 2, 3 or 4.

9.3.7 It would be interesting to relate the RI series more precisely to past inflation, to obtain a better understanding of how the market adjusts its views of prospective inflation, and perhaps to improve my model for the CI term in my consols yield model; but this has not yet been done.

Table 9.3. Statistics and autocorrelation coefficients for various *m/h* series for implied inflation, May 1981 to October 1994

m/h	Number of values of $I_{m/h}$	r_1	$r_1^{12/m}$	Standard deviation of residuals	Equivalent annual standard deviation
1/1	162	0.9539	0.5678	0.0035	0.0097
2/1	81	0.9090	0.5640	0.0049	0.0097
2/2	81	0.9039	0.5452	0.0048	0.0094
12/1 (May)	14	0.5942	0.5942	0.0075	0.0075
12/2	14	0.5888	0.5888	0.0072	0.0072
12/3	14	0.5992	0.5992	0.0055	0.0055
12/4	14	0.5021	0.5021	0.0062	0.0062
12/5	14	0.3387	0.3387	0.0065	0.0065
12/6	14	0.3045	0.3045	0.0071	0.0071
12/7	13	0.4038	0.4038	0.0068	0.0068
12/8	13	0.3675	0.3675	0.0069	0.0069
12/9	13	0.4828	0.4828	0.0078	0.0078
12/10	13	0.4611	0.4611	0.0073	0.0073
12/11	13	0.3440	0.3440	0.0078	0.0078
12/12	13	0.3439	0.3439	0.0082	0.0082

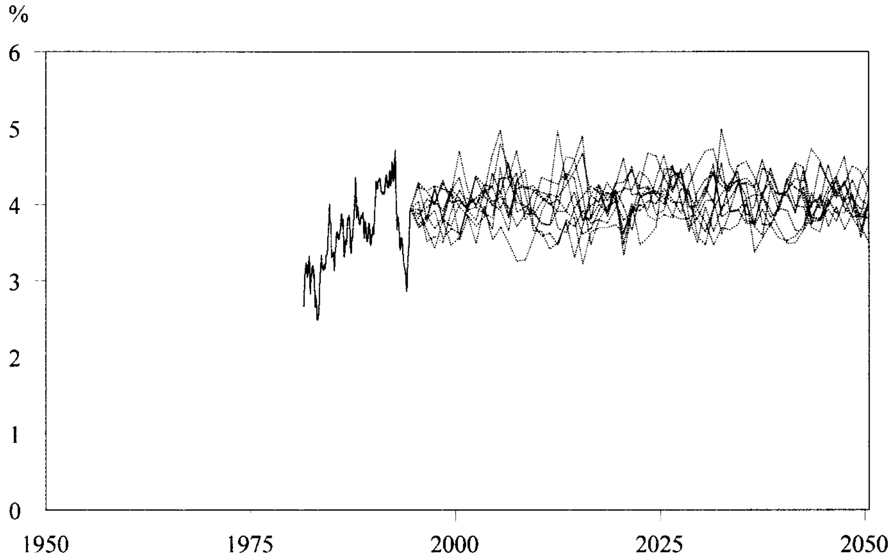

Figure 9.3. Real yield on index-linked stocks, 1981-94, and simulations, 1994-2050

9.4 *Forecasting*

In Figure 9.3, I show a set of ten simulations of $R(t)$ at annual intervals from June 1994 to 2050, along with the past record since 1981, all on a linear scale. These again behave like a typical I(0) series.

10. CURRENCY EXCHANGE RATES

10.1 *Purchasing Power Parity*

10.1.1 In order to link the investment models for a number of different countries, it is necessary to model exchange rates. The model described here is not the only possible way of doing this, but it fits the style of the model for individual countries. I have discussed some aspects of the model before in Wilkie (1992, 1994a).

10.1.2 Denote the exchange rate at time t between countries i and j, expressed as the number of units of currency j for one unit of currency i, as $X_{ij}(t)$. Then $X_{ji}(t) = 1/X_{ij}(t)$. The rate conventionally quoted may be either of these rates.

10.1.3 The purchasing power parity (PPP) approach to exchange rates suggests that the exchange rate between two currencies depends strictly on the relative purchasing powers in the two countries. This could be expressed by putting $X_{ij}(t) = XK.Q_j(t)/Q_i(t)$, where Q_i and Q_j are the consumer price indices of the two countries. It is readily seen that exchange rates do not conform to this pattern, at least not with a constant value of XK. One way of dealing with this is to allow XK to vary with time, denoting it $XK(t)$, and to model $XK(t)$. It can be seen from the data that $XK(t)$ appears to have a roughly constant mean level, but with large deviations away from this level.

10.1.4 It is convenient to change to logarithms and to put:

$$\ln X_{ij}(t) = \ln Q_j(t) - \ln Q_i(t) + \ln XK_{ij}(t)$$

with:

$$\ln XK_{ij}(t) = XMU_{ij} + XN_{ij}(t)$$

where XMU_{ij} is a constant that reflects both the scales of the two currencies and the radices of the two indices used to measure consumer prices, and XN_{ij} has zero mean. XN_{ij} can then be modelled as an AR(1) time series:

$$XN_{ij}(t) \sim \text{AR1}(0, XA_{ij}, XSD_{ij}).$$

10.1.5 This model seems to fit the data reasonably well for many countries. Figures 10.1, 10.2 and 10.3 show graphs for the exchange rates between the U.K. £ and the German mark, Japanese yen and U.S. \$, respectively, showing the actual exchange rate wandering up and down, and the PPP rate cutting through the middle of the wanderings. Provided that one's estimate of the mean is correct, which is reflected in the vertical position of the PPP rate on the graph, one can estimate whether the exchange rate is high or low relative to the PPP rate. However, it may easily be a long time before this apparent anomaly corrects itself, so I would not confidently recommend this method of analysis for short-term trading on the exchange. Nevertheless, it ought to be good in the long term.

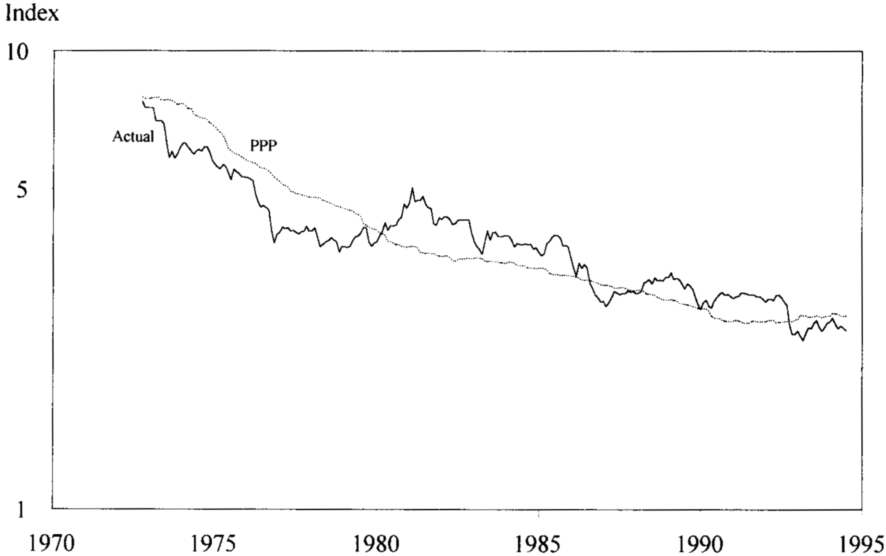

Figure 10.1. Values of actual exchange rate and PPP exchange rate, U.K. £ versus German mark, monthly, 1971-94

10.2 *Cointegration*

10.2.1 The PPP model is equivalent to the suggestion that Q_i, Q_j and X_{ij} are all I(1) series and are cointegrated, with the model:

$$\ln X_{ij}(t) - \ln Q_j(t) + \ln Q_i(t) \sim \text{I}(0).$$

10.2.2 I have not tested every available exchange rate for cointegration, but I have done it for the U.K. £ versus the German mark, Japanese yen and U.S. $. In each case I have used monthly data from September 1972 to June 1994, giving 262 monthly observations. The data are the same as those used in the investigations described in Sections 10.4 to 10.6.

10.2.3 I first test the three consumer price indices to see whether they can reasonably be taken as integrated I(1) series, using the ADF tests for a unit root. Rather oddly the Japanese CPI appears as if it did not have a unit root, but I can see no reason for this, nor is it obvious from a graph of it. I next test the exchange rates to see whether these can also be taken as integrated series, using the same test, and find that all three can be.

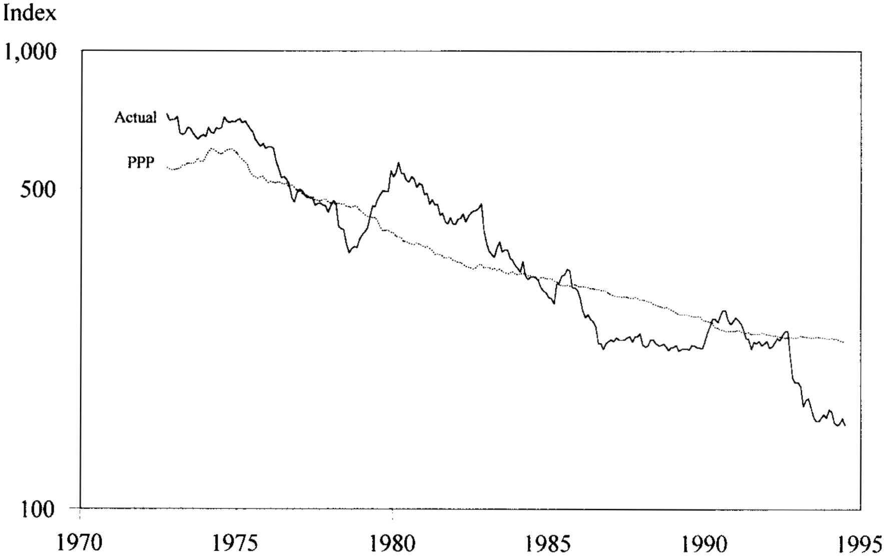

Figure 10.2. Values of actual exchange rate and PPP exchange rate, U.K. £ versus Japanese yen, monthly, 1971-94

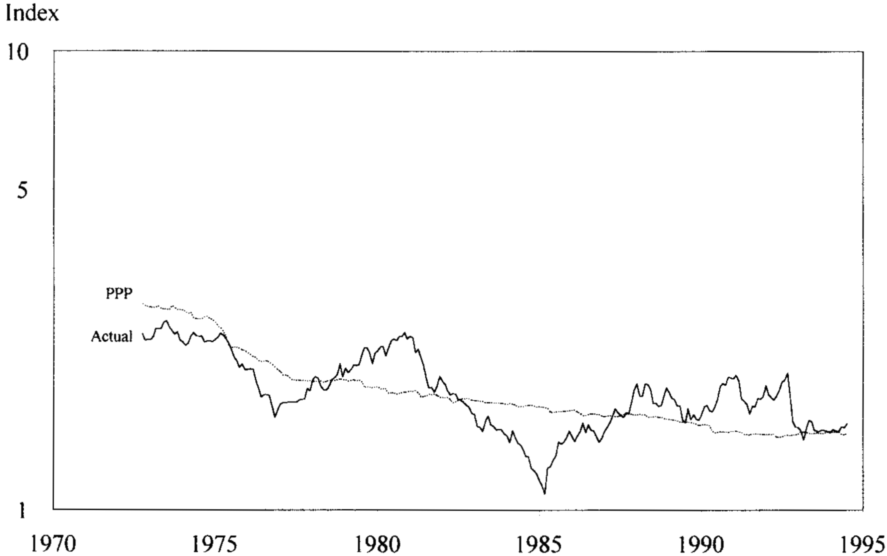

Figure 10.3. Values of actual exchange rate and PPP exchange rate, U.K. £ versus US $, monthly, 1971-94

10.2.4 I then construct the deviations:

$$XD_{ij}(t) = \ln XK_{ij}(t) = \ln X_{ij}(t) - \ln Q_j(t) + \ln Q_i(t)$$

and test these for a unit root. In each case the tests also show that a unit root can be assumed. This does not support my case.

10.2.5 I then test the logarithms of the exchange rate and the logarithms of the two relevant consumer price indices for cointegration. For each country there is very strong evidence of cointegration, with two cointegrating vectors for Germany and Japan, and with one or possibly two cointegrating vectors for the U.S.A. Unfortunately, the coefficients of the cointegrating vectors are nothing like what I was expecting. There is a test whether a specified cointegrating vector is plausible, and I tested all three countries for the vector $(1, -1, 1)$. This is implausible for Germany and Japan, but it might fit the U.S.A. The evidence does not seem very supportive of the PPP approach. Similar results have been found by other authors, whose results I now discuss.

10.3 *Other Investigations*

10.3.1 The PPP hypothesis for exchange rates is an old one, and a number of authors recently have used cointegration methods to test for it. All those I refer to have used monthly data similar to mine, though some have used wholesale price indices as a better indicator of the price of traded goods than consumer prices indices; all have used monthly data, and all have started in 1972 or 1973. However, all have compared exchange rates versus the U.S. $, and all have used a shorter time period, finishing their investigations in 1983, 1986 or 1989.

10.3.2 Baillie & Selover (1987), Enders (1988), and Corbae & Ouliaris (1988) all conclude that there is no evidence for cointegration between the exchange rates and price levels studied, whereas Hamilton (1994, Chapters 19 and 20), who uses evidence about the U.S. $ *v* Italian lira exchange rate as a textbook example, finds the evidence more mixed, but, on balance, considers that the relevant variables are not cointegrated and that the logarithm of a PPP adjusted exchange rate is non-stationary.

10.3.3 A problem with the methods used by all these authors is that they take the random walk hypothesis as the null, and seek convincing evidence, at say a 5% level, that this hypothesis should be rejected. I take the PPP hypothesis, with moderately slow reversion to the mean, as the null, and I need to see convincing evidence that this is not a satisfactory model. The conventional tests do not seem to satisfy my requirement.

10.3.4 Part of the problem is the relatively short term of the available data. Exchange rates have been floating reasonably freely for only a little over 20 years, and this, as I have noted already in ¶4.4.6, may not be long enough to distinguish between two possible models which are close together in the short term, though they have very different long-term characteristics. Even 262 months is too few observations to be able to distinguish the autoregressive coefficient, which is around 0.97 for monthly data (and some of the authors referred to obtain similar results to this) from unity. A longer period is necessary. I am not sure, however, whether the fact that a large number of exchange rates show similar features, with monthly autoregressive coefficients all about 0.97 or yearly coefficients all about 0.7, can be taken as more convincing evidence than tests of a single exchange rate.

10.3.5 Taking all this into account, I would prefer to use the PPP model for future simulation of exchange rates; its short-term properties are similar to those of a random walk model, so those who prefer the random walk for the short term will not find the results very different.

10.4 *Data for Several Countries*

10.4.1 I have available data for exchange rates at monthly intervals from August 1972, when currencies started floating more freely after the breakdown of the post-war Bretton Woods exchange rate system, and I have taken my analysis up to June 1994. For the data sources see Appendix F.10. I also have values of the consumer price indices for these countries, which were analysed in Section 2.7. I have analysed the exchange rates, assuming that the PPP AR(1) model described above is likely to fit. Table 10.1 summarises the results. I have considered both the one series at monthly intervals and also the twelve different series taking observations at yearly intervals.

10.4.2 The values of *XMU* shown are the lowest and highest for any of the twelve yearly series and the overall mean for the monthly series. The values of *XMU* are not important in themselves, but the range is of some interest. For the monthly series the value of *XA* is the first autocorrelation coefficient, and the 'yearly' figure quoted just after it is $XA^{12}_{monthly}$, an estimate of the equivalent annual value. There follow values of *XA* from the yearly series, showing the lowest and highest value and the mean of the 12 values.

10.4.3 For *XSD* similar values are shown: first, the standard deviation of the residuals after fitting a monthly AR(1) model; then the equivalent yearly standard deviation using the formula:

$$XSD^2_{yearly} = XSD^2_{monthly}.(1 - XA^{24})/(1 - XA^2)$$

and then the lowest and highest of the values of *XSD* from the yearly series, and the mean of those values.

10.4.4 The monthly *XA* parameter ranges from 0.947 (New Zealand) to 0.985 (Belgium). Since the value of the exchange rate varies much more within one month than do the values of the consumer price indices, these monthly parameters are similar to what is found if the exchange rate is investigated versus a constant mean rather than a PPP adjusted mean. It is not surprising that others

Table 10.1. Analysis of exchange rates versus U.K. £ for 19 countries from 9/1972 to 6/1994; parameters of AR(1) model for monthly series with 262 values and for 12 yearly series with 21 or 22 values

	XMU overall mean	*XA* monthly series monthly - yearly	*XA* yearly series low - mean - high	*XSD* monthly series monthly - yearly	*XSD* yearly series low - mean - high
Australia	0.5297	0.972 - 0.714	0.552 - 0.654 - 0.747	0.0399 - 0.1195	0.0909 - 0.1206 - 0.1457
Austria	3.1150	0.967 - 0.671	0.429 - 0.551 - 0.684	0.0280 - 0.0819	0.0804 - 0.0951 - 0.1147
Belgium	4.1318	0.985 - 0.831	0.719 - 0.779 - 0.826	0.0292 - 0.0933	0.0907 - 0.1037 - 0.1185
Canada	0.6347	0.971 - 0.707	0.555 - 0.614 - 0.664	0.0342 - 0.1019	0.1046 - 0.1215 - 0.1374
Denmark	2.4312	0.973 - 0.719	0.481 - 0.639 - 0.785	0.0284 - 0.0852	0.0737 - 0.0946 - 0.1205
Finland	2.0160	0.964 - 0.643	0.475 - 0.560 - 0.615	0.0284 - 0.0815	0.0768 - 0.0890 - 0.1046
France	2.2885	0.973 - 0.717	0.550 - 0.686 - 0.759	0.0269 - 0.0806	0.0695 - 0.0804 - 0.0961
Germany	1.1299	0.979 - 0.775	0.575 - 0.680 - 0.760	0.0286 - 0.0886	0.0793 - 0.0984 - 0.1219
Ireland	0.1625	0.957 - 0.592	0.367 - 0.515 - 0.662	0.0218 - 0.0609	0.0508 - 0.0645 - 0.0856
Italy	7.6867	0.956 - 0.582	0.542 - 0.655 - 0.780	0.0298 - 0.0826	0.0607 - 0.0759 - 0.0892
Japan	5.6806	0.971 - 0.705	0.433 - 0.531 - 0.601	0.0342 - 0.1019	0.1263 - 0.1468 - 0.1756
Netherlands	1.2631	0.980 - 0.782	0.584 - 0.699 - 0.787	0.0276 - 0.0859	0.0810 - 0.0977 - 0.1189
New Zealand	0.8416	0.947 - 0.519	0.321 - 0.521 - 0.630	0.0371 - 0.0985	0.0764 - 0.0969 - 0.1217
Norway	2.3036	0.967 - 0.668	0.535 - 0.661 - 0.771	0.0259 - 0.0755	0.0622 - 0.0766 - 0.0982
South Africa	0.3712	0.974 - 0.725	0.584 - 0.677 - 0.754	0.0399 - 0.1204	0.0936 - 0.1234 - 0.1502
Spain	5.2181	0.964 - 0.641	0.380 - 0.550 - 0.728	0.0270 - 0.0776	0.0666 - 0.0847 - 0.1063
Sweden	2.2372	0.977 - 0.757	0.686 - 0.787 - 0.873	0.0279 - 0.0855	0.0558 - 0.0793 - 0.1221
Switzerland	1.0112	0.953 - 0.559	0.284 - 0.384 - 0.495	0.0307 - 0.0837	0.0943 - 0.1016- 0.1139
U.S.A.	0.4439	0.971 - 0.705	0.567 - 0.623 - 0.672	0.0340 - 0.1013	0.0963 - 0.1141- 0.1257

Table 10.2.
Analysis of exchange rates v. U.K. £ for 19 countries, 9/1972 to 6/1994; monthly series with 261 values; correlation coefficients of residuals

	Aus	Ost	Bel	Can	Den	Fin	Fra	Ger	Ire	Ita
Australia	**1.0**									
Austria (Ost)	.36	**1.0**								
Belgium	.32	**.88**	**1.0**							
Canada	**.72**	.39	.36	**1.0**						
Denmark	.37	**.90**	**.87**	.41	**1.0**					
Finland	.23	.46	**.43**	.33	.47	**1.0**				
France	.37	**.87**	**.82**	.40	**.86**	.43	**1.0**			
Germany	.34	**.97**	**.88**	.37	**.92**	.46	**.88**	**1.0**		
Ireland	.28	**.75**	**.69**	.36	**.73**	.51	**.77**	**.75**	**1.0**	
Italy	.29	**.61**	.58	.40	**.61**	.55	**.65**	**.61**	**.61**	**1.0**
Japan	.42	.52	.51	.44	.54	.31	.54	.51	.44	.42
Netherlands	.34	**.94**	**.87**	.36	**.91**	.45	**.87**	**.95**	**.74**	**.62**
New Zealand	**.69**	.32	.28	.57	.34	.19	.34	.31	.31	.26
Norway	.41	**.79**	**.74**	.47	**.77**	.51	**.78**	**.78**	**.62**	.54
South Africa	.48	.43	.39	.51	.44	.28	.42	.41	.40	.40
Spain	.35	**.63**	.59	.44	**.62**	.36	**.62**	**.60**	.52	.57
Sweden	.38	**.68**	**.64**	.46	**.70**	.56	**.65**	**.67**	.54	.54
Switzerland	.28	**.84**	**.72**	.28	**.78**	.34	**.76**	**.83**	**.67**	.54
U.S.A.	**.68**	.39	.36	**.92**	.40	.33	.42	.37	.39	.42

Table 10.2 (continued).
Analysis of exchange rates v. U.K. £ for 19 countries, 9/1972 to 6/1994; monthly series with 261 values; correlation coefficients of residuals

	Jap	Net	N.Z.	Nor	S.A.	Spa	Swe	Swi	U.S.A.
Japan	**1.0**								
Netherlands	.51	**1.0**							
New Zealand	.41	.31	**1.0**						
Norway	.52	**.76**	.38	**1.0**					
South Africa	.40	.40	.45	.47	**1.0**				
Spain	.42	**.61**	.26	.59	.40	**1.0**			
Sweden	.44	**.65**	.31	**.69**	.40	.57	**1.0**		
Switzerland	.53	**.81**	.28	**.64**	.37	.49	.56	**1.0**	
U.S.A.	.48	.36	.55	.48	.53	.47	.46	.29	**1.0**

Table 10.3.
Analysis of exchange rates v. U.K. £ for 19 countries, 9/1972 to 6/1994; 12 series with 21 or 22 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Aus	Ost	Bel	Can	Den	Fin	Fra	Ger	Ire	Ita
Australia	**1.0**	.27	.19	**.80**	.28	**.60**	.12	.24	.09	.11
Austria (Ost)	.24	**1.0**	**.92**	.24	**.94**	.38	**.82**	**.98**	**.77**	**.69**
Belgium	.31	**.94**	**1.0**	.29	**.92**	.35	**.85**	**.90**	**.65**	**.75**
Canada	**.85**	.38	.49	**1.0**	.32	**.60**	.16	.25	.14	.25
Denmark	.31	**.97**	**.96**	.43	**1.0**	.41	**.88**	**.94**	**.74**	**.77**
Finland	.32	.58	**.69**	.55	**.67**	**1.0**	.26	.31	.36	.44
France	.34	**.88**	**.87**	.46	**.88**	.56	**1.0**	**.85**	**.60**	**.67**
Germany	.39	**.96**	**.92**	.49	**.96**	.55	**.91**	**1.0**	**.78**	**.69**
Ireland	.17	**.77**	**.74**	.46	**.73**	.52	**.78**	**.75**	**1.0**	**.72**
Italy	.26	**.69**	**.72**	.49	**.68**	**.64**	**.76**	**.69**	**.83**	**1.0**
Japan	.26	.52	.47	.33	.49	.16	.39	.46	.49	.32
Netherlands	.38	**.95**	**.94**	.50	**.96**	**.62**	**.92**	**.98**	**.72**	**.69**
New Zealand	**.62**	**.63**	.54	.52	**.64**	.29	.57	**.72**	.49	.41
Norway	**.61**	**.76**	**.82**	**.73**	**.84**	.77	.72	**.83**	**.60**	.59
South Africa	.54	.43	.42	.46	.45	.09	.36	.54	.18	.26
Spain	.23	**.77**	**.71**	.31	**.78**	.55	**.78**	**.73**	**.65**	**.73**
Sweden	.39	**.62**	**.75**	.56	**.73**	**.81**	**.62**	**.64**	.46	**.64**
Switzerland	.10	**.91**	**.79**	.23	**.85**	.45	**.77**	**.83**	**.70**	.57
U.S.A.	**.78**	.35	.47	**.93**	.38	.41	.48	.46	.47	.41

investigating exchange rates come to the conclusion that they can be well represented by random walks, with XA set equal to unity, but, as I have explained above, I do not think that this is a sensible long-term model.

10.4.5 The equivalent yearly value of XA ranges from 0.519 to 0.831, and is typically in the range 0.6 to 0.8. Much the same is true for the yearly values, with the highs exceeding 0.6 for all countries except Switzerland (0.495), although the lows are almost all below 0.6. Since there are only 20 or 21 observations in the yearly series, the standard errors of the estimates of the value of the autocorrelation coefficient are quite high, of the order of 0.2, so the range shown does not contradict the hypothesis of an overall value of XA of about 0.65 to 0.7. Assuming that cross rates are consistent, i.e. $X_{ij}(t)X_{jk}(t)X_{ki}(t) = 1$, a table such as that shown in Table 10.1 can be calculated for any base exchange rate, and it is necessary to hypothesise a model that is uniformly consistent. One suitable model is that the values of XA are the same for all exchange rates. I first

Table 10.3 (continued).
Analysis of exchange rates v. U.K. £ for 19 countries, 9/1972 to 6/1994; 12 series with 21 or 22 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Jap	Net	N.Z.	Nor	S.A.	Spa	Swe	Swi	U.S.A.
Australia	.29	.29	**.79**	.57	**.62**	.09	.34	.26	**.73**
Austria (Ost)	**.68**	**.97**	.50	**.69**	.40	.54	.36	**.84**	.24
Belgium	.58	**.89**	.38	**.60**	.40	.43	.41	**.77**	.24
Canada	.28	.29	.59	.59	.37	.07	.43	.20	**.89**
Denmark	**.66**	**.93**	.51	**.70**	.37	.51	.45	**.87**	.28
Finland	.10	.39	.55	**.66**	.18	.56	**.73**	.27	.40
France	.50	**.82**	.30	**.61**	.29	.32	.43	**.83**	.17
Germany	**.63**	**.98**	.45	**.68**	.41	.49	.29	**.87**	.26
Ireland	.47	**.78**	.31	.55	.09	.44	.20	**.62**	.20
Italy	.39	**.67**	.30	.58	.32	.40	.46	.54	.15
Japan	**1.0**	.56	.56	.40	.20	.22	.22	**.69**	.36
Netherlands	.44	**1.0**	.45	**.73**	.40	.50	.35	**.82**	.29
New Zealand	.56	**.65**	**1.0**	.56	.47	.23	.25	.48	.54
Norway	.44	**.84**	**.66**	**1.0**	.43	.35	**.65**	.50	.56
South Africa	.30	.51	**.67**	.44	**1.0**	.00	.13	.31	.30
Spain	.29	**.71**	.34	**.60**	.24	**1.0**	.49	.41	−.05
Sweden	.25	**.69**	.28	**.82**	.29	**.70**	**1.0**	.28	.28
Switzerland	.54	**.83**	.55	.59	.32	**.64**	.43	**1.0**	.17
U.S.A.	.40	.47	.46	**.62**	.41	.23	.45	.22	**1.0**

suggested this in my Montréal paper (Wilkie, 1992), but the method suggested there for separating the exchange rates works only when there are three countries, and not when there are more.

10.4.6 The equivalent yearly values of *XSD* range from 0.0609 (Ireland) to 0.1204 (South Africa), and are reasonably consistent with the observed values of *XSD* from the yearly series. The range of these values is much less than when the exchange rates are measured against some other base currency, as in Wilkie (1994a), where I showed figures, for a slightly shorter period, versus the U.S. $ and the German mark. The Canadian $ moves very closely to the U.S. $, and the relative standard deviation is therefore smaller (around 0.04 annually). The same is true for the currencies of Austria and the Netherlands versus the German mark, and likewise their standard deviations are small (less than 0.02 annually).

10.4.7 The picture is not complete without the crosscorrelations between the respective values of XE_{ij}. These are shown in Table 10.2 for the monthly series and in Table 10.3 for the June and December yearly series.

10.5 *Alternative Approaches*

10.5.1 In order that the model for cross-rates be consistent, it is convenient for the values of *XA* to be the same for all countries. I have tried setting the yearly *XA* = 0.7, and the monthly *XA* = 0.97, which are almost equivalent. The standard deviations of the residuals of the monthly series are almost unchanged, none increasing by more than 0.0002. The standard deviations of the yearly series are changed by more than this, but the change is usually in the third decimal place. The correlation coefficients for the monthly series are almost unchanged. Those for the yearly series change by more, but seldom by more than 0.05.

10.5.2 In carrying out the analysis described above, I measured the deviation between the actual and expected exchange rates, assuming that the mean value *XMU* was known. I estimated the value as the average over the whole period. However, if I had been carrying out the investigation at some earlier date I would not have known this average, and if I do the same calculations at some later date I shall be able to calculate a new, possibly different, average. This problem of parameter estimation, depending on the period of observation, applies to all the series I have investigated, but it seems to have more force in this particular case, because of the relatively short observation period, and the relatively large standard error of the estimate of the mean. For an AR(1) model, the standard

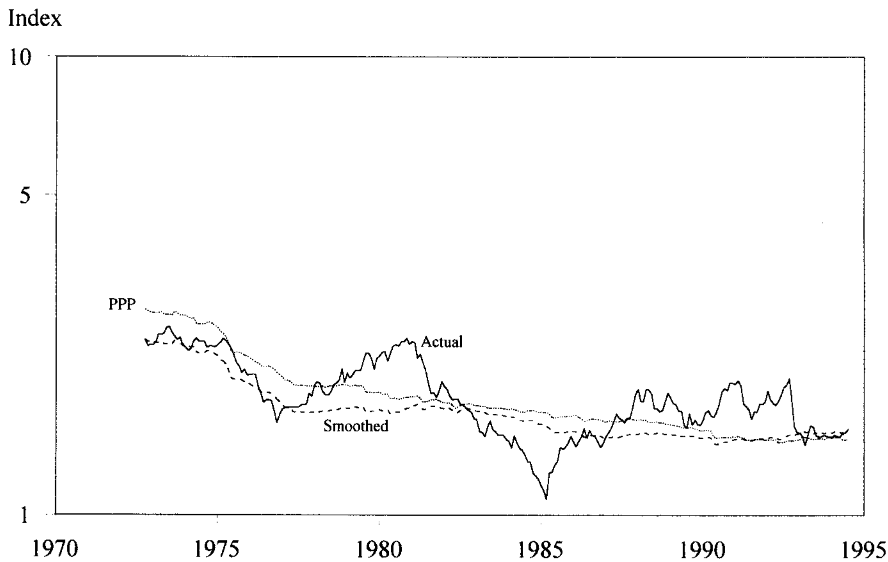

Figure 10.4. Values of actual exchange rate and PPP exchange rate, based (i) on overall mean, (ii) on smoothed mean, U.K. £ versus U.S. \$, monthly, 1971-94

Table 10.4. Analysis of exchange rates versus U.K. £ for 19 countries, basis 2, from 9/1972 to 6/1994; parameters of AR(1) model for monthly series with 262 values and for 12 yearly series with 21 or 22 values, using $XA_{monthly}$ = 0.97 and XA_{yearly} = 0.7, and smoothing *XMU*

	XMU(*t*) in June 1994	*XSD* monthly series monthly - yearly	*XSD* yearly series low - mean - high
Australia	0.6059	0.0395 - 0.1171	0.0862 - 0.1150 - 0.1395
Austria	3.1140	0.0279 - 0.0825	0.0767 - 0.0947 - 0.1175
Belgium	4.1675	0.0291 - 0.0862	0.0898 - 0.1013 - 0.1158
Canada	0.6675	0.0339 - 0.1004	0.1018 - 0.1197 - 0.1379
Denmark	2.4438	0.0282 - 0.0835	0.0720 - 0.0942 - 0.1220
Finland	2.0368	0.0283 - 0.0837	0.0792 - 0.0914 - 0.1072
France	2.3122	0.0267 - 0.0790	0.0679 - 0.0787 - 0.0939
Germany	1.1579	0.0284 - 0.0842	0.0771 - 0.0965 - 0.1207
Ireland	0.1474	0.0217 - 0.0644	0.0489 - 0.0653 - 0.0876
Italy	7.6687	0.0297 - 0.0880	0.0602 - 0.0764 - 0.0921
Japan	5.6098	0.0337 - 0.0998	0.1139 - 0.1349 - 0.1706
Netherlands	1.3012	0.0275 - 0.0814	0.0786 - 0.0960 - 0.1183
New Zealand	0.8581	0.0370 - 0.1095	0.0754 - 0.0987 - 0.1320
Norway	2.3306	0.0257 - 0.0762	0.0615 - 0.0758 - 0.0991
South Africa	0.4318	0.0396 - 0.1173	0.0891 - 0.1201 - 0.1482
Spain	5.2017	0.0269 - 0.0796	0.0648 - 0.0854 - 0.1110
Sweden	2.2746	0.0276 - 0.0818	0.0554 - 0.0765 - 0.1185
Switzerland	1.0083	0.0306 - 0.0907	0.0920 - 0.1057 - 0.1238
U.S.A.	0.4726	0.0338 - 0.1000	0.0939 - 0.1129 - 0.1256

error of the mean is increased by the factor $1/(1 - a)$, where a is the autoregressive parameter. Thus, when the monthly autoregressive parameter is around 0.97, the standard error of the estimate of the mean is around 30 times what it would be were the successive observations to be independent.

10.5.3 One way of dealing with this is to use an adaptive estimate of the mean, so that *XMU* becomes *XMU*(*t*), and it is re-estimated as each new observation is available. One way of doing this, analogous to a Kalman filter approach, is to estimate the mean as an exponentially weighted moving average of the observations. I have arbitrarily chosen a smoothing parameter of 0.01 per month, allowing, therefore, quite slow adaption, and I have started the series by assuming that the exchange rate in September 1972 was spot-on its expected value. There may be ways of justifying alternative smoothing parameters, but I have not investigated these. Figure 10.4 shows the PPP exchange rate based on the smoothed mean superimposed on the values already shown in Figure 10.3.

Table 10.5.
Analysis of exchange rates v. U.K. £ for 19 countries, basis 2, 9/1972 to 6/1994; 12 series with 21 or 22 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Aus	Ost	Bel	Can	Den	Fin	Fra	Ger	Ire	Ita
Australia	**1.0**	.32	.18	**.80**	.30	**.60**	.11	.25	.14	.12
Austria (Ost)	.35	**1.0**	**.91**	.22	**.94**	.40	**.82**	**.98**	**.76**	**.71**
Belgium	.23	**.93**	**1.0**	.14	**.92**	.31	**.85**	**.89**	.54	**.70**
Canada	**.84**	.50	.38	**1.0**	.23	.48	.07	.18	.13	.19
Denmark	.35	**.98**	**.94**	.48	**1.0**	.40	**.88**	**.94**	**.64**	**.74**
Finland	.37	.64	**.64**	.55	**.69**	**1.0**	.21	.28	.36	.35
France	.30	**.91**	**.86**	.43	**.88**	.56	**1.0**	**.85**	.54	**.68**
Germany	.37	**.99**	**.90**	.48	**.96**	.56	**.91**	**1.0**	**.74**	**.70**
Ireland	.32	**.82**	**.68**	.57	**.78**	.50	**.82**	**.80**	**1.0**	**.69**
Italy	.27	**.71**	**.65**	.52	**.69**	.56	**.75**	**.70**	**.80**	**1.0**
Japan	.51	.57	.54	.46	**.60**	.25	.49	.58	.40	.29
Netherlands	.35	**.99**	**.93**	.47	**.97**	**.64**	**.91**	**.98**	**.78**	**.68**
New Zealand	**.61**	**.65**	.47	.51	**.63**	.26	.53	**.69**	.52	.39
Norway	**.59**	**.84**	**.75**	**.69**	**.87**	**.79**	**.69**	**.82**	**.65**	.54
South Africa	.51	.49	.39	.45	.46	.13	.33	.53	.29	.30
Spain	.40	**.77**	**.71**	.46	**.79**	**.64**	**.83**	**.77**	**.68**	**.74**
Sweden	.29	**.68**	**.73**	.47	**.75**	**.78**	**.60**	**.63**	.47	**.61**
Switzerland	.29	**.89**	**.78**	.37	**.87**	.54	**.81**	**.88**	**.73**	.55
U.S.A.	**.79**	.49	.39	**.94**	.47	.44	.47	.48	.59	.44

10.5.4 Using the smoothed mean produces different estimates of the monthly and yearly values of *XA* and *XSD*. In general, the values of both are reduced. One can describe this metaphorically: if the mean moves some distance towards each observation, then there is less to do for the next observation to get back toward the mean. However, the changes in these parameters are not universally in the same direction, i.e. some values of *XA* and *XSD* are larger.

10.5.5 One can use both these modifications together, fixing the value of *XA* to be the same for all countries, and allowing the value of *XMU* to change adaptively. Table 10.4 shows the results of such an analysis. The value in the column headed *XMU*(*t*) is the value of the adapted mean for the last observation, June 1994. Comparison of the values in this column with those of the overall mean in Table 10.1 shows the distance apart the two methods of calculating the

Table 10.5 (continued).
Analysis of exchange rates v. U.K. £ for 19 countries, basis 2, 9/1972 to 6/1994; 12 series with 21 or 22 yearly steps; correlation coefficients of residuals; lower triangle for December series; upper triangle for June series

	Jap	Net	N.Z.	Nor	S.A.	Spa	Swe	Swi	U.S.A.
Australia	.56	.28	**.80**	.58	**.61**	.11	.30	.36	**.75**
Austria (Ost)	**.66**	**.98**	.50	**.74**	.43	.51	.44	**.82**	.23
Belgium	.59	**.90**	.33	.59	.38	.46	.48	**.74**	.11
Canada	.56	.18	**.60**	.54	.39	−.04	.30	.27	**.91**
Denmark	**.66**	**.93**	.47	**.71**	.38	.54	.53	**.89**	.19
Finland	.19	.39	.41	**.69**	.17	.54	.59	.34	.35
France	.54	**.82**	.27	**.61**	.28	.32	.48	**.84**	.10
Germany	**.64**	**.98**	.44	**.68**	.42	.49	.37	**.86**	.21
Ireland	.43	**.73**	.33	.53	.15	.38	.17	**.64**	.17
Italy	.42	**.65**	.27	.56	.35	.41	.49	.58	.08
Japan	**1.0**	**.62**	**.71**	.54	.42	.08	.35	**.66**	**.61**
Netherlands	.59	**1.0**	.42	**.72**	.39	.52	.41	**.85**	.20
New Zealand	**.71**	**.62**	**1.0**	.54	.51	.10	.17	.46	.59
Norway	**.61**	**.84**	**.61**	**1.0**	.41	.39	**.65**	**.61**	.51
South Africa	.47	.49	**.70**	.42	**1.0**	.02	.12	.37	.34
Spain	.31	**.77**	.38	**.71**	.28	**1.0**	.57	.33	−.11
Sweden	.37	**.68**	.18	**.78**	.23	**.77**	**1.0**	.36	.17
Switzerland	.55	**.89**	**.61**	**.74**	.42	**.66**	.49	**1.0**	.26
U.S.A.	.53	.48	.49	**.61**	.41	.38	.38	.38	**1.0**

mean are as at that date; for some countries they are very close, for others further away. The columns for *XSD* for the monthly series and the yearly series are calculated as before.

10.5.6 The correlation coefficients for the monthly and the yearly series (June and December) are shown in Tables 10.6 and 10.5, similar to Tables 10.2 and 10.3.

10.5.7 The numbers in all these tables are further away from those in Tables 10.1, 10.2, and 10.3 than when only one modification is made, but overall there is little major difference. The monthly correlation coefficients are almost the same as in Table 10.2; the yearly ones show bigger changes.

10.5.8 Figure 10.4 shows, for U.K. £ versus U.S. $, the same as Figure 10.3 with the PPP rate based on the smoothed mean added.

Table 10.6.
Analysis of exchange rates v. U.K. £ for 19 countries, basis 2, 9/1972 to 6/1994; monthly series with 261 values; correlation coefficients of residuals

	Aus	Ost	Bel	Can	Den	Fin	Fra	Ger	Ire	Ita
Australia	**1.0**									
Austria (Ost)	.36	**1.0**								
Belgium	.31	**.88**	**1.0**							
Canada	**.72**	.39	.35	**1.0**						
Denmark	.38	**.90**	**.87**	.41	**1.0**					
Finland	.23	.46	**.42**	.32	.48	**1.0**				
France	.37	**.87**	**.82**	.40	**.86**	.43	**1.0**			
Germany	.34	**.97**	**.88**	.36	**.92**	.46	**.88**	**1.0**		
Ireland	.29	**.74**	**.67**	.36	**.72**	.51	**.77**	**.75**	**1.0**	
Italy	.29	**.60**	.56	.40	**.60**	.54	**.65**	**.60**	**.61**	**1.0**
Japan	.44	.52	.51	.45	.55	.32	.55	.51	.43	.42
Netherlands	.33	**.94**	**.87**	.35	**.91**	.45	**.87**	**.95**	**.73**	**.61**
New Zealand	**.69**	.32	.27	.57	.33	.18	.34	.31	.31	.26
Norway	.41	**.79**	**.73**	.46	**.77**	.51	**.78**	**.78**	**.62**	.53
South Africa	.48	.43	.39	.51	.44	.29	.42	.41	.40	.40
Spain	.36	**.63**	.59	.45	**.62**	.36	**.62**	**.60**	.52	.56
Sweden	.38	**.68**	**.64**	.45	**.70**	.56	**.65**	**.67**	.54	.54
Switzerland	.29	**.83**	**.71**	.29	**.78**	.35	**.76**	**.83**	**.67**	.53
U.S.A.	**.68**	.39	.35	**.92**	.40	.33	.42	.37	.40	.43

Table 10.6 (continued).
Analysis of exchange rates v. U.K. £ for 19 countries, basis 2, 9/1972 to 6/1994; monthly series with 261 values; correlation coefficients of residuals

	Jap	Net	N.Z.	Nor	S.A.	Spa	Swe	Swi	U.S.A.
Japan	**1.0**								
Netherlands	.52	**1.0**							
New Zealand	.42	.31	**1.0**						
Norway	.53	**.75**	.37	**1.0**					
South Africa	.41	.39	.45	.47	**1.0**				
Spain	.42	**.61**	.26	**.60**	.41	**1.0**			
Sweden	.45	**.65**	.31	**.69**	.40	.58	**1.0**		
Switzerland	.53	**.81**	.27	**.65**	.37	.49	.57	**1.0**	
U.S.A.	.49	.35	.56	.48	.53	.47	.46	.30	**1.0**

11. SIMULATED RESULTS

11.1 *Simulated Results*

11.1.1 Simulation or 'Monte Carlo' methods are now familiar to actuaries, and are easily implemented, though there is still much that can be learned from books, such as Rubinstein (1981) or Johnson (1987), about how to use more efficient methods, and in particular how to simulate particular distributions. I have used only conventional methods for what follows, but I realise that there are ways in which these methods could be improved, for example by structuring the random unit normal variates so that each sample of 1,000, or whatever number is chosen, has zero mean and unit standard deviation, as described by Tilley (1993).

11.1.2 In order to get a feel for the numerical results provided by the model, I showed, in Wilkie (1986a, 1986b, 1987), the means, variances and correlation coefficients from 1,000 simulations of the processes. I do the same again, including also the items newly modelled. A little notation is necessary. For any variable X, I define:

$$FX(t) = X(t)/X(0)$$

$$GX(t) = 100\{FX(t)^{1/t} - 1\}$$

$$HX(t) = FX(t)/FQ(t)$$

$$JX(t) = 100\{HX(t)^{1/t} - 1\}.$$

11.1.3 Thus $FX(t)$ is the return over t years from an 'investment' of 1 at time 0, and $GX(t)$ is the equivalent compound annual rate of return, expressed as a percentage; $HX(t)$ and $JX(t)$ are defined similarly, but based on 'real' returns relative to the retail prices index Q. I then calculate the mean and standard deviation of each relevant $GX(t)$ and $JX(t)$ for selected values of t, and the correlation coefficients between $GX_1(t)$ and $GX_2(t)$, where X_1 and X_2 are different variables.

11.1.4 In 'The Risk Premium on Ordinary Shares' (Wilkie, 1995) I explain the difference between, in the current notation, $E[GX(t)]$, $100\{\{E[FX(t)]\}^{1/t} - 1\}$ and $100\{\{E[\ln FX(t)/t]\} = 100\{\{E[\ln FX(t)]/t\}$, which, for the lognormal or near lognormal distributions in question, are different, though generally not very far apart. I use the statistics only of $GX(t)$ in what follows.

11.1.5 We are interested in total nominal returns on different classes of asset, which are calculated as:

for shares: $PR(t) = PR(t-1).\{P(t) + D(t).(1 - tax)\}/P(t-1)$

for consols: $CR(t) = CR(t-1).\{1/C(t) + (1-tax)\}.C(t-1)$

for cash: $BR(t) = BR(t-1).\{1+(1-tax).B(t-1)\}$
for index-linked: $RR(t) = RR(t-1).\{1/R(t) + (1-tax)\}.R(t-1).\{Q(t)/Q(t-1)\}$
for property: $AR(t) = AR(t-1).\{A(t) + E(t).(1 - tax)\}/A(t-1)$.

Cash is treated effectively as a one-year bond.

11.1.6 Although tax on income at rate *tax* is allowed for in the formulae, the calculated results are all gross, so *tax* is zero. Note that *CR* here is different from the *CR* in Section 6; I do not think any confusion arises.

11.1.7 I also include *Q* and *W* as variables whose 'returns' we are interested in.

11.2 *Initial Conditions*

11.2.1 In order to start a simulation, one needs a set of initial conditions to represent $\mathscr{F}_0$. There are many possibilities for these. One can use the actual conditions at some chosen date, as I have done for the graphs in Sections 2.11, etc., all of which use the starting conditions as at the end of June 1994. Alternatively, one can use what I call 'neutral' initial conditions, in which the starting values are set at what their long-run means would be if all the standard deviations were zero; in effect I assume that the logged values take their long-run means. Yet another possibility is to use the unconditional means of the values themselves, allowing for the extra $\exp(\frac{1}{2}\sigma^2)$ terms in the means of a lognormal distribution.

11.2.2 For the results quoted below, which are all for the U.K. model, I have used the neutral initial conditions, which are defined as:

$$I(0) = QMU = 0.047$$

$$J(0) = (WW1 + WW2).QMU + JMU = 0.06189$$

$$Y(0)\% = \exp(YW.QMU).YMU\% = 4.0811\%$$

$$C(0)\% = 100QMU + CMU\% = 7.75\%$$

$$B(0)\% = \exp(-BMU).C(0)\% = 6.1576\%$$

$$R(0)\% = RMU\% = 4.0\%$$

$$Z(0)\% = ZMU\% = 7.4\%$$

$$DM(0) = CM(0) = EM(0) = QMU = 0.047$$

$$YE(0) = DE(0) = 0.$$

11.2.3 Values of $Q(0)$, $W(0)$, $D(0)$ or $P(0)$, $E(0)$ or $A(0)$, and also $PR(0)$, $CR(0)$, $BR(0)$, $RR(0)$ and $AR(0)$ can be taken arbitrarily, for example as unity, or 100, or as the values of the indices as at some chosen date.

11.3 *Results for Nominal Returns*

11.3.1 The results of the calculations described above, for nominal returns, GX, for t = 1, 2, 5, 10, 20 and 50, are shown in Table 11.1. M(GX) is the mean of GX for the 1,000 simulations, SD(GX) is the standard deviation of GX and C(GX_1, GX_2) is the correlation coefficient between GX_1 and GX_2.

11.3.2 One can see how the means generally reflect the corresponding means in the model, how the standard deviations reduce with t, and how the correlations vary with t in ways that are, perhaps, at first sight, surprising, but that, on reflection, are reasonable. For example, the return on shares is negatively correlated with inflation over one year, a result that has been observed frequently by financial economists, but is positively correlated with inflation over longer periods, a result that is intuitively acceptable to actuaries. The returns on consols (long-term bonds) are negatively correlated with inflation for a longer period, but in due course the correlation coefficient becomes positive; this is the consequence of reinvestment at positive real rates of interest. Other results can be interpreted similarly.

11.4 *Results for Real Returns*

11.4.1 The results of the same sorts of calculations, for real returns, for t = 1, 2, 5, 10, 20 and 50, are shown in Table 11.2. M(JX) is the mean of JX for the 1,000 simulations, and SD(JX) and C(JX_1, JX_2) are defined similarly. Also shown for each return is C(JX, GQ), the correlation coefficient between JX and the mean rate of inflation over the same period.

11.4.2 The mean results necessarily show broadly the difference between the mean rate of return in nominal terms and the mean rate of inflation. The standard deviations and correlation coefficients deserve careful scrutiny. For short periods, many of the standard deviations increase as compared with the corresponding nominal rate; for longer terms many of them reduce.

11.5 *Results for ARCH Models*

11.5.1 The same sorts of returns could be calculated using the ARCH model for inflation described in Section 2.8. The model suggested there has a different value of QMU from the basic model, so I show the results for two ARCH models, the first model (1) as in ¶2.8.13, and the second model (2) with the same values of QMU, QA and QSC (= QMU) as in the basic model, but with the ARCH parameters QSA and QSB of ARCH model (1). Rather than present a full table, it is sufficient to show only the results for inflation, as in Table 11.3.

11.5.2 In simulating these ARCH models, I discovered that, on occasion, they produced very large values for QSD, which derived from large previous values of $I(t)$, and resulted in excessively large subsequent values, rather like a

Table 11.1 Results for nominal returns from 1,000 simulations, using neutral initial conditions

Term	1	2	5	10	20	50
Mean rate of inflation, *GQ*						
M(*GQ*)	5.00	4.97	4.85	4.74	4.77	4.80
SD(*GQ*)	4.45	4.14	3.71	2.99	2.28	1.47
Mean rate of growth of nominal wages, *GW*						
M(*GW*)	6.56	6.47	6.40	6.35	6.35	6.38
SD(*GW*)	3.75	3.50	3.33	2.71	2.07	1.36
C(*GW*,*GQ*)	0.74	0.87	0.94	0.96	0.96	0.97
Mean rate of nominal total return on shares, *GPR*						
M(*GPR*)	13.20	11.90	11.04	10.91	10.75	10.79
SD(*GPR*)	19.47	12.71	7.41	4.80	3.48	2.31
C(*GPR*,*GQ*)	−0.26	-0.06	0.17	0.34	0.52	0.62
C(*GPR*,*GW*)	−0.20	-0.03	0.19	0.35	0.51	0.61
Mean rate of nominal total return on consols, *GCR*						
M(*GCR*)	8.03	7.86	7.74	7.89	7.92	7.94
SD(*GCR*)	7.92	5.47	2.92	1.70	1.05	1.09
C(*GCR*,*GQ*)	-0.32	-0.39	-0.55	-0.55	-0.16	0.46
C(*GCR*,*GW*)	-0.29	-0.36	-0.51	-0.53	-0.14	0.45
C(*GCR*,*GPR*)	0.30	0.27	0.05	-0.06	0.07	0.33
Mean rate of nominal total return on cash, *GBR*						
M(*GBR*)	6.16	6.22	6.34	6.42	6.48	6.53
SD(*GBR*)	0.0	0.62	1.07	1.28	1.32	1.16
C(*GBR*,*GQ*)	0.0	0.08	0.17	0.33	0.45	0.56
C(*GBR*,*GW*)	0.0	0.07	0.17	0.31	0.43	0.54
C(*GBR*,*GPR*)	0.0	-0.01	-0.00	0.09	0.25	0.35
C(*GBR*,*GCR*)	0.0	-0.19	-0.28	-0.25	0.24	0.77
Mean rate of nominal total return on index-linked, *GRR*						
M(*GRR*)	9.45	9.46	9.01	8.89	8.97	8.99
SD(*GRR*)	8.19	5.78	4.15	3.22	2.39	1.53
C(*GRR*,*GQ*)	0.56	0.75	0.93	0.97	0.99	0.99
C(*GRR*,*GW*)	0.40	0.65	0.88	0.93	0.95	0.96
C(*GRR*,*GPR*)	-0.14	-0.00	0.15	0.33	0.52	0.61
C(*GRR*,*GCR*)	0.30	0.06	-0.34	-0.43	-0.09	0.49
C(*GRR*,*GBR*)	0.0	0.02	0.14	0.32	0.45	0.58
Mean rate of nominal total return on property, *GAR*						
M(*GAR*)	13.97	13.66	13.22	13.16	13.07	13.16
SD(*GAR*)	14.76	8.92	4.59	3.21	2.80	2.31
C(*GAR*,*GQ*)	0.06	0.09	0.25	0.49	0.61	0.59
C(*GAR*,*GW*)	0.04	0.07	0.24	0.48	0.58	0.57
C(*GAR*,*GPR*)	-0.00	0.04	0.07	0.21	0.35	0.38
C(*GAR*,*GCR*)	0.02	-0.03	-0.13	-0.27	-0.01	0.35
C(*GAR*,*GBR*)	0.0	0.01	0.08	0.16	0.27	0.39
C(*GAR*,*GRR*)	0.11	0.09	0.23	0.47	0.60	0.59

Table 11.2 Results for real returns from 1,000 simulations, using neutral initial conditions

Term	1	2	5	10	20	50
Mean rate of growth of real wages, *JW*						
M(*JW*)	1.56	1.48	1.50	1.54	1.52	1.52
SD(*JW*)	2.91	1.99	1.22	0.86	0.60	0.37
C(*JW*,*GQ*)	−0.57	−0.56	−0.49	−0.49	−0.50	−0.46
Mean rate of real total return on shares, *JPR*						
M(*JPR*)	8.21	6.80	5.99	5.93	5.72	5.72
SD(*JPR*)	20.25	13.06	7.41	4.57	2.89	1.73
C(*JPR*,*GQ*)	−0.46	−0.37	−0.34	−0.32	−0.19	−0.07
C(*JPR*,*JW*)	0.26	0.23	0.23	0.22	0.13	0.06
Mean rate of real total return on consols, *JCR*						
M(*JCR*)	3.18	2.99	2.94	3.11	3.06	3.01
SD(*JCR*)	9.89	7.79	5.68	4.08	2.60	1.34
C(*JCR*,*GQ*)	−0.68	−0.78	−0.91	−0.94	−0.92	−0.72
C(*JCR*,*JW*)	0.34	0.42	0.46	0.46	0.48	0.35
C(*JCR*,*JPR*)	0.47	0.44	0.38	0.36	0.25	0.09
Mean rate of real total return on cash, *JBR*						
M(*JBR*)	1.29	1.35	1.54	1.67	1.67	1.66
SD(*JBR*)	4.29	3.99	3.57	2.76	1.99	1.22
C(*JBR*,*GQ*)	−1.00	−0.99	−0.96	−0.91	−0.82	−0.66
C(*JBR*,*JW*)	0.57	0.55	0.48	0.45	0.42	0.29
C(*JBR*,*JPR*)	0.46	0.37	0.32	0.29	0.17	0.05
C(*JBR*,*JCR*)	0.69	0.76	0.84	0.84	0.84	0.84
Mean rate of real total return on index-linked, *JRR*						
M(*JRR*)	4.24	4.28	3.96	3.96	4.01	4.00
SD(*JRR*)	6.47	3.64	1.46	0.76	0.38	0.17
C(*JRR*,*GQ*)	−0.01	0.00	0.00	0.01	−0.03	−0.04
C(*JRR*,*JW*)	−0.01	−0.00	−0.00	−0.02	0.03	0.02
C(*JRR*,*JPR*)	0.01	0.06	−0.02	0.00	0.05	−0.07
C(*JRR*,*JCR*)	0.45	0.36	0.23	0.15	0.17	0.25
C(*JRR*,*JBR*)	0.01	−0.01	−0.02	−0.00	0.08	0.19
Mean rate of real total return on property, *JAR*						
M(*JAR*)	8.70	8.42	8.07	8.08	7.94	7.99
SD(*JAR*)	14.55	9.14	5.03	3.12	2.24	1.79
C(*JAR*,*GQ*)	−0.25	−0.38	−0.54	−0.51	−0.32	−0.11
C(*JAR*,*JW*)	0.13	0.20	0.26	0.30	0.14	0.03
C(*JAR*,*JPR*)	0.12	0.18	0.21	0.21	0.10	0.02
C(*JAR*,*JCR*)	0.20	0.30	0.49	0.48	0.34	0.15
C(*JAR*,*JBR*)	0.25	0.37	0.52	0.46	0.26	0.14
C(*JAR*,*JRR*)	0.08	0.03	−0.00	−0.02	0.02	0.02

hyperinflation. This occurred twice in the 1,000 simulations. Reasonable looking values for all the *GX* functions were produced, but the mean values of some of the *FX* functions were beyond a reasonable range; for example, the value of $100\{\{E[FQ(50)]\}^{1/50} - 1\}$ for model (1) was 28.37% and for model (2) was 15.88%.

11.5.3 The ARCH models show a rather lower standard deviation than the basic model for short periods, but higher for long periods. The higher standard deviations flow through into the results for other variables, which are not shown.

Table 11.3 Results for nominal returns from 1,000 simulations, using ARCH models (1) and (2), with neutral initial conditions

Term	1	2	5	10	20	50
Mean rate of inflation, *GQ*, ARCH model (1)						
M(*GQ*)	4.17	4.17	4.14	4.03	4.10	4.09
SD(*GQ*)	2.66	2.72	3.09	2.95	3.98	3.39
Mean rate of inflation, *GQ*, ARCH model (2)						
M(*GQ*)	4.90	4.90	4.87	4.76	4.81	4.81
SD(*GQ*)	2.68	2.70	2.95	2.70	3.14	2.67

12. CONCLUSION

12.1 I have shown a great many formulae, tables and figures, and I am sure that anyone who has tried to read this paper from beginning to end will have become satiated. I hope, however, that those who are interested will return to the paper to consider yet another aspect of the models I have described, discover something that neither they nor I have noticed before, and will be stimulated to carry the investigations further. I am convinced that this type of modelling is more realistic for the long term than the random walk, efficient market, style of model so well established among financial economists. I realise that it conflicts with their style of model, but only in the longer run. It is quite consistent over short periods.

12.2 Areas where more research could be done include:

(a) investigating other distributions for the residuals of all the series, including α-stable distributions and others that I mention in ¶2.9.3;

(b) investigating cointegrated models for retail prices, wages and dividends, as indicated in Section 5.4, and including possibly also property rentals;

(c) investigating cointegrated or other models for share prices, dividends and earnings, as indicated in Section 5.9;

(d) investigating the link between expected inflation, nominal yields and index-linked bond yields, and their stochastic behaviour, in ways other than I have done;

(e) investigating other variables to represent short-term interest rates, which could better reflect short-term fluctuations than those I have used;

(f) investigating yield curve models and their stochastic behaviour;
(g) investigating wage indices in a wider range of other countries;
(h) investigating long-term and short-term interest rates in a wider range of other countries;
(i) producing specimen simulations for several countries together, using the exchange rate models described in Section 10, and allowing for simultaneous correlation;
(j) investigating the distribution of the sum of a normally distributed random variable and a lognormally distributed one, as in the consols yield model (see Appendix E.5);
(k) completing the formulae for the forecast means and variances as shown in Appendix E, but for the variables I have not covered;
(l) investigating other similar properties of the models analytically, and perhaps confirming them by simulation, or comparing them with the observed facts;
(m) investigating the econometrics literature for other studies of the long-run relationships between wages and prices;
(n) including some measure of Gross National Product as a variable; this might well be cointegrated with both real wages and real dividends, and it would provide a useful tool for long-term economic planning; and
(o) exploring why, if the models described in this paper are reasonable, share prices and other variables fluctuate more than would be reasonable if markets were efficient.

Some of these investigations would take us beyond pure investment models, into wider economic modelling, or into the psychology of markets, or into the inefficiencies of markets.

12.3 Further, besides the obvious actuarial applications of these models, there are many possible applications of them to all sorts of other financial, commercial, industrial and public sector topics, such as the assessment of capital projects, the testing of solvency of financial and other institutions, the analysis of long-term financial derivatives, and any sort of financial planning. I await the future with interest.

REFERENCES

BAILLIE, R.T. & SELOVER, D.D. (1987). Cointegration and models of exchange rate determination. *International Journal of Forecasting*, **3**, 43-51.

BANKER'S ALMANAC (1988). Reid Information Services, East Grinstead.

BOLLERSLEV, T., CHOU, R.Y. & KRONER, K.F. (1992). ARCH modelling in finance: a review of the theory and empirical evidence. *Journal of Econometrics*, **52**, 5-59.

BONSDORFF, H. (1991). A model for investment returns: asymptotic behaviour. *Transactions of the 2nd AFIR International Colloquium*, Brighton, **4**, 27-37.

BONSDORFF, H., PENTIKÄINEN, T., PESONEN, M., PUKKILA, T., RANNE, A., RANTALA, J., RUOHONEN, M. & SARVAMAA, S. (FIM GROUP) (1994). On the asset models as part of all company insurance analysis. *Transactions of the 4th International AFIR Colloquium*, Orlando, **3**, 1471-1501.

BOX, G.E.P. & JENKINS, G.M. (1976). *Time series analysis, forecasting and control.* Holden-Day, San Francisco.

BROCKWELL P.J. & DAVIS, R.A. (1991). *Time series: theory and methods.* Second edition, Springer-Verlag, New York.

CARTER, J. (1991). The derivation and application of an Australian stochastic investment model. *Transactions of the Institute of Actuaries of Australia.* 1991, **1**, 315-428.

CENTRAL STATISTICAL OFFICE (1972). *Financial statistics: notes and definitions.* H.M.S.O., London.

CENTRAL STATISTICAL OFFICE (1972-94). *Financial statistics*, monthly. H.M.S.O., London.

CHATFIELD, C. (1980). *The analysis of time series: an introduction.* Second edition, Chapman and Hall, London.

CLARKSON, R.S. (1991). A non-linear stochastic model for inflation. *Transactions of the 2nd AFIR International Colloquium*, Brighton, **3**, 233-253.

CORBAE, D. & OULIARIS, S. (1988). Cointegration and tests of purchasing power parity. *Review of Economics and Statistics*, **70**, 508-511.

COX, D.R. & MILLER, H.D. (1965). *The theory of stochastic processes.* Methuen, London.

CRAIGHEAD, S. (1994). Chaotic analysis on U.S. Treasury interest rates. *Transactions of the 4th AFIR International Colloquium*, Orlando, **2**, 497-536.

CRYER, J.D. (1986). *Time series analysis.* Duxbury Press, Boston.

DAYKIN, C.D. & HEY, G.B. (1990). Managing uncertainty in a general insurance company. *J.I.A.* **117**, 173-277.

DAYKIN, C.D. & HEY, G.B. (1991). A stochastic cash-flow model of a general insurance company. *Transactions of the 2nd AFIR International Colloquium*, Brighton, **4**, 113-137.

DAYKIN, C.D., PENTIKÄINEN, T. & PESONEN, M. (1994). *Practical risk theory for actuaries.* Chapman & Hall, London.

DEAVES, R. (1993). *Modelling and predicting Canadian inflation and interest rates.* Canadian Institute of Actuaries.

DEPARTMENT OF EMPLOYMENT (1971). *British labour statistics: historical abstract 1886-1968.* H.M.S.O., London.

DEPARTMENT OF EMPLOYMENT (1986). *Retail price indices, 1914-1985.* H.M.S.O., London.

DICKEY, D.A. & FULLER, W.A. (1979). Distributions of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association*, **75**, 427-431.

DICKEY, D.A. & FULLER, W.A. (1981). Likelihood ratio statistics for autoregressive time series with a unit root. *Econometrica*, **49**, 1057-1072.

DOUGLAS, C.M. (1930). The statistical groundwork of investment policy. *T.F.A.* **12**, 173-228.

DUFFIE, D. (1992). *Dynamic pricing theory.* Princeton University Press, Princeton, N.J.

ENDERS, W. (1988). ARIMA and cointegration tests of PPP under fixed and flexible exchange rate regimes. *Review of Economics and Statistics*, **70**, 504-508.

ENGLE, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, **50**, 987-1007.

ENGLE, R.F. & GRANGER, C.W.J., Editors (1991). *Long-run economic relationships; readings in cointegration.* Oxford University Press.

EVERITT, B.S. & HAND, D.J. (1981). *Finite mixture distributions.* Chapman and Hall, London.

FAMA, E.F. & FRENCH, R.K. (1988). Dividend yields and expected stock returns. *Journal of Financial Economics*, **22**, 3-25.

FISHER, I. (1907). *The rate of interest.* The Macmillan Company, New York.

FISHER, I. (1930). *The theory of interest.* The Macmillan Company, New York. Reprinted (Kelley), 1955.

GEOGHEGAN, T.J., CLARKSON, R.S., FELDMAN, K.S., GREEN, S.J., KITTS, A., LAVECKY, J.P., ROSS, F.J.M., SMITH, W.J. & TOUTOUNCHI, A. (1992). Report on the Wilkie stochastic investment model. *J.I.A.* **119**, 173-228.

GRANGER, C.W.J. (1983). *Co-integrated variables and error-correcting models.* Discussion Paper, 83-13, University of California, San Diego.

GRANGER, C.W.J. & TERÄSVIRTA, T. (1993). *Modelling nonlinear economic relationships.* Oxford University Press.

HAMILTON, J.D. (1994). *Time series analysis.* Princeton University Press, Princeton, N.J.

HARDY, M.R. (1993). Stochastic simulation in life office solvency assessment. *J.I.A.* **120**, 131-151.

HARDY, M.R. (1994). Incorporating individual company variation in simulated equity returns. *Transactions of the 4th International AFIR Colloquium*, Orlando, **3**, 1147-1162.

HARVEY, A.C. (1989). *Forecasting, structural time series models and the Kalman filter.* Cambridge University Press.

HOMER, S. (1963). *A history of interest rates: 2000 B.C. to the present.* Rutgers University Press, New Brunswick.

HUA, V. (1994). Modelling equity returns using a simple ARCH model. *Transactions of the 4th International AFIR Colloquium*, Orlando, **2**, 537-565.

HUBER, P.P. (1995). *A review of Wilkie's stochastic investment model.* Actuarial Research Paper No. 70, City University, London.

HÜRLIMANN, W. (1993). Numerical evaluation of the Wilkie inflation model. *Insurance: Mathematics & Economics*, **11**, 311-314.

IBBOTSON, R.G. & SINQUEFIELD, R.A. (1989). *Stocks, bonds, bills and inflation: historical returns (1926-1987).* The Research Foundation of the Institute of Chartered Financial Analysts, Charlottesville, Virginia.

INGERSOLL, J.E., JR. (1987). *Theory of financial decision making.* Rowman & Littlefield, Totowa, N.J.

JANICKI, A. & WERON, A. (1994). *Simulation and chaotic behaviour of α-stable stochastic processes.* Marcel Dekker Inc, New York.

JARQUE, C.M. & BERA, A.K. (1981). An efficient large sample test for normality of observations and regression residuals. *Working papers in Economics and Econometrics*, No. 47. Australian National University.

JOHANSEN, S. & JUSELIUS, K. (1990). Maximum likelihood estimation and inference on cointegration — with applications to the demand for money. *Oxford Bulletin of Economics and Statistics*, **52**, 169-210.

JOHNSON, M.E. (1987). *Multivariate statistical simulation.* John Wiley, New York.

JOHNSON, N.L. & KOTZ, S. (1970). *Continuous univariate distributions.* Houghton Mifflin, Boston.

KARATZAS I. & SHREVE S.E. (1991). *Brownian motion and stochastic calculus.* Second edition, Springer-Verlag, New York.

KENDALL, SIR MAURICE & STUART, A. (1977). *The advanced theory of statistics.* Volume 1, Fourth edition, Charles Griffin, London.

KITTS, A. (1988). *Applications of stochastic financial models: a review.* Department of Social Statistics, University of Southampton.

KITTS, A. (1990). Comments on a model of retail price inflation. *J.I.A.* **117**, 407-412.

MACDONALD, A.S. (1991). On investment strategies using the Wilkie model. *Proceedings of the 2nd AFIR International Colloquium*, Brighton, **3**, 413-427.

MACDONALD, A.S. (1993). What is the value of a valuation? *Transactions of the 3rd International AFIR Colloquium*, Rome, **2**, 723-743.

MACDONALD, A.S. (1994). Appraising life office valuations. *Transactions of the 4th International AFIR Colloquium*, Orlando, **3**, 1163-1183.

MACDONALD, A.S. (1995). *A stochastic evaluation of solvency valuations for life offices.* Ph.D. Thesis, Heriot-Watt University, Edinburgh.

MCLEOD, G. (1982). *Box Jenkins in practice.* Gwilym Jenkins & Partners Ltd, Lancaster.

MADDOCKS, N.E., NISBET, M.J., NISBET, R.M. & BLYTHE, S.P. (1991). Determinism and chaos in long financial series. *Transactions of the 2nd AFIR International Colloquium*, Brighton, **3**, 267-283.

MATURITY GUARANTEES WORKING PARTY (1980). Report of the Maturity Guarantees Working Party. *J.I.A.* **107**, 101-212.

METZ, M. & ORT, M. (1993). Stochastic models for the Swiss consumer's price index and the cost of the adjustment of pensions to inflation for a pension fund. *Transactions of the 3rd AFIR International Colloquium*, Rome, **2**, 789-806.

MILLS, T.C. (1994). *The econometric modelling of financial time series.* Cambridge University Press.

MITCHELL, B.R. (1975). *European Historical Statistics.* Macmillan Press, London.

MITCHELL, B.R. & DEANE, P. (1962). *Abstract of British historical statistics.* Cambridge University Press.

MITCHELL, B.R. & JONES, H.G. (1971). *Second abstract of British historical statistics.* Cambridge University Press.

O.E.C.D. (monthly). *Main economic indicators.* O.E.C.D., Paris.

ONG, A. (1994). *A stochastic model for Treasury bills; an extension to Wilkie's model.* Actuarial Research Report No 68, City University, London.

PEDERSEN, H., SHIU, E. & THORLACIUS, A. (1989). Arbitrage-free pricing of interest rate contingent claims. *Transactions of the Society of Actuaries*, **41**, 231-265.

PENTIKÄINEN, T., BONSDORFF, H., PESONEN, M., RANTALA, J. & RUOHONEN, M. (1989). *Insurance solvency and financial strength.* Insurance Publishing Company, Helsinki.

PESARAN, M.H. & PESARAN, B. (1991). *Microfit 3.0 user manual.* Oxford University Press.

PHELPS BROWN, E.H. & HOPKINS, S.V. (1956). Seven centuries of the prices of consumables, compared with builder's wages. *Economica*, **23**, 296-312.

PRAETZ, P.D. (1972). The distribution of share price changes. *Journal of Business*, **45**, 49-55.

PRIESTLEY, M.B. (1988). *Non-linear and non-stationary time series analysis.* Academic Press, London.

PUKKILA, T., RANNE, A. & SARVAMAA, S. (1994). On stochastic modelling of inflation. *Transactions of the 4th International AFIR Colloquium*, Orlando, **2**, 589-608.

PURCHASE, D.E., FINE, A.E.M., HEADDON, C.P. *et al.* (1989). Reflections on resilience: some considerations of mismatching tests, with particular reference to non-linked long-term insurance business. *J.I.A.* **116**, 347-452.

RAO, B.B., Editor (1994). *Cointegration for the applied economist.* St Martin's Press, New York.

ROSS, M.D. (1989). Modelling a with-profits life office. *J.I.A.* **116**, 691-715.

RUBINSTEIN, R.Y. (1981). *Simulation and the Monte Carlo method.* John Wiley, New York.

SAMORODNITSKY G. & TAQQU, M.S. (1994). *Stochastic non-gaussian random processes.* Chapman & Hall, New York.

SHARP, K.P. (1988). Stochastic models of interest rates. *Transactions of the 23rd International Congress of Actuaries*, Helsinki, **5**, 247-261.

SOLVENCY WORKING PARTY (1986). The solvency of life assurance companies. *T.F.A.* **39**, 251-340.

STRANG, G. (1980). *Linear algebra and its applications.* Second edition, Academic Press, New York.

TAYLOR, S. (1986). *Modelling financial time series.* John Wiley, Chichester.

THOMSON, R.J. (1994). *A stochastic investment model for actuarial use in South Africa.* Presented to the 1994 Convention of the Actuarial Society of South Africa.

TILLEY, J.A. (1990). A stochastic yield curve model for asset/liability simulation. *Transactions of the 1st AFIR International Colloquium*, Paris, **2**, 65-80.

TILLEY, J.A. (1993). Stochastic yield curve models. *Seminar Proceedings of the Canadian Institute of Actuaries*, **4**, Toronto, 62-85.

TILLEY, J.A. & MUELLER, M. (1991). Managing interest rate risk for long liabilities. *Transactions of the 2nd AFIR International Colloquium*, Brighton, **1**, 169-184.

TONG, H. (1990). *Non-linear time series: a dynamical system approach.* Clarendon Press, Oxford.

TOUTOUNCHI, A. (1984). *A comprehensive time-series analysis of the dividends and dividend yields of sectors of the Financial Times-Actuaries and the Standard and Poor's share indices.* Ph.D. Thesis, Heriot-Watt University, Edinburgh.

WALTER, C. (1990). Mise en évidence de distributions Lévy-stables et d'une structure fractable sur le marché de Paris. *Transactions of the 1st AFIR International Colloquium*, Paris, **3**, 241-259.

WILKIE, A.D. (1981). Indexing long term contracts. *J.I.A.* **108**, 299-360 and *T.F.A.* **38**, 55-126.

WILKIE, A.D. (1986a). A stochastic investment model for actuarial use. *T.F.A.* **39**, 341-403.

WILKIE, A.D. (1986b). Some applications of stochastic investment models. *J.I.A.S.S.* **29**, 25-51.

WILKIE, A.D. (1987). Stochastic investment models — theory and applications. *Insurance: Mathematics and Economics*, **6**, 65-83.

WILKIE, A.D. (1992). Stochastic investment models for XXIst Century actuaries. *Transactions of the 24th International Congress of Actuaries*, Montréal, **5**, 119-137.

WILKIE, A.D. (1993). Can dividend yields predict share price changes? *Transactions of the 3rd International AFIR Colloquium*, Rome, **1**, 335-347.

WILKIE, A.D. (1994a). A stochastic model for consumer price indices and exchange rates in several countries. *Transactions of the 4th International AFIR Colloquium*, Orlando, **2**, 633-658.

WILKIE, A.D. (1994b). Keynote speech of Proceedings of the Investment Seminar. *Seminar Proceedings of the Canadian Institute of Actuaries*, **4**, Toronto, 4-40.

WILKIE, A.D. (1995). The risk premium on ordinary shares. *B.A.J.*, **1**, 251-293.

WORKING, H. (1960). Note on the correlation of first differences of averages in a random chain. *Econometrica*, **28**, 916-918. Reprinted in COOTNER, P.H., Editor (1964). *The random character of stock market prices*. MIT Press, Cambridge, Mass.

APPENDIX A

UNIVARIATE TIME SERIES MODELS

A.1 *Textbooks on Time Series Analysis*

A.1.1 Since Box & Jenkins in their seminal work (1976) introduced the concept of autoregressive integrated moving average (ARIMA) time series models, many books based on their approach have appeared, ranging from introductory texts such as those by Chatfield (1980) or Cryer (1986) to more comprehensive works such as those by Harvey (1989), Brockwell & Davis (1991) or Hamilton (1994). Non-linear time series modelling has been discussed by Priestley (1988), Tong (1990) and Granger & Teräsvirta (1993). Autoregressive conditional heteroscedastic (ARCH) models have been considered in many papers, and they have been applied to econometric time series by Taylor (1986) and Mills (1993). Cointegration has been considered in the readings edited by Engle & Granger (1991) and in the introductory series of readings edited by Rao (1994).

A.1.2 These works all address discrete time series modelling; there are a great many others that discuss continuous time series, usually with a title that includes the words 'stochastic processes'; the examples that I refer to are Cox & Miller (1965) and Karatzas & Shreve (1991), but there are many others. However, the analysis of empirical series must necessarily be carried out on observations taken at discrete intervals, although there are many other uses of continuous processes in financial economics, notably the pricing of options.

A.1.3 It can be seen that there is plenty of material to read, and I cannot claim to have read thoroughly all the works noted; but it may be helpful to students of time series modelling to have these books drawn to their attention.

A.2 *Stationary Autoregressive Models*

A.2.1 A time series model is a way of describing statistically the behaviour of a series referred to in general as X, which consists of observations denoted x_t or $x(t)$ with t as an unlimited (or theoretically infinite) series of time points at unit intervals. When considering a specific sample of n observations, it is convenient to identify the times at which they are recorded as $t = 1, 2, ..., n$. When considering forecasting from some given date, it is convenient to call that date 'now' and to put $t = 0$ at that date, so that everything relevant for $t \leq 0$ is known, for as far back as is needed in the circumstances, and everything for $t \geq 1$ is unknown and is to be forecast.

A.2.2 Time series can be divided into those that are 'stationary' and those that are not. For a stationary series the long-term forecast mean and variance are constant. A series might be stationary about a trend, in which case the mean may increase (or decrease) with the trend, with the variance remaining constant. For a non-stationary series the mean and variance are not constant; they might oscillate, though this is not the case with the sort of series we are considering. More commonly the mean and variance both increase indefinitely as one forecasts further ahead.

A.2.3 The simplest sort of time-series models are linear ones; their properties are well understood. Non-linear models are also of interest, and they have received some attention, as noted in ¶A.1.1. A problem about non-linear models is that there are so many to choose from, and it is impossible to be sure that some more elaborate non-linear model might not suit one's data rather better than a simpler non-linear one or even a linear one. Most of the models I have used are linear, but there are some specific non-linearities, for example in my model for nominal interest rates.

A.2.4 The general type of linear, stationary, autoregressive model for a series X is:

$$x_t = \mu + \Sigma_{j=1,p}\, a_j\,(x_{t-j} - \mu) + e_t$$

so that the value of x_t depends on the previous p values of X, plus a random error term e_t, which introduces an 'innovation' at each step. It is assumed that each e_t has zero mean and variance σ^2, and that e_t and e_u are independent for $t \neq u$. It is often convenient to assume that each e_t is normally distributed. If normality of the innovations can be assumed, then the properties of various estimators of the parameters of the model are well known, and estimation is easier to carry out. For example, if we take the first few values of the series as given, then we can use ordinary least squares estimation of the parameters, simply by treating the model as a multiple regression of x_t on x_{t-1} to x_{t-p}.

A.2.5 It may be helpful to transform the data so that the innovations or residuals appear to be approximately normally distributed, for example by taking logarithms of the original series. This turns out to be desirable for all the series investigated that are expressed as indices, such as the consumer price index, wages index, share price index, etc.

A.2.6 A first order autoregressive model, denoted an AR(1) model, is the simplest, and it can be expressed as:

$$x_t = \mu + a(x_{t-1} - \mu) + e_t.$$

The deviation of x_t from the mean μ is proportionate to the deviation of x_{t-1} from μ, plus an innovation e_t, which has zero mean and variance σ^2. The series is stationary if and only if $-1 < a < 1$.

A.2.7 Certain properties of an AR(1) model are easy to establish. The correlation coefficient between x_t and x_{t-k} is a^k. If $0 < a < 1$, the autocorrelation coefficients decline exponentially. A useful way to identify such a series is to calculate the empirical autocorrelation coefficients of the sample data and to see whether they do, in fact, die away approximately exponentially.

A.2.8 The forecast values of x_t, given that 'now' is at $t = 0$, depend only on the value of x_0 and not on any previous values. An AR(1) series, thus, has a 'Markov' property; all we need to know about the current 'state' is expressed in the value of X at the current time.

A.2.9 If we write $z_t = x_t - \mu$, so that $z_0 = x_0 - \mu$, we find that the forecast mean and variance of x_t are given by:

$$E[x_t] = E[z_t] + \mu = a^t z_0 + \mu$$

and

$$V[x_t] = V[z_t] = \sigma^2(1 - a^{2t})/(1 - a^2).$$

Thus the expected value of x_t moves exponentially (decreasing or increasing according to the sign of z_0) from x_0 to its long-run constant value of μ, and the variance tends asymptotically towards its long-run constant value of $\sigma^2/(1 - a^2)$.

A.3 *Stationary Moving Average Models*

A.3.1 The general form of linear stationary moving average model is:

$$x_t = \mu + \Sigma_{j=1,q}\, b_j\, e_{t-j} + e_t$$

so that each value of X is a weighted average of previous values of the innovations. Such a model is generally stationary, provided the coefficients, the b_js, are finite. It may also be possible to express a moving average model as a (possibly infinite) autoregressive model, but this puts certain restrictions on the coefficients. However, it is much more convenient to consider the simpler models with a few parameters rather than an infinite number.

A.3.2 The simplest moving average model is an MA(1) model:

$$x_t = \mu + be_{t-1} + e_t.$$

Each innovation has two effects, one immediately and a second in the following period. The forecast mean and variance of x_t, conditional on knowing everything for $t \leq 0$, are:

$$E[x_1] = \mu + be_0 \text{ and } V[x_1] = \sigma^2$$

and

$$E[x_t] = \mu \text{ and } V[x_t] = \sigma^2(1 + b^2) \text{ for } t > 1.$$

Thus an MA(1) model very quickly reaches its long-run position.

A.3.3 Moving average models do not seem to be necessary for the investment models described, expect in one instance, the model for dividends.

A.4 *Integrated Models*

A.4.1 If the separate terms of a stationary model are successively summed, we get what is called an 'integrated' model. An ARIMA(p,d,q) model is one in which the dth differences of x_t are stationary, with a mixed ARMA(p,q) model. It has not been necessary in the investment model to use more than one level of summation, nor to use mixed ARIMA or ARMA models. It is, however, of interest to consider the properties of an integrated AR(1) model and of an integrated MA(1) model.

A.4.2 The simplest integrated model is a 'random walk', ARIMA(0,1,0), where successive increments are independent, possibly with a non-zero mean, in which case one speaks of a 'random walk with drift'. Given a starting value x_0, the mean and variance of the forecast values are:

$$E[x_t] = x_0 + t\mu$$

and

$$V[x_t] = t\sigma^2.$$

Thus, the mean increases or decreases (according to the sign of μ) linearly and the variance also increases linearly, so the standard deviation increases in proportion to the square root of time $\sqrt{t}$.

A.4.3 If the differences of x_t are generated by an AR(1) model, so that X is ARIMA(1,1,0):

$$x_t - x_{t-1} = z_t = \mu + a(z_{t-1} - \mu) + e_t$$

then the mean and variance of the forecast values of x_t are:

$$E[x_t] = x_0 + t\mu + a(1-a^t)/(1-a).z_0$$

and

$$V[x_t] = \sigma^2\{t - 2a(1 - a^t)/(1 - a) + a^2(1 - a^{2t})/(1 - a^2)\}/(1 - a^2).$$

Thus, the forecast mean, besides adjusting for the short-term position, increases (if $\mu > 0$) linearly, and the variance expands more rapidly than for a random walk with the same value of σ. This is shown in Figure 2.7 for the inflation model.

A.4.4 If x_t is generated as the sum of an MA(1) process, so that it is ARIMA(0,1,1):

$$x_t - x_{t-1} = z_t = \mu + be_{t-1} + e_t$$

then the expected mean and variance are:

$$E[x_t] = x_0 + t\mu + be_0$$

and

$$V[x_t] = \sigma^2\{1 + (t-1)(1 + b)^2\}.$$

Thus after an immediate short-term 'blip', the series behaves very like a random walk with a higher variance $\sigma^2(1 + b)^2$.

A.5 *Unit Roots*

A.5.1 It is convenient to introduce the 'backward step' operator B, defined by: $Bx_t = x_{t-1}$. Using this operator we can write the general AR(p) model, putting $z_t = x_t - \mu$, as:

$$z_t - \Sigma_{j=1,p}\, a_j\, z_{t-j} = (1 - \Sigma_{j=1,p}\, a_j\, B^j)\, z_t = e_t$$

or

$$A(B)z_t = e_t$$

where A(B) is a polynomial in B. The roots of this polynomial, say $r_1, r_2, ..., r_p$ are of relevance in describing the properties of the model. For the model to be stationary, all the roots must have modulus greater than unity, or lie 'outside the unit circle'. Note that some of the roots might be complex.

A.5.2 If X is modelled as an integrated AR model, so that first differences of X are AR(p), then X can be described as an AR(p + 1) model, with a 'unit root', that is the corresponding polynomial for X has one root equal to unity, corresponding to the factor $(1 - B)x_t = x_t - x_{t-1}$.

A.5.3 Given a sample from a particular series, it is of relevance to decide whether it is reasonable to model it as an integrated series or as an AR model with a root near to unity. For example, there is rather little difference in the short term between the random walk:

$$x_t = x_{t-1} + e_t$$

and the AR(1) model;

$$x_t = a.x_{t-1} + e_t$$

if a is close to unity. The long-term properties of these two models are, however, quite different. The closer a is to unity, the larger the number of observations necessary for the models to be distinguished.

A.5.4 A method of testing for the presence of unit roots has been developed by Dickey & Fuller (1979, 1981); this is described also in Engle & Granger (1991), Pesaran & Pesaran (1991) and Rao (1994). Using the simple model above one can write:

$$x_t - x_{t-1} = (a-1)x_{t-1} + e_t$$

and in a more elaborate model (the Augmented Dickey-Fuller or ADF model) one can write:

$$x_t - x_{t-1} = (a-1)x_{t-1} + \Sigma_{j=1,p}\, a_j\,(x_{t-j} - x_{t-j-1}) + e_t$$

carry out a linear regression of $\nabla x_t = x_t - x_{t-1}$ on the various factors on the right hand side, and then test whether the coefficient of x_{t-1}, namely $(a-1)$, is significantly different from zero. A complication is that the distribution of this coefficient is not the usual t-distribution, and special significance values have to be used. These are implemented in the programming package *MICROFIT 3.0* (Pesaran & Pesaran, 1991).

A.5.5 Stationary series are also described as I(0) series, and series that have been integrated once as I(1) series, and I use this notation in the paper.

A.6 *Different Frequencies of Sampling an AR(1) Series*

A.6.1 Consider a series $x(t)$, $t = 0, 1, 2, \ldots$. Consider a second series formed by sampling every mth case of the first series. Call it $x_m(u)$, where $u = mt$, i.e.:

$$x_m(0) = x(0)$$

$$x_m(1) = x(m)$$

$$x_m(2) = x(2m), \text{ and so on.}$$

Assume that $x(t)$ is an AR(1) series, so that:

$$x(t) = \mu + a(x(t-1) - \mu) + e(t)$$

with:

$$V[e(t)] = \sigma^2.$$

A.6.2 A little algebra then shows that $x_m(u)$ is also an AR(1) series, namely:

$$x_m(u) = \mu + A(x_m(u-1) - \mu) + e_m(t)$$

where:

$$A = a^m$$

and

$$V[e_m(t)] = \sigma^2(1 - a^{2m})/(1 - a^2)$$

so that the sampled series has the same mean, a different autoregressive parameter (smaller if $0 < a < 1$), and a larger variance than the original series.

A.6.3 Assume now that $x(t)$ is generated by an AR(p) model. The forecast means depend on the solution of a recurrence equation, and depend on the roots of the polynomial in B described in ¶A.5.1:

$$1 - \Sigma_{j=1,p}\, a_j\, B^j = 0.$$

If the AR(p) model is stationary, then the absolute values of all these roots are greater than unity, and the absolute values of the reciprocals of the roots are less than unity.

A.6.4 The behaviour of the forecast means of $x(t)$ depends on these reciprocals, and, in the long run, on the reciprocal with the largest absolute value. Assume that this is real, positive and unique. Then, in the long run, the series behaves like an AR(1) model with the 'a' parameter equal to this largest reciprocal. The variance is a (possibly complicated) function of the coefficients multiplied by the original variance. See Section 4.4 for an example of this.

A.7 *The Ornstein-Uhlenbeck Process*

A.7.1 Just as Brownian motion or a Wiener process is the continuous equivalent of a random walk, so the Ornstein-Uhlenbeck process is the continuous equivalent of an AR(1) model. One can imagine a continuous series which, when sampled at however short an interval one chooses, behaves like an AR(1) series.

A.7.2 Such series are discussed, for example, by Cox & Miller (1965, pp 225-228) and Karatzas & Shreve (1991, p 358). If $x(t)$ has zero mean, then its derivative can be described as:

$$dx = -\beta x.dt + \rho.dz$$

where dz is the derivative of a Wiener process. Given $x(0) = x_0$ we find:

$$\text{E}[x(t)] = \text{e}^{-\beta t}x_0$$

and

$$\text{V}[x(t)] = \rho^2(1 - \text{e}^{-2\beta t})/2\beta.$$

A.7.3 We can compare these with the mean and variance of an AR(1) model, also with zero mean:

$$\text{E}[x_t] = a^t.x_0$$

and

$$\text{V}[x_t] = \sigma^2(1 - a^{2t})/(1 - a^2)$$

and we see that they correspond if we put:

$$\text{e}^{-\beta} = a$$

or

$$\beta = -\ln a$$

and

$$\rho^2(1 - \text{e}^{-2\beta})/2\beta = \sigma^2$$

or

$$\rho^2 = \sigma^2(-2\ln a)/(1 - a^2).$$

APPENDIX B

MULTIVARIATE TIME SERIES MODELS

B.1 *Vector Autoregressive (VAR) Models*

B.1.1 One way of extending a univariate model into a multivariate one is to treat x as a vector $(x_1, x_2, ..., x_n)'$. The mean of the series becomes a vector, the autoregressive or moving average coefficients become matrices, and instead of a single variance there is a matrix of variances and covariances.

B.1.2 The simplest form of two-factor VAR(1) model can be written as:

$$\boldsymbol{x}_t = \boldsymbol{\mu} + A(\boldsymbol{x}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{e}_t$$

where $\boldsymbol{x}_t$, $\boldsymbol{\mu}$ and $\boldsymbol{e}_t$ are two-element vectors and A is a 2 by 2 matrix. This can be written out in full as:

$$x_1(t) = \mu_1 + a_{11}(x_1(t-1) - \mu_1) + a_{12}(x_2(t-1) - \mu_2) + e_1(t)$$

and

$$x_2(t) = \mu_2 + a_{21}(x_1(t-1) - \mu_1) + a_{22}(x_2(t-1) - \mu_2) + e_2(t)$$

where $e_1(t)$ and $e_2(t)$ have variances σ_1^2 and σ_2^2 respectively, and covariance $\sigma_{12} = \rho\sigma_1\sigma_2$. One can represent the innovations in terms of two independent random variables in many ways; a convenient method is to choose $e_1(t)$ as the primary variable and to set:

$$e_2(t) = b.e_1(t) + e_3(t)$$

where $e_1(t)$ and $e_3(t)$ are independent, with:

$$b = \sigma_{12}/\sigma_1^2 = \rho\sigma_2/\sigma_1$$

and

$$\sigma_3^2 = \sigma_2^2 - b^2.\sigma_1^2 = (1 - \rho^2)\sigma_2^2.$$

See also Section B.4 for the generalisation of this method to more then two random residuals.

B.1.3 We can express $x_2(t)$ explicitly in terms of $x_1(t)$ rather than $e_1(t)$ by writing:

$$e_1(t) = x_1(t) - \mu_1 - a_{11}(x_1(t-1) - \mu_1) - a_{12}(x_2(t-1) - \mu_2)$$

whence:

$$x_2(t) = \mu_2 + b(x_1(t)-\mu_1) + (a_{21}-b.a_{11})(x_1(t-1)-\mu_1)$$
$$+ (a_{22}-b.a_{12})(x_2(t-1)-\mu_2) + e_3(t).$$

If a_{12}, the coefficient relating $x_1(t)$ to $x_2(t)$, is zero, we have a univariate model for $x_1(t)$, and a model for $x_2(t)$ that depends on current and previous values of $x_1(t)$ and on previous values of itself, as well as on one independent innovation. This is an example of a 'transfer function' model (see Section B.2).

B.1.4 It would be possible to investigate several of the investment series using VAR methods, as I have done for prices and wages in Section 3.5. However, I have found it more convenient to use 'transfer functions' for simulation. There is a good reason for this: while there may be some advantages in describing, for example, changes in the price index and the wages index with a VAR model, it means that it is necessary always to forecast both together. It seems a little inconvenient, if one just wishes to forecast prices, to also have to consider a model for wages. However, in the sort of applications with which actuaries are likely to be dealing, we are unlikely to wish to forecast wages without also considering prices. VAR models have, therefore, generally not been used.

B.2 *Transfer Function Models*

B.2.1 Transfer functions models allow models to be built successively for different time series, in what I have described previously as a 'cascade' fashion. The model for $x_2(t)$ in ¶B.1.3 is an example of such a model. In more general terms, two stationary series X and Y, can be related by:

$$y_t = C(\mathrm{B})/D(\mathrm{B}).x_t + yn_t$$

where $C(\mathrm{B})$ and $D(\mathrm{B})$ are polynomials in B, and yn_t follows a univariate ARMA(p,q) model.

B.2.2 The $C(\mathrm{B})$ polynomial allows each y_t to depend on a finite number of values of x_t. For example, putting $C(\mathrm{B}) = c_0 + c_1\mathrm{B}$, and putting $D(\mathrm{B}) = 1$ gives:

$$y_t = c_0.x_t + c_1.x_{t-1} + yn_t$$

whereas the $D(\mathrm{B})$ polynomial allows each y_t to depend on an indefinitely long run of past values of X. For example, putting $D(\mathrm{B}) = 1 - d\mathrm{B}$ and $C(\mathrm{B}) = 1$ gives:

$$y_t = x_t + d.x_{t-1} + d^2.x_{t-2} + \ldots + yn_t.$$

More complex transfer function models for several series can easily be devised. Transfer functions are easily recognised in my models for wages, dividend yields, share dividends, consols yields, etc.

B.2.3 For a transfer function model the 'gain' is calculated as $C(1)/D(1)$ (see McLeod, 1982). This is the amount by which y changes for a unit change in x. If this is unity, then the model is said to have 'unit gain'.

B.3 *Cointegrated Models*

B.3.1 Two integrated or I(1) series X and Y, are said to be 'cointegrated' if some linear combination of them, say:

$$Z = X - gY$$

can be represented as a stationary, I(0), time series. This can be extended to more than two variables, and there may be more than one 'cointegrating vector'.

B.3.2 Assume, for example, that Z is a stationary AR(1) series, with mean zero, so that:

$$z(t) = a.z(t-1) + e_z(t)$$

and that Y is a random walk with drift, so that:

$$y(t) = y(t-1) + \mu + e_y(t).$$

Then X is given by:

$$x(t) = z(t) + g.y(t) = a.z(t-1) + e_z(t) + g\{y(t-1) + \mu + e_y(t)\}$$

and putting:

$$z(t-1) = x(t-1) - g.y(t-1)$$

we get:

$$x(t) = x(t-1) + g.\mu - (1-a)\{x(t-1) - g.y(t-1)\} + e_z(t) + g.e_y(t).$$

Thus X behaves rather like a random walk with drift $g.\mu$, but, in addition, it has a tendency to move back towards gY through the 'error-correcting' term $(1-a)\{x(t-1) - g.y(t-1)\}$. Cointegrated series are, therefore, sometimes described as 'error-correcting' series.

B.3.3 Cointegrated series were first so called by Granger (1983). The collections of readings edited by Engle & Granger (1991) and by Rao (1994) are helpful. Methods for testing for cointegration have been developed by Johansen (see particularly Johansen & Juselius, 1990) [Søren Johansen is the son of a distinguished Danish actuary, Paul Johansen, who is an Honorary Overseas Member of the Institute]. The Johansen methods are conveniently implemented in the computer programme *MICROFIT 3.0* (Pesaran & Pesaran, 1991).

B.3.4 If two series are cointegrated, then their movements are constrained. They tend not to get 'too far away' from each other. Share dividends and share prices are a simple example of cointegrated series, where the cointegration vector is the simple one: $(1, -1)$ i.e. $\ln D - \ln P = \ln Y \sim I(0)$, so that the dividend yield is a stationary I(0), series.

B.4 *Cholesky Decomposition*

B.4.1 Consider n time series $X_1, ..., X_n$, all with zero mean, and all independent for successive time periods, but simultaneously correlated, so that they have covariance matrix $\boldsymbol{\Sigma}$, with terms σ_{ij} such that:

$$V[x_i(t)] = \sigma_{ii}$$

and

$$\mathrm{Cov}[x_i(t).x_j(t)] = \sigma_{ij}.$$

It may be convenient to express the Xs as functions of n independent random variables $E_1, ..., E_n$, each of which has zero mean and unit variance, for example for random simulation, or for the transformation described in ¶B.1.2. There are many ways in which the matrix $\boldsymbol{\Sigma}$ can be expressed in the form $\boldsymbol{AA}'$, but a convenient method is to choose $\boldsymbol{A}$ to be a lower triangular matrix $\boldsymbol{L}$ with $\boldsymbol{A}'$ the upper diagonal matrix $\boldsymbol{L}'$. This is known as the 'Cholesky decomposition' of the matrix $\boldsymbol{\Sigma}$ (see any text book on linear algebra, e.g. Strang, 1980). For the Cholesky decomposition to have real (rather than complex) terms, it is necessary that $\boldsymbol{\Sigma}$ be positive definite, as any valid covariance matrix necessarily is.

B.4.2 Denote the terms of $\boldsymbol{L}$ by c_{ij}, where $c_{ij} = 0$ if $j < i$. We start by putting $c_{11} = \sqrt{\sigma_{11}}$. Then we can calculate terms successively by:

$$c_{12} = \sigma_{12}/c_{11}$$

$$c_{22} = \sqrt{(\sigma_{22} - c_{12}^2)}$$

$$c_{13} = \sigma_{13}/c_{11}$$

$$c_{23} = (\sigma_{23} - c_{12}.c_{13})/c_{22}$$

$$c_{33} = \sqrt{(\sigma_{33} - c_{13}^2 - c_{23}^2)}$$

and so on. This is described in my Montréal paper (Wilkie, 1992), but there is an error therein in the formula for c_{23}.

APPENDIX C

PARAMETER ESTIMATION, MODEL FITTING AND DIAGNOSTIC TESTING

C.1 *Parameter Estimation*

C.1.1 There are two main methods for estimating the optimal parameters for a time-series model: maximum likelihood estimation and least squares estimation. Maximum likelihood estimation requires an assumption to be made about the distribution of the residuals. If they are assumed to be normally distributed, then the two methods produce similar answers. For an ordinary regression of Y on X, assuming that both are normally distributed, they produce identical answers, but for a time series model the answers are only the same by accident.

C.1.2 Consider the series x_i, $i = 1, ..., n$. The mean of the whole series is:

$$m = \Sigma_{i=1,n} x_i / n$$

and the variance is:

$$s^2 = \Sigma_{i=1,n}(x_i - m)^2/n.$$

If the values are independent, an estimate of the population variance is the same with the divisor $(n-1)$. The first autocorrelation coefficient is conventionally defined as:

$$r_1 = \Sigma_{i=1,n-1}(x_i - m)(x_{i+1} - m)/\Sigma_{i=1,n}(x_i - m)^2$$

where there are $(n-1)$ terms in the numerator and n terms in the denominator.

C.1.3 Now consider the ordinary regression of x_i on x_{i-1}, $i = 2, ..., n$. The mean and variance of the first $(n-1)$ values of X are, in general, not the same as the mean and variance of the last $(n-1)$ values. Define these as:

$$m_1 = \Sigma_{i=1,n-1} x_i/(n-1)$$

and

$$m_2 = \Sigma_{i=2,n} x_i/(n-1)$$

with corresponding definitions for s_1^2 and s_2^2. Then the correlation coefficient between the two subsets of X is given by:

$$r = \Sigma_{i=1,n-1}(x_i - m_1)(x_{i+1} - m_2)/(s_1.s_2.(n-1))$$

which, in general, is not the same as r_1.

C.1.4 Now consider the least squares fit of an AR(1) model to the series X, with:

$$x_t = \mu + \alpha(x_{t-1} - \mu) + e(t).$$

We may choose to minimise the sum of squares of the last $(n-1)$ values of the residuals:

$$S = \Sigma_{i=2,n}(x_i - \mu - \alpha(x_{i-1} - \mu))^2.$$

Our estimate of μ is not, in general, equal to any of m, m_1 or m_2 and our estimate of α is not, in general, the same as either r_1 or r. Whether we estimate σ as S/n, $S/(n-1)$ or $S/(n-2)$ (and any is defensible), none of these estimates is, in general, the same as s, s_1 or s_2.

C.1.5 The method just described omits any estimate of x_1, which would be calculated as $\mu + \alpha(x_0 - \mu)$, since we have assumed that we do not know the value of x_0. We can deal with this in a variety of ways: we can omit it, starting with x_2 as just described; we can use the value of x_0 if it is known (as it is for many of my series); we can assume that its value is its unconditional mean μ; we can use the 'backcasting' method of Box & Jenkins (1976), which makes use of the fact that, if any AR(p) series is reversed, it retains many of the same properties; or we can introduce x_0 as another parameter to be estimated, and calculate the values of μ, α and x_0 that give a least squares fit. This last method applied to an AR(1) model gives a value of x_0 that makes the first residual equal to zero, so the estimates of μ and α are the same as in the first method, but the estimate of σ may be different.

C.1.6 Similar considerations apply to the starting values for an MA(q) or ARMA(p,q) model. The estimate of x_1 depends on the just previous residual, say e_0, and possibly on earlier residuals. These are, in general, not known. Some assumption must be made about their value. Possibilities include: setting their values to their expected values, zero; including the unobserved residuals as additional parameters to be estimated; or the backcasting method.

C.1.7 If maximum likelihood estimation is used, which, in general, is preferable to least squares estimation, then there are similar choices to be made about the starting values, and, in addition, one must make suitable assumptions about their variance. One can assume that, for example, x_1 is distributed according to the unconditional mean and variance of the series, and proceed from there; or one can make use of one's knowledge of prior values, as I have generally done; or make other assumptions.

C.1.8 I do not wish to discuss the relative merits of any of these methods. It is sufficient to point out that different statisticians and different computer packages may use different methods, and, therefore, may obtain different parameter estimates. The more terms the series has, the less difference these end effects have, since, in general, the estimators are asymptotically equivalent, i.e. they give the same answers if the series is sufficiently long.

C.1.9 In general, I have estimated the parameters using least squares estimates calculated by a non-linear optimisation method (generally the Nelder-Meade simplex method). Where possible I have allowed for the previous known starting values. This has been done for the previous values in my estimation of the models for inflation, wages, dividend yields and consols yields. For the dividend model and consols yield model, I have used a very long run-in period for the calculation of *DM* and *CM*, since values of the inflation rate are available for what is, in this context, an effectively infinite past period. For the dividend model I have assumed that the values of *YE* and *DE* in year '0' (e.g. 1923, or whatever year is appropriate) are zero. For the calculations in Sections 2.7, 4.8 and 10.4, where I fit AR(1) models to a great many series for different countries and at different frequencies, I use a simple method and estimate the mean μ (*QMU* or ln *YMU*, etc., as appropriate), by the mean of the observations m and the autoregressive parameter α (*QA* or *YA*, etc.), by the first autocorrelation coefficient r_1.

C.2 *Model Testing*

C.2.1 In principle, one would like to know which model to fit before estimating the parameters for it. In practice, one has to fit one or more plausible models, and then decide which one is the most satisfactory amongst those that have been tried. Several criteria are available for deciding between alternative models.

C.2.2 As for an ordinary linear regression, one can calculate the standard errors of the parameter estimates. This can conveniently be done for the general non-linear maximum likelihood method by calculating the 'information matrix' at the optimum point. This is calculated from the inverse of the Hessian matrix of second derivatives of the log likelihood function at the point where it is maximised (see any standard statistical text book for the principles).

C.2.3 Each parameter estimate can then be compared with, say, twice the calculated standard error; this gives a roughly 2½% probability level at one side or a roughly 5% level for a two-sided test. I do not believe that one should stick rigidly to a precise level of significance for this, or any other, test. The probability level is a guide, and any figure such as 5%, 1% or twice the standard error is a 'rule of thumb' rather than a rigid prescription. The standard error of the parameter estimate is only one piece of evidence to be taken into account. Another is whether the parameter has any sensible 'economic' significance; is it plausible that the implied influence of one variable on another should exist?

C.2.4 I do not use one measure that is common in ordinary regression analysis, that of the R^2 of the regression, or the reduction in the original variance produced by the regression as compared with the original variance. In order for this to be a useful measure, one needs to define the 'null' model, against which the reduction in variance is to be compared. In time-series analysis this is not so clear as with ordinary regression, where independence of the variables is usually at least plausible. However, when considering, for example, dividend yields, it is unreasonable to assume that yields from day to day are wholly unconnected, so the null of no correlation is quite unrealistic. An alternative might be to consider the random walk model as the null, but this assumes that what is being modelled as a stationary series is not stationary, which is also unrealistic. In these circumstances I avoid using R^2 at all.

C.2.5 A test appropriate for two models, one of which is a generalisation of another, is the 'likelihood ratio' test (again see any standard statistical text book). If a least squares method of estimation has been used, the log likelihood can be estimated by $n \ln \sigma$. Twice the difference between the log likelihoods of two models that differ, in that k parameters of the more general model are set to zero in the simpler model, is distributed as χ_k^2. In practice, this works out that, if the models differ by only one parameter, the log likelihood needs to improve by more than 2 (i.e. χ^2 by more than 4) for the additional parameter to be significant at a 5% level. Again, this is not an absolute rule, but another piece of evidence. This is an appropriate test for time series.

C.2.6 Thirdly, one can consider diagnostic tests on the residuals, which are described more fully in Section C.3. If the residuals from any model fail these diagnostic tests conspicuously, one may wish to consider a more elaborate model whose residuals would pass the tests; but this depends on which tests have been failed. Significant autocorrelation in the residuals suggests that a higher order AR(p) or MA(q) model should be tried. Significant crosscorrelation between the residuals for one variable and the residuals for another suggests that additional transfer function terms should be included. Significant autocorrelations or crosscorrelations between the squares of the residuals or the variables themselves suggest that an ARCH model should be considered (see Section D.1). Significant evidence of non-normality may suggest either an ARCH model or some non-normal model for the residuals.

C.3 *Diagnostic Testing of the Residuals*

C.3.1 Having chosen or estimated the parameters of a model, one can calculate the residuals using that set of parameters. Diagnostic testing of the parameters means applying a number of tests to show:

— whether the residuals appear to be independent of each other and of the residuals for other variables; and
— whether the residuals appear to be distributed according to the hypothesis of the model, which in general means normally.

The same questions are asked about whether a mortality graduation fits the observed data, and many of the tests used are the same. Actuaries should, therefore, be familiar with the principles.

C.3.2 The first test to use is to calculate the autocorrelation coefficients of the residuals. If the residuals are independent of one another, then the autocorrelation coefficients will not be significantly different from zero; the standard error (assuming normality) of any correlation coefficient in these circumstances is $1/\sqrt{n}$, where n is the number of observations (residuals) in the series. In addition, the partial autocorrelation coefficients are tested (see any time series text book).

C.3.3 Since the models for the different variables are interconnected, it is appropriate also to test for significant crosscorrelations between the residuals for different variables, both simultaneously and at various lags. The crosscorrelation coefficients between series X and Y, at lags ..., -1, 0, 1, ... are defined in the obvious way. Assuming normality and independence, the standard error of each coefficient is also $1/\surd n$.

C.3.4 A third test for independence is to calculate the observed spectrum (or periodogram) of the series, which is the Fourier transform of the observations, usually smoothed in some way. This is particularly appropriate if specific periodic effects are suspected. I have not found regular periodic effects in any of my annual series, though they are clearly there for certain of the monthly series (e.g. the inflation series).

C.3.5 A test familiar from mortality graduations is the Wald-Wolfowitz runs test, which is the same as Steven's change of signs test. Although I apply this test to all the series of residuals, it is less sensitive than the autocorrelation function for identifying lack of independence.

C.3.6 The first test I use for the normality of residuals is to construct a frequency table using a convenient number of equal intervals of the distribution function, and to count how many observations fall into each interval. If the residuals are normally distributed, then the expected numbers in each 'cell' are equal. A χ^2 test for 'actual minus expected' can then be carried out.

C.3.7 Unless the number of observations is quite large, I find that this grouping test is less sensitive than a test of the skewness and kurtosis coefficients. For a normal distribution the theoretical skewness and kurtosis coefficients are zero and 3, respectively. The observed coefficients are defined by:

$$\surd b_1 = m_3/m_2^{3/2}$$

and

$$b_2 = m_4/m_2^2$$

where m_j is the jth moment about the observed mean m.

C.3.8 If a sample of n cases is drawn from a normal distribution, then the observed skewness coefficient $\surd b_1$ is distributed $N(0,6/n)$ and the observed kurtosis coefficient b_2 is distributed $N(3, 24/n)$ (see, for example, Kendall & Stuart, 1977, p 258). Jarque & Bera (1981) therefore suggest a composite test, since:

$$J = n\{b_1/6 + (b_2 - 3)^2/24\}$$

is distributed as χ_2^2.

C.3.9 If the number of observations is large enough (say over 100), then the Kolmogorov-Smirnov test of the maximum deviation of the observed cumulative distribution from the normal distribution function can be used. I have not used this test, because, in general, the sample sizes for the annual series have been too small.

APPENDIX D

ALTERNATIVE MODELS

D.1 *Autoregressive Conditional Heteroscedastic (ARCH) Models*

D.1.1 Autoregressive conditional heteroscedastic (ARCH) models were introduced by Engle (1982), and there have been literally hundreds of papers about or using them since then. Bollerslev, Chou & Kroner (1992) give a good review of various models; Hamilton (1994, Chapter 21) provides a useful text book introduction; Taylor (1986) and Mills (1994) apply them to the investigation of stock market series. Tong (1990) and Granger & Teräsvirta (1993) discuss ARCH models along with many other forms of non-linear time series models.

D.1.2 There seems to be varying terminology for ARCH models; I shall not use anything more specific than 'ARCH'. The essential feature is that the variance of the innovations of a time-series model, for say $x(t)$, is not treated as a constant $\sigma^2 = V$, but as a stochastic variable $V(t)$. Usually the innovation $e(t)$, is calculated from:

$$e(t) = \sqrt{V(t)}.w(t)$$

where the ws are IID with zero mean and unit variance, and are often assumed to be normally distributed.

D.1.3 The models differ in how $V(t)$ depends on the information at time $(t-1)$. One option is to put:

$$V(t) = a_0 + \Sigma_{i=1,p}\, a_i.(x(t-i) - \gamma)^2$$

with $a_0 > 0$, and all the other $a_i \geq 0$, to ensure that the variance is positive. The variance of $e(t)$ is assumed to depend on the squared values of the deviations of the p previous observations of x from some value γ. One could take this value as the mean of x, μ. These are the forms I have used in Section 2.8, with $p = 1$. If $\gamma = \mu$, one can write this model as:

$$V(t) = a_0 + \Sigma_{i=1,p}\, a_i.e(t-i)^2$$

so that $V(t)$ depends on the squares of the p previous values of the innovations.

D.1.4 Another form is to put:

$$V(t) = a_0 + \Sigma_{i=1,p}\, a_i.e(t-i)^2 + \Sigma_{i=1,q}\, b_i.V(t-i)$$

analogous to an ARMA(p,q) process.

D.1.5 A different approach is to put:

$$e(t)^2 = a_0 + \Sigma_{i=1,q}\, a_i.e(t-i)^2 + w(t)$$

where the square of the actual innovation at time t depends on the squares of the q previous innovations, plus a further random innovation $w(t)$, which has a suitable distribution so that $e(t)^2$ does not become negative.

D.1.6 When there are several variables, connected either by transfer function models or by vector autoregressive models, there are even more options, because the variance of the innovations for one series could be made to depend on the values of any of the other series, or of their innovations, as well as of their own.

D.1.7 For the ARCH AR(1) model that I have used in Section 2.8, namely:

$$V(t) = QSD(t)^2 = QSA + QSB.(I(t-1) - QSC)^2$$

with $I(t)$, also AR(1):

$$I(t) = QMU + QA.(I(t-1) - QMU) + QSD(t).QZ(t)$$

one can derive the unconditional expected value of $V(t)$, which is:

$$\{QSA + QSB.(QMU - QSC)^2\}/(1 - QSB/(1 - QA^2))$$

provided that $QSB < (1 - QA^2)$, which it is with the parameters suggested, $QSB = 0.55$ and $QA = 0.6$. If this were not the case, the variance would increase without limit as t increased; this would be an undesirable feature of such a model. However, see Section 11.5 for the problems encountered even with the suggested parameters.

APPENDIX E

FORECASTING THE MODELS

E.1 *Principles*

E.1.1 It is possible to calculate the forecast means, variances and covariances of parts of my model analytically. This can be done exactly for those parts of the model that are linear, and can be done approximately for some other parts. In my earlier papers I had dealt with forecasting by simulation, a method that can be applied in almost any circumstances. I still use simulation for the forecast results given in Section 11. In this Appendix I describe the method of forecasting analytically, and give some results. Some of the results have been developed by Kitts (1988), Bonsdorff (1991), Hürlimann (1993) and Huber (1995).

E.1.2 We denote 'now' by $t = 0$, and we assume that we know $\mathscr{F}_0$, i.e. all the relevant facts for $t \leq 0$. We wish to calculate the mean and variance of the future values of some series, say X, i.e. to calculate $E[X(t)]$ and $V[X(t)]$, for any $t > 0$. The general principle is to express $X(t)$ in terms of elements included in $\mathscr{F}_0$ and future values of the relevant innovations, including the *XE*s, and possibly also other innovations such as *YE*s and *ZE*s.

E.1.3 A specific example will help. The model for inflation is a simple AR(1) model:

$$I(t) = QMU + QA.(I(t-1) - QMU) + QE(t)$$

so:

$$I(1) = QMU + QA.(I(0) - QMU) + QE(1).$$

$I(0)$ is known; we denote $I(0) - QMU$ as $QN(0)$, and put:

$$I(1) - QMU = QA.QN(0) + QE(1).$$

We can continue:

$$I(2) = QMU + QA.(I(1) - QMU) + QE(2)$$

$$= QMU + QA.\{QA.QN(0) + QE(1)\} + QE(2)$$

$$= QMU + QA^2.QN(0) + QA.QE(1) + QE(2).$$

Continuing similarly, we get:

$$I(t) = QMU + QA^t.QN(0) + QA^t.QE(1) + QA^{t-1}.QE(2) + \ldots$$

$$+ QA.QE(t-1) + QE(t).$$

E.1.4 This is an example of a general formula, where we express, say, $X(t)$ as:

$$X(t) = f_t(\mathscr{F}_0) + \Sigma_{i=1,t}\psi_i.XE(t-i+1)$$

where $f_t(\mathscr{F}_0)$ is some function that contains only known terms included in $\mathscr{F}_0$. Now $\mathrm{E}[XE(i)] = 0$ for $i > 0$, so we can put:

$$\mathrm{E}[X(t)] = f_t(\mathscr{F}_0).$$

Also $\mathrm{E}[XE(i)^2] = XSD^2$ for $i > 0$, and $\mathrm{E}[XE(i).XE(j)] = 0$ for $i \neq j$. Now we can put:

$$\mathrm{V}[X(t)] = \mathrm{E}[(X(t) - \mathrm{E}[X(t)])^2]$$

$$= \mathrm{E}[(\Sigma_{i=1,t}\psi_i.XE(t-i+1))^2]$$

$$= (\Sigma_{i=1,t}\psi_i^2).XSD^2$$

$$= \Psi_t.XSD^2.$$

The term 'ψ-weights' is that of Box & Jenkins (1976).

E.1.5 In the case of $I(t)$ we have:

$$\mathrm{E}[I(t)] = f_t(\mathscr{F}_0) = QMU + QA^t.QN(0)$$

and:

$$\psi_i = QA^{i-1}$$

so:

$$\mathrm{V}[I(t)] = (\Sigma_{i=1,t}\psi_i^2).QSD^2$$

$$= (\Sigma_{i=1,t}QA^{2(i-1)}).QSD^2$$

$$= (1 + QA^2 + \ldots + QA^{2(t-2)} + QA^{2(t-1)}).QSD^2$$

$$= \{(1 - QA^{2t})/(1 - QA^2)\}.QSD^2, \text{ provided that } QA \neq 1.$$

E.1.6 It is convenient to introduce a notation borrowed from actuarial compound interest terminology for the sum of a geometric series. We put:

$$\ddot{a}(x,t) = 1 + x + \ldots + x^{t-2} + x^{t-1}$$

$$= (1 - x^t)/(1 - x) \qquad \text{if } x \neq 1$$

$$= t \qquad \text{if } x = 1.$$

So we can write:

$$V[I(t)] = \ddot{a}(QA^2,t).QSD^2.$$

E.1.7 The expected values and the ψ-weights can also be calculated recursively. A simulation of future values with *QSD* set to zero provides the expected values, and suitable formulae allow the ψ-weights to be calculated. In the case of *I* we get:

$$E[I(1)] - QMU = QA.QN(0)$$

$$E[I(t)] - QMU = QA.\{E[I(t - 1)] - QMU\}$$

and also:

$$\psi_1 = 1$$

$$\psi_i = QA.\psi_{i-1}.$$

E.1.8 In this case these recurrence relations are easily solved to give explicit results. In other cases the recursive method is easier to implement. However, it may not be easy to see the asymptotic properties unless explicit formulae are found. Doing it both ways also provides a check on the results.

E.2 *Retail Prices*

E.2.1 We have shown that:

$$E[I(t)] = QMU + QA^t.QN(0)$$

and

$$V[I(t)] = \ddot{a}(QA^2,t).QSD^2.$$

E.2.2 By similar methods we get:

$$E[\ln Q(t)] = \ln Q(0) + t.QMU + QA.\ddot{a}(QA,t).QN(0)$$

and

$$V[\ln Q(t)] = QSD^2.\{t - 2QA.\ddot{a}(QA,t) + QA^2.\ddot{a}(QA^2,t)\}/(1 - QA)^2.$$

E.2.3 Since it is assumed that $\ln Q(t)$ is distributed normally, $Q(t)$ is distributed lognormally. For the general lognormal distribution, where $\ln X \sim N(\mu,\sigma^2)$:

$$E[X^r] = \exp(r\mu + \tfrac{1}{2}r\sigma^2)$$

so:

$$E[X] = \exp(\mu + \tfrac{1}{2}\sigma^2)$$

and

$$V[X] = \{E[X]\}^2.(\exp(\sigma^2) - 1).$$

Using these formula we can calculate the mean and variance of $Q(t)$.

E.2.4 We can also find asymptotic results as $t \to \infty$. For I we get:

$$E[I(t \to \infty)] \to QMU$$

and

$$V[I(t \to \infty)] \to QSD^2/(1 - QA^2)$$

provided, in both cases, that $|QA| < 1$. For ln Q we get:

$$E[\ln Q(t \to \infty)] \to \infty$$

provided that $QMU \neq 0$, and:

$$V[\ln Q(t \to \infty)] \to \infty.$$

E.2.5 This demonstrates one difference between a stationary I(0) series and an integrated I(1) series: the asymptotic mean and variance of the former are often finite, and of the latter are often infinite (but, for example, the asymptotic mean would not be finite in this case if QMU were zero).

E.2.6 We can also compare the variance of ln Q with the variance it would have if it were generated by a random walk model, by putting $QA = 0$. Call the variable Q^*, so that:

$$\ln Q^*(t) - \ln Q^*(t-1) = I^*(t) = QMU + QE(t).$$

Then:

$$E[\ln Q^*(t)] = \ln Q^*(0) + t.QMU$$

and

$$V[\ln Q^*(t)] = t.QSD^2.$$

E.2.7 Thus, the expected value of ln Q^* follows a straight course from ln $Q^*(0)$ with slope QMU, independently of the value of $QN^*(0)$, whereas the expected value of ln Q moves asymptotically towards a parallel track, affected by the value of $QN(0)$. The variance of ln Q^* depends linearly on t, and hence its standard deviation depends linearly on $\sqrt{t}$, whereas the variance of ln Q is larger (if $QA > 0$). The ratio of the two variances is given by:

$$VRQ(t) = \mathrm{V}[\ln Q(t)]/\mathrm{V}[\ln Q^*(t)]$$

$$= \{1 - 2QA.\ddot{a}(QA,t)/t + QA^2.\ddot{a}(QA^2,t)/t\}/(1 - QA)^2$$

which tends asymptotically to $1/(1 - QA)^2$. Thus, although the variances are the same for $t = 1$, in the long run the variance of ln Q is larger, equivalent to a random walk with standard deviation $QSD/(1 - QA)$. This is shown in Figure 2.7.

E.3 *Wages*

E.3.1 I have described two models for wages in Section 3, a transfer function model and a VAR model. For the transfer function model we get:

$$\mathrm{E}[J(t)] = (WW1 + WW2).QMU + WQ.QA^{t-1}.QN(0) + WMU + WA^t.WN(0)$$

and

$$\mathrm{V}[J(t)] = (WQ^2.\ddot{a}(QA^2,t-1) + WW1^2).QSD^2 + \ddot{a}(WA^2,t).WSD^2$$

where:

$$WQ = WW1.QA + WW2$$

and

$$WN(0) = J(0) - WW1.I(0) - WW2.I(-1) - WMU.$$

E.3.2 As $t \rightarrow \infty$:

$$\mathrm{E}[J(t \rightarrow \infty)] \rightarrow (WW1 + WW2).QMU + WMU$$

and

$$\mathrm{V}[J(t \rightarrow \infty)] \rightarrow (WQ^2/(1 - QA^2) + WW1^2).QSD^2 + WSD^2/(1 - WA^2).$$

E.3.3 The next step is to find, after considerable manipulation:

$$\mathrm{E}[\ln W(t)] = \ln W(0) + t.(WW1 + WW2).QMU + WQ.\ddot{a}(QA,t).QN(0)$$

$$+ t.WMU + WA.\ddot{a}(WA,t).WN(0)$$

and

$$\mathrm{V}[\ln W(t)] = \{t.WW1^2 + 2.WW1.WQ.(t - 1 - QA.\ddot{a}(QA,t-1))/(1 - QA)$$

$$+ WQ^2.(t - 1 - 2.QA.\ddot{a}(QA,t-1) - QA^2.\ddot{a}(QA^2,t-1))/(1 - QA)^2\}.QSD^2$$

$$+ (t - 2WA.\ddot{a}(WA,t) + WA^2.\ddot{a}(WA^2,t))/(1 - WA)^2.WSD^2.$$

E.3.4 As $t \rightarrow \infty$, both E[ln $W(t \rightarrow \infty)$] and V[$W(t \rightarrow \infty)$] $\rightarrow \infty$. However, it is relevant to compare the asymptotic behaviour of V[ln $W(t)$] with a random walk with the same one-year variance. This can be done by considering the ratio:

$$VRW(t) = \mathrm{V}[\ln W(t)]/(t.\mathrm{V}[\ln W(1)]).$$

This is shown in Figure 3.4.

E.3.5 The VAR(1) model for prices and wages together is, in principle, the same as the model for prices alone, and the results are of the same form as shown in Section E.2 for $I(t)$ and ln $Q(t)$, with vectors $[I(t), J(t)]'$ replacing $I(t)$ alone, [ln $Q(t)$, ln $W(t)$]$'$ replacing ln $Q(t)$, $[QMU, WMU]'$ replacing QMU, the matrix $\boldsymbol{A}$ of ¶¶B.1.2 and 3.5.1 replacing QA, and a suitable covariance matrix $\boldsymbol{\Sigma}$ replacing QSD.

E.3.6 The ψ-weights, which are now matrices, show the impulse response function, that is, the responses of ln $Q(t)$ and ln $W(t)$ to 'spikes' of unity in $I(0)$ and $J(0)$ (see Hamilton, 1994, Chapter 11), and the ultimate response function is the value of $\psi(t)$ as $t \rightarrow \infty$, i.e. the long-run response of ln $Q(t)$ and ln $W(t)$ to spikes in $I(0)$ and $J(0)$. For the VAR(1) model it is given by:

$$\boldsymbol{G} = (\boldsymbol{I} - \boldsymbol{A})^{-1}$$

where $\boldsymbol{I}$ is the identity matrix.

E.4 *Share Dividend Yields, Dividends and Prices*

E.4.1 The model for dividend yields is basically an AR(1) model, but there is an additional influence from inflation, so there are also terms in QE, and hence QSD^2 as well as in YE, and hence YSD^2. Note that E[$QE(i).YE(j)$] = 0 for all i and j. We get:

$$\mathrm{E}[\ln Y(t)] = YW.\{QMU + QA^t.QN(0)\} + \ln YMU + YA^t.YN(0)$$

where:

$$YN(0) = \ln Y(0) - YW.I(0) - YMU$$

and

$$\mathrm{V}[\ln Y(t)] = YW^2.\ddot{a}(QA^2,t).QSD^2 + \ddot{a}(YA^2,t).YSD^2.$$

E.4.2 As $t \to \infty$:

$$E[\ln Y(t \to \infty)] \to YW.QMU + \ln YMU$$

and

$$V[\ln Y(t \to \infty)] \to YW^2.QSD^2/(1 - QA^2) + YSD^2/(1 - YA^2).$$

E.4.3 The model for dividends contains an influence from inflation, and thereafter is basically an MA(1) model, with a short-term additional influence from dividend yields, so there are terms in QE and QSD^2, YE and YSD^2, as well as in DE and DSD^2. After a great deal of manipulation, we get:

$$E[DM(t)] = DD.\ddot{a}(DDC,t).QMU + DDC^t.DM(0) + DD.DDC^{t-1}.QA.\ddot{a}(QD,t).QN(0)$$

where:

$$DDC = 1 - DD$$

and

$$QD = QA/DDC$$

and

$$V[DM(t)] = DD^2.\{\ddot{a}(DDC^2,t) - 2QD.\ddot{a}(DDC.QA,t)$$

$$+ QD^2.\ddot{a}(QA^2,t)\}.QSD^2/(1 - QD)^2$$

provided that $QD \neq 1$, which is the case with the suggested parameters, but could easily not have been the case.

E.4.4 As $t \to \infty$:

$$E[DM(t \to \infty)] \to QMU$$

and

$$V[DM(t \to \infty)] \to DD^2.\{1/(1 - DDC^2) - 2QD/(1 - DDC.QA)$$

$$+ QD^2/(1 - QA^2)\}.QSD^2/(1 - QD^2).$$

E.4.5 We then get:

$$E[K(1)] = (DW.DD + DX).QMU + DW.DDC.DM(0)$$

$$+ (DW.DD + DX).QA.QN(0) + DMU + DY.YE(0) + DB.DE(0)$$

and

$$V[K(1)] = (DW.DD + DX)^2.QSD^2 + DSD^2$$

and for $t > 1$:

$$E[K(t)] = \{DW.DD.\ddot{a}(DDC,t) + DX\}.QMU + DW.DDC^t.DM(0)$$

$$+ \{DW.DD.DDC^{t-1}.\ddot{a}(QD,t) + DX.QA^{t-1}\}.QA.QN(0) + DMU$$

and

$$V[K(t)] = \{f_1^2.\ddot{a}(DDC^2,t) - 2.f_1.f_2.\ddot{a}(DDC.QA,t) + f_2^2.\ddot{a}(QA^2,t)\}.QSD^2$$

$$+ DY^2.YSD^2 + (1 + DB^2).DSD^2$$

where:

$$f_1 = DW.DD/(1 - QD)$$

and

$$f_2 = DW.DD.QD/(1 - QD) - DX.$$

E.4.6 As $t \to \infty$:

$$E[K(t \to \infty)] \to (DW + DX).QMU + DMU$$

and

$$V[K(t \to \infty)] \to \{f_1^2/(1 - DDC^2) - 2f_1.f_2/(1 - DDC.QA)$$

$$+ f_2^2/(1 - QA^2)\}.QSD^2 + DY^2.YSD^2 + (1 + DB^2).DSD^2.$$

E.4.7 We next find:

$$E[\ln D(t)] = \ln D(0) + \{DW.(t - DDC.\ddot{a}(DDC,t)) + t.DX\}.QMU$$

$$+ DW.DDC.\ddot{a}(DDC,t).DM(0)$$

$$+ \{DW.DD.(\ddot{a}(DDC,t) - \ddot{a}(QA,t).QD)/(1 - QD) + DX.\ddot{a}(QA,t)\}.QA.QN(0)$$

$$+ t.DMU + DY.YE(0) + DB.DE(0)$$

and

$$V[\ln D(t)] = \{f_1^2.(t - 2\ddot{a}(DDC,t) + \ddot{a}(DDC^2,t))$$

$$- 2f_1.f_2.(t - \ddot{a}(DDC,t) - \ddot{a}(QA,t) + \ddot{a}(DDC.QA,t))/(1 - QA)$$

$$+ f_2^2.(t - 2\ddot{a}(QA,t) + \ddot{a}(QA^2,t))/(1 - QA)^2\}.QSD^2$$

$$+ (t - 1).DY^2.YSD^2 + \{(t - 1).(1 + DB)^2 + 1\}.DSD^2$$

both these formulae being valid for $t = 1$ as well as $t > 1$.

E.4.8 As $t \to \infty$, both E[ln $D(t \to \infty)$] and V[$D(t \to \infty)$] $\to \infty$. Again, we can compare the asymptotic behaviour of V[ln $D(t)$] with a random walk with the same one-year variance, by considering the ratio:

$$VRD(t) = \mathrm{V}[\ln D(t)]/(t.\mathrm{V}[\ln D(1)]).$$

This is shown in Figure 5.5.

E.4.9 Finally, noting that $\ln P(t) = \ln D(t) - \ln Y(t)$, we get:

$$\mathrm{E}[\ln P(t)] = \mathrm{E}[\ln D(t)] - \mathrm{E}[\ln Y(t)]$$

$$= \ln D(0) + \{DW.(t - DDC.\ddot{a}(DDC,t)) + t.DX\}.QMU$$

$$+ DW.DDC.\ddot{a}(DDC,t).DM(0)$$

$$+ \{DW.DD.(\ddot{a}(DDC,t) - \ddot{a}(QA,t).QD)/(1 - QD)$$

$$+ DX.\ddot{a}(QA,t)\}.QA.QN(0) + t.DMU + DY.YE(0) + DB.DE(0)$$

$$- \{YW.QMU + QA^t.QN(0) + \ln YMU + YA^t.YN(0)\}$$

and

$$\mathrm{V}[\ln P(t)] = \mathrm{V}[\ln D(t)] + \mathrm{V}[\ln Y(t)] - 2YW.\{(f_1 - f_2).\ddot{a}(QA,t)/(1 - QA)$$

$$- DW.DDC.\ddot{a}(DDC.QA,t)/(1 - QD) + f_2.QA.\ddot{a}(QA^2,t)/(1 - QA)\}.QSD^2$$

$$- 2DY.YA.\ddot{a}(YA,t-1).YSD^2$$

where the extra terms represent the covariance between ln $D(t)$ and ln $Y(t)$.

E.4.10 As $t \to \infty$, both E[ln $P(t \to \infty)$] and V[$P(t \to \infty)$] $\to \infty$. Again it is of relevance to compare the asymptotic behaviour of V[ln $P(t)$] with a random walk with the same one-year variance, by considering the ratio:

$$VRP(t) = \mathrm{V}[\ln P(t)]/(t.\mathrm{V}[\ln P(1)]).$$

This is shown in Figure 5.7.

E.5 *Long-Term Interest Rates*

E.5.1 The model for long-term interest rates is more difficult, since it is not wholly linear. The consols yield splits into two parts:

$$C(t) = CW.CM(t) + CR(t).$$

CM is linear in *QE*, hence normally distributed, whereas ln *CR* is linear in *CE* and *YE*, so *CR* is lognormally distributed. Since no innovation appears in the formulae for both parts, they are independent. The sum of a normal and a lognormal is distributed neither normally nor lognormally, but presumably somewhere in between. However, the two parts can be investigated separately.

E.5.2 The formula for *CM* is the same as that for *DM*, with different parameters. The same results, therefore, apply, namely:

$$E[CM(t)] = CD.\ddot{a}(CDC,t).QMU + CDC^t.CM(0)$$

$$+ CD.CDC^{t-1}.QA.\ddot{a}(QC,t).QN(0)$$

where:

$$CDC = 1 - CD$$

and

$$QC = QA/CDC$$

and

$$V[CM(t)] = CD^2.\{\ddot{a}(CDC^2,t) - 2QC.\ddot{a}(CDC.QA,t)$$

$$+ QC^2.\ddot{a}(QA^2,t)\}.QSD^2/(1 - QC)^2$$

provided that $QC \neq 1$.

E.5.3 The formula for *CR* is essentially that for an AR(3) model, with a modified residual that is correlated with *YE*(*t*). This is one case where recurrence relations make things much easier. We first put:

$$CN(t) = \ln CR(t) - \ln CMU$$

and

$$CE^*(t) = CY.YE(t) + CE(t).$$

Then:

$$CN(t) = CA1.CN(t-1) + CA2.CN(t-2) + CA3.CN(t-3) + CE^*(t).$$

We start with:

$$CN(1) = CA1.CN(0) + CA2.CN(-1) + CA3.CN(-2) + CE^*(1)$$

so:

$$E[CN(1)] = CA1.CN(0) + CA2.CN(-1) + CA3.CN(-2)$$

and

$$\psi_1 = 1.$$

Then:

$$CN(2) = CA1.CN(1) + CA2.CN(0) + CA3.CN(-1) + CE^*(2)$$

so:

$$E[CN(2)] = CA1.E[CN(1)] + CA2.CN(0) + CA3.CN(-1)$$

and

$$\psi_2 = CA1.$$

Next:

$$CN(3) = CA1.CN(2) + CA2.CN(1) + CA3.CN(0) + CE^*(3)$$

so:

$$E[CN(3)] = CA1.E[CN(2)] + CA2.E[CN(1)] + CA3.CN(0)$$

and

$$\psi_3 = CA1^2 + CA2.$$

Thereafter:

$$E[CN(t)] = CA1.E[CN(t-1)] + CA2.E[CN(t-2)] + CA3.E[CN(t-3)]$$

and

$$\psi_t = CA1\psi_{t-1} + CA2\psi_{t-2} + CA3\psi_{t-3}.$$

Then:

$$E[\ln CR(t)] = E[CN(t)] + \ln CMU$$

and

$$V[\ln CR(t)] = (\Sigma_{i=1,t}\psi_i^2).(CY^2.YSD^2 + CSD^2).$$

E.5.4 Noting the independence of $CM(t)$ and $CR(t)$ we can then put:

$$E[C(t)] = CW.E[CM(t)] + E[CR(t)]$$

and

$$V[C(t)] = CW^2.V[CM(t)] + V[CR(t)]$$

calculating $E[CR(t)]$ and $V[CR(t)]$ by the formulae for a lognormal distribution shown in ¶E.2.3.

E.5.5 Provided that $V[\ln CR(t)]$ is fairly small, say less than 0.01, then the distribution of $CR(t)$ is quite close to normal, and one could assume, therefore, that $C(t)$ is almost normally distributed. As $V[\ln CR(t)]$ increases, it may be a better approximation to assume that $C(t)$ is lognormally distributed.

E.6 *Short-Term Interest Rates, Property, Index-Linked Yields and Exchange Rates*

E.6.1 The models for these are all relatively simple, being either AR(1) models, or similar to other models already shown. The reader is invited to develop the formulae him or herself. The only complication is in combining the lognormal distribution for the ratio B/C with the non-standard distribution for C.

APPENDIX F

DATA SOURCES

F.1 *U.K. Retail Price Indices*

F.1.1 The source of the U.K. monthly data is Department of Employment (1986) (DE) and regular DE publications since then. The monthly series starts in August 1914, and is taken from:
- — 1914-1947: cost of living index numbers, 1914-1947;
- — 1947-1994: general index of retail prices, various base dates, 1947-55, 1956-61, 1962-73, 1974-86, 1987-94.

F.1.2 The annual data for earlier centuries have been constructed by taking several indices and splicing them together; since 1914 June values of the monthly series have been used (assuming that June 1914 had the same value as August 1914). Most of the older indices have been taken from Mitchell & Deane (1962) (MD), but the first is from Phelps Brown & Hopkins (1956) (PBH). The series are:
- — 1264-1661: seven centuries of the prices of consumables (PBH);
- — 1661-1696: Schumpeter-Gilboy price indices 1661-1823, A, consumers' goods (MD);
- — 1696-1790: Schumpeter-Gilboy price indices 1661-1823, B, consumers' goods (MD);
- — 1790-1850: indices of British commodity prices 1790-1850, based on the Gayer, Rostow & Schwarz monthly indices (MD);
- — 1850-1871: the Rousseaux price indices, 1800-1913, overall index (MD);
- — 1871-1914: Board of Trade wholesale price indices, 1871-1938, total index (MD); and
- — 1914-1994: the monthly data noted above, June values.

F.2 *U.K. Wages Indices*

F.2.1 The source of the U.K. monthly data is Department of Employment (1971) and regular DE publications since then. From 1920 to 1934 the series is available only for June and December, and intermediate monthly values have been interpolated. The series used are:
- — 1920-1967: indices of basic weekly wage rates, 1920-1968; and
- — 1967-1994: index of average earnings: all employees; Great Britain, various base dates, 1968-75, 1976-79, 1980-82, 1983-87, 1988-89, 1990-94.

F.2.2 The annual data for earlier centuries have been constructed, like those for prices, by taking several indices and splicing them together; since 1920, June values of the monthly series have been used. The older series have been taken from Mitchell (1975) (M) and Mitchell & Jones (1971) (MJ). The series are:

- — 1809-1850: money wages in industry, U.K. 1800-1959 (M);
- — 1850-1880: G.H. Wood, wages and earnings, 1850-1902, United Kingdom, average money wages (not allowing for unemployment) (MJ); and
- — 1880-1920: A.L. Bowley, wages, United Kingdom, 1880-1936 (MJ);
- — 1920-1994: the monthly data noted above, June values.

F.3 *U.K. Ordinary Shares*

F.3.1 The price and yield indices have been based on:

- — 1919-1923: yearly, the BZW Index;
- — 1924-1928: monthly, the share index given by Douglas (1930);
- — 1929-1962: the Actuaries Investment Indices, published by the Institute of Actuaries and the Faculty of Actuaries and privately circulated; and
- — 1962-1994: the FTA (now FTSEA) All-Share Index.

F.3.2 In each case a dividend index is calculated from the product of the price index and the dividend yield, with adjustments to ensure continuity of the dividend index (see Wilkie, 1995).

F.3.3 The FTA 500-Share Index, now the FTA Non-Financials Index, from the *Financial Times*-Actuaries series has also been used.

F.4 *U.K. Long-Term Fixed-Interest (Consols)*

F.4.1 This is represented by the yield on 2½% Consols, annually from 1756 to 1900, and monthly thereafter, until December 1977; thereafter the yield on the *Financial Times*-Actuaries (FTA) British Government Securities (BGS) Irredeemables Index has been used. The sources are:

- — 1797-1900: 2½% Consols, yearly, Mitchell & Deane (1962);
- — 1900-1929: 2½% Consols, monthly, BZW Gilts book (42nd edition);
- — 1930-1962: 2½% Consols, monthly, Actuaries Investment Index;
- — 1963-1977: 2½% Consols, monthly, *Financial Times*-Actuaries Investment Indices; and
- — 1978-1994: FTA BGS Irredemables Index, monthly, *Financial Times*.

F.5 *U.K. Short-Term Fixed-Interest (Cash)*

F.5.1 The values and dates of change of Bank rate and its successors are recorded, and monthly and yearly series derived from these. The sources are:

- — 1797-1939: Bank rate, Mitchell & Deane (1962);
- — 1939-1972: Bank rate, Central Statistical Office (1972);
- — 1972-1981: minimum lending rate, *Banker's Almanac* (1988); and
- — 1981-1994: Bank base rates, Central Statistical Office (1972-94).

F.6 *U.K. Property Indices*

F.6.1 The Jones Lang Wootton indices of net income and income yield have been used. These are available yearly from June 1967 to June 1977, and quarterly thereafter, but only the June values have been used.

F.7 *U.K. Index-Linked Yields*

F.7.1 The yields on the FTA Index-Linked All Stocks Index from May 1981 to December 1985, and thereafter the yields on the 'over 5 years' Index, in both cases assuming 5% inflation, have been used.

F.8 *Other Countries Consumer Price Indices*

F.8.1 Consumer price indices from January 1969 to June 1994, published by OECD (monthly), are used, except for the U.K. where the Retail Prices Index has been used. For Australia, Ireland and New Zealand only quarterly values (attributed to February, May, August and November in each case) are published, and monthly values have been calculated by interpolation.

F.9 *Other Countries Share Indices*

F.9.1 The Morgan Stanley Capital International share indices are available from January 1970 to March 1987. These are not the most convenient indices to use; they provide a price index and a total return index, which assumes that dividends net of tax, as if for a Luxembourg pension fund, were reinvested. One can calculate the implied dividend, and hence yield, from the differences between the monthly changes of the two series, but these are not always consistent (negative values occur), nor do they produce smooth dividend indices, because of a shortage of decimal places in the original indices, so that the monthly values of the dividend index are erratic.

F.9.2 From March 1987 the *Financial Times*-Actuaries World Indices, which give the values one would like, have been used.

F.10 *Exchange Rates*

F.10.1 From August 1972 to June 1993 exchange rates supplied by Quantec Investment Technology Limited, whose help is acknowledged, have been used, and thereafter the implied relative exchange rates derived from the local currency and sterling indices from the *Financial Times*-Actuaries World Indices.

APPENDIX G

NOTATION

G.1 The notation I use for describing my model avoids the usual mathematical style of Greek letters, subscripts and superscripts, but denotes each variable or parameter by a 'name' consisting of one or more letters. There is some consistency about the naming, which is described in this appendix.

G.2 Basic variables, Q, W, P, etc. have single letters.

G.3 The rates of growth (logarithmic) are also given single letters. Thus I represents the force of inflation, the difference between the logarithms of Q in successive years. J and K represent the force of growth of wages and dividends respectively.

G.4 Parameters associated with the basic variable commence with the single letter of that variable, so that, e.g., QMU, QA, QSD, etc. relate to the Retail Prices Index, Q.

G.5 Suffixes to a variable x have the following meaning:

xMU represents a mean value (not necessarily the unconditional mean of the variable);

xSD represents a standard deviation;

$xE(t)$ represents a residual in year t; and

$xZ(t)$ represents a unit normal variable.

All variables include at least these parameters and variables. For all x:

$xE(t) = xSD.xZ(t)$.

G.6 Further parameters include:

xA, an autoregressive parameter, e.g. QA, YA;

$CA1$, $CA2$, $CA3$, three autoregressive parameters for consols;

xW, a transfer function parameter, e.g. YW, WW;

$WW1$, $WW2$, two transfer function parameters whereby retail prices influence wages;

xB, a moving average parameter, e.g. DB;

xD, another transfer function parameter, expressing a moving average effect;

$xM(t)$, an exponentially weighted moving average of previous values, e.g. $DM(t)$, $CM(t)$; and

$xN(t)$, part of a variable that has zero mean, though not necessarily independent.

G.7 The notation is not used entirely consistently. Thus $BD(t)$ represents the difference between $\ln B(t)$ and $\ln C(t)$. $ZN(t)$ does not have zero mean; and there are other irregularities. However, I have attempted to be uniform at times.

ABSTRACT OF THE DISCUSSION

Dr A. Kitts, F.I.A. (opening the discussion): Originally this model came to be developed through the author's involvement in the Maturity Guarantees Working Party in the late 1970s, and was first presented to the Faculty of Actuaries in November 1984 (*T.F.A.* **39**, 341). Since then the author has been a prolific producer of many further papers on the subject, greatly outnumbering those produced by the few of us who have been bold enough to follow him into this area of research and development and to criticise his work.

The purpose of this paper is to provide a technical update and further enhancements to the Wilkie investment model. The original model explored the relationships between consumer prices, share dividend yields and share prices, and long-term fixed-interest Government Bond yields. In this paper the author re-visits these relationships, for example by comparing projections generated from the original model with actual results for the period from 1983 to 1994. Further, the author goes on to explore the relationships between the additional variables of wages, short-term interest rates, property yields and values and currency exchange rates. This last area, of currency exchange rates, allows the model to be applied internationally, based on consumer price differentials; that is to say purchasing power parities.

The author provides many suggestions for further research and applications of the model, such as: assessment of capital projects; testing of solvency of financial institutions; and the analysis of long-term financial derivatives.

I now turn to some key issues and questions. In the literature, one finds a great deal of non-constructive and non-specific criticisms exchanged, between various authors, on stochastic financial models. I think that the underlying cause is a communication failure. This prompts the question as to whether the purpose and terms of reference are defined precisely enough to allow a logical and rational discussion of the models we are estimating.

On the methodology adopted, anyone who has been involved in formal academic research and development is likely to need to come to terms with the more pragmatic approach often adopted by actuaries. This prompts the question of whether the data analysis and model testing are sufficiently rigorous to ensure that all important issues are highlighted sufficiently.

On the question of actuarial judgement, the author makes no apologies for the selection and manipulation of data sets to provide what he considers to be a more sensible series and, consequently, more sensible results. This prompts the question as to when actuarial judgement should be allowed to over-ride statistical technical analysis and when it should not.

On the issue of stationarity, there is little doubt that the series being modelled, such as consumer prices, are non-stationary. Of course, within what we might call a stationary sub-period, models can, and do, appear stationary. The author accepts this non-stationarity, for example, by embracing a fundamental secular shift in productivity. Consequently, while he examines inflation over 730 years, he chooses to fit a reasonable model to the last stationary sub-period of some less than 10% of the whole period. The model does produce some credible projections for the period from 1982 to 1994, as demonstrated in the paper. However, this begs the question of when the current paradigm will be broken, and when this current stationary sub-period will end and another different stationary sub-period begin. Also, how will we know when this event occurs?

Next, on the treatment of increased uncertainty with deviation from the mean and shocks, there is little doubt that the distributions of residuals are non-normal, and that they have fat tails. Apart from the modelling being more complicated, and suggesting that extreme caution must be exercised when working with the tails of the distributions of results, this also raises the question of the distinction between fundamental secular change and shocks.

Then, on the application of the model, there are many difficult questions that remain to be answered satisfactorily in the development and application of stochastic financial models. It is important to realise that these questions are important if models are to be applied in an intelligent and critical manner, rather than as a 'black box'.

If we are to accept that these financial time series are non-stationary, then it may be appropriate to re-examine the fundamental concept of the actuary's role in risk diversification over the medium to long term, if we are to avoid lulling our clients into a false sense of security. For encouraging the development and application of these models, and for prompting the discussion of all these important questions, the author is to be thanked.

Professor R. S. Clarkson, F.F.A.: I begin my remarks by commenting on Section 12:

(1) The author states in ¶12.1: "I am convinced that this type of modelling is more realistic for the long term than the random walk, efficient market, style of model so well established among financial economists." With this I agree strongly.

(2) I agree with the very perceptive last area of suggested research in ¶12.2; namely, to explore why, if the models of this paper are realistic, share prices and other variables fluctuate more than would be reasonable if markets were efficient. Shiller's 1989 book, *Market Volatility*, is, in my view, an excellent documentary of this phenomenon; but it seems to me that very few financial economists have faced up to this important issue impartially.

(3) I also agree with the author's observation, at the end of ¶12.2: "Some of these investigations would take us beyond pure investment models, into wider economic modelling, or into the psychology of markets, or into the inefficiencies of markets." The psychology of markets should, in my opinion, be the key to financial economics, but I have grave doubts as to the validity of the simple models of rational behaviour that are universally employed at present.

Having, as many of you know, criticised numerous aspects of financial economics at various times, I have given considerable thought as to how a better way forward might be found. My conclusion is that a better model for human behaviour in the face of uncertainty is required. This links in very well with what I think is a very perceptive statement at the end of the paper.

The main observation I should like to make is that, based on what I would call my 'sharp end' experience of investment management over around a quarter of a century, I was forced to abandon linear models of the ARIMA or Box Jenkins type in favour of custom-built non-linear models that seemed to me to reflect real world behaviour in a better fashion. For instance, my gilts model, as described in *T.F.A.* **36**, 85 and *J.I.A.* **106**, 85, can be regarded as an obvious extension of Pepper's excellent yield curve model, once we relax the condition that price is a linear function of coupon. In a similar fashion, my equity model, as described in *T.F.A.* **37**, 439 and *J.I.A.* **110**, 17, can be regarded as an extension of the Weaver & Hall model, once we replace linear regression and the analysis of variance by non-linear formulations.

I have grave doubts as to whether it is fair to regard the time series we are examining as stationary over the long term. It seems to me that the major movements in investment markets are a change from one consensus to another. In the discussion of the Geoghegan *et al.* paper (*J.I.A.* **119**, 173), it was suggested that, at that time, there had been a fundamental change in the economic background. Clearly this referred to the fact that the U.K. was then a member of the ERM. There were different policies being pursued by both the Treasury and the Bank of England, and it would seem to me unreasonable to expect the same model of inflation to apply both to that period and to, say, the period from 1970 to 1980.

I have a great regard for all the detailed work that the author has put into the paper; I found it most interesting and most stimulating. However, the work that was done by the Geoghegan *et al.* working party, of which I was a member, drew attention to some of the problems of non-linearity, and I would have been happier if the author had addressed these questions in this paper rather than working mainly in the context of linear models. In that paper there was comment on the main inflation series, that provides for negative and positive movements in inflation with equal probability, and provides for a significant probability of negative inflation. The working party concluded that, in practice, such behaviour was unlikely.

The paper that I produced for the AFIR colloquium in 1991 was a first attempt at a better model. However, I was somewhat unhappy about the author's description of it in this paper. I introduced an intrinsic rate below which inflation rarely fell, since it is clearly skewed to the upside. In his description of my model, the author talks of *QMU* as the mean value, whereas that should actually be the intrinsic value. The author suggests that I had not investigated the mean and variance of my non-linear inflation series. However, I was running out of space, as my AFIR paper was limited to 20 pages, and my final paper was 22, so I added an appendix of 200 values as a 200-year simulation. One can easily produce a histogram in five percentage points of my simulation. If you do the same for inflation in the post-war period, 1951 to 1994, you obtain a histogram which looks rather similar to mine. It is skewed very strongly to the upside.

The author mentions that there are problems of estimation in my non-linear model. With a mortality table, you could argue as to whether the true select period was two, three, four or five years; but you might find, from practical experience, that using two years as a select period was a reasonable, practical way forward. Similarly, many of the parameters in my model were estimated from 'actuarial judgement', as was said by the working party. My first guess of the parameters was the one I used for the 200-year simulation, and the mean rate of inflation over the period came out at 6%, which I regarded as a very reasonable average value.

The main criticism that I would make of the paper is that it does not use downside risk. Variance, the symmetric measure, is not an adequate measure of risk. If you look at the very skewed distributions of my rate of inflation and also at the actual experience, they are quite different from the symmetric picture of the author's simple ARIMA measure. That, however, is my only major criticism of the paper.

Mr A. D. Smith: This paper presents models which describe a number of financial markets, and I asked myself to what extent these markets are efficient, as described by the model. In other words, is there a relationship between risk and return in this model or are they unrelated?

I begin by considering an investor who can switch investments each year, according to some rule. The rule I used was to invest everything in whatever investment class does best for the next year, under the hypothesis that all the normal error terms turn out to be zero. I then simulated, for the year, to see what actually happens, and sometimes I made the best choice, but sometimes, with the benefit of hindsight, I could have done better. I kept on rolling this up for fifty years, and performed the whole exercise 200 times, resulting in 10,000 simulated investment decisions. Over all the scenarios, the average 50-year return for my switching rule is 17.16% p.a. This compares with 13.04% for the single best class, which is property. In this paper the author gets 13.16%, which is not significantly different, given the number of simulations. Furthermore, the switching strategy reduces the standard deviation by a small margin, so we have a marginally reduced risk, and an out-performance of 4.1% p.a. compound. This suggests that the market is rather inefficient.

I have not really taken a proper account of risk, but just gone for the highest expected return. One way to adjust for risk is to assume a utility function. I assumed a log utility function. If this were a typical investor, we would expect to see equities outperforming gilts by around 3% p.a. in the long run, which seems about right and also fits with the model that we have here. So, let us suppose that we have a log utility function, with a 50-year time horizon, but can still trade every year. If we are to believe the textbooks, we expect to see some trade-off between risk and return, and some justification for diversification. This happens only to a small degree with the Wilkie model. Most of the time, it is still optimal to be fully invested in a single class; the most popular being property, and then equities. Some degree of diversification, where more than one asset class is held, can be justified in about one year in six. I could have assumed a more concave utility function to describe a lower risk tolerance and justify more frequent diversification, but this would not be consistent with the observed risk premium of equities over gilts.

One notable feature of the Wilkie model is the popularity of property investment. It is noticeable that, over a 50-year time horizon, Table 11.1 shows property outperforming equity by an average of 2.37%, the level of risk, as measured by the standard deviation, being identical for the two classes. Although, at first sight, these conclusions appear to arise from historic data, it seems to me that this degree of outperformance cannot be sustainable in the long run. Indeed, the property income projection is extrapolated from a relatively short period of exceptional growth. I do not expect this to continue, and neither does the market, judging by the recent rise in rental yields. However, the property model does not seem to allow for any connection between high yields and reduced income growth, and so a combination of high yields and generous income growth is projected into the future.

It is particularly interesting that the author himself had suggested some parameters for the property model before he had examined the data, and these parameters were quoted in the Daykin & Hey paper (*J.I.A.* **117**, 173). Rerunning the model with these parameters, I find that equities outperform property by about 1.9% p.a., on average, over 50 years, roughly on a par with index-linked gilts, which seems more reasonable to me. I guess that the Daykin & Hey model was based on the subjective assessment of the returns which might be expected on property, given the level of risk involved. It seems that an actuary's hunch has produced a more plausible model than detailed analysis of historic data. It would be interesting to examine whether the same principle applies to the other economic series described in the paper.

I have, therefore, rerun my optimisation based on the Daykin & Hey parameters. Sadly, I find again that diversification only makes sense one year in six. The switching strategy outperforms equities, which is now the best single class, by 5.6% p.a. compounded over a 50-year time horizon. There is little extra risk involved; in every one of my 200 scenarios the active strategy outperformed the best of the rest by at least 40 basis points p.a. over 50 years. Although the Daykin & Hey parameters improve matters slightly from the point of efficiency, I still cannot detect a meaningful risk return relationship in this model.

The models I use most of the time are broadly consistent with efficient markets, and so it is useful for me to have a different kind of model, such as the one in this paper, for comparison in practical investigations.

Professor H. Tong (a visitor): My comments on the technical side have to do with the non-linear aspect of time series analysis. There are several aspects of linear models which limit one's horizon. For example, linear models tell you that you do not respect your current position. In a sense, it does not matter where you are; if you are asked to make a forecast, you will give exactly the same prediction interval regardless of your current position. Whether it is a bull market or a bear market, you would get the same prediction interval if you used a linear model to produce any forecast. This, to my mind, is unrealistic. If you want to have a more realistic model to respect your current position, then it would be unavoidable that non-linear models would have to be used.

The second point is that the current position is not always known precisely, because of information delay, for example. There is always some relevance in looking at the sensitivity of your model to the initial value. Initial value here means your current position; so it is important to see how sensitive your model is to any perturbation of the initial condition. Of course, if your model is linear, then this is a trivial exercise, and it will not throw much light. So, it is in this sense that the non-linear model will also be quite important.

Another aspect is that, if you look at some time series, in particular univariate time series, then, because of incomplete information, you may wish to use a linear model which may be valid over a certain area of the state space. If you bring in some exogenous variables, then some non-linearity may be thrown up. For example, in hydrology data, it is well known that, if you look at the amount of water passing through a certain point of the river, you get the river flow data. If you model on the basis of a single time series a linear time series model would be adequate, but, if you bring in the temperature or the rainfall, then there is a different story. Suppose you live somewhere to the north of Great Britain, say Iceland. If the temperature is above 2°C you have an enormous amount of river flow, simply because of the melting of the ice and snow from the glaciers, hence the non-linearity. So it is very clear that non-linearity will come in a very natural way.

If that is the case, then non-linearity will be inevitable. Some earlier speakers mentioned a phenomenon which is related to time reversibility, and also to some non-linear model, which looks to me as if it tried to take account of the conservation of energy in an abstract sense.

I point these features out only to encourage the author to look further ahead in his next paper by incorporating non-linearity.

Mr M. H. D. Kemp, F.I.A.: I am pleased that the model is now extended to property and overseas equities. However, I do sometimes wonder whether the model is a little too complicated for the types of applications to which it is often put. I shall concentrate on pension fund asset/liability studies. Such studies essentially provide some estimate of the range of likely outcomes in the future for key features of the pension fund. You then identify an investment strategy that gives you the best spread of outcomes in terms of the risks involved. So, you certainly need assumptions about how the future might evolve, that is the sorts of assumptions that are involved in a stochastic model. However, the question is: do you need a model as complicated as this one?

My doubts in this area coalesced into something tangible when I read paragraphs like 2.11.4 and 5.11.2, and Figure 5.5 and others similar to it. What they effectively show is the spread of the confidence intervals for the dividend index, the RPI, and so on, looking out into the future. The author rightly makes the point that the graphs are different from those that might arise with a 'random walk'. You can see this, for example, in Figure 5.5, where the solid lines are the results that come from the author's model, and the dotted lines are a 'random walk' model. However, looking at this type of graph, it did strike me that, if you just doubled the standard deviation that was being used for the 'random walk', you would get close to the results that came out of the Wilkie model. Indeed, looking through the paper, it seems that, provided that you use the right adjustments, you can always arrange to get roughly the same kind of spread of results from these two ostensibly different models. You might get the 'envelope' spreading in a slightly different way, but it seems that you can get a similar picture to the Wilkie model by adopting what might be called an adjusted 'random walk' type of approach.

Having noticed this, my next thought was that, given that we can produce the same sorts of results by using a different model with different assumptions, which are the key assumptions in a stochastic investment model which drive the results that come out of an asset/liability study? Here I think that, perhaps, a 'random walk' or some kind of adjusted version as I have just described, does score over a more complicated model. At least with a 'random walk', the key assumptions that you need to make are intuitively obvious, that is the expected returns on the different assets; how volatile those assets are going to be; and also what kind of linkage they have to each other. It is much more difficult to work out what are the key underlying assumptions within the Wilkie model.

One other thing that also concerns me about the Wilkie model is rather more fundamental, and looks forward to the days when, as actuaries, we will concentrate on dynamic investment strategies rather than static ones. The Wilkie model implies the existence of profitable trading strategies. Mr Smith appeared to imply that you can get 4% p.a. excess return just by exploiting anomalies in the model. I am quite prepared to believe that such anomalies do exist. However, I think that it is dangerous, when you are making stochastic projections, to assume that your fund manager will actually be able to take advantage of those anomalies. You need to be quite careful, when you employ models like this, to avoid them homing in on such anomalies. Again, a random walk type of model has the advantage of robustness. It does not build in any of these types of anomalies.

Dr M. R. Hardy, F.I.A.: Stochastic simulation is regarded by many actuaries, particularly life office actuaries, as an esoteric exercise, which is deemed superfluous or unnecessary to the financial management of their companies. Only a handful of companies use stochastic simulation techniques on a day-to-day basis. I do not believe that this situation can last. Increasingly, stochastic simulation will be recognised as an essential tool in all areas of life office management — particularly in solvency control. In the not-too-distant future, life offices will have to pick up the techniques of stochastic simulation; and for almost every U.K. company the most crucial element in the stochastic asset/liability model is the investment model. At the moment the only useful model in the public domain is the Wilkie model. The author has shown that his model has been rigorously tested, and where there are on-going problems these are clear, and there are discussions for further development.

Other long-term asset models exist, and are used in the profession, but they are not in the public domain. None of them has been subject to the kind of scrutiny that the Wilkie model has been subjected to. Actuaries continue to use the models, not because they are not aware of its flaws, but because they are aware of them and feel that they do not invalidate the results, provided that the model is used appropriately and the results are interpreted intelligently. In particular, in the work I do, I find that I am more interested in how results move in relative terms, comparing strategies or inputs, than in the absolute level of the results. Even actuaries who eschew the author's model, and use their own, tend to borrow elements from it and use it as a benchmark for their own models.

Stochastic simulation is an exciting technique that provides new insights into patterns and distributions of cash flows in insurance and pension funds. Broadly, in my experience of exploring the Wilkie model, the results appear feasible. The links that do not appear in a straight random walk model seem credible and important in, for example, life office solvency work.

That the Wilkie model is much better than nothing has been demonstrated in this paper, if you work through the statistical testing of the parameters, and, I think, also by the very informative graphs. I particularly like the graphs which show the past patterns followed by simulated future patterns, for example in Figures 2.6 and 2.7. The paper includes the original model and some extensions. The much vaunted inflation residual problem has been dealt with, or considered, by using an ARCH model. I am not entirely convinced that this is going to turn out to be the best approach. As the standard deviation of the annual inflation rate increases, and this increase is passed through to all the other variables, this increases their standard deviations significantly. Further, we have been introduced to two extra parameters. The main problem that the ARCH model addresses — that of non-independence of residuals — is not apparently necessary, according to ¶2.8.7, which says that there does not seem to be much evidence of autocorrelation. So the conditional hetero-schedastic element of the model has been introduced solely to allow for the fat tails of the residuals. My instinct — and I have not tested this — is that something like Gamma residuals would be a simpler solution, and would be at least one parameter less.

I wonder whether the p values in Table 2.9 make sense. If the ARCH models give fat-tailed residuals and the p values are testing normality, then, presumably, you would not expect an ARCH model to pass a test of normality.

I do not know the answer to the point raised earlier about stationarity, but I do know that to ignore this model, because of possible long-term stationarity problems, and to stick with a deterministic model, will mean that actuaries are missing out on insights which, I think, will prove essential, and they will find themselves falling behind the offices who are using stochastic simulation to make decisions.

Mr P. P. Huber, F.I.A.: The author is to be congratulated on introducing the actuarial profession to cointegration, unit root tests, and vector autoregressive models. These are very useful techniques for the analysis of time series. However, I have reservations about the methodology proposed in Appendix C and in ¶1.4.1. It appears that the author is recommending that asset models should be developed by establishing a linear relationship based on economic theory (or 'common sense'), fitting it to the data, and then testing whether this relationship satisfies various goodness-of-fit tests. If the tests are not satisfied, then parameters should be added until they are satisfied or the result should be ignored on theoretical grounds. This methodology ignores the problems associated with multiple hypothesis testing (which can lead to data-mining). It basically restricts the choice of models to the ARIMA class, and it does not allow 'common sense' to be influenced by the data (which would allow us to improve our understanding of the economy). This methodology only ensures that the 'common sense' used to develop the model is used consistently in applications of the model.

The problems associated with the Wilkie model stem primarily from this methodology, and have been illustrated by Kitts (1990), by Geoghegan *et al.* (1992) and by myself (1995), all of which are referred to in the paper. As the important problems have not been addressed in this paper, I will attempt to summarise them. Evidence of data-mining can be obtained by considering the fit of the model over all the available out-of-sample data. Over this interval, the variance of the inflation and consols residuals are significantly less than, and greater than, QSD and CSD, respectively; and the cross-correlation between the inflation residuals and the dividend growth and consols residuals are 0.6 and 0.7, respectively (which are significant at the 5% level). These results suggest that the variance and covariance structure of the model is inappropriate.

In ARIMA modelling, non-stationarity can only be removed by differencing the data. However, differencing is not a suitable method for dealing with all types of non-stationarity. Therefore, in

¶10.3.3, the author chooses to ignore the test result on theoretical grounds. This approach can lead to biased parameter estimates. Only considering standard ARIMA models also causes problems when confronted with outliers. The significance of *YW* (in the dividend yield model) and *CY* (in the consols model) depend entirely on outliers, and it is not statistically valid to use the models suggested to account for them.

In addition, the stability of the model over different time intervals was not adequately examined. The Chow test rejects the hypothesis that the inflation model is stable. The value of *BA* (in the short-term rate model) is affected by the period during and after the Second World War in which interest rates were constant. Over the period 1955-93, Ong (1994) obtained a value of 0.43 for *BA*. This is significantly less than 0.74. This result is not significantly affected by the differences in the transformations used. The dividend yield and consols models are also unstable as a result of outliers.

The methodology adopted is inflexible when dealing with prior theory. This is illustrated in the fitting of the inflation transfer functions in the dividend growth and consols models. The values of *CW* and *CD* (in the consols model) are fixed and *DD* (in the dividend growth model) is included, even though it is not significant. Surprisingly, in Section 5.3, the author did not consider setting *DW* to zero and including *DX*. He, uncharacteristically, ignores prior theory in determining the consols model's transformations by allowing negative nominal yields, but preventing negative real yields.

In conclusion, these are very significant problems that need to be properly addressed. I am aware that it is easier to criticise than to create, but criticism is an essential part of the scientific process. It is sometimes necessary to take a few steps back before any meaningful progress can be made.

Mr G. S. Finkelstein: At the start of the discussion, the opener predicted that the apparent non-stationarity and non-normality of some of the data used by the author were likely to be among the key issues discussed. It seems that his prediction was correct, given some of the comments of previous speakers. For example, concern has been expressed about the skewness of the data and, therefore, the assumption of normality; about considering using a Gamma distribution instead of the Normal; and about the apparent non-stationarity of the inflation data. I believe that these issues are related; the connection being that the underlying probability distributions are stable non-Gaussian. I was pleased to see that the author, in ¶12.2, placed stable distributions at the top of his list for further research, since I have been conducting research into their use.

Mr Huber has recently showed that the apparent non-stationarity of the inflation data is due to occasional random shocks over the whole time period. This leads to sub-periods, demarcated by the occasions when the shocks occurred, over which the data appeared to be stationary. I suggest that these random shocks are consistent with a stable non-Gaussian distribution. This is because all stable distributions, apart from the Gaussian, have infinite variance, and give rise to occasional extreme stochastic fluctuations. In addition, I think that the stationarity requirement changes if the distributions are stable non-Gaussian. This is because it does not seem sensible to expect the same mean and variance to apply at all points of time, when the variances are infinite or do not exist. It is also possible to allow for skewness with stable non-Gaussian distributions.

A word of warning to anyone attempting to fit stable distributions to the author's share price data — the Actuaries Investment Index before 1962 was a geometric one. Geometric indices are likely to lead to more kurtosis and skewness than arithmetic ones, since the former are more sensitive to downward movements in the price of their constituents. In the extreme case, if any one of the constituent companies becomes insolvent, the whole of the geometric index collapses, while the impact on an arithmetic index will be much less.

I have one minor reservation with the paper, and that is the author's revised choice of starting date for his time series in Section 2.3. By treating the period 1919 to 1923 as an outlier, and excluding it from the analysis, this leads to much lower kurtosis (see ¶2.3.9). This is likely to lead to a model with much less extreme stochastic fluctuations. Whether or not this is appropriate depends on the purpose for which it is intended to use the model. It could be inappropriate in applications where the user is interested in extreme tail probabilities, that is very low probabilities of insolvency for a life fund.

The author's model is just that — a model; and by definition of the word it must be imperfect. However, in my view, it is the best model that is publicly available.

Professor S. Haberman, F.I.A.: Like Dr Hardy, I think we should acknowledge the debt that we owe to the author for presenting his model to the profession in full detail, and then sharing it with us on successive updatings. This openness is to be applauded. I know from anecdotal comments that there are a number of similar models being used by practitioners, without their having been exposed to rigorous scrutiny by their peers in the profession. I find this possibility rather worrying.

I first look at the nature of the model. It is just one of many that could be fitted to the past data available and then used for predicting the future. Any such model is essentially, in broad terms, a combination of statistics and economics, and we can regard the set of models as lying on some sort of spectrum connecting the extremes of a time series model with no economic input whatsoever and an econometric model. The Wilkie model lies somewhere between these extremes. This characteristic should be recognised, and we should note that there are, and doubtless will be, criticisms of the model from both a statistical viewpoint and from an economic or econometric viewpoint.

Mr Huber has spoken about his own work. I think that he has demonstrated that the model has deficiencies; it does provide a poor fit in statistical terms to the past data; in some sense it is over-parameterised; and perhaps it is too complex. Its structure is sensitive to outliers in the data set, and, as others have said, the predictions do depend heavily on certain values of certain parameters.

In econometric terms, one could make criticisms of the model. There are deficiencies, in that exogenous variables that might contribute towards explaining the data are not present. For example, if we look at the recent period of economic history of the U.K., there is no item in the model to represent the oil price shock of the early 1970s.

My second point concerns what the actuarial profession expects of a stochastic asset model, and the uses to which it is to be put. In ¶1.2.2. the author mentions some of the many papers and reports that have used the Wilkie model. In general, these papers have dealt with the forecasting of future contingent cash flows, and the availability of the Wilkie model has provided an extra dimension relative to using the traditional deterministic asset return model. The Wilkie model enables variability in the future to be estimated. Let us consider such an application, and I pose the following question: do we believe the resulting estimates of means, variances, percentiles, probabilities of ruin, etc., that come out of using the Wilkie model? Your answer should be 'no'. What the calculations based on the model do is to give us an indication.

If your answer to this hypothetical question that I posed were 'yes', then I believe that you are expecting too much of any stochastic asset model. The estimates from any model cannot be taken blindly. I have drawn your attention to the implications of a thorough statistical analysis, but we should also note various other points: the high and unpredictable volatility of financial markets; and the difficulty of making estimates for the long-term future in relation to the credible data sets that we have available now. We also note that, for estimating quantities like the probability of insolvency, we would need a model with satisfactory properties in the tails of the relevant asset return distributions.

So, the second question that I would pose is the following: does this weight of criticism and caution matter? I do not think that it does in certain circumstances. Providing that we have the right philosophical approach, I believe that models like that of the author can be extremely useful, despite the difficulties to which I have alluded. The model provides a self-contained and consistent way of generating simulations or scenarios. The estimates of moments, percentiles, etc. cannot be regarded as unbiased, but they can be informative as general indications of results, especially if conducted alongside a thorough sensitivity analysis and monitoring of the results as they emerge. Further, if the model is used in an application that is repeated at regular intervals (for example in valuation work in pensions or insurance), then the use of such a methodology for carrying out calculations means that the series of updating estimates themselves provide valuable information, providing that the approach is reinforced by the appropriate checking. I believe that models like the author's will continue to be of great practical value, providing that they are appreciated for what they can achieve and not regarded as an astrologer's crystal ball.

Professor S. M. Schaefer (a visitor): The paper makes an important contribution to the interface between actuarial science and financial economics, which is a fruitful area, and one which would merit a great deal more study.

Most of the analysis that the author does is on rates of various kinds, such as the rate of inflation, the rate of return, and so on. I assume that you are interested in assessing the distribution of the levels connected with those rates — for example, the level of the stock market, the level of the inflation index, and so on. The latter are non-linear transformations of the former. This means that the distribution of, for example, the retail price index will depend, not just on the first moments of the distribution of the inflation rate (or, correspondingly, the distribution of the rate of return on the equity market), but also on second moments. When you do your simulations, this will come out in the numbers, as it were, perfectly automatically; but it means that, in projecting these distributions, it is also interesting to think about the stochastic behaviour of second moments, which is something that the author raises in his paper. Its direct impact on these levels may be something which might be worth more thought.

I now make some comments connected, in one way or another, with efficient markets. Some considerable time ago I was told it was probably a mistake to go into financial economics, because "more or less all the interesting problems had been solved." We had efficient markets theory; we had the capital assets pricing model; we had option pricing theory; we had Modigliani & Miller, and that was that. It was a period when financial economists were almost shrill about the efficient markets hypothesis. I think that that era has now passed, and financial economists have had to face up to all sorts of anomalies in the data that do not fit the theory.

One point worth stressing is that, for a long time, people have understood that there is really no such thing as 'the efficient markets hypothesis' that is independent of some theory about pricing. In fact, you cannot actually state what the efficient markets hypothesis is without some statement about pricing. So, when we talk about the applicability or usefulness of the efficient markets model, what we are really talking about is whether or not one should be using some sort of pricing model. From my perspective, I think that this type of approach is very helpful. One reason is this: faced with a very large amount of data, and without some a priori structure on these data, it is virtually impossible to say anything at all.

I would argue that the author is using 'theory' when he says that he thinks some specifications are a priori reasonable or not. In doing this, he is excluding certain possibilities based on his experience. The only difference is whether or not, as a matter of judgement, one thinks that it is worth taking a priori reasoning further in placing certain constraints, via a model, on the way one analyses the data. From my own experience, I think that, on the whole, this is rather a fruitful thing to do. On occasions extremely useful results emerge from essentially a priori bits of analysis. I mention two examples. First, option pricing theory has been hugely constructive, whether it is correct or incorrect in detail, and it certainly did not come out of a 'free form' analysis of the data.

As a second example, modern term structure theory predicts that interest rate volatility will have an impact on bond prices. This also emerged from the theory, and was subsequently confirmed by looking at the data. This is a very respectable line of approach (e.g. in physics), and so on. I do not think we should discount it altogether.

Leading on from that, there are a number of aspects of theory which would be helpful in the context of the specific model that the author is looking at. When we look at asset prices — take bonds, once again — potentially they contain quite a lot of information about the market's long-term predictions of certain variables, and the relationship between the prices of long-term bonds and short-term bonds once again provides potentially useful information. I am sure that you will have seen the analysis that the Bank of England is now producing on a regular basis, looking at the relationship between index-linked and conventional bonds, in order to derive long-term projections of inflation. This information might be a very useful adjunct to the short-term projection techniques that the author is using.

Mr T. J. Sheldon, F.I.A.: I was disappointed to read that, in a paper as extensive this, there would be no consideration of the applications of the author's model. Maybe that will be the subject of a further paper — it certainly merits one, since, as actuaries, we should be at least as interested in the applications of the model as in the model itself.

One potential application of the author's model is the study of the interaction between solvency and investment strategy in a life office. The regulations governing solvency valuations relate the valuation interest rates to gross redemption yields on fixed-interest assets and running yields on equities and property. A life office can, therefore, improve its published solvency position by switching into higher yielding assets. If we consider an office writing predominantly conventional life business, with an excess of investment income over expenses, other things being equal, it will prefer to receive capital gains rather than investment income. An office which switches into higher-yielding assets for short-term solvency considerations may well find that, in the longer term, any advantage is outweighed by an increased tax bill.

How can an actuary set about formulating some suitable rules for investment strategy subject to solvency constraints? The obvious way is to try out a few deterministic scenarios. Such an exercise will generally show that it makes little sense switching into higher running yield assets, unless the solvency position is particularly acute.

An alternative approach would be to use the Wilkie model to tackle the problem. If you do so, you may well find that switching to higher-yielding assets not only improves statutory solvency in the short term, but also enhances longer-term investment performance, despite the adverse tax consequences. This occurs because, under the Wilkie model, income streams (such as dividends or property rents) are relatively stable, so price movements in the short term are determined mainly by changes in yield. Yields follow autoregressive processes, and so revert to their averages. Consequently, when an asset class has a high yield, its yield is likely to fall and its price is likely to rise (and vice versa). This is, of course, the rationale of high income funds, but clearly not everyone believes in this principle. The alternative view, based on discounted cash flow, is that a high yield reflects low expected growth in income. The conclusion one would draw from applying the Wilkie model to the problem is that a yield-based switching rule is beneficial both for short-term solvency and for net-of-tax long-term investment returns.

Should we believe this result? Can such a simple strategy really work in practice? It may well be true that the model is a faithful representation of history. Is it, however, prudent for this particular application to accept that such a strategy will succeed in future; for this is what we would be doing if we unquestioningly accept the results of our modelling work? There is a danger that a well-intentioned asset/liability study, such as this, could lure the office into a false sense of security. Is the model really suitable for this type of investigation, or should we place more faith in simple deterministic scenarios?

Mr A. F. Wilson, F.I.A.: The minimum funding requirement is one area where this paper is going to be important. When we are dealing with the question of setting minimum funding standards and cash equivalent terms, including taking into account 'equity' investments, we do have to set appropriate parameters, and these should probably not be too far removed from the parameters which the author talks about as the neutral initial parameters in the paper. It is, therefore, comforting that the work that has been going on behind the scenes on this question is coming out with figures not very different from those parameters, although the actual fit of them is somewhat different. This leads me to wonder whether, when one is putting together a comprehensive model from all the various models that the author has in this paper, one should ensure that the periods over which one is testing to get the parameters are consistent; otherwise one finds that some of the parameters, when compared one against the other, do not fit together as well as one might expect.

This model, as has been said, had its origin in the Maturity Guarantees Working Party. I remember thinking at the time, and ever since, that one of the most important prerequisites of all models that we produce is to make sure we have the correct null hypothesis, because, in most of the series we deal with, the ability to refute the null hypothesis is often very limited. In other words, when you look at the white noise inevitable in these various runs of data, you can find that it is actually quite difficult to take the null hypothesis and say that we must reject it. The reason why I think that this is very important is that I believe that the choice for the null hypothesis is between some variation of a random walk model and some variation of a cyclical model. Very often the two look very similar over short periods, but when one looks over longer periods they look very different. If one goes to sea, all around looks flat, but that does not mean that the whole world is flat; if we go far enough we have to consider curvature. Thus, for example, when I read Mr Huber's paper, I wondered whether this was not a case of looking too closely at the locally flat surrounds, and not trying to think what happens when one goes further away.

Much of human life is dictated by cycles — not much by linearity. I feel that, in many aspects, one has to start from looking at whether or not there are, if I can put it this way, cyclical models with fluctuations around means, that ought to be taken as the primary model, only taking into account random walk models and revisions to means if the cyclical model can be shown to be inappropriate. This is the strength of much of the author's work; choosing his null hypotheses and testing them thoroughly. This is where the financial economics comes in, and why it is not purely statistical. It is a question of getting the null hypotheses right.

Mr Smith highlighted another interesting point; the possibility of finding a strategy which, historically, would have given you consistent out-performance to choose some asset class that is demonstrably relatively cheap, then you should not be diversifying, but concentrating on that asset. One of the problems is that, if everybody used the strategy, or if an increasing proportion of people did so, then all the various parameters and graphs would gradually change to prevent the outperformance continuing. You would then gradually lose the ability to make that sort of extra money.

This illustrates one of the problems we have if we are looking backwards: to what extent have there been secular changes which we should, or should not, take into account; and to what extent should we extrapolate those into the future? A further problem is that discrete changes do happen. One does have to remember that many of these price series involve only marginal buyers and sellers, with the vast majority of stock remaining untouched. In those circumstances, a move from one consensus to another can, therefore, happen very rapidly. It does not surprise me to find that we have fat tails.

One of the most interesting aspects is the extension of the paper to overseas markets. I hope that we shall get some discussion on whether what the author has done on overseas markets is appropriate, or whether better parameters could be put forward. That is the area where minimum funding requirements could give rise to significant problems. There has been the suggestion that just using U.K. equities as an equity content is wrong, and one should use overseas equities. To do so, it is very important that we know what we are doing, and that we have appropriate parameters to use.

Dr A. J. G. Cairns, F.F.A. (in a written contribution that was read to the meeting): In the field of stochastic investment modelling, any model can only be an approximation to a much more complex reality. Such models can range from the very simple Geometric Brownian Motion models, often used in the pricing of derivatives, to complex asset models, such as the one now under discussion. Finding what one regards as a good model is a difficult process. The author readily admits that there are alternatives which may be just as good, and, indeed, makes a number of suggestions at the end of his paper indicating where improvements could be made. Inevitably there will be a number of discussants who will criticise the Wilkie model. This is because there are many criteria, some conflicting, which must be satisfied by a good model. Many of these critics will see that one or two of these criteria are not wholly satisfied, and use this as conclusive evidence that the Wilkie model should not be used under any circumstances. However, I rarely see these critics putting forward alternative models which are anything like as comprehensive or as good as the Wilkie model. I challenge them to make their models available for scrutiny.

In applications, it is important for one to be able to quantify, for example, the level of risk inherent in a certain investment strategy. It is only possible to do this using stochastic investment models; deterministic scenarios tell us nothing in this respect. Even if the model is flawed, we can at least give ourselves a good feel for the level of risk, and be confident that the numerical results generated by such a model reflect this risk with reasonable accuracy.

I now make some points of a rather statistical nature, which I consider to be of relevance to the whole field of stochastic modelling. In the present paper, much time is spent discussing the standard errors of parameter estimates. This is a very important point, because, not only is a model an approximation to reality, but we do not know what the 'true' set of parameters should be for this model. It is, therefore, essential as part of any simulation exercise, to repeat the exercise many times using a range of parameter values which is consistent with the past data and with the standard errors of the parameters and their correlations. One can carry out such a process in a more rigorous way by using a Bayesian approach, as I have shown in my paper 'Uncertainty in the Modelling Process', that is to be presented to the 25th International Congress of Actuaries.

The models, themselves, are also not known with certainty. There may be a range of models which all fit the data equally, or almost as well, as the Wilkie model. In all applications, one should, as a matter of course, entertain a range of such models. If it is found that our conclusions are not sensitive to the choice of stochastic investment model used, then there is no problem. If, instead, it is found that our conclusions do depend significantly on the choice of model, then one can use the model averaging approach, described in the papers: 'Model Uncertainty, Data Mining and Statistical Inference', by C. Chatfield and 'Assessment and Propagation of Model Uncertainty' by D. Draper, both to appear in the *Journal of the Royal Statistical Society*, and in my paper that I have already referred to, which allows one to take account of model uncertainty. It is important, under such circumstances, that one does not provide the results of an analysis of uncertainty based on a single model when one knows that other plausible models would give rise to different conclusions.

I consider that it is inappropriate to stick dogmatically with a single parameter set and with a single model. Failure to take parameter and model uncertainty into account could result in a significant underestimation of the level of risk.

Mr P. Stanyer (a visitor): I am from the railways pension fund. The railways pension fund has about half of its assets accounted for by a closed fund for pensioners. One of the major policy issues that we have to confront is the likely performance of bonds compared with equities. This is a problem of interest to all institutional investors.

There are a number of ways in which one tries to address this particular policy problem. However, it is clear that one particular stark difference that arises from the author's work and the work of financial economists is in the different treatment of the statistical properties of bond and equity returns, and specifically in how plausible the random walk hypothesis is considered to be. If I think about long-term investment strategy, I am quite content to think of equities as following entirely a random path, particularly for the 'duration' (the effective horizon), for a very mature scheme (eight years for the railway pension fund). I am quite aware of all the literature on why that is not quite right, but, as a broad assumption in setting strategy for a mature fund, that is fine.

However, do I really think that returns from bond investments are going to follow a random path? I do not know. The literature seems to be divided into two. The paper that comes to my mind is one by Martin Leibowitz (from the *Financial Analyst Journal*). This is a simple, easy-to-read short paper, which extrapolated annualised volatility. In terms of an equity market, I am quite happy with this as a starting assumption for a strategy over eight years. For bonds, I am not so sure that we should assume, when setting long-term strategy, a random walk with returns in one period independent of returns in the preceding period.

The explicit modelling of these two different approaches is encourgaged by the author's work. For pension funds to be presented with alternative estimates for the likelihood of equities outperforming bonds over different time horizons, according to both the 'actuarial' approach and, separately, the 'random walk' hypothesis, would be a service to pension funds and to the investors.

Mr D. J. Parsons, F.I.A.: What use is the Wilkie model to the ordinary working actuary? Very few are using it yet. I have heard from some that this is because it is too academic, and that it is only really useful to high-powered investment people with high-powered computers, and that it would not add anything to the advice that we give to our clients, customers or investors.

Looking at the Securities and Investments Board report on pension transfers and opt-outs recently, I found the following: "If an investor is exposed to the risks of adverse fund performance without understanding their nature, scale and likelihood, then the investor can be said to have been harmed if financial loss has actually occurred or is reasonably likely to occur". "Reasonably likely to occur". Who defines that? We, as actuaries, are uniquely able to identify such likelihoods, particularly with this tool provided by the author, but do we do it? Would there be any advantage to the investors if we did?

In practice, advice on the likelihood of risk tends to comprise, simply, words such as: "values can go down as well as up". However, we never put a probability rating on this. We hold ourselves out to be experts in the field of financial probabilities, and yet we cannot, or do not, give a straight answer to the question of how likely it is that values will go down.

Standard disclosure projections show the proceeds of savings and insurance products with assumed investment returns of 6%, 9% and 12% for pensions products, and 5%, $7\frac{1}{2}$% and 10% for insurance business. The general public could be led to believe that the central rate is the one which is expected to happen, and the other two rates are outliers — levels between which the investment return will fall. Should they be allowed to believe this, and do we know better? We could give odds of twenty to one against the money which is returned at the end of the term being less than the amount invested; four to one against it being less than the lower projection; and so on. With appropriate use of the author's methodology, each and every investment provider could assess the appropriate odds for each of its products. I am sure that the investors would appreciate this advice, as well as the salesmen. The regulators may have to prescribe appropriate values for *QMU*, etc., to ensure that advertising was fair, but I am sure that we can cope with that.

Mr J. P. Ryan, F.I.A.: I want to issue one or two notes of caution. This follows on some of the ideas introduced by Mr Sheldon and Professor Haberman. Professor Haberman, in particular, said that the Wilkie model was a cross between a general stochastic model and an econometric type model. Indeed, if one had an extremely good econometric model or used only non-economic variables in order to forecast interest rates, inflation rates and stock market prices accurately, one could significantly reduce the overall uncertainty in asset forecasts. However, the very idea that that concept might exist needs to be tailored into some other thinking when we use the model for forecasting things other than purely long-term asset variability. If we are going to use it in broader model office solvency calculations — I am thinking here particularly of non-life ones — then we may need to take some of these factors into account, in particular some other means of how inflation impacts on share prices and other financial variables. Other short-term factors come into play on the other side of the balance sheet. We are not necessarily going to get the correlations right if we combine a pure Wilkie-type model together with estimates of the overall uncertainty derived from the liability side, because our claims reserves are based on a forecast of 5% inflation, or whatever. There will be some other correlations that are not necessarily picked up.

Clearly, that is outside the scope of this paper, and does not in any way invalidate anything that the author has written in this paper, which I think is extremely good in terms of evaluating the asset side of the balance sheet. These issues need to be taken into account if they are then being used in another context, such as model office solvency. This applies, to a lesser extent, on the life assurance side, where there are probably fewer variables than on the non-life side, where many very complex variables come into play. Here relationships and the types of models used are fundamentally different, and may lead to some different conclusions. I do not have any simple answers to this, but I would like to add it on as a point (p) in the author's list of recommended further work in ¶12.2.

Mr P. J. Lee, F.I.A.: I now, after this paper, think that I understand how the author produces a model for a particular country. He has done the profession a great service in publishing what, effectively, amounts to a 'Do It Yourself' guide to producing a stochastic investment model. If anyone out there wants to try and do it, I think that they now have the information available in this paper. I also think that the author is setting standards, in asset/liability modelling work, of transparency and quality for others in the profession to follow. The work of the profession in this area is expanding, as asset/liability modelling is becoming increasingly common and accepted as part of the due diligence work, and actuaries have an increasingly influential role to play in this area. We have already heard Mr Sheldon refer to an application in the area of life insurance, Mr Stanyer has just mentioned pension fund applications and Mr Ryan mentioned non-life. I could also mention charities, where some work has been done in assessing, not just investment strategy, but expenditure strategy. How does a charity make sure that it does not over-spend and eat into its capital too fast?

Next, the extension of the Wilkie model to most of the major developed countries and to exchange rates is not only a testament to its resilience and applicability, but should be very useful in helping actuaries and others to assess the risks of international and currency risk in a more scientific manner. It is now possible, using the Wilkie model, with the exchange rates and the interest rates for different countries, to make a modelling assessment of the relative risks of hedging or non-hedging.

Then, to echo comments made by Mr Smith, Professor Schaefer and a couple of others, rather like Galileo, the author has had the courage to challenge the orthodoxy of his time and to produce what seems to be strong evidence that, however efficient markets ought to be in theory, in practice they have been 'travelling hopefully', and have not arrived yet. Whoever says markets are inefficient also says that there is a trading opportunity, and we, for our part, look forward to some very interesting discussions on this point with the investment management community.

Mr P. M. Booth, F.I.A. (closing the discussion): This paper is a thorough appraisal of stochastic investment modelling, which has updated previous work to take into account more recent developments in statistical techniques. It has also provided the profession with plenty of leads for further research. A number of different approaches to stochastic modelling and testing models have been presented. The author should be thanked by the profession for this paper, and I am sure that those who have criticised aspects of the models which have been proposed would be happy to recognise the contribution that the author has made to this field by exposing his work to rigorous analysis. The author has also repaid his debt to the profession by not allowing narrow commercial interests to prevent him bringing forward his stochastic investment modelling ideas for criticism. We need to use the advantage of a professional body to subject our ideas to appraisal before they are used in practice. This process is in our joint interest, and, ultimately, in the interest of us all as individuals.

I now refer to some difficulties that I and other speakers have with the inflation model. Models have been produced for the U.K. based on long data series, and for other countries using shorter periods. The skewness of the residuals in the U.K. model appears to be removed to some extent by the use of ARCH models in Section 2.8. It is pleasing to see this subject introduced, but the reader is left wondering whether the author believes that they are better than the ARMA type approach, as no comparative conclusion appears to be drawn. Tests of the data indicate an absence of stationarity in the inflation data. It would appear that the use of the ARMA straightjacket, without any allowance for shocks, leads to an incomplete representation of the process. It is probably impossible, however, to model completely the complex economic and political factors that lead to the fall in the purchasing power of money to produce a model that captures either all the features of the data or the underlying economic interactions, never mind both.

Points made by many speakers and the author's own analysis indicate the importance of doing further work to produce better models or to understand the limitations of existing models. An example of how the estimated models could be used out of context, with some danger, can be seen in the inflation models for New Zealand, estimated from one rather turbulent 25-year period. The *QMU* value in Table 2.7 is about 9%.

Both the author and his critics make a contribution to the development of our knowledge in this field, and I hope that they all mutually recognise the contribution that they all make. It should always be remembered that it is hardly the fault of the author that there is an inherent non-stationarity about the inflation data; and the model used, in its proper practical context, is still likely to be valuable, as long as the user understands the difficulties. Actuaries should always test the robustness of their modelling to the assumptions underlying the model.

The model for wages seems sensible. I think statisticians may be concerned about the remark, in Appendix B.1.4, that the vector autoregressive model may be inconvenient for wages and prices, because both would have to be simulated together. However, for practical actuarial use it may be a satisfactory approach simply to use transfer functions from prices to wages. The dividend yield and dividend series are examples of models where the fundamental model structure appears sound, but where the author gives us a number of leads for further research and discussion on detail. The strong positive correlation between dividend residuals and yield residuals at a later stage, mentioned in ¶5.3.5, could result from an increase in the level of real dividends being recognised by the market in a later year as unsustainable, or as involving a reduction in dividend cover, and thus causing a later fall in capital values compensating for the earlier rise in capital values at lag zero, when dividends appear to have been followed up by share prices. This would, of course, be an indication of market inefficiency. The reverse process may take place when, for example, dividend controls were imposed. It would be useful to investigate the effects of changes in payout ratios and the imposition and subsequent removal of dividend controls on the residuals in greater detail. As Mr Smith mentioned, the dividend yield model does not assume market efficiency. This means that dividend yields are expected to fall when dividend yields are too high and vice versa. This would probably accord with the views of most actuaries.

I am not familiar with Professor Brennan's work, mentioned in ¶5.10.6, but Fama & French, mentioned in ¶5.10.5, produce a relationship between dividend yields and returns which does not contradict the efficient market hypothesis. Fama & French argue that a shock, for example a rise in long-term real interest rates, decreases capital values and increases yields. The share has a higher expected return from this high dividend yield position, not because the market is inefficient in a way which would cause the dividend yield to return to its previous lower level through a rise in capital values, but because the long-term income stream provides a higher internal rate of return on the purchase price. When looking at relationships between dividend yields and returns, it is important to distinguish between market inefficiency and what are, in effect, compound interest effects. This is a rich area for further work by the author and others, and may resolve some of the minor disagreements between the author and Mr Kemp.

I am nervous about estimating the long-term bond model using the whole data set. It is clear, from Section 6.2, that there was a fundamental change in the 1970s, when the market stopped being 'surprised' by inflation. The author mentioned this, but then went ahead to estimate the model from the whole data set. Perhaps, in practice, greater judgement could be used in selecting the estimating period, perhaps estimating the bond model from more recent history. Perhaps, as Professor Haberman mentioned, dummy variables would be needed to deal with the effects of the 1970s oil shock and monetary shocks.

Since the author estimated the original model, a whole mass of literature has appeared on rational expectations, and, perhaps, this could have been referred to, or could be brought in in future work. In Section 9.3 the author discusses the issue of market implied inflation rates. These may, as was pointed out by Professor Schaefer, enable us to produce better models for the effect of recent inflation on consols yields. However, we would not expect any revised model for market expected inflation to be universally appropriate for the last 70 years. Thus, stochastic modelling requires actuarial judgement in the choice of the estimation period and in other aspects of modelling, and does not negate the need for such judgements, as is often thought. It is not a black box.

Judgement should also be used when modelling short-term interest rates. It has already been mentioned that the author's inclusion of the war-time period, when short-term, but not long-term, interest rates were fixed, makes a very significant difference to the parameter estimates; much more significant than that caused by any difference in model forms, discussed in Section 7.5. In fact, the inclusion of the war-time period leads to the almost doubling of the parameter estimate. Again, we see that the data period chosen for the estimation of the model can greatly affect the parameters. Users of models should use their experience and knowledge of the data in choosing periods over which to fit models and in interpreting the results of applying models.

It is on this note, on the philosophy of model use and on construction, on which I would like to end. The author is to be thanked for introducing the profession to a wide range of statistical techniques and applications. It is a most comprehensive paper. It provides a fine review which will assist the profession in developing practical models. There are, of course, difficulties which have been brought out by a number of speakers, but, as we have heard from Dr Hardy, the models give intuitive appeal and provide useful results in practical situations; although the author has, perhaps, left himself open to greater criticism than he deserves by not giving sufficient attention to some of the real statistical and economic difficulties which exist.

To take up Mr Wilson's challenge, the foreign currency model, in Sections 10.1 to 10.4, appears to accord with economic common sense. However, this model, together with the property model, should only be used with great care, given the short experience period and enormous data difficulties, particularly on the property side. They may help us to obtain an understanding of the problems we seek to solve, but they should never be allowed, as certain econometric models have often led economists to do, to obscure the fundamental issues, as they could easily do if not used with great care. The answer to dealing with some of these difficulties may lie in improving modelling techniques. However, the management of these difficulties lies in the hands of the user: in the way he models; in the estimation period used; in the applications chosen; and in the reporting to clients.

The President (Mr C. D. Daykin, C.B., F.I.A.): The author is to be congratulated on another thoroughly researched and comprehensively argued paper. A decade or so ago he led the way for the profession in the development of a consistent set of stochastic models of inflation and investments. The models have been widely used by many members of the profession in a number of applications, the description of which could certainly fill a book. However, they also came under some criticism, partly from members of our own profession and partly from North American financial economists, who were implacably opposed to the use of autoregressive investment models to represent stock market behaviour. Patient explanation of the important differences between the role of a long-term model, such as that proposed by the author, and the ability to predict short-term investment behaviour for the purposes of tactical investment decisions, which was thought by many as being best approached by the random walk model, led to a degree of reluctant acceptance, at least as being possibly appropriate for the U.K.

This paper, extending some limited results that the author presented at the 1994 AFIR Colloquium, demonstrates that the modelling approach can be used quite effectively for a whole range of countries; it is certainly not just a U.K. phenomenon. This paper is invaluable in extending the scope of the earlier work to other countries and to other asset classes, and, as has been mentioned, this has been particularly topical in relation to the discussions taking place currently on the minimum funding requirement.

I believe that stochastic modelling is of fundamental importance to our profession. How else can we seriously advise our clients and our wider public on the consequences of managing uncertainty in the different areas in which we work? It is important for all actuaries to come to grips with this type of modelling work, and we have much to learn about the alternative ways in which such modelling can be approached.

The author is a shining example of an actuary who, not only carries out excellent research and comes up with a multitude of new ideas, but also publishes it and exposes it to view. Several speakers, including the closer, have referred to the importance of this for the intellectual health of our profession. I am sure that you will all want to join with me in congratulating the author for the fine paper which he has presented.

Professor A. D. Wilkie, F.F.A., F.I.A. (replying): Mr Huber said that it was important, in time series forecasting, to difference series until they became stationary. In his own paper he suggested that certain of the series should be differenced more than I have done. The problem about differencing an already stationary series is that, when you integrate it again, the series ceases to be stationary. In effect, if you say that the change in interest rates in each period is stationary, that means that interest rates themselves are not. This may be fine if all that you are interested in is one-step-ahead forecasting, which is essentially what the Box Jenkins ARIMA models were developed for. They were primarily interested in forecasting the next value. The whole purpose of actuarial models is that we are not interested in forecasting just the next value, but we are interested in the structure for very many years ahead. Therefore, the over-differencing, as I would call it, which is recommended for time series modelling, is a mistake for this sort of modelling.

Mr Smith and others made a point about the fact that this model conflicts, in certain respects, with an efficient markets model. I think, in a sense, that my model is a very bad model — bad in the sense that it correctly represents an economy which is behaving badly. It would be very much better if people believed this model, and then behaved accordingly, and it changed. Share prices would not go up to the silly heights that they got to in the middle of 1987. There would not then have been the subsequent crash. There would not then have been the response of the Government which destabilised the economy for about the next three years. It would be much better if share prices remained fairly constant multiples of dividends or fairly constant P/E ratios. People who make their money dealing in the stock market might not like it, but it would be beneficial for all the rest of us. Similarly, when we look at currencies; it would be much better if currencies retained an approximate purchasing power parity. The yen would now be falling instead of rising. It would again be much better for international trade to have more stable exchange rates rather than less stable ones. If people go out from this meeting with any message, it is this: please believe the model and make it come untrue.

WRITTEN CONTRIBUTION

The author subsequetly wrote: I should like to thank all those who contributed to a most stimulating discussion. I hope that the various suggestions made will be taken up, either by the speakers or by others, in order to advance the frontiers of this vast, and for me most exciting, subject. I should like to comment on a few of the remarks made.

The opener asked when we might know when a current paradigm is broken and the current stationary sub-period ends, and another different stationary sub-period begins. Prices have risen in Britain almost every year since 1934. I do not think that it was until the late 1950s, about 25 years later, that people recognised this as a permanent feature. I therefore think that we only notice that a different sub-period has begun many years after it has happened. For example, it is possible that we have entered a new phase of low and stable inflation, which will last for many years, but we will not know this until it has, in fact, lasted for a couple of decades.

Professor Clarkson defended what I described as his *ad hoc* model. I do not dispute that such *ad hoc* models are useful. They are easy to use for simulation, and I have used them myself for certain purposes. My point is that I would like to see the statistical characteristics of the model better investigated.

Mr Smith made some pertinent observations about how one could use my model to trade profitably. I made much the same points in Wilkie (1986b). I think it is sensible for investment managers or individuals to act as if such trading were profitable, but it would be unsafe to rely on the potential profits. See also my final oral remarks at the meeting.

Professor Tong made some remarks about non-linear models. Indeed, I think these are very much worthy of further investigation, and his own book on the subject, Tong (1990), is invaluable. Unfortunately, many of our data series are rather too short to allow clear phases of different types to be distinguished; but it is worth trying.

Mr Kemp suggested that one can do a lot by using a random walk model with a suitable standard deviation. There is something to be said for this approach, but much against it. The modifications to the standard deviation need to be different in the short and the long term; that is the point of the several graphs in the paper that show the random walk confidence interval along with the model confidence interval (e.g. Figures 2.7, 3.4, 5.5, etc.). Further, the cross correlations are not uniform, as shown in particular by the results in Tables 11.1 and 11.2. So I think one may get broadly right results in some cases, but miss a lot of relevant, and possibly important, details by using a pure random walk model.

Dr Hardy asked whether the p values in Table 2.9 make sense. Yes, they do. ARCH residuals are still normally distributed, even though the standard deviation for each residual is different. Thus, dividing the observed residual by the appropriate standard deviation gives what should be a series of unit normal variates. Practitioners in the options market appear to believe that 'volatility' varies from time to time. This suggests that some form of conditional heteroscedastic distribution might have advantages. Presumably Dr Hardy's gamma residuals would be homoscedastic. It is worth investigating whether, with a homoscedastic fat-tailed distribution, one can be fooled into believing that it is heteroscedastic, because occasionally some large 'blips' turn up.

However, it is nice to have a model that suits both short intervals of observations and longer intervals, and it seems to me that only the stable distributions mentioned by Mr Finkelstein do this nicely. I hope that Mr Finkelstein will carry on with his investigations into stable distributions.

Mr Huber raised pertinent questions about my model and my parameter estimation. I shall await his improved model with interest.

Professor Haberman wondered whether additional exogenous variables (such as the oil price shock of the early 1970s) might not help to explain the observed variables better. They might well, especially if one were writing an economic history. However, for the purpose of random simulation of many years into the future, all the variables must be endogenous. So, if I include price shocks, I need to have a stochastic model to forecast how frequently these will occur in the future, and of what size. I prefer to leave such shocks as part of the random residuals.

Professor Haberman also makes the very pertinent point that, if one is simulating in order to estimate probabilities from the tails of the distributions of the simulated variables, one need to have a model with satisfactory properties in the tails of the underlying distributions.

Professor Schaefer had done a lot of work himself on the term structure of interest rates. I have done no more than fit models to the very short and very long end of the term structure, and I have not attempted to model the intervening curve. There have been many valuable academic papers about the term structure of interest rates, some from Professor Schaefer himself, but I have not yet seen one that has investigated what I think are the minimum of four necessary parameters: short and long nominal rates and short and long real rates. Many academic papers set up a theoretical model and investigate the mathematics of it without testing it against observed data at all. I have started from the other end, with no preconceived theory about the term structure, but plenty of observations. There still seems a lot to do to bridge the gap.

One approach, however, is the analysis done by the Bank of England, to which Professor Schaefer referred, in relation to expectations for inflation, the quantity of *RI* that I plot in Figure 9.2. Instead of my single value at each time, the bank is able to construct a 'yield curve' of the market's expectation of future inflation over all future periods. However, nowhere have I seen substantial academic theory about the investigation in a time-series manner of a complete curve, such as a yield curve or a mortality table. The best approach, so far, seems to be to be that of Tilley (1990).

If Dr Cairns had not already suggested the paper by Chatfield (1995), discussing model uncertainty, I would have done so myself. I would recommend both this paper and Cairns's own paper (1995) on this subject.

The closer asked whether I believe that ARCH models are better than simply using normally distributed residuals. I am not sure whether I do, but they are one way of dealing with the fat-tailedness of the observed residuals, a problem that I think needs to be tackled somehow. A different problem arises with the ARCH model I have used for inflation. As I note in ¶11.5.2, the ARCH model I use produces occasional exceptionally high values similar to hyperinflations. This creates both computational and interpretational difficulties; but perhaps an ARCH model with different parameters would be all right.

The closer also drew attention to the problems of interpreting values such as those shown in Table 2.7, which relate to a relatively short and 'turbulent' 25-year period. I quite agree. The value of *QMU* for the U.K. in that table is 8.38%, and I would not think that this is a good value to use for future simulation; the same is true for other countries.

The closer also discussed the periods over which parameters are estimated. In general, I prefer to use the longest period that seems to make sense, and for which the model appears to have been uniform (if not statistically stationary). The longer the period one uses the smaller the standard errors of the parameter estimates. However, if you go too far back, it is likely that the model will have changed. The change from gold to paper currency was a significant change in the mechanisms affecting inflation, and it seems reasonable therefore, to use the post-1920 data for inflation rather than pre-First World War data. However, if data were available for share dividend yields pre-1914, I would expect that they would show much the same pattern as nowadays. It seems to me reasonable to use the evidence of Consols in the 19th Century to assist in modelling index-linked stocks nowadays, and so on.

In earlier papers, (e.g. Wilkie, 1986a) I discussed estimating parameters over different periods. However, to do this again would have made the paper even longer than it is already. I leave this as a task for others.

References

CAIRNS, A.J.G. (1995). Uncertainty in the modelling process. *Transactions of the 25th International Congress of Actuaries*, **1**, 67–94.

CHATFIELD, C. (1995). Model uncertainty, data mining and statistical inference. *Journal of the Royal Statistical Society, Series A*, **158**, 419–466.