Adv. Appl. Prob. **36,** 747–773 (2004)
Printed in Northern Ireland

THE LOG-NORMAL APPROXIMATION IN FINANCIAL AND OTHER COMPUTATIONS

DANIEL DUFRESNE,* *University of Melbourne*

Abstract

Sums of log-normals frequently appear in a variety of situations, including engineering and financial mathematics. In particular, the pricing of Asian or basket options is directly related to finding the distributions of such sums. There is no general explicit formula for the distribution of sums of log-normal random variables. This paper looks at the limit distributions of sums of log-normal variables when the second parameter of the log-normals tends to zero or to infinity; in financial terms, this is equivalent to letting the volatility, or maturity, tend either to zero or to infinity. The limits obtained are either normal or log-normal, depending on the normalization chosen; the same applies to the reciprocal of the sums of log-normals. This justifies the log-normal approximation, much used in practice, and also gives an asymptotically exact distribution for averages of log-normals with a relatively small volatility; it has been noted that all the analytical pricing formulae for Asian options perform poorly for small volatilities. Asymptotic formulae are also found for the moments of the sums of log-normals. Results are given for both discrete and continuous averages. More explicit results are obtained in the case of the integral of geometric Brownian motion.

Keywords: Sums of log-normal variables; Brownian motion; Asian option; basket option; exponential functional of Brownian motion

2000 Mathematics Subject Classification: Primary 60J65; 60F05

1. Introduction

The log-normal distribution has had a very large number of applications; the book of Crow and Shimizu (1988) lists reliability (lifetime distribution), biology (growth models), ecology, atmospheric sciences and geology and other applications. One reason for the appeal of the log-normal in modelling is obvious: if a quantity is positive, then assuming that the logarithm of the quantity is normally distributed yields a tractable model which is relatively easy to estimate statistically. Another reason is the stability of the log-normal when taking products. As Dennis and Patil (1988) wrote (in Crow and Shimizu (1988)), ‘Whenever quantities grow multiplicatively, the log-normal becomes a leading candidate for a statistical model of such quantities.’ This explains in good part the persistence of the geometric Brownian motion model for security prices in economics and finance.

An unfortunate problem arises when sums of log-normals are considered, in that the distribution of sums of log-normals is never log-normal; moreover, the convolution of log-normal distributions does not have a simple explicit expression. The sum of log-normals arises naturally in a variety of models; two specific examples are (i) mobile radio cellular systems, where they appear in the signal-to-noise ratio (see Slimane (2001)), and (ii) the pricing of average or basket

Received 12 May 2003; revision received 28 May 2004.
* Postal address: Centre for Actuarial Studies, Department of Economics, University of Melbourne, VIC 3010, Australia. Email address: dufresne@unimelb.edu.au

options in finance (references are given below). Some approximations and bounds have been suggested. This paper concerns the asymptotic distributions of the sums of log-normals.

This author has encountered sums of log-normals in two related contexts: (i) the pricing of Asian and basket options and (ii) the study of an integral functional of Brownian motion (given in (1.1) below). The second problem is particularly interesting because explicit expressions for the law of the integral do exist, but are relatively complicated. In finance, the integral (1.1) occurs in the pricing of so-called Asian options (see below); the integral is itself an approximation of the actual quantity which defines the payoff of average options, which is of course a discrete average of prices of the security. The integral in (1.1) also arises in other models, for instance in physics (disordered systems); see for instance Comtet *et al.* (1998). It is also involved in the solution of the stochastic differential equation

$$\mathrm{d}X_t = (a_1 X_t + a_2)\,\mathrm{d}t + a_3 X_t\,\mathrm{d}B_t$$

(with a_1, a_2, a_3 constants), which is

$$X_t = X_0 \mathrm{e}^{\tilde{B}_t} + a_2 \mathrm{e}^{\tilde{B}_t} \int_0^t \mathrm{e}^{-\tilde{B}_s}\,\mathrm{d}s, \qquad \tilde{B}_t = \left(a_1 - \frac{a_3^2}{2}\right)t + a_3 B_t.$$

It is known that the variables

$$\mathrm{e}^{\tilde{B}_t} \int_0^t \mathrm{e}^{-\tilde{B}_s}\,\mathrm{d}s, \qquad \int_0^t \mathrm{e}^{\tilde{B}_s}\,\mathrm{d}s$$

have the same distribution for each fixed $t \geq 0$ (Dufresne (1989), Carmona *et al.* (1997)).

Some details will now be given regarding Asian and basket options. The payoffs of Asian (or average) options are expressed in terms of the average price of some security (stock, market index) or commodity. Basket options have payoffs which depend on linear combinations of the prices of several securities. In options on commodities (such as crude oil or natural gas), the price of the underlying security is often replaced with an average in order to decrease volatility, or else to reduce the possibility of manipulating prices close to expiration. If, as in the Black–Scholes model, the underlying securities are modelled as geometric Brownian motions, then the pricing of Asian or basket options is intimately related to finding the distribution of the sum or of the integral of geometric Brownian motions; some explicit results are known in the particular case where an Asian option has continuous averaging with equal weights, see Geman and Yor (1993) and Dufresne (2000) for details. The case of continuous averaging is, of course, an idealization of reality, but more explicit results have so far been found regarding continuous averages than for discrete ones; the continuous averaging formulae (with appropriate corrections) are good approximations of the discrete ones when the averaging dates are numerous enough and spread evenly through time; however, for other types of averages, there are no explicit formulae for option prices. Moreover, the explicit formulae known so far in the continuous case are not simple. The consequence is that practitioners rely on approximate formulae (mostly the log-normal approximation and Edgeworth series) or on Monte Carlo simulations. The log-normal approximation is sometimes very accurate, a fact which has apparently not been justified mathematically so far; Taleb (1997, Chapters 22 and 23) mentioned the log-normal approximation, but, with regard to Asian options, recommended the use of Monte Carlo simulations whenever volatility exceeds 30%. This empirical observation relates directly to the conclusions of this paper, as it will be shown that the limit distribution of the sums or averages involved in Asian or basket options are either normal or log-normal as volatility tends to 0. With the exception of two brief numerical examples in Section 7, this paper deals exclusively with the mathematical derivations of the limit distributions; the numerical comparison of option prices with their approximation is left for subsequent contributions. The combinations of geometric Brownian motions considered are general enough to include all Asian or basket options, whether discrete, continuous, or mixed.

Some required facts will now be recalled with respect to the only particular case where explicit formulae are known for the density of sums of log-normals. As explained by Geman and Yor (1993) and others, in the Black–Scholes model the random variable of interest in the pricing of Asian options with continuous averaging is

$$\int_0^T S_0 \mathrm{e}^{ms+\sigma B_s}\,\mathrm{d}s, \tag{1.1}$$

where S_0 is the initial price of the underlying security and B is standard Brownian motion under the risk-neutral measure. The drift m is, for example, equal to $r - \sigma^2/2$ when the underlying security does not pay dividends and the risk-free rate of interest is r. In this and in other situations m may be positive or negative.

Geman and Yor (1993) used the following transformation of (1.1): let

$$t' = \frac{\sigma^2 T}{4} \quad \text{and} \quad \mu = \frac{2m}{\sigma^2}; \tag{1.2}$$

then, by the scaling property of Brownian motion, the random variable in (1.1) has the same distribution as $4S_0/\sigma^2$ multiplied by

$$A_{t'}^{(\mu)} = \int_0^{t'} \mathrm{e}^{2(\mu s+B_s)}\,\mathrm{d}s.$$

This transformation has been used elsewhere by Yor (including many of the papers reproduced in Yor (2001)) and by this author, Dufresne (2000), (2001a), (2001b). This parametrization is advantageous in many ways, as shown especially by the work of Marc Yor.

However, the above transformation may not be the most natural one for the purpose of finding the asymptotic distribution of (1.1) when T tends to zero or infinity. We will instead use the following one: let

$$t = \sigma^2 T \quad \text{and} \quad \nu = \frac{m}{\sigma^2}; \tag{1.3}$$

then

$$\int_0^T S_0 \mathrm{e}^{ms+\sigma B_s}\,\mathrm{d}s \overset{\mathrm{D}}{=} \frac{S_0}{\sigma^2} M_t^{\nu}, \quad \text{where } M_t^{\nu} = \int_0^t \mathrm{e}^{\nu s + B_s}\,\mathrm{d}s$$

(here $\overset{\mathrm{D}}{=}$ denotes equality in distribution). It can be seen that t is the cumulative variance (or quadratic variation) of the logarithm of B over the time period $[0, t]$. The standardized drift ν may be positive or negative.

Both parametrizations (1.2) and (1.3) remove σ from the algebraic manipulations. Letting t' or t tend to 0 can mean either leaving the maturity T fixed while letting σ decrease to 0, or else leaving the volatility σ fixed while letting maturity decrease to 0. We can go from one set of parameters to the other by employing the identity in distribution

$$M_t^{\nu} \overset{\mathrm{D}}{=} 4A_{t/4}^{(2\nu)}. \tag{1.4}$$

A complete list of references on Asian option pricing will not be given here; the reader is referred to Dufresne (2000) and Linetsky (2001). The greater difficulty of pricing Asian options with short maturities, or small volatilities, was noticed by Rogers and Shi (1995, p. 1087), who solved the associated PDE numerically, and also by Fu *et al.* (1999), who inverted Geman and Yor's (1993) Laplace transform for Asian calls. Dufresne (2000) was unable to compute Asian option prices for t smaller than approximately 0.1, while the Laguerre series performed better as t increased (the number of required terms decreases with increasing t). Linetsky (2001) also noticed that more terms of his series expression for Asian option prices are required for small t; he was able to get an accurate price in a case where $t = 0.09$ at the cost of computing 57 terms of the series, and 400 terms are required in one case where $t = 0.01$, while larger t require less computational effort. Now, an option with a maturity of one year on an underlying security with a volatility $\sigma = 0.30$ has a normalized maturity of $t = 0.09$. A one-month averaging period in an oil or gas price with 60% annual volatility yields $t = 0.03$. Much shorter standardized maturities t result when the original maturity or volatility are smaller. A maturity of $T = \frac{1}{12}$ (one month) and a 10% annual volatility means that $t = 0.000833$. Standardized volatilities of 0.0001 or less arise in practice. Therefore, it would seem that the analytical expressions known so far for Asian options, as well as some of the numerical procedures, are good mostly for relatively large values of t, which are not very common in practice.

The conclusion is that there is a clear need for better approximations for small t. Observe that simulation does not seem to suffer from the small-t problem, but has, however, its own difficulties when used to price Asian options; see for instance Vázquez-Abad and Dufresne (1998), Fu *et al.* (1999), and Su and Fu (2000).

The same phenomenon is observed for the known formulae for the density of $A_t^{(\mu)}$. Yor (1992) derived the joint law of $(B_t, A_t^{(\mu)})$,

$$\mathrm{P}(A_t^{(\mu)} \in \mathrm{d}u \mid B_t + \mu t = x) = \frac{\sqrt{2\pi t}}{u} \exp\biggl(\frac{x^2}{2t} - \frac{1}{2u}(1 + \mathrm{e}^{2x})\biggr)\theta_{\mathrm{e}^x/u}(t)\,\mathrm{d}u,$$

where

$$\theta_r(t) = \frac{r}{\sqrt{2\pi^3 t}} \exp\biggl(\frac{\pi^2}{2t}\biggr) \int_0^\infty \mathrm{d}y \exp\biggl(-\frac{y^2}{2t}\biggr) \exp(-r\cosh y)\sinh(y)\sin\biggl(\frac{\pi y}{t}\biggr).$$

The trigonometric function in this expression causes numerical problems because of the increasing oscillations of the integrand when t gets smaller. Observe that the factor $\exp(\pi^2/2t)$ is getting larger at the same time. The Laguerre series obtained in Dufresne (2000) also suffer from the small-t problem, though the reason is apparently that the required moments of $1/A_t^{(\mu)}$ get very large when t is small. Dufresne (2001b) obtained the following expression for the density of $1/2A_t^{(\mu)}$:

$$\begin{aligned} f_\mu(x,t) &= \mathrm{e}^{-\mu^2 t/2} * 2^{-\mu} x^{-(\mu+1)/2} \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{e}^{-x\cosh^2 y} q(y,t) \cos\biggl(\frac{\pi}{2}\biggl(\frac{y}{t} - \mu\biggr)\biggr) H_\mu(\sqrt{x}\sinh y)\,\mathrm{d}y \qquad (1.5) \end{aligned}$$

for $x > 0$, where

$$q(y,t) = \frac{\mathrm{e}^{\pi^2/8t - y^2/2t}}{\pi\sqrt{2t}} \cosh y$$

and $H_\mu(\cdot)$ is the Hermite function (Lebedev (1972, p. 290)). Again there is a trigonometric function with an argument in $1/t$ and a factor $\exp(\pi^2/8t)$, which cause numerical instability when t is small.

Sections 2 and 3 deal with continuous averaging. The limit distributions are normal or log-normal when t tends to 0, and log-normal when t tends to infinity. The log-normal approximation is given what may be its first rigorous justification. (As far as this author knows, the only prior justification of the log-normal approximation, though imperfect, is the result given in Theorem 3.3(b) below, which shows that, as t tends to infinity, the normalized logarithm of M_t^0 tends to the law of the absolute value of a normal variable; Revuz and Yor (1999, p. 48) traced this result back to Durrett (1982).) Section 2 looks at limits of M_t^ν when t tends to 0, while Section 3 is concerned with limits as t tends to infinity. The results in Section 2 should be contrasted with the (seemingly) very different ones obtained by Barrieu *et al.* (2004).

Readers interested in discrete sums (as opposed to integral functionals of Brownian motion) may wish to skip to Sections 4 and 5. Section 4 defines a general integral functional of several geometric Brownian motions, which includes the combinations or averages involved in all Asian or basket options, and studies its distribution as the volatilities tend to 0. Again, normal and log-normal distributions are obtained in the limit. This paper does not say which of the two approximations, normal or log-normal, will be best for pricing Asian options or in other situations; this topic is left for further investigation.

Section 5 compares two slightly different log-normal approximations, and also shows that the difference of two log-normals approximates combinations with both positive and negative weights. Section 6 looks at the limits of *processes* related to Asian option pricing when volatility tends to 0. Section 7 concludes the paper with ideas for further study, including two numerical examples of Asian option prices with their normal and log-normal approximation. Appendix A derives some asymptotic formulae for the moments of $1/A_t^{(\mu)}$ that are used in some of the proofs, but which are of interest in their own right.

The 'big oh' and 'small oh' symbols have their usual meanings:

$$a(t) = \mathcal{O}(t^k) \quad \text{as } t \to 0+$$

if $|a(t)/t^k|$ remains bounded as t decreases to 0, and

$$a(t) = o(t^k) \quad \text{as } t \to 0+$$

means that

$$\lim_{t\to 0+} \frac{a(t)}{t^k} = 0.$$

We denote by N_{m,s^2} a random variable with a normal $\mathrm{N}(m, s^2)$ distribution. We denote convergence in distribution by $\xrightarrow{\mathrm{D}}$ and almost sure convergence by $\xrightarrow{\text{a.s.}}$. We will use the following general results related to convergence in distribution (Billingsley (1999, p. 27)):

1. Suppose that $\|\cdot\|$ is a norm (in what follows, either the Euclidean norm on $\mathbb{R}^d$ or the supremum norm on $\mathrm{C}[0, T]$), and that $X_n \xrightarrow{\mathrm{D}} X^*$; if $\|X_n - Y_n\| \xrightarrow{\text{a.s.}} 0$, then $Y_n \xrightarrow{\mathrm{D}} X^*$.

2. Suppose that $X_n \xrightarrow{\mathrm{D}} X^*$ and that $\{X_n\}$ is uniformly integrable; then $\mathrm{E}\, X_n \to \mathrm{E}\, X^*$. A sufficient condition for uniform integrability is that $\sup_n \mathrm{E}\,|X_n|^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$; another is that $Y_1 \le X_n \le Y_2$ almost surely for all n, where Y_1 and Y_2 are integrable.

Finally, B is one-dimensional standard Brownian motion, with

$$\underline{B}_t = \inf_{0\le u\le t} B_u, \qquad \bar{B}_t = \sup_{0\le u\le t} B_u,$$

and we write $\underline{B} = \underline{B}_1$ and $\bar{B} = \bar{B}_1$. Each element of the vector $(B^{(1)}, \dots, B^{(d)})$ is one-dimensional standard Brownian motion, and the above notation is also used for its running maximum and minimum, but it is *not* assumed that these Brownian motions are independent.

2. Limit distribution of M_t^ν as t tends to 0

Theorem 2.1. (Normal limit as $t \to 0+$.) *Let*

$$m(t) = t \quad or \quad m(t) = \mathrm{E}\, M_t^\nu$$

and

$$v(t) = \frac{t^3}{3} \quad or \quad v(t) = \sqrt{\mathrm{var}(M_t^\nu)}.$$

Then, as $t \to 0+$,

$$\frac{M_t^\nu - m(t)}{\sqrt{v(t)}} \xrightarrow{\mathrm{D}} N_{0,1}$$

and, for $k \in \mathbb{N}$,

$$\mathrm{E}\left(\frac{M_t^\nu - m(t)}{\sqrt{v(t)}}\right)^k \to \mathrm{E}\, N_{0,1}^k. \tag{2.1}$$

Proof. First, let $m(t) = t$ and $v(t) = t^3/3$. An obvious change of variable yields that

$$M_t^\nu = t \int_0^1 \mathrm{e}^{\nu t u + B_{ut}}\, \mathrm{d}u.$$

The distribution of M_t^ν is the same as that of

$$\tilde{M}_t^\nu = t \int_0^1 \mathrm{e}^{\nu t u + \sqrt{t} B_u}\, \mathrm{d}u. \tag{2.2}$$

We find that

$$t^{-3/2}(\tilde{M}_t^\nu - t) = t^{-1/2} \int_0^1 (\mathrm{e}^{\nu t u + \sqrt{t} B_u} - 1)\, \mathrm{d}u. \tag{2.3}$$

Now

$$\frac{1}{\sqrt{t}}(\mathrm{e}^{x\sqrt{t}} - 1) = x + \frac{x^2\sqrt{t}}{2}\mathrm{e}^{\zeta},$$

where ζ lies between 0 and $x\sqrt{t}$. Apply this with $x = \nu u \sqrt{t} + B_u$: since the trajectories of Brownian motion are almost surely continuous, they are also almost surely bounded over finite intervals, and the ζ above almost surely tends to 0 uniformly in u. We thus have

$$t^{-1/2}(\mathrm{e}^{\nu t u + \sqrt{t} B_u} - 1) \xrightarrow{\text{a.s.}} B_u.$$

Moreover, the function on the left-hand side is uniformly bounded when $0 < t, u < 1$ (considering a single continuous trajectory of B). Hence,

$$t^{-3/2}(\tilde{M}_t^\nu - t) \xrightarrow{\text{a.s.}} \int_0^1 B_u\, \mathrm{d}u \quad \text{as } t \to 0+.$$

It is well known that the distribution of the integral on the right-hand side is normal with mean 0 and variance $\frac{1}{3}$.

Finally, it is possible to replace $m(t) = t$ with $\mathrm{E}\, M_t^\nu$ because

$$\frac{t - \mathrm{E}\, M_t^\nu}{t^{3/2}} \to 0$$

as t decreases to 0 (see (2.5) below). Similarly, $v(t) = t^3/3$ may be replaced with the standard deviation of M_t^ν because of (2.7) below.

For the convergence of moments (in (2.1)), we give two possible proofs, (i) and (ii). The first one is more straightforward, but incomplete.

(i) Suppose that $m(t) = \mathrm{E}\, M_t^\nu$ and $v(t) = t^3/3$. Recall the formula for the moments of M_t^ν (Ramakrishnan (1954), Dufresne (1989), Yor (1992)):

$$\mathrm{E}(M_t^\nu)^n = n! \sum_{k=0}^{n} \mathrm{e}^{\alpha_k t} \Bigg[\prod_{j=0,\, j\neq k}^{n} (\alpha_k - \alpha_j) \Bigg]^{-1}, \tag{2.4}$$

where $\alpha_k = k\nu + k^2/2$ for $k \in \mathbb{N}$. In particular,

$$\mathrm{E}\, M_t^\nu = \frac{\mathrm{e}^{(\nu+1/2)t} - 1}{(\nu + \frac{1}{2})} = t + \tfrac{1}{2}(\nu + \tfrac{1}{2})t^2 + \mathcal{O}(t^3) \tag{2.5}$$

and

$$\begin{aligned} \mathrm{E}(M_t^\nu)^2 &= \frac{2}{(\nu+1)(2\nu+3)}\mathrm{e}^{(2\nu+2)t} - \frac{2}{(\nu+\frac{1}{2})(\nu+\frac{3}{2})}\mathrm{e}^{(\nu+1/2)t} + \frac{2}{(\nu+1)(2\nu+1)} \\ &= t^2 + (\nu + \tfrac{5}{6})t^3 + \mathcal{O}(t^4), \end{aligned} \tag{2.6}$$

which implies (by subtracting the square of (2.5)) that

$$\lim_{t\to 0+} \frac{\operatorname{var} M_t^\nu}{t^3/3} = 1. \tag{2.7}$$

We have thus proved (2.1) for $k = 1, 2$. The author has checked the cases $k = 3, \dots, 6$ in the same way, that is, by considering the Taylor series of the moments up to the required order, and then simplifying (the reader is spared the messy details). The case of arbitrary k has not been proved in this fashion, though this appears feasible.

(ii) Suppose that $m(t) = t$ and $v(t) = t^3/3$. Since $1 - \mathrm{e}^{-x} \leq x$ for nonnegative x, we find that (see (2.3))

$$\begin{aligned} t^{-1/2} \int_0^1 (\mathrm{e}^{\nu t u + \sqrt{t} B_u} - 1)\, \mathrm{d}u &\geq t^{-1/2} \int_0^1 (\mathrm{e}^{\nu t u + \sqrt{t}\underline{B}} - 1)\, \mathrm{d}u \\ &= t^{-1/2}(\mathrm{e}^{\sqrt{t}\underline{B}} - 1)\frac{\mathrm{e}^{\nu t} - 1}{\nu t} + \frac{\mathrm{e}^{\nu t} - 1 - \nu t}{\nu t^{3/2}} \\ &\geq \underline{B}\frac{\mathrm{e}^{\nu t} - 1}{\nu t} + \frac{\mathrm{e}^{\nu t} - 1 - \nu t}{\nu t^{3/2}}. \end{aligned} \tag{2.8}$$

Observe that the last expression converges to $\underline{B}$ as $t \to 0+$. Similarly,

$$\begin{aligned} t^{-1/2}\int_0^1 (\mathrm{e}^{\nu t u+\sqrt{t}B_u} - 1)\,\mathrm{d}u &\le t^{-1/2}\int_0^1 (\mathrm{e}^{\nu t u+\sqrt{t}\bar{B}} - 1)\,\mathrm{d}u \\ &= t^{-1/2}(\mathrm{e}^{\sqrt{t}\bar{B}} - 1)\frac{\mathrm{e}^{\nu t}-1}{\nu t} + \frac{\mathrm{e}^{\nu t}-1-\nu t}{\nu t^{3/2}} \\ &\le \bar{B}\mathrm{e}^{\sqrt{t}\bar{B}}\frac{\mathrm{e}^{\nu t}-1}{\nu t} + \frac{\mathrm{e}^{\nu t}-1-\nu t}{\nu t^{3/2}}. \end{aligned} \tag{2.9}$$

The last inequality follows since

$$\frac{1}{\sqrt{t}}(\mathrm{e}^{x\sqrt{t}} - 1) = x\mathrm{e}^{\zeta} \le x\mathrm{e}^{x\sqrt{t}},$$

where ζ lies between 0 and $x\sqrt{t}$, which is valid for $x > 0$.

Noting that $\underline{B}$ and $\bar{B}\mathrm{e}^{\sqrt{t}\bar{B}}$ are both integrable, we have thus shown that the variables in (2.3) (when $0 < t < 1$) are bounded below and above by integrable random variables; they are hence uniformly integrable. Since (2.3) converges in distribution as $t \to 0+$, those inequalities imply convergence of first moments to the first moment of the limit distribution (see the end of Section 1). The same reasoning works for higher moments, only raise the inequalities to the appropriate power and note that the variables $\underline{B}^k$ and $\bar{B}^k\mathrm{e}^{k\sqrt{t}\bar{B}}$ are integrable for any $k \in \mathbb{N}$.

The same results (in (i) or (ii)) are correct if $m(t) = \mathrm{E}\,M_t^{\nu}$ instead of $m(t) = t$, since

$$\mathrm{E}\biggl(\frac{M_t^{\nu} - \mathrm{E}\,M_t^{\nu}}{\sqrt{v(t)}}\biggr)^k - \mathrm{E}\biggl(\frac{M_t^{\nu} - t}{\sqrt{v(t)}}\biggr)^k = \sum_{j=0}^{k-1}\binom{k}{j}\mathrm{E}\biggl(\frac{M_t^{\nu} - t}{\sqrt{v(t)}}\biggr)^j\biggl(\frac{t - \mathrm{E}\,M_t^{\nu}}{\sqrt{v(t)}}\biggr)^{k-j},$$

which is seen to tend to 0 by a recursive argument. The same limits (2.1) hold if $v(t)$ is replaced with var M_t^{ν} because of (2.7), which completes the proof.

Remark 2.1. Yor (2001, p. 54) recently found that the formula (2.4) for the moments of M_t^{ν} had previously appeared in Ramakrishnan (1954), in relation to an astronomical model.

Remark 2.2. When one of the constants $\{\alpha_k;\ k \ge 1\}$ equals 0, the expressions for the moments are slightly different, as explained by Dufresne (1989). This does not affect the results above.

Theorem 2.2. (Log-normal limit as $t \to 0+$.) *Let $m(t)$ and $v(t)$ be as in Theorem 2.1. Then, as $t \to 0+$,*

$$\frac{m(t)}{\sqrt{v(t)}}\log\biggl(\frac{M_t^{\nu}}{m(t)}\biggr) \xrightarrow{\mathrm{D}} N_{0,1} \tag{2.10}$$

and, for $k \in \mathbb{N}$,

$$\mathrm{E}\biggl(\frac{m(t)}{\sqrt{v(t)}}\log\biggl(\frac{M_t^{\nu}}{m(t)}\biggr)\biggr)^k \to \mathrm{E}\,N_{0,1}^k. \tag{2.11}$$

We will use the following lemma.

Lemma 2.1. *Assume that, as n tends to infinity, the constants $\{a_n;\ n \ge 1\}$ tend to* 0.

(a) *Suppose the sequence of random variables $\{Z_n;\ n \ge 1\}$ converges in distribution to Z^*. Then*

$$\frac{1}{a_n}\log(1 + a_n Z_n)\mathbf{1}_{\{1+a_n Z_n>0\}} \xrightarrow{\mathrm{D}} Z^* \quad \textit{as } n \to \infty.$$

(b) *Conversely, suppose that* $\{U_n;\ n \geq 1\}$ *converges in distribution to* U^*. *Then*

$$\frac{\mathrm{e}^{a_n U_n} - 1}{a_n} \xrightarrow{\mathrm{D}} U^* \quad \text{as } n \to \infty.$$

Proof. To prove (a), apply Skorokhod's representation theorem (Billingsley (1999, p. 70)): there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathrm{P}})$ on which variables $\{\tilde{Z}_n;\ n \geq 1\}$ and $\tilde{Z}^*$ are defined, such that $\tilde{Z}_n \stackrel{\mathrm{D}}{=} Z_n$ for all n, $\tilde{Z}^* \stackrel{\mathrm{D}}{=} Z^*$ and Z_n converges $\tilde{\mathrm{P}}$-almost surely to $\tilde{Z}^*$. Clearly, $\mathbf{1}_{\{1+a_n\tilde{Z}_n>0\}}$ converges almost surely to 1 as n tends to infinity, and so

$$\frac{1}{a_n}\log(1 + a_n\tilde{Z}_n)\mathbf{1}_{\{1+a_n\tilde{Z}_n>0\}} = \tilde{Z}_n \frac{1}{a_n\tilde{Z}_n}\int_0^{a_n\tilde{Z}_n} \frac{\mathrm{d}u}{1+u}\mathbf{1}_{\{1+a_n\tilde{Z}_n>0\}} \xrightarrow{\text{a.s.}} \tilde{Z}^*.$$

Part (b) of the lemma is proved similarly.

Proof of Theorem 2.2. The limit distribution (2.10) follows at once from Lemma 2.1(a) and

$$\frac{m(t)}{\sqrt{v(t)}}\log\left(\frac{M_t^\nu}{m(t)}\right) = \frac{m(t)}{\sqrt{v(t)}}\log\left(1 + \frac{\sqrt{v(t)}}{m(t)}\frac{M_t^\nu - m(t)}{\sqrt{v(t)}}\right),$$

noting that $\lim_{t\to 0+}\sqrt{v(t)}/m(t) = 0$.

To prove (2.11), recall $\underline{B}$ and $\bar{B}$ from the proof of Theorem 2.1 and note that

$$\mathrm{e}^{\sqrt{t}\underline{B}}\frac{\mathrm{e}^{\nu t} - 1}{\nu t}t \leq \tilde{M}_t^\nu \leq \mathrm{e}^{\sqrt{t}\bar{B}}\frac{\mathrm{e}^{\nu t} - 1}{\nu t}t. \tag{2.12}$$

Hence,

$$g(t)\underline{B} + h(t) \leq \frac{m(t)}{\sqrt{v(t)}}\log\left(\frac{\tilde{M}_t^\nu}{m(t)}\right) \leq g(t)\bar{B} + h(t), \tag{2.13}$$

where

$$g(t) = \frac{\sqrt{t}m(t)}{\sqrt{v(t)}} \to \sqrt{3}, \qquad h(t) = \log\left(\frac{\mathrm{e}^{\nu t} - 1}{\nu t}\frac{t}{m(t)}\right) \to 0$$

as $t \to 0+$. Hence, the variables in the middle of (2.13), raised to a power $k \geq 1$, are uniformly integrable, and thus all moments converge to those of the limit distribution. This completes the proof.

Theorem 2.2 implies in particular that

$$\mathrm{E}(\log M_t^\nu) = \log t + o(\sqrt{t}), \qquad \operatorname{var}(\log M_t^\nu) \sim \frac{t}{3}$$

as $t \to 0+$.

Theorem 2.3. (Normal limit for reciprocal average as $t \to 0+$.) *Let* $m(t)$ *and* $v(t)$ *be as in Theorem 2.1. Then, as* $t \to 0+$,

$$\frac{m(t)}{\sqrt{v(t)}}\left(\frac{m(t)}{M_t^\nu} - 1\right) \xrightarrow{\mathrm{D}} N_{0,1}$$

and, for $k \in \mathbb{N}$,

$$\mathrm{E}\left[\frac{m(t)}{\sqrt{v(t)}}\left(\frac{m(t)}{M_t^\nu} - 1\right)\right]^k \to \mathrm{E}\,N_{0,1}^k.$$

Proof. There are at least two ways to prove convergence in distribution. One is to use Lemma 2.1(b): let

$$U_t = -\frac{m(t)}{\sqrt{v(t)}}\log\left(\frac{M_t^\nu}{m(t)}\right) \xrightarrow{\text{D}} U^* = N_{0,1}, \qquad a_t = \frac{\sqrt{v(t)}}{m(t)}.$$

A second more direct proof also yields convergence of moments. Initially, let $m(t) = t$, and recall (2.2), (2.8), (2.9) and (2.12). Then

$$\frac{t}{\sqrt{v(t)}}\left(1 - \frac{t}{M_t^\nu}\right) \stackrel{\text{D}}{=} \frac{t}{\sqrt{v(t)}}\left(1 - \frac{t}{\tilde{M}_t^\nu}\right) = \frac{t}{\sqrt{v(t)}}\,\frac{\int_0^1 (\mathrm{e}^{\nu t u + \sqrt{t} B_u} - 1)\,\mathrm{d}u}{\int_0^1 \mathrm{e}^{\nu t u + \sqrt{t} B_u}\,\mathrm{d}u}. \tag{2.14}$$

The last expression is easily seen to converge to

$$\sqrt{3}\int_0^1 B_u\,\mathrm{d}u \sim \mathrm{N}(0, 1),$$

while it is bounded below by

$$\frac{t^{3/2}}{\sqrt{v(t)}}\left[\underline{B}\mathrm{e}^{-\sqrt{t}\underline{B}} + \left(\frac{\mathrm{e}^{\nu t} - 1}{\nu t}\right)^{-1}\frac{\mathrm{e}^{\nu t} - 1 - \nu t}{\nu t^{3/2}}\mathrm{e}^{-\sqrt{t}\underline{B}}\right]$$

and bounded above by

$$\frac{t^{3/2}}{\sqrt{v(t)}}\left[\bar{B}\mathrm{e}^{\sqrt{t}(\bar{B} - \underline{B})} + \left(\frac{\mathrm{e}^{\nu t} - 1}{\nu t}\right)^{-1}\frac{\mathrm{e}^{\nu t} - 1 - \nu t}{\nu t^{3/2}}\mathrm{e}^{-\sqrt{t}\underline{B}}\right].$$

Those two bounds converge to $\sqrt{3}\underline{B}$ and $\sqrt{3}\bar{B}$ respectively, and are each uniformly bounded (when $0 < t < 1$, say) by variables which have all moments finite.

The proof for $m(t) = \mathrm{E}\,M_t^\nu$ is obtained as follows. Denote by X_t the right-hand side of (2.14) and let

$$Y_t = \frac{m(t)}{\sqrt{v(t)}}\left(1 - \frac{m(t)}{\tilde{M}_t^\nu}\right).$$

Then

$$Y_t - X_t = \frac{m(t) - t}{\sqrt{v(t)}}\left(\frac{m(t) + t}{\tilde{M}_t^\nu} - 1\right),$$

which tends to 0 almost surely as $t \to 0+$. This shows that Y_t has the same limit distribution as X_t. Finally, turn to moments: for $k \geq 1$,

$$\mathrm{E}\,X_t^k - \mathrm{E}\,Y_t^k = \sum_{j=1}^{k}\binom{k}{j}\mathrm{E}[X_t^{k-j}(Y_t - X_t)^j],$$

where the expectations on the right all tend to 0 as $t \to 0+$, and so the kth moments of X_t and Y_t have the same limit. This completes the proof.

Theorem 2.3 implies that

$$\mathrm{E}\left(\frac{1}{M_t^\nu}\right) = \frac{1}{t} + o\left(\frac{1}{\sqrt{t}}\right), \qquad \mathrm{var}\left(\frac{1}{M_t^\nu}\right) \sim \frac{1}{3t}$$

as $t \to 0+$. These could also be obtained (with a little more effort) from the integral formulae for the moments of $1/M_t^\nu$; see Dufresne (2000, p. 417).

As a last comment, observe that Lemma 2.1 may be reformulated as follows.

Corollary 2.1. *Suppose that $a_n, b_n, X_n > 0$ and $b_n/a_n \to 0$. Then the following are equivalent:*

(i)
$$\frac{X_n - a_n}{b_n} \xrightarrow{\mathrm{D}} U^*,$$

(ii)
$$\frac{a_n}{b_n} \log\left(\frac{X_n}{a_n}\right) \xrightarrow{\mathrm{D}} U^*,$$

(iii)
$$\frac{a_n}{b_n}\left(\frac{a_n}{X_n} - 1\right) \xrightarrow{\mathrm{D}} -U^*.$$

When considering limit distributions as $t \to 0+$, the normal and log-normal limits occur simultaneously because $\sqrt{v(t)}/m(t) \to 0$ as $t \to 0+$. However,

$$\frac{\sqrt{\operatorname{var} M_t^\nu}}{\operatorname{E} M_t^\nu} \not\to 0 \quad \text{as} \quad t \to \infty,$$

which explains why there is a log-normal limit distribution in the next section, but no normal limit.

3. Limit distributions of M_t^ν as t tends to infinity

Recall (Dufresne (1990)) that $\lim_{t\to\infty} M_t^\nu = M_\infty^\nu$ is finite if and only if $\nu < 0$, and that, moreover,

$$\frac{2}{M_\infty^\nu} \sim \operatorname{gamma}(-2\nu, 1), \qquad \nu < 0. \tag{3.1}$$

Theorem 3.1. (No normal limit for average as $t \to \infty$.) *Let $m(t) = \operatorname{E} M_t^\nu$ and $v(t) = \operatorname{var} M_t^\nu$. For any $\nu \in \mathbb{R}$, the reduced variable*

$$\frac{M_t^\nu - m(t)}{\sqrt{v(t)}} \tag{3.2}$$

does not converge to a normal distribution as $t \to \infty$. If $\nu \geq -1$, then it tends to 0 almost surely.

Proof. First, suppose that $\nu < 0$. Then M_t^ν converges almost surely to an inverse gamma variable, while the denominator tends either to a positive constant or to $+\infty$. A normal limit distribution is impossible. If $-1 \leq \nu < 0$, then, by (2.5), (2.6) and (3.1), the variance of M_t^ν tends to infinity, while the squared mean either tends to a constant or tends to infinity at a slower rate than the variance. Hence, (3.2) tends to 0 almost surely if $-1 \leq \nu < 0$.

Suppose next that $\nu \geq 0$. From (2.5) and (2.6),

$$m(t) \sim \frac{1}{\nu + \frac{1}{2}} \mathrm{e}^{(\nu+1/2)t}, \qquad \nu > -\tfrac{1}{2},$$
$$\frac{1}{\sqrt{v(t)}} \sim \mathrm{e}^{-(\nu+1)t}\sqrt{(\nu+1)(\nu+\tfrac{3}{2})}, \qquad \nu > -1,$$

and so $m(t)/\sqrt{v(t)} \to 0$; it is thus sufficient to consider the limit of $M_t^\nu/\sqrt{v(t)}$. We get

$$\mathrm{e}^{-(\nu+1)t}\int_0^t \mathrm{e}^{\nu s+B_s}\,\mathrm{d}s \le \mathrm{e}^{-t+\bar{B}_t}\int_0^t \mathrm{e}^{\nu(s-t)}\,\mathrm{d}s,$$

which tends to 0 almost surely as t tends to infinity.

Theorem 3.2. (No normal limit for reciprocal average as $t \to \infty$.) *For any $\nu \in \mathbb{R}$, the distribution of*

$$\frac{1/M_t^\nu - \mathrm{E}(1/M_t^\nu)}{\sqrt{\mathrm{var}(1/M_t^\nu)}} \tag{3.3}$$

does not converge to a normal distribution as $t \to \infty$. If $\nu \ge 0$, then the above variable tends to 0 *almost surely.*

Proof. If $\nu < 0$, then the limit distribution is obviously a gamma$(-2\nu, 1)$ distribution minus its mean and divided by its standard deviation.

For $\nu \ge 0$, it is perhaps easier to consider the expression (3.3) with M_t^ν replaced with $A_t^{(\mu)}$, with $\mu = \nu/2$. Refer to Appendix A for the asymptotic behaviour of the first two moments of $A_t^{(\mu)}$ (see (A.6)–(A.9), (A.13)–(A.18) and the comment after (A.18) in Appendix A). For all $\mu \ge 0$, it can be seen that

$$\frac{\mathrm{E}(1/A_t^{(\mu)})}{\sqrt{\mathrm{var}(1/A_t^{(\mu)})}} \to 0$$

as t tends to infinity. Thus, it only remains to show that

$$\frac{1}{A_t^{(\mu)}\sqrt{\mathrm{var}(1/A_t^{(\mu)})}} \to 0$$

almost surely. In the case $\mu = 0$, this follows from Theorem 3.3(b) (see below), which will now be seen to imply that

$$\lim_{t\to\infty}\frac{A_t^{(0)}}{t^p} = \infty \quad \text{a.s.}$$

for all p. Suppose that there exist $C, p > 0$ and a set E of positive probability such that

$$\liminf_{t\to\infty}\frac{A_t^{(0)}}{t^p} \le C$$

on E. Then it follows that

$$\liminf_{t\to\infty}\frac{1}{\sqrt{t}}\log A_t^{(0)} \le 0$$

on E, which is a contradiction since Theorem 3.3(b) says that

$$\lim_{t\to\infty}\mathrm{P}\left(\frac{1}{\sqrt{t}}\log A_t^{(0)} > 0\right) = 1.$$

Next, consider $\mu > 0$. First, note that

$$\mathrm{e}^{pt}A_t^{(\mu)} \xrightarrow{\text{a.s.}} \infty, \qquad p > -2\mu,$$

as t tends to infinity, since the above may be rewritten as

$$\mathrm{e}^{(p+2\mu)t+2B_t}[\mathrm{e}^{-2\mu t-2B_t}A_t^{(\mu)}];$$

the first factor tends to ∞ almost surely, while the second has a strictly positive limit in distribution. The equations (A.6)–(A.9) and (A.13)–(A.18) (see Appendix A) show that

$$\sqrt{\mathrm{var}\bigg(\frac{1}{A_t^{(\mu)}}\bigg)} \sim Kt^p\mathrm{e}^{a(\mu)t}$$

as $t \to \infty$, where K and p are constants and

$$a(\mu) = \begin{cases} -\dfrac{\mu^2}{4} & \text{if } 0 < \mu \le 4, \\ 4 - 2\mu & \text{if } \mu > 4. \end{cases}$$

Obviously, $a(\mu) > -2\mu$ for all $\mu > 0$, which ends the proof.

Theorem 3.3. (Limits of $\log M_t^\nu$ as $t \to \infty$.) *The following limits hold when $t \to \infty$.*

(a) *Suppose that $\nu < 0$. Then*

$$\frac{1}{\sqrt{t}}(\log M_t^\nu - \nu t) \xrightarrow{\text{a.s.}} \infty, \qquad \frac{1}{\sqrt{t}}\log M_t^\nu \xrightarrow{\text{a.s.}} 0.$$

(b) *Suppose that $\nu = 0$. Then*

$$\frac{1}{\sqrt{t}}\log M_t^0 \xrightarrow{\mathrm{D}} |N_{0,1}|$$

and

$$\mathrm{E}\bigg(\frac{1}{\sqrt{t}}\log M_t^0\bigg)^k \to \mathrm{E}\,|N_{0,1}|^k, \qquad k \in \mathbb{N}.$$

(c) *Suppose that $\nu > 0$. Then*

$$\frac{1}{\sqrt{t}}[\log(M_t^\nu) - \nu t] \xrightarrow{\mathrm{D}} N_{0,1}$$

and

$$\mathrm{E}\bigg(\frac{1}{\sqrt{t}}[\log(M_t^\nu) - \nu t]\bigg)^k \to \mathrm{E}\,N_{0,1}^k, \qquad k \in \mathbb{N}.$$

Proof. Part (a) is an obvious consequence of (3.1). The limit distribution in (b) has a well-known proof; see Comtet *et al.* (1998, Section 3.1) or Revuz and Yor (1999, Exercise 1.18, p. 23). We will give another proof, based on Bougerol's identity (for more details on the results used below, see Bougerol (1983) and Alili *et al.* (1997)). This identity says that, if (V, W) is two-dimensional Brownian motion, then

$$\int_0^t \mathrm{e}^{V_s}\,\mathrm{d}W_s \stackrel{\mathrm{D}}{=} \sinh(W_t)$$

for each fixed $t > 0$. This is equivalent to

$$\sqrt{A_t^{(0)}}N_{0,1} \stackrel{\mathrm{D}}{=} \sinh(\sqrt{t}N_{0,1})$$

if $N_{0,1}$ is independent of $A_t^{(0)}$. This implies that

$$\frac{1}{\sqrt{t}}\log A_t^{(0)} + \frac{1}{\sqrt{t}}\log(N_{0,1}^2) \overset{\mathrm{D}}{=} \frac{1}{\sqrt{t}}\log[\sinh^2(\sqrt{t}N_{0,1})].$$

The second term on the left-hand side tends to 0 almost surely as t tends to infinity, and $\sinh^2 y$ behaves like $\mathrm{e}^{2|y|}/4$ as $y \to \pm\infty$, which yields that

$$\frac{1}{\sqrt{t}}\log A_t^{(0)} \overset{\mathrm{D}}{\to} 2|N_{0,1}|.$$

This is the result sought, by (1.4).

Convergence of moments in (b) results from the same uniform integrability argument as in the proof of Theorem 2.1 after noting that

$$\frac{1}{\sqrt{t}}\log t + \underline{B} \le \frac{1}{\sqrt{t}}\log \tilde{M}_t^0 \le \frac{1}{\sqrt{t}}\log t + \bar{B}.$$

In (c), time reversal implies that, for any ν (Dufresne (1989)),

$$M_t^\nu \overset{\mathrm{D}}{=} \mathrm{e}^{\nu t + B_t}\int_0^t \mathrm{e}^{-\nu u - B_u}\,\mathrm{d}u.$$

Take logarithms on either side, subtract νt and divide by $\sqrt{t}$ to get

$$\frac{1}{\sqrt{t}}(\log M_t^\nu - \nu t) \overset{\mathrm{D}}{=} \frac{B_t}{\sqrt{t}} + \frac{1}{\sqrt{t}}\log\int_0^t \mathrm{e}^{-\nu u - B_u}\,\mathrm{d}u.$$

The first summand on the right-hand has an $\mathrm{N}(0, 1)$ distribution, while the second one converges to 0 almost surely. To prove convergence of moments, it is sufficient to show that the last expression is uniformly integrable. This is done by noting that it has lower and upper bounds

$$\frac{B_t - \bar{B}_t}{\sqrt{t}} + \frac{1}{\sqrt{t}}\log\biggl(\frac{1-\mathrm{e}^{-\nu t}}{\nu}\biggr) \quad \text{and} \quad \frac{B_t - \underline{B}_t}{\sqrt{t}} + \frac{1}{\sqrt{t}}\log\biggl(\frac{1-\mathrm{e}^{-\nu t}}{\nu}\biggr)$$

respectively. Those bounds are uniformly integrable, because

$$\frac{B_t - \bar{B}_t}{\sqrt{t}} \overset{\mathrm{D}}{=} B_1 - \bar{B}_1, \qquad \frac{B_t - \underline{B}_t}{\sqrt{t}} \overset{\mathrm{D}}{=} B_1 - \underline{B}_1.$$

Part (b) of Theorem 3.3 implies that, as $t \to \infty$,

$$\mathrm{E}(\log M_t^0) \sim \sqrt{\frac{2t}{\pi}}, \qquad \operatorname{var}(\log M_t^0) \sim t\biggl(1 - \frac{2}{\pi}\biggr),$$

while part (c) means that, for $\nu > 0$,

$$\mathrm{E}(\log M_t^\nu) = \nu t + o(\sqrt{t}), \qquad \operatorname{var}(\log M_t^\nu) \sim t.$$

Observe that exact integral expressions can be found for the moments of $\log M_t^\nu$, using the density (1.5). Comtet *et al.* (1998) gave other formulae regarding the first moment of $\log M_t^\nu$.

4. Limits of more general sums of log-normals

The continuous averages with equal weights studied in the previous sections are important mathematically, as they allow explicit formulae for many quantities of interest. However, financial (and other) computations concern discrete averages with weights which are not necessarily equal, and these are not always well approximated by continuous averages with equal weights. In this section, we consider more general averages involving any number of different securities. This includes all Asian and basket payoffs, as well as hybrids of the two types. For instance, an option's payoff might be based on the sum of the time-weighted averages of two securities $S^{(1)}$ and $S^{(2)}$, say

$$\sum_{j=1}^{n_1} w_j^{(1)} S_{t_j}^{(1)} + \sum_{j=1}^{n_2} w_j^{(2)} S_{t_j}^{(2)}. \tag{4.1}$$

Here $(S^{(1)}, S^{(2)})$ would often be correlated log-normal processes.

It is not possible to formulate the limit distribution problem in the same way here as it was in Section 2. In order to appreciate this, suppose that Brownian motion is sampled at times t_1 and t_2, with $0 < t_1 < t_2$, yielding a weighted average

$$w_1 \mathrm{e}^{mt_1+\sigma B_{t_1}} + w_2 \mathrm{e}^{mt_2+\sigma B_{t_2}} = \int_0^t \mathrm{e}^{ms+\sigma B_s}\,\mathrm{d}F(s), \qquad t \geq t_2, \tag{4.2}$$

where F is the combination of Dirac measures $w_1\delta_{t_1} + w_2\delta_{t_2}$. Then, letting t decrease to 0, Y_t loses one Dirac mass when $t \in (t_1, t_2)$, and is equal to 0 if $t < t_1$, which does not yield a very interesting limit, whichever normalization is chosen. Some other way must then be used to find an approximation which preserves the measure F. There is an obvious choice: let σ tend to 0, rather than t. For example, denote the expression in (4.2) by Y_σ, and consider the limit distribution

$$\frac{Y_\sigma - \mathrm{E}\,Y_\sigma}{\sigma} = \int_0^t \frac{1}{\sigma}(\mathrm{e}^{ms+\sigma B_s} - \mathrm{e}^{ms+(\sigma^2/2)s})\,\mathrm{d}F(s) \xrightarrow{\text{a.s.}} \int_0^t \mathrm{e}^{ms} B_s\,\mathrm{d}F(s).$$

The last variable having a normal distribution, it is then simple to check that

$$\frac{Y_\sigma - \mathrm{E}\,Y_\sigma}{\operatorname{var} Y_\sigma} \xrightarrow{\text{D}} N_{0,1}.$$

(In other words, $\operatorname{var} Y_\sigma \sim \sigma^2 \operatorname{var}[\int \mathrm{e}^{ms} B_s\,\mathrm{d}F(s)]$ as $\sigma \to 0$.)

Let us compare this with the limits obtained in Section 2. Consider (1.1), before the transformation (1.3) is performed. Here F is the Lebesgue measure restricted to $[0, T]$ and, if

$$Y_\sigma = \int_0^T \mathrm{e}^{ms+\sigma B_s}\,\mathrm{d}s,$$

then $(Y_\sigma - \mathrm{E}\,Y_\sigma)/\sqrt{\operatorname{var} Y_\sigma}$ tends to the standard normal as σ tends to 0, which is the same as Theorem 2.1. The counterparts of Theorems 2.2 and 2.3 then follow from Lemma 2.1, as before. In conclusion, letting t or σ tend to 0 lead to the same asymptotic distributions in the case of the integral in (1.1). From an intuitive point of view this is not surprising, since the variance of σB_t decreases to 0 as either σ or t tends to 0. (We remark that an alternative to letting σ tend to 0 would be to rescale the measure F such that its whole mass is concentrated increasingly close to 0. In light of the above comments, this possibility will not be considered, as it is equivalent to letting σ tend to 0.)

Consequently, in this section we consider a vector of n correlated geometric Brownian motions $(S^{(1)}, \dots, S^{(n)})$ ('log-normal processes') and look at the limit distributions of general averages (such as (4.1) above) when the volatilities of all the securities tend to 0. Rather than letting all the separate volatilities tend to 0, we simplify the algebra by introducing a factor p in all the volatilities:

$$\text{volatility of security } k = p\sigma_k, \qquad k = 1, \dots, n.$$

As p decreases to 0, all the volatilities tend to 0. We assume that

$$S_t^{(k)} = S_0^{(k)} \exp(\mu_k t + p\sigma_k B_t^{(k)}), \qquad k = 1, \dots, n.$$

Here $(B^{(1)}, \dots, B^{(n)})$ is, under the risk-neutral measure, a vector of (possibly correlated) standard Brownian motions.

We now describe the notation to be used for the averages. Rather than writing averages as in (4.1), we prefer writing any time-weighted combination of security k as an integral of the process $\exp(p\sigma_k B_t^{(k)})$ with respect to a signed measure $F^{(k)}$:

$$\text{combination of prices of security } k = \int_0^T \mathrm{e}^{p\sigma_k B_t^{(k)}} \, \mathrm{d}F_t^{(k)}.$$

This notation accommodates both discrete and continuous averages, or combinations of these. A discrete combination of the security $S^{(k)}$, with weights $w_j^{(k)}$ at time t_j, $j = 1, \dots, n_k$, is therefore written as

$$\sum_{j=1}^{n_k} w_j^{(k)} S_{t_j}^{(k)} = \int_0^T \mathrm{e}^{p\sigma_k B_t^{(k)}} \, \mathrm{d}F_t^{(k)},$$

where the measure $F^{(k)}$ assigns mass $S_0^{(k)} \mathrm{e}^{\mu_k t_j} w_j^{(k)}$ to the time point t_j for $j = 1, \dots, n_k$. A continuous average over $[0, T]$ is written as the right-hand side of the last equation, but now

$$F^{(k)}(s_1, s_2) = \frac{S_0^{(k)}}{T} \int_{s_1}^{s_2} \mathrm{e}^{\mu_k t} \, \mathrm{d}t$$

for any interval (s_1, s_2) with $0 \le s_1 < s_2 \le T$. To avoid trivialities, we assume that, *for each k, $F^{(k)}$ is not the zero measure, $F^{(k)}[0, T]$ is finite and σ_k is strictly greater than* 0.

In order to include all the above types of combinations of geometric Brownian motions, we consider random variables of the form

$$X_p = \sum_{k=1}^{n} \int_0^T \mathrm{e}^{p\sigma_k B_t^{(k)}} \, \mathrm{d}F_t^{(k)},$$

where $F^{(1)}, \dots, F^{(n)}$ are signed measures, and look for limit distributions of X_p (suitably normalized) as p tends to 0. First (Theorem 4.1), we consider normal limit distributions for

$$\frac{X_p - \mathrm{E}\, X_p}{p}.$$

Observe that signed measures may assign a negative mass to a set, and so, in this case, negative weights $w_j^{(k)}$ are allowed in (4.1). Next (Theorem 4.2), we restrict the analysis to proper measures (that is, all weights must now be nonnegative), and look for the limit distribution of

$$\frac{1}{p}\log\biggl(\frac{X_p}{\mathrm{E}\,X_p}\biggr).$$

It will turn out that $\mathrm{E}\,X_p$ can always be replaced with X_0 in the above expressions. Similar results will be obtained for $1/X_p$ as well (Theorem 4.3).

Theorem 4.1. *Suppose that $F^{(1)},\dots,F^{(n)}$ are signed measures. Then, as p tends to* 0*,*

$$\frac{X_p-\mathrm{E}\,X_p}{p}\xrightarrow{\text{a.s.}} Y=\sum_{k=1}^{n}\sigma_k\int_0^T B_t^{(k)}\,\mathrm{d}F_t^{(k)}\sim N_{0,v^2}$$

and, for $k\in\mathbb{N}$,

$$\mathrm{E}\biggl(\frac{X_p-\mathrm{E}\,X_p}{p}\biggr)^k\to v^k\,\mathrm{E}\,N_{0,1}^k,$$

where

$$\begin{aligned}v^2&=\operatorname{var}(Y)\\&=\sum_{k=1}^{n}\sigma_k^2\int_0^T\!\!\int_0^T(t_1\wedge t_2)\,\mathrm{d}F_{t_1}^{(k)}\,\mathrm{d}F_{t_2}^{(k)}+2\sum_{1\le j<k\le n}\mathrm{E}(B_1^{(j)}B_1^{(k)})\int_0^T\!\!\int_0^T(t_1\wedge t_2)\,\mathrm{d}F_{t_1}^{(j)}\,\mathrm{d}F_{t_2}^{(k)}.\end{aligned}$$

These results also hold if $\mathrm{E}\,X_p$ is replaced with X_0.

Proof. No generality is lost by assuming that $p>0$. We find that

$$\frac{X_p-\mathrm{E}\,X_p}{p}=\sum_{k=1}^{n}\int_0^T\frac{\mathrm{e}^{p\sigma_k B_t^{(k)}}-1}{p}\,\mathrm{d}F_t^{(k)},$$

and the almost-sure limit follows from dominated convergence, given that

$$\frac{\mathrm{e}^{p\sigma_k B_t^{(k)}}-1}{p}\xrightarrow{\text{a.s.}}\sigma_k B_t^{(k)}.$$

The variance of Y is found by expanding Y^2 and then taking expectations.

Convergence of moments is established by noting that, when $0<p<1$,

$$\biggl|\frac{\mathrm{e}^{p\sigma_k B_t^{(k)}}-1}{p}\biggr|\le\sigma_k(\bar{B}_t^{(k)}-\underline{B}_t^{(k)})\mathrm{e}^{\sigma_k\bar{B}_t^{(k)}},$$

where $\bar{B}_t^{(k)}=\max_{0\le s\le t}B_s^{(k)}$ and $\underline{B}_t^{(k)}=\min_{0\le s\le t}B_s^{(k)}$.

The same results hold if $\mathrm{E}\,X_p$ is replaced with X_0, because, as p tends to 0,

$$\frac{\mathrm{E}\,X_p-X_0}{p}\to 0.$$

Theorem 4.2. *Suppose that $F^{(1)}, \dots, F^{(n)}$ are measures. Then, as p tends to 0,*

$$\frac{1}{p}\log\biggl(\frac{X_p}{\mathrm{E}\,X_p}\biggr) \xrightarrow{\text{a.s.}} \frac{Y}{X_0}$$

and, for $k \in \mathbb{N}$,

$$\mathrm{E}\biggl[\frac{1}{p}\log\biggl(\frac{X_p}{\mathrm{E}\,X_p}\biggr)\biggr]^k \to \biggl(\frac{v}{X_0}\biggr)^k \mathrm{E}\,N_{0,1}^k,$$

where Y and v are as in Theorem 4.1. These results also hold if $\mathrm{E}\,X_p$ is replaced with X_0.

Proof. The first claim results from

$$\frac{1}{p}\log\biggl(\frac{X_p}{\mathrm{E}\,X_p}\biggr) = \frac{1}{p}\log\biggl(1 + \frac{p}{\mathrm{E}\,X_p}\frac{X_p - \mathrm{E}\,X_p}{p}\biggr) \xrightarrow{\text{a.s.}} \frac{Y}{X_0}.$$

Moreover,

$$\frac{1}{p}\log\biggl(\frac{X_p}{X_0}\biggr) - \frac{1}{p}\log\biggl(\frac{X_p}{\mathrm{E}\,X_p}\biggr) = \frac{1}{p}\log\biggl(1 + \frac{p}{X_0}\frac{\mathrm{E}\,X_p - X_0}{p}\biggr) \to 0.$$

Convergence of moments results from dominated convergence, after noting that

$$X_0 \exp\Bigl[p \min_k(\sigma_k \underline{B}_T^{(k)})\Bigr] \le X_k \le X_0 \exp\Bigl[p \max_k(\sigma_k \bar{B}_T^{(k)})\Bigr].$$

This completes the proof.

Theorem 4.2 implies that, as $p \to 0$,

$$\mathrm{E}(\log X_p) = \log X_0 + o(p), \qquad \mathrm{var}(\log X_p) \sim \frac{p^2 v^2}{X_0^2}.$$

Theorem 4.3. *Suppose that $F^{(1)}, \dots, F^{(n)}$ are measures. Then, as p tends to 0,*

$$\frac{1}{p}\biggl[\frac{1}{X_p} - \mathrm{E}\biggl(\frac{1}{X_p}\biggr)\biggr] \xrightarrow{\text{a.s.}} -\frac{Y}{X_0^2}$$

and, for $k \in \mathbb{N}$,

$$p^{-k}\,\mathrm{E}\biggl[\frac{1}{X_p} - \mathrm{E}\biggl(\frac{1}{X_p}\biggr)\biggr]^k \to \biggl(\frac{v}{X_0^2}\biggr)^k N_{0,1}^k,$$

where Y and v are as in Theorem 4.1. The results above also hold if $\mathrm{E}(1/X_p)$ is replaced with $1/X_0$ or with $1/\mathrm{E}(X_p)$.

Proof. From

$$Z_p = \frac{1}{p}\log\biggl(\frac{X_p}{X_0}\biggr) \xrightarrow{\text{a.s.}} \frac{Y}{X_0},$$

it follows that

$$\frac{1}{p}\biggl(\frac{1}{X_p} - \frac{1}{X_0}\biggr) = \frac{1}{pX_0}(\mathrm{e}^{-pZ_p} - 1) \xrightarrow{\text{a.s.}} -\frac{Y}{X_0^2}.$$

In the expressions on the left-hand side, $1/X_0$ may be replaced with $\mathrm{E}(1/X_p)$, since, when $0 < p < 1$,

$$\begin{aligned}
\frac{X_0}{p}\left(\frac{1}{X_0} - \frac{1}{X_p}\right) &\le \exp\Big[-\min_k(\sigma_k \underline{B}_T^{(k)})\Big]\Big(\exp\Big[\max_k(\sigma_k \bar{B}_T^{(k)})\Big] - 1\Big), \\
\frac{X_0}{p}\left(\frac{1}{X_0} - \frac{1}{X_p}\right) &\ge \exp\Big[-\min_k(\sigma_k \underline{B}_T^{(k)})\Big]\min_k(\sigma_k \underline{B}_T^{(k)}).
\end{aligned} \tag{4.3}$$

These two bounds are integrable, the left-hand side tends to 0 almost surely, and so

$$\frac{1}{p}\left[\frac{1}{X_0} - \mathrm{E}\left(\frac{1}{X_p}\right)\right] \to 0.$$

Similarly, $1/X_0$ may be replaced with $1/\mathrm{E}\,X_p$ because

$$\frac{1}{p}\left(\frac{1}{X_0} - \frac{1}{\mathrm{E}\,X_p}\right) = \frac{1}{X_0\,\mathrm{E}\,X_p}\,\frac{\mathrm{E}\,X_p - X_0}{p} \to 0.$$

Convergence of moments results from the bounds (4.3).

Theorem 4.3 implies that, as $p \to 0$,

$$\mathrm{E}\left(\frac{1}{X_p}\right) = \frac{1}{X_0} + o(p), \qquad \mathrm{var}\left(\frac{1}{X_p}\right) \sim \frac{p^2 v^2}{X_0^4}.$$

5. Some comments on log-normal approximations

We first compare two log-normal approximations in the case where each $F^{(k)}$ is a measure, and then discuss a log-normal difference approximation when at least one $F^{(k)}$ is a signed measure.

The usual way to find a log-normal approximation for a nonnegative distribution is to match first and second moments, which implies that

$$X_p \approx \text{log-normal}(m_{1p}, s_{1p}^2),$$

where

$$s_{1p}^2 = \log\left[\frac{\mathrm{E}\,X_p^2}{(\mathrm{E}\,X_p)^2}\right], \qquad m_{1p} = \log(\mathrm{E}\,X_p) - \frac{s_{1p}^2}{2}.$$

Theorem 4.2 now suggests a different log-normal approximation:

$$X_p \approx X_0 \mathrm{e}^{pY/X_0} \sim \text{log-normal}(m_{2p}, s_{2p}^2),$$

with

$$m_{2p} = \log X_0, \qquad s_{2p}^2 = \frac{p^2 v^2}{X_0^2}.$$

The following result shows that the two sets of log-normal parameters are close when volatilities are small.

Theorem 5.1. *As $p \to 0$,*

$$m_{1p} - m_{2p} = \mathcal{O}(p^2), \qquad s_{1p}^2 - s_{2p}^2 = \mathcal{O}(p^4).$$

Proof. First, there exist ξ_p and η_p, both between 0 and p^2, such that

$$\begin{aligned}\frac{1}{p^4}(s_{1p}^2 - s_{2p}^2) &= \frac{1}{p^4}\log\biggl[\mathrm{e}^{-p^2 v^2/X_0^2}\frac{\mathrm{E}\,X_p^2}{(\mathrm{E}\,X_p)^2}\biggr] \\ &= \frac{1}{p^4}\log\biggl[1 - \frac{p^2 v^2}{X_0^2} + \frac{p^4 v^4}{2X_0^4}\mathrm{e}^{-\xi_p v^2/X_0^2} \\ &\qquad + \frac{p^2}{(\mathrm{E}\,X_p)^2}\biggl(1 - \frac{p^2 v^2}{X_0^2}\mathrm{e}^{-\eta_p v^2/X_0^2}\biggr)\frac{\mathrm{E}\,X_p^2 - (\mathrm{E}\,X_p)^2}{p^2}\biggr].\end{aligned}$$

The expression inside the square brackets may be rewritten as $1 + p^4 K_p$, where K_p can be shown to have a finite limit as p tends to 0. For the first parameters, the situation is simpler:

$$\frac{1}{p^2}(m_{1p} - m_{2p}) = \frac{1}{p^2}\log\biggl(\frac{\mathrm{E}\,X_p}{X_0}\biggr) - \frac{s_{1p}^2}{2p^2} \to \frac{1}{2X_0}\sum_k \sigma_k^2 \int t\,\mathrm{d}F_T^{(k)} - \frac{v^2}{2X_0^2}.$$

This completes the proof.

Next, turn to the case where at least one of the $F^{(k)}$ is not a proper measure, that is, where there are positive and negative weights in the combination of securities. Theorem 4.1 suggests a normal approximation for X_p, but numerical computations (not shown here) reveal that a better approximation in this case might be a difference of log-normals: separate the positive and negative components and express X_p as the difference of two positive sums, and apply Theorem 4.2 to each sum separately; the following result justifies the approximation of X_p by the difference of two log-normals (the simple proof is omitted).

Theorem 5.2. *Suppose that $X_p^{(1)}$ and $X_p^{(2)}$ are as in Theorem 4.2, with*

$$\frac{1}{p}\log\biggl(\frac{X_p^{(j)}}{X_0^{(j)}}\biggr) \to \frac{Y^{(j)}}{X_0^{(j)}}, \qquad j = 1, 2.$$

Then

$$\lim_{p\to 0}\frac{1}{p}\Bigl[X_p^{(1)} - X_p^{(2)} - \Bigl(X_0^{(1)}\mathrm{e}^{pY^{(1)}/X_0^{(1)}} - X_0^{(2)}\mathrm{e}^{pY^{(2)}/X_0^{(2)}}\Bigr)\Bigr] = 0 \quad \textit{a.s.}$$

This justifies considering the approximation

$$X_p^{(1)} - X_p^{(2)} \approx X_0^{(1)}\mathrm{e}^{pY^{(1)}/X_0^{(1)}} - X_0^{(2)}\mathrm{e}^{pY^{(2)}/X_0^{(2)}}.$$

6. Limits of some related stochastic processes

The following results, given without proof, concern some stochastic processes which arise in the study of Asian options with continuous averaging. We let, for $\sigma > 0$ and $\nu \in \mathbb{R}$,

$$M_t^{\nu,\sigma} = \int_0^t \mathrm{e}^{\nu s + \sigma B_s}\,\mathrm{d}s, \qquad S_t^{\nu,\sigma} = x\mathrm{e}^{\nu t + \sigma B_t} + \mathrm{e}^{\nu t + \sigma B_t}\int_0^t \mathrm{e}^{-\nu s - \sigma B_s}\,\mathrm{d}s,$$

and

$$X_t^{\nu,\sigma} = \frac{M_t^{\nu,\sigma} - M_t^{\nu,0}}{\sigma}, \qquad Y_t^{\nu,\sigma} = \frac{S_t^{\nu,\sigma} - S_t^{\nu,0}}{\sigma},$$
$$\tilde{X}_t^{\nu,\sigma} = \frac{1}{\sigma}\log\biggl(\frac{M_t^{\nu,\sigma}}{M_t^{\nu,0}}\biggr), \qquad \tilde{Y}_t^{\nu,\sigma} = \frac{1}{\sigma}\log\biggl(\frac{S_t^{\nu,\sigma}}{S_t^{\nu,0}}\biggr).$$

It is known that, if $x = 0$, then $M_t^{\nu,\sigma}$ and $S_t^{\nu,\sigma}$ have the same distribution for fixed t; however, the second process is Markov, while the first one is not. The theorem shows that both processes have Gaussian limits, when suitably normalized, as $\sigma \to 0+$.

Theorem 6.1. *In each of the following, convergence is almost sure in the supremum norm over* $[0, T]$ *for any* $T < \infty$.

(a) *The process* $X^{\nu,\sigma}$ *converges to* $X^{\nu,0}$, *where*

$$X_t^{\nu,0} = \int_0^t e^{\nu s} B_s \, ds.$$

(b) *The process* $Y^{\nu,\sigma}$ *converges to* $Y^{\nu,0}$, *where*

$$\begin{aligned} Y_t^{\nu,0} &= x e^{\nu t} B_t + \int_0^t e^{\nu(t-s)}(B_t - B_s) \, ds, \\ dY_t^{\nu,0} &= \nu Y_t^{\nu,0} \, dt + S_t^{\nu,0} \, dB_t. \end{aligned}$$

(c) *The process* $\tilde{X}^{\nu,\sigma}$ *converges to* $\tilde{X}^{\nu,0}$, *where*

$$\tilde{X}_t^{\nu,0} = \frac{X_t^{\nu,0}}{M_t^{\nu,0}}, \qquad \tilde{X}_0^{\nu,0} = 0.$$

(d) *If* $x \geq 0$, *the process* $\tilde{Y}^{\nu,\sigma}$ *converges to* $\tilde{Y}^{\nu,0}$, *where*

$$\tilde{Y}_t^{\nu,0} = \frac{Y_t^{\nu,0}}{S_t^{\nu,0}}, \qquad \tilde{Y}_0^{\nu,0} = 0.$$

7. Conclusion

The main conclusions of this paper are:

1. For combinations of geometric Brownian motions with small volatilities or short durations, the limit distributions may be normal or log-normal, depending on the normalization chosen; the normal and log-normal are equivalent because, intuitively, the standard deviation of the sums are small relative to the mean, as volatilities tend to 0.

2. When maturities tend to infinity, log-normal limit distributions are sometimes obtained, but no instance of a normal limit has been found.

Further theoretical and numerical work is required to determine the value of these results for pricing Asian and basket options; in order to keep this paper from becoming too long, this will be done in subsequent contributions. As a preview, however, two numerical examples are briefly presented below.

Example 7.1. Consider case 1 in Example 7.2 of Dufresne (2000), which had also been used in other papers. An at-the-money Asian call option, with continuous averaging, has maturity T of 1 year, the volatility is $\sigma = 0.10$, the risk-free rate of interest is 0.02 and the initial stock price is 2. Monte Carlo simulations (with 200 000 replications) give a 95% confidence interval for the price of 0.05602 ± 0.00017. The Laguerre series studied in the same paper work when $t = \sigma^2 T$ is large enough, but they fail here, because $t = 0.01$ is too small. The improved

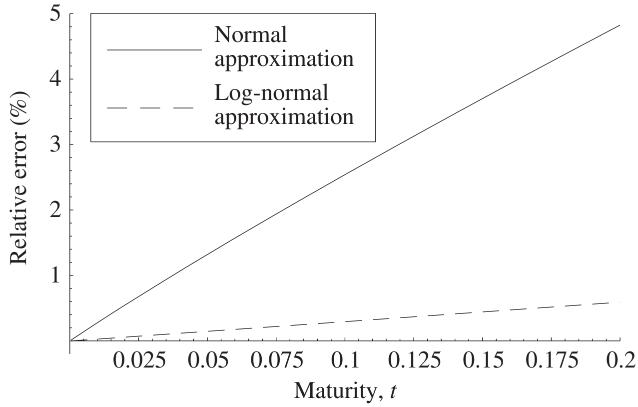

FIGURE 1.

Laguerre series of Schröder (2002) may give an accurate answer (this particular computation has not been performed), but the required programming and computing are far from trivial. The expansion given by Linetsky (2001), with 400 terms and very significant programming and computing efforts, yields 0.055986.

The normal approximation gives 0.0557, and the usual (moment-matching) log-normal approximation yields 0.0560537, with, in each case, an insignificant computing effort. The relative errors are 0.005 and 0.001 respectively. The log-normal approximation is well within the 95% confidence interval found by simulation.

Example 7.2. Figure 1 shows the relative errors (as percentages of the prices obtained by Monte Carlo simulation) of normal and log-normal approximations for the prices of at-the-money Asian call options (again with continuous averaging) for different maturities. The quantities approximated are

$$c(t) = \mathrm{e}^{-rt}\,\mathrm{E}\left(\frac{1}{t}M_t^0 - 1\right)_+ .$$

(As explained in Section 1, here t stands for $\sigma^2 T$. For instance, $t = 0.04$ might correspond to $\sigma = 20\%$ and $T = 1$, or to $\sigma = 40\%$ and $T = 0.25$.) It is seen that, for both approximations, the relative errors tend to zero as t tends to 0, but that the log-normal approximation produces relative errors which are about 10 times smaller than those of the normal approximation. The relative errors are roughly linear in t, and tend to 0 as t tends to 0.

Appendix A. Asymptotic expressions for the first two moments of $1/2A_t^{(\mu)}$

In this appendix, we find asymptotic formulae for the first two moments of $1/2A_t^{(\mu)}$ as t tends to infinity. We use results from Dufresne (2000), (2001b),

$$\begin{aligned}\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right) &= \frac{\mathrm{e}^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}\int_0^\infty y\mathrm{e}^{-y^2/2t}\,\frac{\cosh[(\mu-1)y]}{\sinh(y)}\,\mathrm{d}y \\ &= \frac{\mathrm{e}^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}\int_0^\infty y\mathrm{e}^{-y^2/2t}\,\frac{\mathrm{e}^{(\mu-2)y}+\mathrm{e}^{-\mu y}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y \qquad \text{(A.1)}\end{aligned}$$

and

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu+2)}}\right) = \mathrm{e}^{-(2\mu+2)t}\left[\mu + \mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)\right] \tag{A.2}$$

for all $\mu \in \mathbb{R}$, and from Dufresne (2001a),

$$\frac{1}{2A_t^{(-\mu)}} \overset{\mathrm{D}}{=} \frac{1}{2A_t^{(\mu)}} + G_\mu \tag{A.3}$$

for all $\mu > 0$, where G_μ is independent of $A_t^{(\mu)}$ and has a gamma$(\mu, 1)$ distribution.

It is enough to find an asymptotic formula when $0 \le \mu < 2$, and then use (A.2) and (A.3) for the other values of μ.

First, suppose that $0 < \mu < 2$. Then both $\mu - 2$ and $-\mu$ are strictly negative, and (A.1) is a function of t times the sum of two integrals of the form

$$\int_0^\infty y\mathrm{e}^{-y^2/2t}\,\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y, \tag{A.4}$$

with $a > 0$. For $n \ge 1$, there is a $\zeta(y)$, between 0 and y^2, such that

$$\begin{aligned}
(2t)^n n!\bigg[\int_0^\infty y\mathrm{e}^{-y^2/2t}\,\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y - \sum_{k=0}^{n-1}\frac{(-1)^k}{(2t)^k k!}\int_0^\infty y^{2k+1}\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y\bigg] \\
= \int_0^\infty y^{2n+1}\mathrm{e}^{-\zeta(y)/2t}\,\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y \\
\to \int_0^\infty y^{2n+1}\,\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y
\end{aligned}$$

as $t \to \infty$. The last integral is related to the logarithmic derivative of the gamma function, $\psi(z)$, which has the following expression (Lebedev (1972, p. 7)):

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \Gamma'(1) + \int_0^\infty \frac{\mathrm{e}^{-u} - \mathrm{e}^{-zu}}{1-\mathrm{e}^{-u}}\,\mathrm{d}u, \qquad \mathrm{Re}(z) > 0.$$

Hence,

$$\int_0^\infty y^{2n+1}\,\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y = 2^{-2n-2}\int_0^\infty u^{2n+1}\,\frac{\mathrm{e}^{-au/2}}{1-\mathrm{e}^{-u}}\,\mathrm{d}u = 2^{-2n-2}\psi^{(2n+1)}\left(\frac{a}{2}\right),$$

and (A.4) has the asymptotic expansion

$$\int_0^\infty y\mathrm{e}^{-y^2/2t}\,\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-2y}}\,\mathrm{d}y \sim \sum_{k=0}^\infty \frac{(-1)^k}{4(8t)^k k!}\psi^{(2k+1)}\left(\frac{a}{2}\right).$$

Finally,

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right) \sim \frac{\mathrm{e}^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}\sum_{k=0}^\infty \frac{\alpha_k^{(\mu)}}{t^k}, \qquad 0 < \mu < 2, \tag{A.5}$$

as $t \to \infty$, with

$$\alpha_k^{(\mu)} = \frac{(-1)^k}{2^{3k+2}k!}\left[\psi^{(2k+1)}\left(\frac{\mu}{2}\right) + \psi^{(2k+1)}\left(1 - \frac{\mu}{2}\right)\right].$$

Now turn to the case $\mu = 0$. Since

$$\frac{e^{-2y}+1}{1-e^{-2y}} = 1 + 2\frac{e^{-2y}}{1-e^{-2y}}$$

and

$$\int_0^\infty y e^{-y^2/2t}\,dy = t,$$

the preceding considerations yield that

$$E\left(\frac{1}{2A_t^{(0)}}\right) \sim \frac{1}{\sqrt{2\pi t}}\left(1 + \frac{1}{t}\sum_{k=0}^\infty \frac{\alpha_k^{(0)}}{t^k}\right),$$

with

$$\alpha_k^{(0)} = \frac{(-1)^k}{2^{3k+1}k!}\psi^{(2k+1)}(1), \qquad k \geq 0.$$

Using (A.2) and (A.3), these formulae allow the derivation of asymptotic expressions for the first moment of $1/2A_t^{(\mu)}$ for any $\mu \in \mathbb{R}_+$. For example,

$$E\left(\frac{1}{2A_t^{(2)}}\right) \sim \frac{e^{-2t}}{\sqrt{2\pi t}}\left(1 + \frac{1}{t}\sum_{k=0}^\infty \frac{\alpha_k^{(0)}}{t^k}\right),$$

$$E\left(\frac{1}{2A_t^{(\mu)}}\right) \sim -\mu + \frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}\sum_{k=0}^\infty \frac{\alpha_k^{(-\mu)}}{t^k}, \qquad -2 < \mu < 0,$$

$$E\left(\frac{1}{2A_t^{(\mu)}}\right) \sim (\mu-2)e^{-(2\mu-2)t} + \frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}\sum_{k=0}^\infty \frac{\alpha_k^{(\mu-2)}}{t^k}, \qquad 2 < \mu < 4.$$

For the purposes of this paper, the first terms in the asymptotic expressions are required, which are seen to be

$$E\left(\frac{1}{2A_t^{(0)}}\right) \sim \frac{1}{\sqrt{2\pi t}}, \tag{A.6}$$

$$E\left(\frac{1}{2A_t^{(\mu)}}\right) \sim \frac{e^{-\mu^2 t/2}\alpha_0^{(\mu)}}{\sqrt{2\pi t^3}}, \qquad 0 < \mu < 2, \tag{A.7}$$

$$E\left(\frac{1}{2A_t^{(2)}}\right) \sim \frac{e^{-2t}}{\sqrt{2\pi t}}, \tag{A.8}$$

$$E\left(\frac{1}{2A_t^{(\mu)}}\right) \sim (\mu-2)e^{-(2\mu-2)t}, \qquad \mu > 2. \tag{A.9}$$

Next, consider the second moment of $1/2A_t^{(\mu)}$. From Dufresne (2000, p. 417),

$$E\left(\frac{1}{2A_t^{(\mu)}}\right)^2 = \int_0^\infty \phi_\mu(2,t,y)\frac{\cosh[(\mu-1)y]}{\sinh(y)}\,dy,$$

where

$$\phi_\mu(2,t,y) = \left[\left(1-\frac{\mu}{2}\right)^2 + \frac{3}{4t} - \frac{y^2}{4t^2}\right]\frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}ye^{-y^2/2t}. \tag{A.10}$$

The second moment of $1/2A_t^{(\mu)}$ is then the sum of three integrals, and finding the asymptotic expansion of each of these integrals yields that (when $0 < \mu < 2$)

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2 \sim \frac{\mathrm{e}^{-\mu^2 t/2}}{\sqrt{2\pi t^3}} \sum_{k=0}^{\infty} \frac{\beta_k^{(\mu)}}{t^k}, \tag{A.11}$$

with $\beta_0^{(\mu)} = (1-\mu/2)^2\alpha_0^{(\mu)}$. From Corollary 3.4 of Dufresne (2001b), with $r = n = 1$ in the first formula there,

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu+2)}}\right)^2 = \mathrm{e}^{-(2\mu+2)t}\left[(\mu-1)\,\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right) + \mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2\right]. \tag{A.12}$$

This formula implies, in view of (A.5), that (A.11) holds also when $2 < \mu < 4$ (the constants $\{\beta_k^{(\mu)};\ k \geq 0\}$ are again combinations of derivatives of $\psi(\cdot)$). For the same values of μ, (A.9) then implies that

$$(\mu-1)\,\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right) + \mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2 \sim (\mu-1)(\mu-2)\mathrm{e}^{-(2\mu-2)t},$$

which in turn gives

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2 \sim (\mu-3)(\mu-4)\mathrm{e}^{-(4\mu-8)t}, \qquad 4 < \mu < 6.$$

It can be checked by induction that the same formula holds for $\mu \in (2n, 2n+2)$ for all $n \geq 2$. Now suppose that μ is an even, nonnegative integer. From (A.10),

$$\mathrm{E}\left(\frac{1}{2A_t^{(0)}}\right)^2 = \frac{1}{\sqrt{2\pi t^3}} \int_0^\infty \left[1 + \frac{3}{4t} - \frac{y^2}{4t^2}\right] y\mathrm{e}^{-y^2/2t}\left[1 + 2\frac{\mathrm{e}^{-2y}}{1-\mathrm{e}^{-2y}}\right]\mathrm{d}y.$$

Proceeding as for the first moment, we find that

$$\mathrm{E}\left(\frac{1}{2A_t^{(0)}}\right)^2 = \left[1 + \frac{3}{4t}\right]\mathrm{E}\left(\frac{1}{2A_t^{(0)}}\right) - \frac{1}{\sqrt{2\pi t^3}}\left\{\frac{1}{4t^2}\int_0^\infty y^3\mathrm{e}^{-y^2/2t}\left[1 + 2\frac{\mathrm{e}^{-2y}}{1-\mathrm{e}^{-2y}}\right]\mathrm{d}y\right\}.$$

The expression in braces has the asymptotic expansion

$$\frac{1}{4t^2}\int_0^\infty y^3\mathrm{e}^{-y^2/2t}\left[1 + 2\frac{\mathrm{e}^{-2y}}{1-\mathrm{e}^{-2y}}\right]\mathrm{d}y \sim \frac{1}{2} + \frac{1}{32t^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(8t)^k k!}\psi^{(2k+3)}(1),$$

and so

$$\mathrm{E}\left(\frac{1}{2A_t^{(0)}}\right)^2 \sim \frac{1}{\sqrt{2\pi t}}\left(1 + \frac{1}{t}\sum_{k=0}^{\infty}\frac{\beta_k^{(0)}}{t^k}\right),$$

where the constants $\{\beta_k\}$ are combinations of the derivatives of $\psi(z)$ at $z = 1$. In particular,

$$\beta_0^{(0)} = \alpha_0^{(0)} + \tfrac{1}{4}.$$

Using (A.12), this yields that

$$\mathrm{E}\left(\frac{1}{2A_t^{(2)}}\right)^2 \sim \frac{\mathrm{e}^{-2t}}{\sqrt{2\pi t^3}}\left(\frac{1}{4}+\sum_{k=1}^{\infty}\frac{\beta_k^{(0)}-\alpha_k^{(0)}}{t^k}\right).$$

In the same fashion, it is seen that (A.17) below holds:

$$\mathrm{E}\left(\frac{1}{2A_t^{(0)}}\right)^2 \sim \frac{1}{\sqrt{2\pi t}}, \tag{A.13}$$

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2 \sim \frac{\mathrm{e}^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}\beta_0^{(\mu)}, \qquad 0<\mu<2, \tag{A.14}$$

$$\mathrm{E}\left(\frac{1}{2A_t^{(2)}}\right)^2 \sim \frac{\mathrm{e}^{-2t}}{4\sqrt{2\pi t^3}}, \tag{A.15}$$

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2 \sim \frac{\mathrm{e}^{-\mu^2 t/2}}{\sqrt{2\pi t^3}}\beta_0^{(\mu)}, \qquad 2<\mu<4, \tag{A.16}$$

$$\mathrm{E}\left(\frac{1}{2A_t^{(4)}}\right)^2 \sim \frac{\mathrm{e}^{-8t}}{\sqrt{2\pi t}}, \tag{A.17}$$

$$\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2 \sim (\mu-3)(\mu-4)\mathrm{e}^{-(4\mu-8)t}, \qquad \mu>4. \tag{A.18}$$

By subtracting the squares of (A.6)–(A.9), it is seen that, in all cases, the first term of the asymptotic expansion of $\mathrm{var}(1/A_t^{(\mu)})$ is also given by the right-hand sides of (A.13)–(A.18).

Asymptotic formulae for $\mu<0$ can be found by appealing to (A.3), which yields that

$$\mathrm{E}\left(\frac{1}{2A_t^{(-\mu)}}\right)^2 = \mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right)^2 + 2\mu\,\mathrm{E}\left(\frac{1}{2A_t^{(\mu)}}\right) + \mu(\mu+1). \tag{A.19}$$

Acknowledgements

Support from the National Science and Engineering Research Council of Canada is gratefully acknowledged. The author also thanks the Centre for Actuarial Studies, Department of Economics, University of Melbourne, where most of this research was conducted.

References

Alili, L., Dufresne, D. and Yor, M. (1997). Sur l'identité de Bougerol pour les fonctionelles exponentielles du mouvement brownien avec drift. In *Exponential Functionals and Principal Values Related to Brownian Motion*. Biblioteca de la Revista Matemática Iberoamericana, Madrid, pp. 3–14.

Barrieu, P., Rouault, A. and Yor, M. (2004). A study of the Hartman–Watson distribution motivated by numerical problems related to the pricing of Asian options. To appear in *J. Appl. Prob.* **41,** No. 4.

Billingsley, P. (1999). *Convergence of Probability Measures*, 2nd edn. John Wiley, New York.

Bougerol, P. (1983). Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré B (N.S.)* **19,** 369–391.

Carmona, P., Petit, F. and Yor, M. (1997). On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential Functionals and Principal Values Related to Brownian Motion*. Biblioteca de la Revista Matemática Iberoamericana, Madrid, pp. 73–121.

Comtet, A., Monthus, C. and Yor, M. (1998). Exponential functionals of Brownian motion and disordered systems. *J. Appl. Prob.* **35,** 255–271. (Reproduced in Yor (2001).)

CROW, E. L. AND SHIMIZU, K. (eds) (1988). *Lognormal Distributions: Theory and Applications*. Marcel Dekker, New York.

DENNIS, B. AND PATIL, G. P. (1988). Applications in ecology. In *Lognormal Distributions: Theory and Applications*, eds E. L. Crow and K. Shimizu, Marcel Dekker, New York, pp. 303–330.

DUFRESNE, D. (1989). Weak convergence of random growth processes with applications to insurance. *Insurance Math. Econom.* **8,** 187–201.

DUFRESNE (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuarial J.* **1990,** 39–79.

DUFRESNE (2000). Laguerre series for Asian and other options. *Math. Finance* **10,** 407–428.

DUFRESNE, D. (2001a). An affine property of the reciprocal Asian process. *Osaka J. Math.* **38,** 379–381.

DUFRESNE, D. (2001b). The integral of geometric Brownian motion. *Adv. Appl. Prob.* **33,** 223–241.

DURRETT, R. (1982). A new proof of Spitzer's result on the winding of two-dimensional Brownian motion. *Ann. Prob.* **10,** 244–246.

FU, M. C., MADAN, D. B. AND WANG, T. (1999). Pricing continuous Asian options: a comparison of Monte Carlo and Laplace transform inversion methods. *J. Comput. Finance* **2,** 49–74.

GEMAN, H. AND YOR, M. (1993). Bessel processes, Asian options and perpetuities. *Math. Finance* **3,** 349–375.

LEBEDEV, N. N. (1972). *Special Functions and Their Applications*. Dover, New York.

LINETSKY, V. (2001). Exact pricing of Asian options: an application of spectral theory. To appear in *Operat. Res.*

RAMAKRISHNAN, A. (1954). A stochastic model of a fluctuating density field. *Astrophys. J.* **119,** 682–685.

REVUZ, D. AND YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd edn. Springer, New York.

ROGERS, L. G. C. AND SHI, Z. (1995). The value of an Asian option. *J. Appl. Prob.* **32,** 1077–1088.

SCHRÖDER, M. (2002). On the valuation of arithmetic-average Asian options: Laguerre series and theta integrals. Preprint, Department of Mathematics, University of Mannheim.

SLIMANE, B. S. (2001). Bounds on the distribution of a sum of independent lognormal random variables. *IEEE Trans. Commun.* **49,** 975–978.

SU, Y. AND FU, M. C. (2000). Importance sampling in derivative securities pricing. In *Proc. 2000 Winter Simulation Conf.*, eds J. A. Joines *et al.*, Society for Computer Simulation International, San Diego, CA, pp. 587–596.

TALEB, N. (1997). *Dynamic Hedging: Managing Vanilla and Exotic Options*. John Wiley, New York.

VÁZQUEZ-ABAD, F. AND DUFRESNE, D. (1998). Accelerated simulation for pricing Asian options. In *Proc. 1998 Winter Simulation Conf.*, eds D. J. Medeiros *et al.*, IEEE, Los Alamitos, CA, pp. 1493–1500.

YOR, M. (1992). On some exponential functionals of Brownian motion. *Adv. Appl. Prob.* **24,** 509–531. (Reproduced in Yor (2001).)

YOR, M. (2001). *Exponential Functionals of Brownian Motion and Related Processes.* Springer, New York.