Can a coherent risk measure be too subadditive?

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Abstract

We consider the problem of determining appropriate solvency capital requirements in an insurance or financial environment. We demonstrate that the property of subadditivity that is often imposed on risk measures can lead to the undesirable situation where the expected shortfall increases by a merger. We propose to replace the subadditivity property by a 'regulator's condition'. We find that for an explicitly specified confidence level, the Value-at-Risk coincides with the regulator's desires and is the 'most efficient' capital requirement in the sense that it minimizes some reasonable cost function. Within the framework of the so-called coherent risk measures, we find that, again for an explicitly specified confidence level, the Tail-Value-at-Risk is the optimal capital requirement, satisfying the regulator's condition.

Keywords: Risk measures; Solvency capital requirements; (Tail-) Value-at-Risk; Diversification; Subadditivity.

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1 The required solvency capital

Consider a set Γ of real-valued random variables defined on a given probability space $(\Omega, \mathcal{F}, P)$. We will assume that $X_1, X_2 \in \Gamma$ implies that $X_1+X_2 \in \Gamma$, and also $aX_1 \in \Gamma$ for any $a > 0$ and $X_1 + b \in \Gamma$ for any b. Any function $\rho : \Gamma \to R$ that assigns a real number to any element of Γ is called a *risk measure* (with domain Γ).

In the sequel, we will interpret Ω as the set of states of nature at the end of some fixed reference period, for instance one year. The set Γ can be interpreted as the extended set of all losses, over the reference period, related to insurance policies that a regulatory authority wants to control.

Let X be a particular element of Γ. In case all claims are settled at the end of the insurance year and all premiums are paid at the beginning of this year, then X can be defined as claims minus premiums. In a more general setting where not all claims are settled within one year, we can define X as the sum of the claims to be paid over the year and the reserves to be set up at the end of that year, minus the sum of the reserves available at the beginning of the year and the premiums paid over the year.

In an insurance business, the production cycle is inverted, meaning that premiums are paid by the policyholder before claims are paid by the insurer. A portfolio may get into problems in case its loss X is positive, or equivalently, its gain $-X$ is negative, because the liabilities to the insureds cannot be fulfilled completely in this case. Solvency reflects the financial capacity of a particular risky business to meet its contractual obligations. In order to protect the policyholders, the regulatory authority imposes a *solvency capital requirement* $\rho[X]$. This means that the regulator requires that the available capital in the company, this is the surplus of assets over liabilities (reserves), has at least to be equal to $\rho[X]$. This capital is used as a buffer against the risk that the premiums and reserves combined with the investment income will turn out to be insufficient to cover future policyholder claims. In principle, $\rho[X]$ will be chosen such that one can be 'fairly sure' that the event $'X > \rho[X]'$ will not occur.

Although we will stick to the definition of loss as introduced in the insurance example above, many of the results in this paper also hold for other interpretations of the elements of Γ, such as the losses related to the decrease in value of assets minus liabilities over a fixed reference period.

In case of a retail bank for instance, we could define X as the random variable that reflects how the difference between the market values of the assets and the market value of the liabilities might change. The market value of the assets (typically loans) decreases due to changes in interest rates, spreads and the occurrence of default during the reference period, whereas the market value of the liabilities (in this case mostly savings accounts) depends on the level of interest rates and also embeds operational risk that the company faces.

Two well-known model-dependent risk measures used for setting solvency capital requirements are *Value-at-Risk* and *Tail-Value-at-Risk*. For a given probability level p they are denoted by Q_p and TVaR_p, respectively. They are defined by

$$Q_p[X] = \inf\{x \mid P[X \leq x] \geq p\}, \qquad 0 < p < 1, \tag{1}$$

where $\inf\{\phi\} = \infty$ by convention, and

$$\mathrm{TVaR}_p[X] = \frac{1}{1-p}\int_p^1 Q_q[X]\ dq, \qquad 0 < p < 1. \tag{2}$$

The *shortfall* for the portfolio with loss X and solvency capital requirement $\rho[X]$ is defined by

$$\max(0, X - \rho[X]) = (X - \rho[X])_+ . \tag{3}$$

The shortfall can be interpreted as that part of the liabilities that can not be paid by the insurer. It could also be referred to as the *residual risk,* the *insolvency risk* or also the *policyholder deficit.*

Notice that $\mathrm{TVaR}_p[X]$ can be written as a linear combination of the corresponding quantile and its expected shortfall:

$$\mathrm{TVaR}_p[X] = Q_p[X] + \frac{1}{1-p}E\left[(X - Q_p[X])_+\right], \tag{4}$$

where the expectation is taken with respect to the base probability measure P.

Properties of risk measures have been investigated extensively, see e.g., Goovaerts, De Vylder & Haezendonck (1984), or more recently, Artzner, Delbaen, Eber & Heath (1999) and Szegö (2004). Some well-known properties that risk measures may or may not satisfy are monotonicity, positive homogeneity, translation invariance, subadditivity and additivity for comonotonic risks. They are defined as follows:

- *Monotonicity*: for any $X_1, X_2 \in \Gamma$, one has that $X_1 \leq X_2$ implies $\rho[X_1] \leq \rho[X_2]$.
- *Positive homogeneity*: for any $X \in \Gamma$ and $a > 0$, one has that $\rho[aX] = a\rho[X]$.
- *Translation invariance*: for any $X \in \Gamma$ and $b \in R$, one has that $\rho[X + b] = \rho[X] + b$.
- *Subadditivity*: for any $X_1, X_2 \in \Gamma$, one has that $\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$.
- *Additivity for comonotonic risks*: for any $X_1, X_2 \in \Gamma$ which are comonotonic, one has that $\rho[X_1 + X_2] = \rho[X_1] + \rho[X_2]$.

Notice that the random couple (X_1, X_2) is said to be comonotonic if

$$(X_1, X_2) \stackrel{d}{=} (Q_U[X], Q_U[Y]), \tag{5}$$

where $\stackrel{d}{=}$ stands for 'equality in distribution' and U is a random variable that is uniformly distributed on the unit interval $(0, 1)$. Theoretical and practical aspects of the concept of comonotonicity in insurance and finance, are considered in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b). In the sequel, when we consider losses X_j, we assume that they are elements of Γ. Also, when we mention that a risk measure satisfies a certain property, it has to be interpreted as that it satisfies this property on Γ.

The desirability of the subadditivity property of risk measures has been a major topic for research and discussion. As is well-known, the Value-at-Risk does not in general satisfy the subadditivity property, whereas for any p the Tail-Value-at-Risk does.

In Artzner, Delbaen, Eber & Heath (1999), a risk measure that satisfies the properties of monotonicity, positive homogeneity, translation invariance and (most noticeably) subadditivity is called a *coherent* risk measure. The terminology 'coherent' can be somewhat misleading in the sense that it may suggest that any risk measure that is not 'coherent' is always inadequate.

In general, the properties that a risk measure should satisfy depend on the risk preferences in the economic environment under consideration. The coherent set of axioms should be regarded as a typical (and appealing) set. The 'best set of axioms' is however non-existing, as any axiomatic setting is based on a 'belief' in the axioms. Different sets of axioms for risk measurement may represent different schools of thought.

Consider a portfolio with loss X. The regulator wants the solvency capital requirement related to the loss X to be large enough, to ensure that the shortfall is sufficiently small. In order to reach this goal, the regulator introduces a risk measure for the shortfall, which we will denote by φ:

$$\varphi\left[(X-\rho\,[X])_+\right]. \tag{6}$$

From (6), we see that the process of setting capital requirements, requires two different risk measures: the risk measure ρ that determines the solvency capital and the risk measure φ that measures the shortfall.

It seems reasonable to assume that φ satisfies the following condition:

$$\rho_1\,[X] \leq \rho_2\,[X] \Rightarrow \varphi\left[(X-\rho_1\,[X])_+\right] \geq \varphi\left[(X-\rho_2\,[X])_+\right]. \tag{7}$$

A sufficient condition for (7) to hold is that φ is monotonic and hence preserves stochastic dominance.

Clearly, the regulator wants $\varphi\left[(X-\rho\,[X])_+\right]$ to be sufficiently small. The assumption (7) implies that the larger the capital, the better from the viewpoint of minimizing $\varphi\left[(X-\rho\,[X])_+\right]$. On the other hand, holding capital has a cost. The regulator can avoid requiring an excessive solvency capital by taking this cost of capital into account. The capital requirement ρ could be determined as the solution to the following minimization problem:

$$\min_{\rho[X]}\left\{\varphi\left[(X-\rho\,[X])_+\right]+\rho\,[X]\,\varepsilon\right\}, \qquad 0<\varepsilon<1, \tag{8}$$

which balances the two conflicting criteria of low residual risk and low cost of capital. Here, ε can be interpreted as a measure for the extent to which the cost of capital is taken into account. The regulatory authority can decide to let ε be company-specific or risk-specific. In case $\varepsilon=0$, the cost of capital is not taken into account at all and a solvency capital $\rho\,[X] = \max\,[X]$ results. Increasing the value of ε means that the regulator increases the relative importance of the cost of capital and hence, will decrease the optimal solution of the problem.

Take as an example $\varphi[X]=E[X]$. The expected shortfall can be interpreted as the theoretical stop-loss premium that has to be paid to insure the insolvency risk. The following result is proven in Dhaene, Goovaerts & Kaas (2003):

Theorem 1.1 *The smallest capital* $\rho[X]$ *that is a solution of*

$$\min_{\rho[X]} \left\{ E\left[(X - \rho[X])_+ \right] + \rho[X]\, \varepsilon \right\}, \qquad 0 < \varepsilon < 1 \tag{9}$$

is given by

$$\rho[X] = Q_{1-\varepsilon}[X]. \tag{10}$$

Proof.
We introduce the cost function

$$C[X, d] = E\left[(X - d)_+ \right] + d\varepsilon. \tag{11}$$

Let us first assume that $Q_{1-\varepsilon}[X] > 0$.
When $d > 0$, the function $C[X, d]$ corresponds with the surface between the distribution function of X and the horizontal line $y = 1$, from d on, together with the surface $d\varepsilon$, see Figure 1. A similar interpretation for $C[X, d]$ as a surface holds when $d < 0$. One can easily verify that $C[X, d]$ is decreasing in d if $d \leq Q_{1-\varepsilon}[X, d]$ while $C[X, d]$ is increasing in d if $d \geq Q_{1-\varepsilon}[X, d]$. We can conclude that the cost function $C[X, d]$ is minimized by choosing $d = Q_{1-\varepsilon}[X]$.
Let us now assume that $Q_{1-\varepsilon}[X] < 0$. A similar geometric reasoning leads to the conclusion that also in this case, the cost function is minimized by $Q_{1-\varepsilon}[X]$.
Note that the minimum of (9) is uniquely determined, except when $(1 - \varepsilon)$ corresponds to a flat part of the distribution function. In the latter case, the minimum is obtained for any x for which $F_X(x) = 1 - \varepsilon$. Determining the capital requirement as the smallest amount for which the cost function in (9) is minimized leads to the solution (10). ∎

Theorem 1.1 provides a theoretical justification for the use of Value-at-Risk to set solvency capital requirements. Hence, to some extent the theorem supports the current regulatory regime established by the Basel Capital Accord, which has put forward a Value-at-Risk-based capital requirement approach (see Basel Committee (1988, 1996, 2003)). However, it is important to emphasize that the Value-at-Risk is not used to 'measure risk' here; it (merely) appears as an optimal capital requirement. Therefore, the well-known problems of Value-at-Risk-based risk management, see among many others e.g., Basak & Shapiro (2001), do not apply to our context.

The risk that we measure and want to keep under control is the shortfall $(X - \rho[X])_+$. This shortfall risk is measured by $E\left[(X - \rho[X])_+\right]$. This approach corresponds to the classical actuarial approach of measuring or comparing risks by determining or comparing their respective stop-loss premiums.

It is worthwhile to mention that (8) may also represent the problem of determining an appropriate solvency capital from the viewpoint of a financial institution itself (typically referred to as the problem of *economic* capital). In that case, the risk measure φ would represent the risk preferences of the financial institution and ε would represent its capital cost. In practice, it is well understood that economic capital is different from regulatory capital. If e.g., a high rating is desired, the financial institution is willing to reduce its shortfall risk further then as required by the regulator, even though it may considerably increase the capital cost.

From (4) it follows that the minimal value of the cost function in (9) can be expressed as

$$C\left[X, Q_{1-\varepsilon}\left[X\right]\right] = E\left[\left(X - Q_{1-\varepsilon}\left[X\right]\right)_+\right] + Q_{1-\varepsilon}\left[X\right]\varepsilon = \varepsilon \text{ TVaR}_{1-\varepsilon}\left[X\right]. \quad (12)$$

A more general version of the minimization problem (8) with φ being a *distortion risk measure*, is considered in Dhaene, Goovaerts & Kaas (2003), Laeven & Goovaerts (2003) and Goovaerts, Van den Borre & Laeven (2003).

2 Diversification and the subadditivity axiom

Consider two portfolios with respective losses X_1 and X_2. Assume that the solvency capital requirement imposed by the regulator is given by the risk measure ρ. When each of the portfolios is not liable for the shortfall of the other one, the capital requirement for each portfolio is given by $\rho\left[X_j\right]$. When the two portfolios are together liable for the eventual shortfall of the aggregate loss $X_1 + X_2$, we will say that the portfolios are merged. The solvency capital requirement imposed by the supervisory authorities will in this case be equal to $\rho\left[X_1 + X_2\right]$. Merging the two portfolios will lead to a decrease in shortfall given by

$$\sum_{j=1}^{2}\left(X_j - \rho\left[X_j\right]\right)_+ - \left(X_1 + X_2 - \rho\left[X_1 + X_2\right]\right)_+. \quad (13)$$

As mentioned in Dhaene, Goovaerts & Kaas (2003), the following inequality holds with probability 1:

$$(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq \sum_{j=1}^{2} (X_j - \rho[X_j])_+ . \tag{14}$$

Inequality (14) states that the shortfall of the merged portfolio is always smaller than the sum of the shortfalls of the separate portfolios, when adding the capitals. It expresses, that from the viewpoint of the regulatory authorities a merger with adding the capitals is to be preferred in the sense that the shortfall decreases. The underlying reason is that within the merged portfolio, the shortfall of one of the entities can be compensated by the gain of the other one. This observation can be summarized as: '*a merger decreases the shortfall risk*'.

It is important to remark that inequality (14) does not necessarily express that merging is advantageous for the owners of the business related to the portfolios (the shareholders). Evaluating whether a merger is advantageous or not for them can be done by comparing the returns on capital of the two situations. Consider portfolio j. Let X_j be the loss (claim payments minus premiums) related to that portfolio over the reference period and let K_j be its available capital. If the loss X_j is smaller than the capital K_j, the capital at the end of the reference period will be given by $K_j - X_j$, whereas in case the loss X_j exceeds K_j, the business unit related to this portfolio gets ruined and the end-of-the-year capital equals 0. Hence, for portfolio j the end-of-the-year capital is given by $(K_j - X_j)_+$. It is straightforward to prove that

$$(K_1 + K_2 - X_1 - X_2)_+ \leq \sum_{j=1}^{2} (K_j - X_j)_+ \tag{15}$$

holds. Hence, in terms of maximizing the end-of-the-period capital, it is advantageous to keep the two portfolios separate. This situation may be preferred from the shareholders point of view, essentially because in this case fire walls are built in, ensuring that the ruin of one portfolio will not contaminate the other one. Notice that the optimal strategy from the owners point of view is now just the opposite of the optimal strategy from the regulators point of view. Inequality (15) justifies the well-know advice '*don't put all your eggs in one basket*'. If the shareholders have a capital $K_1 + K_2$ at their disposal, if the riskiness of the business is given by (X_1, X_2), and if their goal is to maximize the return on capital, then splitting the risks over two separate entities is always to be preferred.

To conclude, when the regulator talks about diversification, he means the decrease in shortfall caused by merging. When the shareholders talk about diversification, they are talking about the increase in return caused by building in firewalls.

Let us now come back to equation (14). We found that, from the viewpoint of minimizing the shortfall (this is the viewpoint of the regulator) it is better to merge and adding up the stand-alone capitals. Moreover, only taking into account the criterion of minimizing the shortfall, inequality (14) indicates that the capital of the merged portfolios can, to a certain extent, be smaller than the sum of the capitals of the two separate portfolios, as long as the merged shortfall does not become larger than the sum of the separate shortfalls. This observation has led to the belief (by researchers and practitioners) that a risk measure for setting capital requirements should be subadditive. In axiomatic approaches to capital allocation, the property of subadditivity is often considered as one of the axioms. Important to notice is that the requirement of subadditivity implies that

$$(X_1 + X_2 - \rho[X_1 + X_2])_+ \geq (X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ . \qquad (16)$$

Hence, from (14) and (16), we see that when adapting a subadditive risk measure in a merger, one could end up with a larger shortfall than the sum of the shortfalls of the stand-alones.

In the remainder of this paper we will investigate the problem that risk measures can be too subadditive, which can lead to an increase in shortfall.

3 Avoiding that a merger increases the shortfall

As we observed in the previous section, any theory that postulates that risk measures are subadditive should at least constraint this subadditivity, ensuring that merging, which leads to a lower aggregate capital requirement, does not increase the shortfall risk. In this section, we will investigate a number of requirements that could be imposed by the regulator in addition to the subadditivity requirement, in order to ensure that the merger will indeed lead to a less risky situation.

A first additional condition required by the regulator could be as follows: For any couple (X_1, X_2), the capital requirement ρ has to fulfill the condition

$$(X_1 + X_2 - \rho[X_1 + X_2])_+ \leq \sum_{j=1}^{2} (X_j - \rho[X_j])_+ \,. \tag{17}$$

This condition means that the regulator requires that the shortfall of any two merged portfolios with losses X_1 and X_2 respectively, is never allowed to be larger than the sum of the shortfalls of the stand-alones.

Theorem 3.1 *Consider a couple* (X_1, X_2) *for which*

$$\Pr\left[X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right] > 0 \tag{18}$$

holds. If the capital requirement ρ *fulfills the condition* (17) *for this couple* (X_1, X_2)*, then one has that*

$$\rho\left[X_1 + X_2\right] \geq \rho\left[X_1\right] + \rho\left[X_2\right]. \tag{19}$$

Proof. Consider the couple (X_1, X_2) that fulfills the condition (18). Let us assume that $\rho\left[X_1 + X_2\right] \leq \rho\left[X_1\right] + \rho\left[X_2\right]$. From the condition (17), we find that

$$\begin{aligned} & E\left[(X_1 + X_2 - \rho[X_1 + X_2])_+ \mid X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right] \\ \leq & \sum_{j=1}^{2} E\left[(X_j - \rho[X_j])_+ \mid X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right]. \end{aligned} \tag{20}$$

From this inequality, one immediately finds that $\rho\left[X_1 + X_2\right] = \rho\left[X_1\right] + \rho\left[X_2\right]$ must hold. ■

An immediate consequence of the theorem is that any capital requirement ρ that is subadditive and that fulfills the condition (17) must necessarily be additive for all couples (X_1, X_2) for which (18) holds true. Hence, any such capital requirement is 'almost' additive. Only couples (X_1, X_2) that are 'enough' negative dependent, in the sense that $\Pr\left[X_1 > \rho\left[X_1\right], X_2 > \rho\left[X_2\right]\right] = 0$, may lead to a merged capital requirement that is strictly smaller than the sum of the stand-alone requirements.

The theorem above illustrates the fact that the subadditivity axiom and condition (17) are in fact not compatible. If the regulator requires that a merge of portfolios will never increase the shortfall, then he cannot propose a subadditive risk measure.

Note that from the proof of Theorem 3.1, we see that the condition (17) in that theorem can be weakened to the condition (20).

Let us now weaken the condition (17) in another way. We assume that the subadditive capital requirement ρ is such that the expected shortfall of the merger does not exceed the sum of the expected shortfalls of the separate portfolios. Hence, we will assume that the capital requirement ρ has to satisfy the following additional condition for all couples (X_1, X_2):

$$E\left[(X_1 + X_2 - \rho[X_1 + X_2])_+\right] \leq \sum_{j=1}^{2} E\left[(X_j - \rho\left[X_j\right])_+\right]. \tag{21}$$

The subadditivity condition together with the condition (21) ensures that the capital will be decreased in case of a merger, but only to such an extent that the situation becomes on average not more risky after the merge.

In the following theorem we prove that in case of bivariate normal random variables, the condition (21) is fulfilled for a broad class of risk measures ρ.

Theorem 3.2 *For any translation invariant and positive homogenous risk measure ρ and any bivariate normal distributed random couple (X_1, X_2), we have that the condition (21) is fulfilled, which means that merging always decreases the expected shortfall in this case.*

Proof. Assume that (X_1, X_2) is bivariate normal with $var\left[X_j\right] = \sigma_j^2$ and $var\left[X_1 + X_2\right] = \sigma^2$.
Let Z be a standard normal distributed random variable. Then we immediately find

$$E\left[(X_j - \rho\left[X_j\right])_+\right] = \sigma_j E\left[(Z - \rho\left[Z\right])_+\right]$$

and

$$E\left[(X_1 + X_2 - \rho\left[X_1 + X_2\right])_+\right] = \sigma E\left[(Z - \rho\left[Z\right])_+\right].$$

From

$$\sigma \leq \sigma_1 + \sigma_2$$

we find the stated result. ■

The theorem states that in a 'normal world' a translation invariant and positive homogenous risk measure can never be too subadditive. This result is independent of the fact whether ρ is subadditive or not. Hence, it also holds for the Value-at-Risk for instance. The result can easily be generalized to the class of bivariate elliptical distributions, this is the class of couples (X_1, X_2) of which the characteristic function can be expressed as

$$E\left[\exp\left(i\left(t_1 X_1 + t_2 X_2\right)\right)\right] = \exp\left(i\mathbf{t}^T\boldsymbol{\mu}\right) \cdot \phi\left(\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right), \qquad \mathbf{t} = (t_1\ t_2)^T, \quad (22)$$

for some function ϕ, a 2-dimensional vector $\boldsymbol{\mu}$ and a 2×2 matrix $\boldsymbol{\Sigma}$. The function ϕ is called the *characteristic generator* of (X_1, X_2). Notice that the characteristic generator of the bivariate normal distribution is given by $\phi(u) = \exp(-u/2)$. A standard reference for the theory of elliptical distributions is Fang, Kotz & Ng (1987). For applications of elliptical distributions in insurance and finance, see Landsman & Valdez (2002) and Valdez & Dhaene (2004).

Theorem 3.2 could give the impression that under very general conditions, the requirement (21) holds true. However, this is not the case, even not for Tail-Value-at-Risk, which is undoubtedly the best-known *subadditive* risk measure for setting capital requirements. In the following examples we illustrate that Tail-Value-at-Risk does not in general fulfill condition (21).

Example 3.1 (TVaR can be too subadditive: discrete distributions) *Consider two mutually independent losses X_1 and X_2 that are independent and identically distributed Bernoulli random variables with*

$$\Pr\left[X_j = 1\right] = \theta,$$

so that $\Pr\left[X_j = 0\right] = 1 - \theta$. It is straightforward to show that the quantiles are given by

$$Q_p\left[X_j\right] = \begin{cases} 0 & : 0 < p \leq 1 - \theta, \\ 1 & : 1 - \theta < p < 1. \end{cases}$$

By considering the two cases, it is not difficult to verify that

$$TVaR_p\left[X_j\right] = \begin{cases} \frac{\theta}{1-p} & : 0 < p \leq 1 - \theta, \\ 1 & : 1 - \theta < p < 1. \end{cases}$$

Now, consider the sum $S = X_1 + X_2$. Its distribution function is given by

$$F_S(s) = \begin{cases} 0 & : s < 0, \\ (1-\theta)^2 & : 0 \leq s < 1, \\ 1 - \theta^2 & : 1 \leq s < 2, \\ 1 & : s \geq 2. \end{cases}$$

and its quantiles are given by

$$Q_p[S] = \begin{cases} 0 & : 0 < p \leq (1-\theta)^2, \\ 1 & : (1-\theta)^2 < p \leq 1-\theta^2, \\ 2 & : 1-\theta^2 < p < 1. \end{cases}$$

By considering three cases, it is straightforward to show that

$$TVaR_p[S] = \begin{cases} \frac{2\theta}{1-p} & : 0 < p \leq (1-\theta)^2, \\ 1 + \frac{\theta^2}{1-p} & : (1-\theta)^2 < p \leq 1-\theta^2, \\ 2 & : 1-\theta^2 < p < 1. \end{cases}$$

Furthermore, note that

$$E\left[(S - TVaR_p(S))_+\right] = \theta^2\left(1 - \frac{\theta^2}{1-p}\right) \ : \ \textit{if} \ (1-\theta)^2 < p \leq 1-\theta^2.$$

Using the previous results, we find that

$$E\left[(X_j - TVaR_p(X_j))_+\right] = 0 \ : \ \textit{if}\ p \geq 1-\theta.$$

Therefore,

$$E\left[(S - TVaR_p(S))_+\right] > \sum_{j=1}^{2} E\left[(X_j - TVaR_p(X_j))_+\right],$$

whenever $(1-\theta)^2 < p < 1-\theta^2$ *and* $p > 1-\theta$*, which is equivalent to the case where*

$$1-\theta < p < 1-\theta^2.$$

Take as an example $\theta = 0.1$*. Then we find that for any* p *in* $]0.9; 0.99[$ *we have that* $E\left[(S - TVaR_p(X))_+\right] > 0$ *while* $\sum_{j=1}^{2} E\left[(X_j - TVaR_p(X_j))_+\right] = 0$.

Example 3.2 (TVaR can be too subadditive: continuous distributions) *Suppose that* X_1 *is uniformly distributed on the unit interval* $(0,1)$*. Let* X_2 *be the random variable defined by*

$$X_2 = \begin{cases} 0.9U & \textit{if}\ 0 < X_1 \leq 0.9, \\ X_1 & \textit{if}\ 0.9 < X_1 < 1, \end{cases}$$

where U *is uniformly distributed on* $(0,1)$ *and independent of* X_1.
It is straightforward to prove that X_2 *is also uniformly distributed on the unit*

interval.
For the uniformly distributed random variables X_j we have that

$$TVaR_p[X_j] = \frac{1+p}{2},$$

and

$$E[X_j - TVaR_p[X_j]]_+ = \frac{(1-p)^2}{8}.$$

For $p = 0.85$, we find that the Tail-Value-at-Risk and the expected shortfall are given by

$$TVaR_{0.85}[X_j] = 0.925$$

and

$$E[X_j - TVaR_{0.85}[X_j]]_+ = 0.0028125,$$

respectively.
Consider now the sum $S = X_1 + X_2$.
For $0 < s < 2$, we find

$$\begin{aligned} F_S(s) &= \Pr\left[S \leq s,\ 0 < X_1 \leq 0.9\right] + \Pr\left[S \leq s,\ X_1 > 0.9\right] \\ &= \int_0^{0.9} \Pr\left[U \leq \frac{s - x_1}{0.9}\right] dx_1 + \Pr\left[0.9 < X_1 \leq 0.5s\right] \end{aligned}$$

Hence, the distribution function of S is given by

$$F_S(s) = \begin{cases} \frac{s^2}{1.8} & : 0 < s \leq 0.9, \\ -\frac{s^2}{1.8} + 2s - 0.9 & : 0.9 < s \leq 1.8, \\ \frac{s}{2} & : 1.8 < s < 2. \end{cases}$$

For $0.9 < d \leq 1.8$ we have that

$$\begin{aligned} E\left[(S-d)_+\right] &= \int_d^{1.8} [1 - (2s - \frac{s^2}{1.8} - 0.9)]ds + \int_{1.8}^{2} (1 - \frac{s}{2})ds \\ &= -\frac{d^3}{5.4} + d^2 - 1.9d + 1.27. \end{aligned}$$

For $p = 0.85$, we find that $Q_{0.85}[S] = 1.5$. This implies that

$$\begin{aligned} TVaR_{0.85}[S] &= Q_{0.85}[S] + \frac{1}{0.15} E\left[(S - Q_{0.85}[S])_+\right] \\ &= 1.8. \end{aligned}$$

Note that $TVaR_{0.85}[S]$ is strictly smaller than $TVaR_{0.85}[X_1]+TVaR_{0.85}[X_2]$. The expected shortfall of S is given by

$$E[(S - TVaR_{0.85}[S])_+] = 0.01.$$

One can verify that the expected shortfall of S is strictly larger than the sum of the expected shortfalls of the X_J:

$$E\left[(S - TVaR_{0.85}\left[S\right])_+\right] > \sum_{j=1}^{2} E\left[(X_j - TVaR_{0.85}\left[X_j\right])_+\right].$$

The two examples above illustrate the fact that subadditive risk measures, in particular Tail-Value-at-Risk, can be too subadditive, in the sense that the expected shortfall of the merged portfolios is larger than the sum of the expected shortfalls of the two separate portfolios.

4 The 'regulator's condition'

In the previous section we considered requirements that could be imposed in addition to the subadditivity axiom in order to ensure that a merger does not lead to a more risky situation. We found some particular results, but we did not find a general satisfying solution. In this section, we will investigate a different approach. We will replace the subadditivity axiom by a new one which can be seen as a compromise between the subadditivity axiom and condition (21).

As we have seen, the subadditivity condition forces to decrease capital requirements after a merger, without constraining the extent to which the capital can be decreased. The regulator wants the expected shortfall to be as small as possible, which means a preference for a high solvency capital requirement. On the other hand, he does not want to decrease the expected shortfall at any price, imposing an extremely large burden on the financial industry. Therefore we propose a 'regulator's condition' that is a compromise between the two conflicting criteria of 'low expected shortfall' and 'low cost of capital'.

We propose to replace the subadditivity requirement and the requirement (21) by the following requirement that a risk measure ρ for determining the solvency capital required for a risky business should fulfill:

For any couple (X_1, X_2), the capital requirement ρ has to fulfill the condition

$$\begin{aligned} & E\left[(X_1 + X_2 - \rho[X_1 + X_2])_+\right] + \rho\left[X_1 + X_2\right]\varepsilon \\ \leq & \sum_{j=1}^{2} \left\{E\left[(X_j - \rho\left[X_j\right])_+\right] + \rho\left[X_j\right]\varepsilon\right\}, \qquad 0 < \varepsilon < 1. \end{aligned} \tag{23}$$

Note that ε could be equal to the cost of capital, but it could also be a number smaller than the cost of capital, depending on to what extent the regulator is willing to take into account this cost. We will call (23) the *regulator's condition.*

Theorem 3.2 above can be adjusted to the following formulation:

Theorem 4.1 *For any translation invariant, positive homogenous and subadditive risk measure ρ and any bivariate normal random couple (X_1, X_2), the regulator's condition (23) is fulfilled.*

The result of Theorem 4.1 can easily be extended to the case of elliptical random couples.

Let us now consider the case of general random loss variables.

Theorem 4.2 *The capital requirement $\rho\left[X\right] = Q_{1-\varepsilon}\left[X\right]$ fulfills the regulator's condition (23). Also, any subadditive capital requirement $\rho\left[X\right] \geq Q_{1-\varepsilon}\left[X\right]$ fulfills the regulator's condition.*

Proof. The regulator's condition (23) can be expressed in terms of the cost function $C\left[X, d\right]$ introduced in the proof of Theorem 1.1:

$$C\left[X_1 + X_2, \rho[X_1 + X_2]\right] \leq C\left[X_1, \rho[X_1]\right] + C\left[X_2, \rho[X_2]\right].$$

The proof for $Q_{1-\varepsilon}$ follows immediately from (12) and the subadditivity of Tail-Value-at-Risk.
Let us now consider a subadditive capital requirement $\rho \geq Q_{1-\varepsilon}$. From

$$Q_{1-\varepsilon}\left[X_1 + X_2\right] \leq \rho\left[X_1 + X_2\right] \leq \rho\left[X_1\right] + \rho\left[X_2\right]$$

and the fact that $C\left[X_1 + X_2, d\right]$ is increasing in d if $d \geq Q_{1-\varepsilon}\left[X_1 + X_2\right]$, we find

$$C\left[X_1 + X_2, \rho\left[X_1 + X_2\right]\right] \leq C\left[X_1 + X_2, \rho\left[X_1\right] + \rho\left[X_2\right]\right].$$

Furthermore, from (14) we find

$$C[X_1 + X_2, \rho[X_1] + \rho[X_2]] \leq C[X_1, \rho[X_1]] + C[X_2, \rho[X_2]],$$

which proves the stated result. ■

Assume that the regulator wants to set the capital requirement ρ as the one that fulfills the regulator's condition (23) and that makes the cost function $E\left[(X - \rho[X])_+\right] + \rho[X]\,\varepsilon$ minimal for any X. Combining Theorems 1.1 and 4.2, we find that the solution to this problem is given by the risk measure $Q_{1-\varepsilon}$.

Let us now assume that the regulator wants to use a coherent risk measure that fulfills the regulator's condition (23). From Theorem 4.2, we have that any risk measure TVaR_p with $p \geq 1 - \varepsilon$ belongs to this class.

Concave distortion risk measures are a subclass of the class of coherent risk measures, see Wang (2000). It can be proven that within the class of concave distortion risk measures that are larger than $Q_{1-\varepsilon}$, the smallest element is $\text{TVaR}_{1-\varepsilon}$, see Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004). Hence, from Theorem 4.2 we find that $\text{TVaR}_{1-\varepsilon}$ is the smallest concave distortion risk measure that is larger than $Q_{1-\varepsilon}$ and fulfills the regulator's condition (23). Notice that the level of the optimal Value-at-Risk or Tail-Value-at-Risk under consideration, depends explicitly on ε, i.e., on the extent to which the cost of capital is taken into account.

Condition (23) can be generalized by replacing the expectation operator by a distortion risk measure. It is not difficult to prove that in that case Theorem 4.2 remains valid, the only difference being that $Q_{1-\varepsilon}$ is now to be calculated with respect to a distorted probability distribution function.

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Figure 1: Geometric Proof of Theorem 1.1

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