Modelling Mortality Risk with Extreme Value Theory:

The Case of Swiss Re's Mortality-Indexed Bonds

Measuring and managing mortality risk is a huge challenge for risk managers. It is particularly hard to get a handle on extreme mortality events, but there is at least one viable modelling tool: extreme value theory. What is EVT and how can it be used to measure the distribution tail of extreme mortality events? Owen Beelders and David Colarossi test the usefulness of EVT by applying this method to Swiss Re's mortality index bond issue.

Extreme events are low-probability events that can have extreme economic consequences. The stock market crash of 1987 was an extreme event for financial markets that wiped out almost $1 trillion of market capitalization. Hurricane Andrew was an extreme event for the insurance industry that caused an insured loss of $17 billion and an economic loss of $30 billion (OECD, 2000). Given the severe consequences of extreme events, how can one manage them efficiently?

In the financial industry, there are only a handful of "far-out-of-the-money" derivatives for static hedging against extreme events, and dynamic hedging has proven to be impossible during times of crisis. The Basel II capital accord seeks to provide risk mitigation measures by forcing financial institutions to both calculate Value-at-Risk at very high percentiles and hold sufficient economic capital to weather low-probability events. For the insurance industry, where much of the risk is typically held on the balance sheet, diversification across geographies and risks, reinsurance markets and loss caps have been the risk mitigants for ensuring that a company is not bankrupted by an extreme event.

The statistical methods for evaluating extreme events require an accurate measure of the tail of the distribution. And one recently introduced method for examining the tail is extreme value theory (EVT). EVT not only models the given sample of observations in the tail, but can also be used to extrapolate the probability of even more extreme, out-of-sample events.

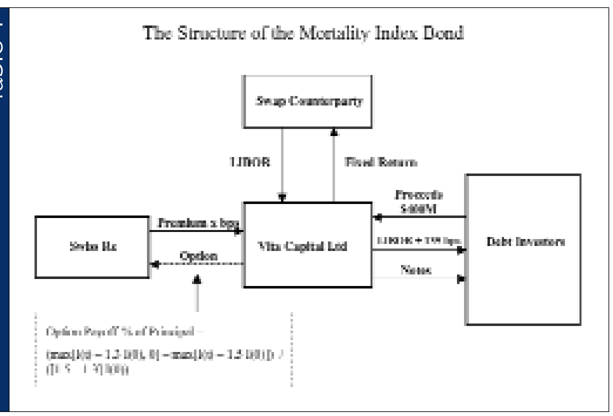
However, given that EVT is based on an asymptotic argument and samples sizes are small, one still has to be very careful with its use. In this article, we apply EVT to Swiss Re's mortality index bond issue and review some of the benefits of the approach.

The Swiss Re Mortality Bond

Swiss Re is the world's largest provider of life and health reinsuranc. It is exposed to very large amounts of mortality risk that cannot effectively be transferred to other life reinsurers who already bear similar risks, so it turned to the capital markets to hedge its mortality exposure.

Swiss Re set up a special purpose vehicle, dubbed Vita Capital Ltd., that issued $400 million in mortality-indexed

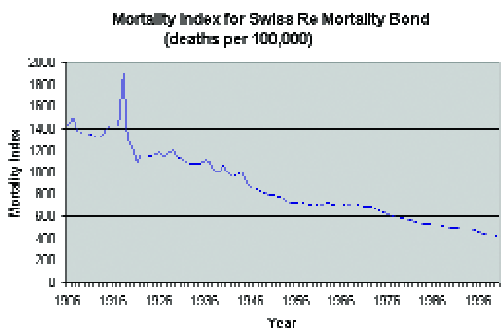
Table 1



notes. The proceeds are placed in a collateral account and invested in high-quality securities whose returns are swapped into a Libor-based return. The notes are expected to mature on January 1, 2007, at par. However, if mortality in five countries increases to pre-defined levels, investors in the notes may receive less than par or even zero at maturity. Note holders will receive quarterly interest payments of Libor plus 1.35% (annualized) in return for bearing extreme mortality risk. Table 1 contains a graphical description of the issuance structure.

The mortality index is based on a weighted average of mortality in the US, UK, France, Italy and Switzerlanand is also weighted by age and gender. The base case index was calculated for 2002. If, in any calendar year, the index is greater than 130% of the 2002 base value, a pre-defined portion of the principal in the collateral account will be paid to Swiss Re under the terms of the derivative contract. If the index reaches 150% of the 2002 base level, 100% of the principal will be paid to Swiss Re. The transaction also allows for partial payments in multiple years (e.g., if the index reached 135% of the 2002 base levels in each of 2004 and 2005, Swiss Re would receive two payments of 25% of principal each). However, given the improvements in healthcare and technology, one would expect future mortality to decrease from year to year. In financial terms, the contract is equivalent to a call

Figure 1



option spread on the index, with a lower strike price of 130% of 2002 mortality and an upper strike price of 150%.

Given that this is mortality risk, the traditional way to value this contract and assess the risk is to use an actuarial approach – i.e., one would simulate future paths of the index over the life of the deal and determine the probability of attachment, P(I(t) >1.3I(0)), the probability of exhaustion, P(I(t) > 1.5I(0)), and the expected loss, where I(0) is the index for 2002 and I(t) is the index for 2003 through 2006. The Swiss Re mortality index is a weighted average of mortality for the US (70%), UK (15%), France (7.5%), Italy (5%) and Switzerland (2.5%). The index is further weighted according to gender — 65% male and 35% female — and according to age bands. By construction, the weighted-average age does not increase over time.

We have graphed the time series of the 94 data points in the index in Figure 1 (above).

Valuation of the Mortality Bond

For simplicity, we chose to begin with the model provided by Milliman USA in the offering circular and then applied extreme value theory. Since the area of concern is only the extreme movements in the index, extreme value theory dictates the results of our analysis. The choice of starting point had little effect on the results of our analysis.

In the Milliman model, a bootstrap simulation approach was used to value the contract. Random draws from the distribution of mortality improvements were used to update the mortality index over a four-year period as follows, I(t) = I(t-1)x (- i(t) /100), where I(0) is the index in 2002 and i(t) is a randomly drawn mortality improvement.

To draw from the distribution of annual mortality improvements, Milliman created five buckets for the 93 observations. The range of the buckets is [-33.2;-2.7], [-2.5;-0.1], [-0.1;2.2], [2.2;4.1] and [4.7;31.6], and the number of observations in each bucket is 9, 18, 38, 18 and 10. Given the inherent serial correlation in the mortality changes, a Markov model was constructed for transitioning between buckets where the transition matrix was calibrated to fit the serial correlation of the data.

Once a bucket was chosen, an observation number was drawn from a uniform distribution, and the simulated mortality improvement was interpolated to match the uniform random number. For example, if bucket 3 was selected, a uniform random number between 1 and 38 was drawn. If the uniform random number is 23.6, then the bootstrapped observation would be interpolated between the 23rd and 24th observation.

The probabilities of attachment, exhaustion and expected loss in Milliman's results are based on 50,000 simulated paths for the four-year risk period. The probabilities of attachment can be compared to the probabilities of other catastrophic events – such as earthquakes in California – and to the probability default on a corporate bond. The expected loss is comparable to the product of: (a) the probability of default on a corporate bond; and (b) one minus the expected recovery rate on the bond, given a default.

The main concern with this method is the way that observations are drawn from bucket 1, the bucket that captures the changes in mortality from -33.2% to -2.7%. This bucket straddles the attachment point and does not reach the exhaustion point.

The problem with the Milliman method is that it uses linear interpolation between the most extreme and second most extreme observation, thus reducing the likelihood of ever reaching the attachment point. In addition, it does not allow for more extreme values than -33.2%. The EVT method, in contrast, can be applied to more frequent and more extreme mortality events; when we applied EVT, we also hoped that it would prove more accurate than Milliman with regard to reflecting the underlying risk of extreme events.

Application of Extreme Value Theory

EVT can be applied to maxima or exceedences[1]. We are in interested in the exceedences in the tail of the distribution because the payoff of the mortality indexed bond occurs when mortality exceeds a certain level. Under certain regularity conditions, the distribution of the exceedences is given by the generalized Pareto distribution (GPD) that is defined as follows:

$$g_{\xi,\sigma}(x) = \frac{1}{\sigma}\left(1+\xi\frac{x}{\sigma}\right)^{-(1+1/\xi)}$$

for $\xi \neq 0$, and

$$g_{0,\sigma}(x) = \frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right)$$

for $\xi=0$, where ξ is the exceedence, ξ is the shape parameter and $\sigma \geq 0$ is the scale parameter. The support is $x \geq 0$ when $x \geq 0$ and $\xi \geq 0$ and $0 \leq x \leq -\sigma/\xi$ for $\xi<0$. When $\xi>0$, we have the Pareto distribution; when $\xi<0$, we have the type II Pareto distribution; and when $\xi=0$, we have the exponential distribution. When $\xi<0$, the tail of the distribution is truncated at $-\sigma/\xi$, when $\xi=0$, the tail is not fat and the underlying distribution of the data could be normal; it is only when $x>0$ that the tail of the distribution can be characterized as fat-tailed. The two benefits of using EVT is that it applies to the tail of the distribution and can be used to extrapolate out of sample.

A very important step in using the GPD is the choice of the threshold or boundary value for the exceedences. Because our focus is on modelling bucket 1 in Milliman's model, we do not choose or estimate the optimal threshold value. Given that the distribution only has two parameters and we have a small sample size, we set up a VBA subroutine in MS Excel for computing the likelihood for each combination of parameters and then used a grid search to find the maximum likelihood estimates. Figure 2 contains a surface plot of the log-likelihood for the parameters where the maximum was attained at $\xi = 1.85$ and $\sigma = 0.585$. Having obtained an estimate of ξ that is greater than 0, we can conclude that the lower tail of the distribution of mortality improvements is fat tailed — i.e., the probability of large increases in mortality, and risk to note holders, would be underestimated if one was simply to use a normal distribution to model the data.

We estimate both the GPD for $\xi \neq\neq 0$ and $\xi = 0$ and report the results in Table 2 (next page). Based on the likelihood ratio test, we should use the Pareto distribution rather than the exponential distribution, although the standard errors of the parameters are quite large. We also include the probability of attachment and exhaustion for the two distributions. The two probabilities are larger for the case where $\xi \neq 0$ because this is the case for a fat-tailed distribution; when $\xi=0$, the tails are not fat and the underlying distribution could be normal. The probability of attachment for the Pareto distribution is close to the historical frequency of 1% based on the one case in the 93 changes of history. In addition, the probability has not declined that much from the attachment to exhaustion point, which once again emphasizes the fatness of the tail. In contrast, the exponential distribution of the tail suggests that the probability of attachment is closer to 0.2% and the probability of exhaustion is closer to a 1-in-10,000 event. In Figure 3, we plot the two densities in the vicinity of the attachment and exhaustion points, and we note how quickly the exponential distribution asymptotes to zero relative to the Pareto distribution.

In Figure 4, we plot the log-likelihood surface where we include the case of $\xi=0$.

It is important to note that the log-likelihood can dive down very steeply as x approaches zero and, as a result, numerical methods sometimes struggle to find the maximum if the starting value is not well chosen. In stark comparison to the precipitous drop in the surface in the vicinity of $\xi=0$ is the relative flatness of the surface near the maximum, so one has to be careful.

Figure 2

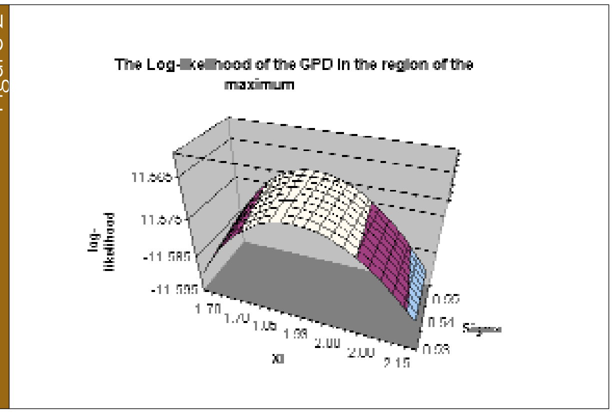


Figure 3

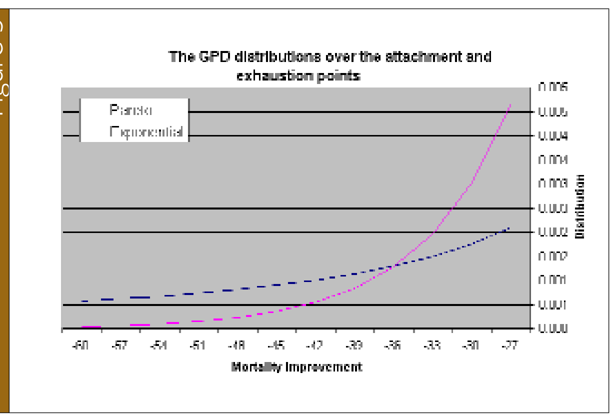
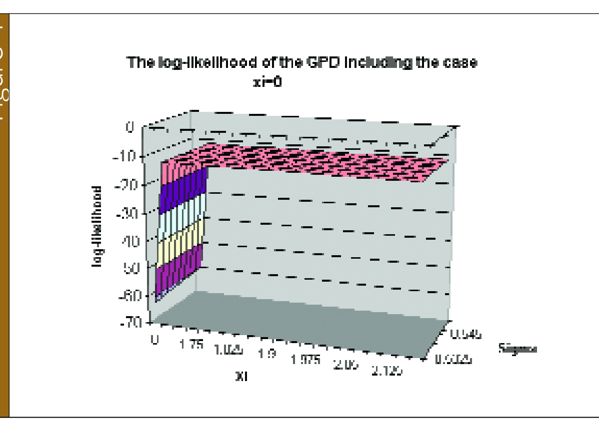


Figure 4



Simulation and Valuation

Having estimated the GPD, we can now follow the same simulation process as in the Milliman model; but whenever we draw an observation from bucket 1, we draw from the GPD distribution. This is relatively easily to do because we can invert the cumulative distribution function. We used 300,000 replications instead of the 50,000 in the Milliman analysis because there is only a 10% chance of drawing from bucket 1. We report the results in Table 3 (below).

The probabilities and expected loss from the stressed model are larger than those in the Milliman analysis. The exponential distribution provides the greatest probability of attachment, but smaller probabilities of exceedence and expected loss because of the thinner tail. Whereas the exponential distribution suggests that an attachment event occurs once every 271 years, the Pareto distribution suggests that such an event only occurs once every 307 years. For the exponential distribution, the exhaustion events only occur once every 2,439 years and once every 685 years for the Pareto distribution. For both distributions, the expected loss is only 14 and 22 basis points (bps), respectively, and the question now is whether the 135 bps per year is sufficient compensation for the risk.

Another interesting point is that the simulated probabilities are greater than the probabilities in Table 2, because two successive mortality increases could lead to a cumulative increase that exceeds the attachment point.

Table 2

Parameter Estimates

Description	$\xi \neq 0$	$\xi = 0$
ξ	1.85	0
std. error	0.967	
σ	0.545	7.000
std. error	0.437	2.333
log-likelihood	-11.569	-14.701
	Test Stat	p-value
LR Test	8.262	0.004
$\Pr(i(L) < 30\%)$	0.008750	0.001959
$\Pr(i(L) < 50\%)$	0.006568	0.000112

Table 3

Annual Average Probabilities and Expected Loss

Model	Attachment Probability (%)	Exhaustion Probability (%)	Expected Loss (%)
Offering Circular	0.077	0.004	0.016
Exponential	0.369	0.041	0.143
EVT	0.326	0.146	0.215

Closing Thoughts

For the Swiss Re mortality indexed bond, EVT provided a very transparent way of stressing the tail of the distribution despite the small sample size and a method for extrapolating beyond the most extreme observation within the sample. After significant stressing, the spread over Libor is still a multiple of at least six times greater than the expected loss.

Rehashing our question above: Is the 135 bps spread sufficient compensation for the risk? According to the market, it is! Swiss Re initially intended to issue $250 million of these bonds. Although the offering was increased to $400 million due to investor interest, the issue was still over-subscribed. During June 2004, the deal was trading at Libor +100 bps – significantly lower than the spread paid to note holders.

EVT does not have to be used only as a statistical distribution; it can also be converted to a risk-neutral distribution. We have used EVT to estimate the lower tail of stock index returns and then used out-of-the-money puts to obtain a risk-adjusted distribution for pricing purposes. Madan and Unal (2003) also used a similar approach to price the Federal Deposit Insurance Corporation's insurance risk. The only constraint on this method is the availability of prices for calibrating the risk premium. ■

FOOTNOTES:

1) For more information about these two approaches, please refer to Embrechts, Kluppelberg and Mikosch (see below).

BIBLIOGRAPHY:

Araya, Rodrigo, and **Ricardo Viloria** (2003). "Vita Capital Ltd.: A Program to Issue Notes Linked to Extreme Mortality," Moody's Structured Finance, Pre-Sale Report, November 14, 2003.

Cuthbertson, K., and **D. Nitzsche** (2001). "Financial Engineering: Derivatives and Risk Management," John Wiley and Sons, Ltd.

Embrechts, Paul, Claudia Kluppelberg and **Thomas Mikosch** (1997). "Modelling External Events for Insurance and Finance [Applications of Mathematics, Vol 33]," Springer Verlag.

Madan, Dilip, and **Haluk Unal** (2003). "Risk-Neutralizing a Loss Distribution: Pricing the FDIC's Reinsurance Risk." Working paper at Robert H. Smith School of Business, University of Maryland.

✎ **OWEN BEELDERS** is a vice president in ABS Research at Citigroup Global Markets; he can be reached at owen.beelders@citigroup.com.

✎ **DAVID COLAROSSI** is a director in the structured products group of Quanta US Holdings; he can be reached at dmcolarossi@quantaholdings.com