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On the Probability Function in the Collective Theory of Risk.

By **Fredrik Esscher** (Stockholm).

1. Upon reading Dr. LUNDBERG's paper »Über die Wahrscheinlichkeitsfunktion einer Risikenmasse»[1] and trying to penetrate it along my own lines of thought, I found another way of deducing some of his formulas, giving the results in a form that directly invites a fairly simple approximation of the probability function. Though time has not permitted my going deeper into the problem, I propose here to give a brief account of the method.

As to the fundamental ideas and the characteristic properties of the collective theory of risk, I refer to a series of papers by Lundberg[2]. Here I give only the definitions necessary for the following deductions.

A group of policies in an insurance business is observed with regard to the occurrence of claims. Let $s(x)$ — in the sequel briefly called the *risk function* — denote for every policy the probability that, when a claim occurs, the sum at risk is $\leqq x$.[3]

Supposing $s(x)$ to be a known function (numerically or analytically defined) we seek the probability $F(x, P)$ that the total amount of the sums at risk under all claims is $\leqq x$ when

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[1] This journal 1930.

[2] Most of them cited in his above-mentioned paper. See also: LAURIN: »An Introduction into Lundberg's Theory of Risk», this journal 1930, and CRAMÉR: »On the Mathematical Theory of Risk», the jubilee volume of Försäkringsaktiebolaget Skandia, Stockholm 1930.

[3] In the collective theory of risk no assumptions are usually made about the separate policies. Cf., however, the generalizations on pp. 176—177.

the *expected* number of claims during the period of observation is equal to P.

Now, the probability that n claims have occurred when the expected number of claims is equal to P is given by the Poisson limit

$$\frac{e^{-P} \cdot P^n}{n\,!}.$$

Hence, denoting by $w_n(x)$ the probability that the total amount of the sums at risk under n claims is $\leqq x$, we get the following expression for the probability sought[1]

$$F(x, P) = \sum_0^\infty{}^n \frac{e^{-P}\, P^n}{n\,!}\, w_n(x), \tag{1}$$

where $w_n(x)$ is defined by the recurrent formula (only positive sums at risk are considered)

$$w_n(x) = \int_0^x w_{n-1}(x - z)\, ds(z) \tag{2}$$

and the conditions

$$w_0(x) = \begin{cases} 0 \text{ when } x \lneqq 0 \\ 1 \text{ when } x \geqq 0. \end{cases} \tag{2*}$$

Formula (1), however, is valid under more general assumptions than those made above. Let P_1, P_2, P_3, etc. be the expected number of claims in different groups of policies $(P_1 + P_2 + \cdots = P)$ and $s_1(x)$, $s_2(x)$, $s_3(x)$, etc. the corresponding risk functions. Then we have for the νth group

$$F_\nu(x, P_\nu) = \sum_0^\infty{}^n \frac{e^{-P_\nu} \cdot P_\nu^n}{n\,!}\, w_{\nu,\,n}(x),$$

[1] Cf. LUNDBERG: Teori för riskmassor, Stockholm 1919.

with the corresponding adjunct

$$\int_0^\infty e^{iux}\, d\,F_\nu(x, P_\nu) = e^{P_\nu \int_0^\infty (e^{iuz}-1)\, ds_\nu(z)}.$$

If now $F(x, P)$ denotes the probability function obtained when all the groups are taken together, we get the following expression for the adjunct of $F(x, P)$

$$\int_0^\infty e^{iux}\, d\,F(x, P) = e^{\Sigma^\nu P_\nu \int_0^\infty (e^{iuz}-1)\, ds_\nu(z)}$$

and thus, putting

$$s(x) = \frac{1}{P}\, \Sigma^\nu\, P_\nu \cdot s_\nu(x), \tag{3}$$

$$\int_0^\infty e^{iux}\, d\,F(x, P) = e^{P \int_0^\infty (e^{iuz}-1)\, ds(z)}, \tag{4}$$

i. e. the same formula that would be obtained if the risk function had been invariable and $= s(x)$. Thus the formula (1) is valid also in cases where the risk function varies from one policy to another or from one part of the period of observation to another. We have, according to (3), only to interpret $s(x)$ as an average of all the risk functions that have been operating.

In the sequel we suppose, for the sake of simplicity, that the sums at risk are measured by the mean value as unit, so that

$$\int_0^\infty x\, ds(x) = 1. \tag{5}$$

Then P — the expected number of claims — becomes equal to the amount of risk premiums that have flowed in to the

Company during the period of observation, and $F(x, P)$ denotes the probability that the total amount of the sums at risk under all claims is $\leqq x$ when the amount of risk premiums has reached the value P.

As being a probability function, $s(x)$ always fulfils the condition $s(\infty) = 1$. In order, however, to avoid superfluous reasoning about the validity of the following deductions, I suppose that $s(x)$ becomes unity already for some finite value of x, as is always the case in practical applications.

Denoting by γ_n the half-invariants of $F(x, P)$, we conclude from (4) that

$$\gamma_n = P \int_0^\infty z^n \, d s(z). \tag{6}$$

As to the function $F(x, P)$ it is further to be observed that

$$F(+0, P) = e^{-P}.$$

In the sequel, therefore, no attention need be paid to the point $x = 0$.

Our problem now will be to find a method for the numerical calculation of $F(x, P)$ when $x > 0$.

2. Put

$$\nu_n = \int_0^\infty z^n e^{hz} \, d s(z), \tag{7}$$

where h is a real number, positive or negative, which for the present is assumed to be finite, and let $\bar{s}(x)$ be a new risk function, defined by

$$d \bar{s}(x) = \frac{e^{hx}}{\nu_0} \, d s(x). \tag{8}$$

Further put, analogous to (2),

$$\bar{w}_n(x) = \int_0^x \bar{w}_{n-1}(x - z) \, d \bar{s}(z).$$

On account of (8) we then get

$$d w_n(x) = \nu_0^n e^{-hx} \, d \bar{w}_n(x),$$

and thus, after differentiation of (1),

$$d F(x, P) = e^{-hx - P(1-\nu_0)} \sum_0^\infty{}_n \frac{e^{-P\nu_0} (P \nu_0)^n}{n!} \, d \bar{w}_n(x)$$

or

$$d F(x, P) = e^{-hx - P(1-\nu_0)} \, d \bar{F}(x, P \nu_0), \tag{9}$$

where $\bar{F}(x, P\nu_0)$ is a new probability function, defined in the same way as $F(x, P)$ but obtained by using $\bar{s}(x)$ instead of $s(x)$ as risk function. Thus, denoting by $\bar{\gamma}_n$ the half-invariants of $\bar{F}(x, P\nu_0)$, we have, analogous to (6),

$$\bar{\gamma}_n = P \nu_0 \int_0^\infty z^n \, d \bar{s}(z)$$

or, on account of (7) and (8),

$$\bar{\gamma}_n = P \nu_n .$$

To other properties of the function $\bar{F}(x, P\nu_0)$ we shall revert later on.

3. Integrating (9) from 0 to x and putting $x = \varkappa P$, we get[1]

$$F(\varkappa P, P) = e^{-P\psi} \int_0^{\varkappa P} e^{-h(z - \varkappa P)} \, d \bar{F}(z, P\nu_0) \tag{10 a}$$

[1] Thus $\varkappa$ denotes the »loss ratio» in relation to the risk premium P, and $F(\varkappa P, P)$ the probability that the »loss ratio» is $\leqq \varkappa$.

$$(10\,b)\qquad = 1 - e^{-P\psi}\int_{\varkappa P}^{\infty} e^{-h(z-\varkappa P)}\, d\,\overline{F}(z, P\nu_0),$$

where

$$(11)\qquad \psi = 1 + \varkappa h - \nu_0 .$$

For a given value of $\varkappa$ the function ψ has only one maximum value, viz. when

$$(12)\qquad \varkappa = \nu_1 = \int_0^{\infty} z\, e^{hz}\, ds(z).$$

If h is determined in this way we always get $\psi \geqq 0$ and, on account of (5), $h \gtreqqless 0$ according as $\varkappa \gtreqqless 1$, at the same time as the integrals in (10), regarded as functions of h, assume their maximum values. It is further to be observed that the integrals become <1 if formula (10 a) is used when $0<\varkappa \leqq 1$, and formula (10 b) when $\varkappa \geqq 1$. Thus we always have

$$F(\varkappa P, P) < e^{-P\psi} \qquad \text{when } 0<\varkappa \leqq 1$$

and

$$1 - F(\varkappa P, P) < e^{-P\psi} \qquad \text{when } \varkappa \geqq 1.$$

In the sequel h is always supposed to be determined according to (12). Then, normalizing the function $\overline{F}(z, P\nu_0)$ by putting

$$(13)\qquad \overline{F}(P\nu_1 + t\sqrt{P\nu_2},\ P\nu_0) = \overline{F}_0(t, P\nu_0),$$

we may write the formulas (10) as follows

$$(14\,a)\qquad F(\varkappa P, P) = e^{-P\psi}\int_{-\infty}^{0} e^{-th\sqrt{P\nu_2}}\, d\,\overline{F}_0(t, P\nu_0) \qquad 0<\varkappa \leqq 1$$

$$(14\,b)\qquad 1 - F(\varkappa P, P) = e^{-P\psi}\int_{0}^{\infty} e^{-th\sqrt{P\nu_2}}\, d\,\overline{F}_0(t, P\nu_0). \qquad \varkappa \geqq 1$$

Though in other forms and other notations, these formulas are to be found in Lundberg's paper.

4. We now revert to the probability function $\overline{F}(x, P\nu_0)$. Analogous to (4) we may write

$$\int_0^{\infty} e^{iux}\, d\,\overline{F}(x, P\nu_0) = e^{P\nu_0\int_0^{\infty}(e^{iuz}-1)\, d\bar{s}(z)},$$

and hence, on account of (13),

$$\int_{-\infty}^{+\infty} e^{iux}\, d\,\overline{F}_0(x, P\nu_0) = e^{-\frac{iuP\nu_1}{\sqrt{P\nu_2}} + P\nu_0\int_0^{\infty}\left(e^{\frac{iuz}{\sqrt{P\nu_2}}}-1\right) d\bar{s}(z)}$$

$$= e^{-\frac{u^2}{2} + \vartheta\frac{u^3\nu_3}{6\nu_2^{3/2}P^{1/2}}},$$

where $|\vartheta| < 1$.

For finite values of h and u the second term in the exponential obviously tends to zero as P tends to infinity. If, further, we remember the assumption made about the risk function $s(x)$ (paragraph 1.) and the relation between $\varkappa$ and h, defined by the formula (12), it is easily shown that $\frac{\nu_3}{\nu_2^{3/2}}$, and consequently also the second term in the exponential tends to zero as $\varkappa$ tends to infinity. Hence, denoting by $\Phi(t)$ the normal (Gauss-Laplace) probability function

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{t^2}{2}}\, dt,$$

we conclude that[1]

[1] See e.g. Cramér: »On the Composition of Elementary Errors», this journal 1928.

$$\bar{F}_0(t, P\nu_0) \to \Phi(t) \qquad (\varkappa > 0) \tag{15}$$

as $P \to \infty$ *or* $\varkappa \to \infty$ (independently of the value of P).

If $s'(x)$ has a uniformly bounded derivative, this is the case also with $\bar{F}'_0(t, P\nu_0)$ except in the point $t = -\frac{P\nu_1}{\sqrt{P\nu_2}}$ where $d\bar{F}_0(t, P\nu_0) = e^{-P}$. For all values of $t > -\frac{P\nu_1}{\sqrt{P\nu_2}}$ it can then be proved that

$$\bar{F}_0'(t, P\nu_0) \to \Phi'(t) \qquad (\varkappa > 0) \tag{16}$$

at the same time as $\bar{F}_0(t, P\nu_0) \to \Phi(t)$.

Put, for that purpose,

$$\bar{F}_0'(t, P\nu_0) - \Phi'(t) = \varDelta(t)$$

and

$$-\varepsilon < \int_t^{t+\delta} \varDelta(u)\, du < \varepsilon$$

or

$$-\frac{\varepsilon}{\delta} < \int_0^1 \varDelta(t + \delta \cdot z)\, dz < \frac{\varepsilon}{\delta}, \tag{16*}$$

where, on account of (15), ε tends to zero as P or $\varkappa$ tends to infinity.

Now we have

$$\int_0^1 \varDelta(t + \delta \cdot z)\, dz = \varDelta(t) + \delta \int_0^1 z \cdot \frac{\varDelta(t + \delta \cdot z) - \varDelta(t)}{\delta \cdot z}\, dz,$$

and hence, under the assumptions made about $s'(x)$,

$$\varDelta(t) - \delta \cdot k < \int_0^1 \varDelta(t + \delta \cdot z)\, dz < \varDelta(t) + \delta \cdot k, \tag{16**}$$

where k is a *finite*, positive number.

Combining (16*) and (16**) and putting $\delta = \sqrt{\varepsilon}$ we get

$$-(1 + k)\sqrt{\varepsilon} < \varDelta(t) < (1 + k)\sqrt{\varepsilon},$$

and thus the formula (16) is proved.

Under these last assumptions about $s(x)$ it follows from (16) that the integrals in (14) for increasing values of P and $\varkappa$ tend to the corresponding integrals, obtained by replacing $\bar{F}_0(t, P\nu_0)$ by $\Phi(t)$, so that

$$F(\varkappa P, P) \sim e^{-P\psi} \int_{-\infty}^{0} e^{-th\sqrt{P\nu_2}}\, d\Phi(t) \tag{17 a}$$

when $P \to \infty$ and $0 < \varkappa \leqq 1$, and

$$1 - F(\varkappa P, P) \sim e^{-P\psi} \int_0^{\infty} e^{-th\sqrt{P\nu_2}}\, d\Phi(t) \tag{17 b}$$

when $P \to \infty$ and $\varkappa \geqq 1$ *or* when $\varkappa \to \infty$.

These formulas seem to be valid also in cases where $s(x)$ has a finite number of discontinuities, provided, however, that the sum of the saltuses at the discontinuities is less than 1. If, on the contrary, $s(x)$ is a »staircase» function, the formulas are *not* valid. These questions, however, will not be discussed here.

Finally, it may be pointed out that, when $\varkappa \neq 1$, the right members of (17) tend to zero in the following way:

$$F(\varkappa P, P) \sim \frac{e^{-P\psi}}{-h\sqrt{2\pi P\nu_2}} \qquad 0 < \varkappa < 1$$

$$1 - F(\varkappa P, P) \sim \frac{e^{-P\psi}}{h\sqrt{2\pi P\nu_2}} \qquad \varkappa > 1$$

Though in other notations, and apart from a factor ν_0 in the numerators[1], these are the asymptotic expressions found by LUNDBERG in the case when $P \to \infty$.

5. On account of the results in the preceding paragraph the formulas (14) directly induce us to seek an approximation of the probability function for finite values of P and $\varkappa$ by means of the asymptotic expressions (17). With regard to the fact that $1 - F(\varkappa P, P)$ — under the conditions given above — independently of the value of P tends to the asymptotic formula as $\varkappa$ tends to infinity, it can be supposed that good results will be obtained, especially for those values of $\varkappa$ that are of greatest interest in practical applications, i. e. values of $\varkappa > 1$. We shall therefore try to use the asymptotic expressions (17) as a first approximation of $F(\varkappa P, P)$ and $1 - F(\varkappa P, P)$ respectively.

Let $y_0(\varkappa P, P)$ be the approximate value of $F(\varkappa P, P)$ determined in this way, and put

$$A_0(u) = \int_0^\infty e^{-tu}\, d\, \Phi(t) = \frac{1 - \Phi(u)}{\sqrt{2\pi}\, \Phi'(u)}.$$

Then we have

$$\text{(18 a)} \qquad y_0(\varkappa P, P) = e^{-P\psi} \cdot A_0(-h\sqrt{P\nu_2}) \qquad 0 < \varkappa \leqq 1$$

$$\text{(18 b)} \qquad 1 - y_0(\varkappa P, P) = e^{-P\psi} \cdot A_0(h\sqrt{P\nu_2}) \qquad \varkappa \geqq 1$$

Especially in the case of small values of P and $\varkappa$ it may be expected, however, that better results will be obtained if $\overline{F}_0(t, P\nu_0)$ in the formulas (14) is approximated by means of the first terms of the so-called *A-series*. Having regard to the first two terms only, we have

$$\overline{F}_0(t, P\nu_0) = \Phi(t) - \beta_3\, \Phi'''(t),$$

where

$$\beta_3 = \frac{\nu_3}{6\,\nu_2\sqrt{P\nu_2}},$$

and hence, putting

$$A_3(u) = \int_0^\infty e^{-tu}\, d\, \Phi'''(t) = u^3 A_0(u) - \frac{u^2 - 1}{\sqrt{2\pi}},$$

$$\text{(19 a)} \qquad y_3(\varkappa P, P) = e^{-P\psi}\left[A_0(-h\sqrt{P\nu_2}) + \beta_3 A_3(-h\sqrt{P\nu_2})\right]$$

when $0 < \varkappa \leqq 1$, and

$$\text{(19 b)} \qquad 1 - y_3(\varkappa P, P) = e^{-P\psi}\left[A_0(h\sqrt{P\nu_2}) - \beta_3 A_3(h\sqrt{P\nu_2})\right]$$

when $\varkappa \geqq 1$.

In these formulas we have, according to (7), (11) and (12),

$$\nu_n = \int_0^\infty z^n e^{hz}\, d\, s(z)$$

$$\psi = 1 + \varkappa h - \nu_0$$

$$\varkappa = \nu_1.$$

For practical use a small table of the values of $A_0(u)$ and $A_3(u)$ is given at the end of this paper. As to $A_3(u)$, the figures in the last decimal place are not always significant.

6. *Numerical examples.* We begin with some simple cases in which the probability function can be exactly calculated according to (1) and (2).

[1] The factor ν_0 in Lundberg's formulas has probably crept in through the normalization.

Ex. 1. Put

$$s(x) = \begin{cases} \frac{1}{2}x & \text{when } 0 \leqq x \leqq 2 \\ 1 & \text{when } x > 2. \end{cases}$$

According to (2) we then get

$$w_n(x) = \frac{1}{n!}\left[\left(\frac{x}{2}\right)^n - \binom{n}{1}\left(\frac{x}{2} - 1\right)^n + \binom{n}{2}\left(\frac{x}{2} - 2\right)^n - \cdots\right]$$

where the series is to be interrupted by the last term in which $\frac{x}{2} - \nu$ is > 0.

In the following table some true values of $1 - F(\varkappa P, P)$ are compared with the corresponding approximate values according to (18) and (19).

Table 1.

$P = 1.$

$\varkappa$	$1-F(\varkappa P,P)$	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$
0.5	0.5342	0.6726	0.5771
1.0	.4238	.5000	.4137
2.0	.1614	.2134	.1612
4.0	.0208	.0242	.0220
6.0	.0017	.0018	.0017
8.0	.0001	.0001	.0001

Ex. 2.

$$s(x) = 1 - e^{-x}$$

$$w_n(x) = \sum_n^\infty {}_i \frac{e^{-x} \cdot x^i}{i!}.$$

In this case the risk function does not fulfil the condition $s(x) = 1$ for any finite value of x, but the fact that the preceding formulas and deductions are nevertheless valid can easily be verified.

Table 2 a.

$P = 1.$

$\varkappa$	$1-F(\varkappa P,P)$	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$
0.5	0.4698	0.6453	0.5038
1.0	.3457	.5000	.3589
2.0	.1826	.2620	.1840
4.0	.0472	.0618	.0482
6.0	.0114	.0141	.0118
8.0	.0026	.0031	.0027
10.0	.0006	.0007	.0006

Table 2 b.

$P = 5.$

$\varkappa$	$1-F(\varkappa P,P)$	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$
0.5	0.7687	0.8077	0.7714
1.0	.4361	.5000	.4369
2.0	.0745	.0851	.0744
3.0	.0075	.0082	.0074
4.0	.0005	.0006	.0005

Ex. 3.

$$s(x) = \begin{cases} \frac{1}{2}(1 - e^{-x}) & \text{when } 0 \leqq x \leqq 1 \\ 1 - \frac{1}{2}e^{-x} & \text{when } x > 1. \end{cases}$$

Here the risk function has a discontinuity at $x = 1$ with the saltus $= \frac{1}{2}$.

The probability function can be numerically calculated from

$$F(n + \xi, 2P) = \sum_0^n {}_\nu F_1(\nu + \xi, P)\frac{e^{-P} \cdot P^{n-\nu}}{(n-\nu)},$$

where $F_1(x, P)$ is the probability function in the preceding example and where n is an integer and $0 \leqq \xi < 1$. In a point of discontinuity we put

$$F(x, P) = \frac{1}{2}[F(x - 0, P) + F(x + 0, P)].$$

Table 3.

$P = 2$.

$\varkappa$	$1-F(\varkappa P, P)$	$1-y_0(\varkappa P, P)$	$1-y_3(\varkappa P, P)$
0.5	0.6916	0.7347	0.6746
1.0	.4247	.5000	.4104
2.0	.1229	.1552	.1215
3.0	.0316	.0389	.0318
4.0	.0077	.0091	.0078
5.0	.0018	.0021	.0018

Ex. 4. All the sums at risk are equal to unity, so that

$$ds(x) = \begin{cases} 1 & \text{when } x = 1 \\ 0 & \text{when } x \neq 1 \end{cases}$$

and

$$w_n(x) = \begin{cases} 1 & \text{when } x \geqq n \\ 0 & \text{when } x < n. \end{cases}$$

$F(\varkappa P, P)$ is in this case a discontinuous function that does not tend to the asymptotic expressions (17) when P or $\varkappa$ tends to infinity. Nevertheless the formulas (18) and (19) seem to give satisfactory results as shown by the following tables 4 a and 4 b.

On account of the small values of P, all these examples are of greater theoretical than practical interest, but they give a good illustration of the accuracy attainable and of the rapidity with which the asymptotic values approach the true ones

Table 4 a.

$P = 2$.

$\varkappa$	$1-F(\varkappa P, P)$	$1-y_0(\varkappa P, P)$	$1-y_3(\varkappa P, P)$
0.5	0.7294	0.7711	0.7329
1.0	.4586	.5000	.4529
2.0	.0978	.0998	.0918
3.0	.0106	.0100	.0095
4.0	.0008	.0006	.0006

Table 4 b.

$P = 10$.

$\varkappa$	$1-F(\varkappa P, P)$	$1-y_0(\varkappa P, P)$	$1-y_3(\varkappa P, P)$
0.2	0.9984	0.9987	0.9986
0.6	.9014	.9094	.9035
1.0	.4795	.5000	.4790
1.4	.1095	.1150	.1091
1.8	.0106	.0107	.0104
2.2	.0004	.0005	.0005

with increasing values of $\varkappa$. Though there is no reason to suppose that the results will be less good for larger values of P, I have tried, however, to test the formulas (18) and (19) in some cases in which $P = 100$, 500 and 1 000 and the risk functions are those used in the examples 1 and 2. As the probability function for such values of P cannot be exactly calculated without an immense amount of labour, it has been necessary to estimate the errors. The results, calculated according to the formulas given below, are collected in Tables 5 and 6.

In spite of the rough methods that have been used in the calculations of the maximum errors, the results must be regarded as on the whole satisfactory. To judge from the examples given above, it seems, therefore, that it would be

Table 5.

$$s(x)=\begin{cases}\frac{1}{2}x & \text{when } 0\leqq x\leqq 2\\ 1 & \text{when } x>2\end{cases}$$

h	$\varkappa$	$P=100$			$P=500$			$P=1\,000$		
		$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$	Max. Error of $y_3(\varkappa P,P)$	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$	Max. Error of $y_3(\varkappa P,P)$	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$	Max. Error of $y_3(\varkappa P,P)$
−0.15	0.82	$1-0.552\times10^{-1}$	$1-0.564\times10^{-1}$	0.058×10^{-1}	$1-0.167\times10^{-3}$	$1-0.168\times10^{-3}$	0.003×10^{-3}	$1-0.188\times10^{-6}$	$1-0.189\times10^{-6}$	0.002×10^{-6}
−0.10	0.88				$1-0.716\times10^{-2}$	$1-0.722\times10^{-2}$	0.010×10^{-2}	$1-0.264\times10^{-3}$	$1-0.265\times10^{-3}$	0.002×10^{-3}
−0.05	0.94				$1-0.105$	$1-0.106$	0.008	$1-0.386\times10^{-1}$	$1-0.388\times10^{-1}$	0.002×10^{-1}
0.00	1.00	0.500	0.491	0.008	0.500 0	0.496 1	0.000 8	0.500 0	0.497 3	0.000 4
0.05	1.07				0.927×10^{-1}	0.919×10^{-1}	0.006×10^{-1}	0.306×10^{-1}	0.305×10^{-1}	0.001×10^{-1}
0.10	1.14				0.830×10^{-2}	0.828×10^{-2}	0.004×10^{-2}	0.622×10^{-4}	0.620×10^{-4}	0.004×10^{-4}
0.15	1.22	0.301×10^{-1}	0.296×10^{-1}	0.047×10^{-1}	0.145×10^{-4}	0.144×10^{-4}	0.002×10^{-4}	0.173×10^{-8}	0.172×10^{-8}	0.002×10^{-8}
0.20	1.31				0.523×10^{-8}	0.522×10^{-8}	0.010×10^{-8}	0.304×10^{-15}	0.303×10^{-15}	0.003×10^{-15}
0.50	2.00	0.270×10^{-13}	0.269×10^{-13}	0.036×10^{-13}	0.142×10^{-62}	0.142×10^{-62}	0.005×10^{-62}			

Table 6.

$$s(x)=1-e^{-x}$$

$\varkappa$	$P=100$			$P=500$			$P=1\,000$		
	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$	Max. Frror of $y_3(\varkappa P,P)$	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$	Max. Error of $y_3(\varkappa P,P)$	$1-y_0(\varkappa P,P)$	$1-y_3(\varkappa P,P)$	Max. Error of $y_3(\varkappa P,P)$
0.82	$1-0.935\times10^{-1}$	$1-0.971\times10^{-1}$	0.348×10^{-1}	$1-0.147\times10^{-2}$	$1-0.149\times10^{-2}$	0.009×10^{-2}	$1-0.125\times10^{-4}$	$1-0.126\times10^{-4}$	0.004×10^{-4}
0.88				$1-0.257\times10^{-1}$	$1-0.260\times10^{-1}$	0.011×10^{-1}	$1-0.289\times10^{-2}$	$1-0.291\times10^{-2}$	0.005×10^{-2}
0.94				$1-0.169$	$1-0.172$	0.005	$1-0.875\times10^{-1}$	$1-0.885\times10^{-1}$	0.010×10^{-1}
1.00	0.500	0.486	0.034	0.500	0.494	0.003	0.500	0.496	0.001
1.07				0.137	0.135	0.004	0.614×10^{-1}	0.608×10^{-1}	0.007×10^{-1}
1.14				0.157×10^{-1}	0.155×10^{-1}	0.006×10^{-1}	0.120×10^{-2}	0.119×10^{-2}	0.002×10^{-2}
1.22	0.676×10^{-1}	0.654×10^{-1}	0.191×10^{-1}	0.453×10^{-3}	0.448×10^{-3}	0.025×10^{-3}	0.140×10^{-5}	0.140×10^{-5}	0.004×10^{-5}
1.31				0.470×10^{-5}	0.466×10^{-5}	0.032×10^{-5}	0.202×10^{-9}	0.202×10^{-9}	0.007×10^{-9}
2.00	0.199×10^{-8}	0.197×10^{-8}	0.135×10^{-8}	0.142×10^{-39}	0.142×10^{-39}	0.022×10^{-39}			

worth while to devote to the method a closer study than I have been able to give it here.

The maximum errors in Tables 5 and 6 have been calculated by means of the following formulas. From

$$F(\varkappa P, P) - y_3(\varkappa P, P) = $$
$$= \begin{cases} e^{-P\psi} \int\limits_{-\infty}^{0} e^{-th\sqrt{P\nu_2}}\, d\, Q(t, P\nu_0) & 0 < \varkappa \leqq 1 \\ -\, e^{-P\psi} \int\limits_{0}^{\infty} e^{-th\sqrt{P\nu_2}}\, d\, Q(t, P\nu_0), & \varkappa \geqq 1 \end{cases}$$

where

$$Q(t, P\nu_0) = \overline{F}_0(t, P\nu_0) - \Phi(t) + \beta_3\, \Phi'''(t),$$

it follows, when $\varkappa = 1$

$$| F(\varkappa P, P) - y_3(\varkappa P, P) | < M,$$

and when $\varkappa \neq 1$

$$| F(\varkappa P, P) - y_3(\varkappa P, P) | < 2\, M\, e^{-P\psi}$$

where M is such a number that

$$| Q(t, P\nu_0) | < M.$$

In the above examples M has been estimated according to the method developed by CRAMÉR in his paper »On the mathematical theory of risk». Using Cramér's notations we have (cf. Cramér's formula (35 a))

$$| I^{(\omega)}\, Q(t, P\nu_0) | < \frac{1}{2}\left(\delta + \frac{\varepsilon}{\omega}\right) \qquad 0 < \omega \leqq 1,$$

where in this case

$$\frac{1}{2}\,\delta = \frac{0.10}{P\lambda_3} + \frac{0.44}{\sqrt[3]{P\lambda_3}}\, e^{-\sqrt[3]{P\lambda_3}}$$

$$\lambda_3 = \nu_2^3 : \nu_3^2$$

and where ε is such a number that

$$\varepsilon > \frac{2}{\pi} \left| \int\limits_{-\infty}^{+\infty} e^{-iut}\, d\overline{F}_0(t, P\nu_0) \right|$$

for all $u > \sqrt{P\lambda_3}$.

Though they are deduced for the special risk functions used in Tables 5 and 6, these formulas are valid in all cases where $\nu_4^3 \leqq 12\, \nu_3^4$ and $P \geqq 100$.

Putting now

$$\omega = \frac{1}{\beta \log \frac{1}{\varepsilon}}$$

and following the method of Cramér we finally get

$$M = \varepsilon + e^{\frac{1}{\beta}}\left(\delta + \beta \cdot \varepsilon \log \frac{1}{\varepsilon}\right),$$

where β is a number that may appropriately be chosen in the vicinity of

$$\beta = \sqrt{\frac{\delta}{\varepsilon \log \frac{1}{\varepsilon}}}.$$

If ε is a sufficiently small number as compared with δ, we then get, approximately,

$$M = \delta.$$

7. Finally, it may be pointed out that, in cases where the formula (16) is valid, the asymptotic expression for the

frequency function given by Lundberg can be directly obtained from the formulas that have been deduced here.

Put, when $x > 0$,

$$F'(x, P) = f(x, P)$$

$$\bar{F}'(x, P\nu_0) = \bar{f}(x, P\nu_0) = \frac{1}{\sqrt{P\nu_2}}\,\bar{f}_0(t, P\nu_0)$$

where $\bar{f}_0(t, P\nu_0)$ is the normalized frequency function obtained by the substitution $t = \dfrac{x - P\nu_1}{\sqrt{P\nu_2}}$. Then, if h is determined according to (12), we find from (9) that

$$f(\varkappa P, P) = \frac{e^{-P\psi}}{\sqrt{P\nu_2}}\,\bar{f}_0(0, P\nu_0) \qquad (\varkappa > 0)$$

and hence, on account of (16), that

$$f(\varkappa P, P) \sim \frac{e^{-P\psi}}{\sqrt{2\pi P\nu_2}} \qquad (\varkappa > 0)$$

as P *or* $\varkappa$ tends to infinity.

As shown by LUNDBERG, this formula gives exceedingly good results also for finite values of P and $\varkappa$.

Values of $A_0(u)$ and $A_3(u)$.

u	$A_0(u)$	$A_3(u)$	u	$A_0(u)$	$A_3(u)$
0.0	0.5000	0.3989	2.1	0.1620	0.1402
.1	.4625	.3954	2.2	.1564	.1329
.2	.4292	.3864	2.3	.1510	.1262
.3	.3997	.3738	2.4	.1460	.1197
.4	.3733	.3590	2.5	.1413	.1139
.5	.3496	.3429	2.6	.1369	.1083
.6	.3283	.3262	2.7	.1327	.1031
.7	.3091	.3095	2.8	.1288	.0981
.8	.2918	.2930	2.9	.1250	.0936
.9	.2760	.2770	3.0	.1215	.0894
1.0	.2616	.2616	3.5	.1063	.0708
1.1	.2484	.2469	4.0	.0944	.0594
1.2	.2364	.2330	4.5	.0848	.0483
1.3	.2253	.2198	5.0	.0769	.0404
1.4	.2152	.2074	6.0	.0648	.0294
1.5	.2058	.1958	7.0	.0559	.0222
1.6	.1971	.1849	8.0	.0491	.0174
1.7	.1890	.1748	9.0	.0438	.0139
1.8	.1816	.1652	10.0	.0395	.0114
1.9	.1746	.1563	15.0	.0265	.0051
2.0	.1681	.1480	20.0	.0199	.0029