8TH INTERNATIONAL CONGRESS ON INSURANCE: MATHEMATICS & ECONOMICS

ACTUARIAL THEORY

FOR DEPENDENT RISKS ★

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(★ title of a book co-authored with J. Dhaene, M. Goovaerts, R. Kaas and D. Vyncke)

Risk measures

Stochastic orders

Dependence structures

Credibility models

Bonus-malus scales

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Risk measures

- A risk measure is a functional ρ mapping a risk X to a non-negative real number $\rho[X]$, possibly infinite.
- The meaning of $\rho[X]$ is as follows: $\rho[X]$ represents the minimum extra cash which has to be added to X to make it “acceptable".
- A large value of $\rho[X]$ indicates that X is “dangerous".
- Risk measures have been extensively studied in the actuarial literature since 1970, in the guise of premium principles; see e.g. GOOVAERTS ET AL. (1984).

Coherent risk measures

A risk measure satisfying

Translativity: $\rho[X + c] = \rho[X] + c$ whatever the risk X and the constant c;

Subadditivity: $\rho[X + Y] \leq \rho[X] + \rho[Y]$ whatever the risks X and Y;

Homogeneity: $\rho[cX] = c\rho[X]$ whatever the risk X and the positive constant c;

Monotonicity: $\Pr[X \leq Y] = 1 \Rightarrow \rho[X] \leq \rho[Y]$ whatever the risks X and Y;

is said to be coherent in the sense of ARTZNER ET AL. (1999).

Comonotonic additivity

- (X, Y) is comonotonic $\Leftrightarrow \exists\, Z$ and $\uparrow$ functions t_1 and t_2 such that

$$(X, Y) =_d \Big(t_1(Z), t_2(Z)\Big);$$

  see DHAENE ET AL. (2002a,b) for theory and applications in insurance and finance.

- The risk measure ρ is comonotonic additive if $\rho[X + Y] = \rho[X] + \rho[Y]$ whatever the comonotonic risks X and Y.

- There is no diversification effect for comonotonic risks when the risk measure is comonotonic additive.

Denneberg representation theorem

- Let $\mathcal{B}$ be the set of bounded risks.
- If $\rho : \mathcal{B} \mapsto \mathbb{R}^+$ is comonotonic additive, monotone and satisfies $\rho[1] = 1$ then there exists a non-decreasing distortion function g satisfying $g(0) = 0$ and $g(1) = 1$, such that

$$\rho[X] \equiv \rho_g[X] = \int_0^{+\infty} g\Big(\Pr[X > t]\Big) dt.$$

- ρ_g is known as a Wang risk measure.
- Moreover,

$$\rho_g \text{ subadditive } \Leftrightarrow g \text{ concave.}$$

Value-at-Risk (VaR)

- Given a risk X and a probability level $p \in (0,1)$, the corresponding VaR, denoted as $\text{VaR}[X;p]$, is defined as

$$\text{VaR}[X;p] = F_X^{-1}(p).$$

- Note that any Wang risk measure can be represented as a mixture of VaR's:

$$\rho_g[X] = \int_0^1 \text{VaR}[X;1-p]dg(p).$$

- VaR is associated with the distortion function $g(x) = \mathbb{I}[x > 1-p]$; it is not coherent (it fails to be subadditive).

Tail-VaR

- Given a risk X and a probability level p,

$$\text{TVaR}[X;p] \;=\; \frac{1}{1-p}\int_p^1 \text{VaR}[X;\xi]\,d\xi,\;\; p\in(0,1)\,.$$

- TVaR is associated with the distortion function $g(x)=\min\left(\frac{x}{1-p},1\right)$; it is coherent.
- If F_X is continuous then

$$\text{TVaR}[X;p]=\mathbb{E}\Big[X\Big|X>\text{VaR}[X;p]\Big],\;\; p\in(0,1)\,,$$

and is the “average loss in the worst $1-p\%$ cases".

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Stochastic orders and risk measures

- Most classical stochastic orderings are associated with particular risk measures.
- Given two risks X and Y,

$$X \preceq_{\text{ST}} Y \quad \Leftrightarrow \quad \rho_g[X] \leq \rho_g[Y] \ \ \forall \uparrow \text{distortions } g$$
$$\Leftrightarrow \quad \text{VaR}[X;p] \leq \text{VaR}[Y;p] \text{ for all } 0 \leq p \leq 1.$$

- Given two risks X and Y,

$$X \preceq_{\text{ICX}} Y \quad \Leftrightarrow \quad \rho_g[X] \leq \rho_g[Y] \ \ \forall \uparrow \text{concave distortions } g$$
$$\Leftrightarrow \quad \text{TVaR}[X;p] \leq \text{TVaR}[Y;p] \text{ for all } 0 \leq p \leq 1.$$

- See e.g. DENUIT ET AL. (2004).

Convex order

- Given two random variables X and Y,

$$X \preceq_{\text{CX}} Y \Leftrightarrow X \preceq_{\text{ICX}} Y \text{ and } \mathbb{E}[X] = \mathbb{E}[Y].$$

- It can be shown that

$$X \preceq_{\text{CX}} Y \Rightarrow \mathbb{V}\text{ar}[X] \leq \mathbb{V}\text{ar}[Y]$$

  so that $\preceq_{\text{CX}}$ expresses the intuitive idea of "X being less variable than Y".

- Separation Theorem: $X \preceq_{\text{ICX}} Y$ iff $\exists\ Z$ such that

$$X \preceq_{\text{ST}} Z \preceq_{\text{CX}} Y.$$

Likelihood ratio order

- Given two random variables X and Y, X is said to be smaller than Y in the likelihood ratio order, denoted as $X \preceq_{\text{LR}} Y$, when

$$\Pr[X \in A] \Pr[Y \in B] \geq \Pr[X \in B] \Pr[Y \in A] \text{ for all } A \leq B.$$

- Let X and Y be two rv's. Then, $X \preceq_{\text{LR}} Y$ if, and only if,

$$[X|a \leq X \leq b] \preceq_{\text{ST}} [Y|a \leq Y \leq b] \text{ for all } a < b \in \mathbb{R}$$

or

$$p \mapsto F_Y(\text{VaR}[X; p]) \text{ is convex.}$$

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Copulas

- A copula is (the restriction to the unit square $[0,1]^2$ of) a joint cdf for a bivariate random vector with unit uniform marginals.
- Let us consider $\boldsymbol{X} = (X_1, X_2)$ with marginals $X_1 \sim F_1$ and $X_2 \sim F_2$.
- Then, there exists a copula $C : [0,1]^2 \to [0,1]$ such that

$$F_{\boldsymbol{X}}(x_1, x_2) = C\big(F_1(x_1), F_2(x_2)\big), \quad \boldsymbol{x} \in \mathbb{R}^2.$$

- $C(\cdot,\cdot)$ is called a copula since it “couples" the marginals $F_1(\cdot)$ and $F_2(\cdot)$ to form the bivariate cdf $F_{\boldsymbol{X}}(\cdot,\cdot)$.

Conditional increasingness

- The random couple $\boldsymbol{X}$ is said to be CI if

$$\Pr[X_2 > x_2 | X_1 = x_1] \text{ is non-decreasing in } x_1$$
$$\Pr[X_1 > x_1 | X_2 = x_2] \text{ is non-decreasing in } x_2.$$

- This is equivalent to

$$[X_2 | X_1 = x_1] \preceq_{\mathsf{ST}} [X_2 | X_1 = x_1'] \text{ for any } x_1 \leq x_1'$$
$$[X_1 | X_2 = x_2] \preceq_{\mathsf{ST}} [X_1 | X_2 = x_2'] \text{ for any } x_2 \leq x_2'.$$

- CI is a property of the copula, that is, if C is a copula for $\boldsymbol{X}$, $\boldsymbol{X}$ CI $\Leftrightarrow C$ CI.

Supermodular functions

- A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is said to be supermodular when

$$\phi(b_1, b_2) - \phi(a_1, b_2) - \phi(b_1, a_2) + \phi(a_1, a_2) \geq 0$$

  for all $a_1 \leq b_1, a_2 \leq b_2$.
- Such a function assigns more weight to points (a_1, a_2) and (b_1, b_2) expressing positive dependence.
- If ϕ is twice differentiable, it is supermodular iff $\frac{\partial^2}{\partial x_1 \partial x_2}\phi \geq 0$ (such a function is called regular supermodular).

Total positivity of order 2 (TP$_2$)

- The random couple $\boldsymbol{X}$ is said to be TP$_2$ if its pdf is log-supermodular, that is, if

$$f_{\boldsymbol{X}}(a_1, a_2) f_{\boldsymbol{X}}(b_1, b_2) \geq f_{\boldsymbol{X}}(a_1, b_2) f_{\boldsymbol{X}}(b_1, a_2)$$

  for any $a_1 \leq b_1$ and $a_2 \leq b_2$.

- This is equivalent to

$$[X_2 | X_1 = x_1] \preceq_{\text{LR}} [X_2 | X_1 = x_1'] \text{ for any } x_1 \leq x_1'$$
$$[X_1 | X_2 = x_2] \preceq_{\text{LR}} [X_1 | X_2 = x_2'] \text{ for any } x_2 \leq x_2'.$$

- $\boldsymbol{X}$ is said to be MTP$_2$ if

$$f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{y}) \leq f_{\boldsymbol{X}}(\boldsymbol{x} \vee \boldsymbol{y}) f_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{y}) \;\; \forall \;\; \boldsymbol{x} \in \mathbb{R}^n.$$

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Notation

- Let N_t be the number of claims reported by a given policyholder during period t, $t = 1, 2, \ldots, T$.
- Being generated by the same individual, the N_t's may be correlated; this serial correlation justifies a posteriori corrections.
- Let
$$N_\bullet = \sum_{t=1}^{T} N_t$$
be the total number of claims reported during the T observation periods.

The model

- Let us denote as $\mathbb{E}[N_t] = \lambda_t$ the expected annual claim number; λ_t contains all the information included in the price list about the policyholder in period t (like age, sex, power of the car, and so on).
- Let Θ be a positive random variable with unit mean; it represents the unexplained heterogeneity.
- Given $\Theta = \theta$, the random variables N_t, $t = 1, 2, \ldots$, are independent and $\sim \mathcal{P}oi(\lambda_t \theta)$, i.e.

$$\Pr[N_t = k | \Theta = \theta] = \exp(-\theta\lambda_t)\frac{(\theta\lambda_t)^k}{k!}, \ \ k \in \mathbb{N}.$$

Intuitive statements

- In this model, we intuitively feel that the following statements are true:

  S1 Θ “increases" in the past claims $N_\bullet$

  S2 N_{T+1} “increases" in the past claims $N_\bullet$

  S3 N_{T+1} and $N_\bullet$ are “positively dependent".
- The meaning of “increases" in S1 and S2, as well as of “positive dependence" involved in S3 has to be precised.
- These statements are true in the classical Poisson-Gamma model if the increasingness is wrt $\preceq_{\text{LR}}$ and the positive dependence is TP_2.

Poisson mixture model

- The results valid in the Poisson-Gamma model remain true in any Poisson mixture model, that is

$$
\begin{aligned}
[\Theta|N_\bullet = n] &\preceq_{\text{LR}} [\Theta|N_\bullet = n'] \text{ for } n \leq n' \\
[N_{T+1}|N_\bullet = n] &\preceq_{\text{LR}} [N_{T+1}|N_\bullet = n'] \text{ for } n \leq n'
\end{aligned}
$$

  but

$$
\mathbb{E}[N_{T+1}|N_\bullet = n] = \lambda_{T+1}\psi(n)
$$

  where ψ is increasing but not necessarily linear.

- $(N_{T+1}, N_\bullet)$ as well as each (N_t, N_s) are TP$_2$. Moreover, $(\Theta, N_1, \ldots, N_T)$ is MTP$_2$.

- SHAKED & SPIZZICHINO (1998), PURCARU & DENUIT (2002a,b, 2003).

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Bonus-malus scales

- In practice, bonus-malus scales are enforced in MTPL, and not credibility models.
- The model for claim numbers is the same as for credibility theory.
- Policyholders are now placed in a scale:

Level	Relativities
s	r_s
$\vdots$	$\vdots$
ℓ	r_ℓ
$\vdots$	$\vdots$
0	r_0

Bonus-malus systems

- Such scales possess a number of levels, $s+1$ say, numbered from 0 to s.
- A specified level is assigned to a new driver (often according to the use of the vehicle).
- Each claim free year is rewarded by a bonus point (i.e. the driver goes one level down).
- Claims are penalized by malus points (i.e. the driver goes up a certain number of levels each time he files a claim).

Bayesian relativities

- Let $L(t)$ be the level occupied by a given policyholder in year t; typically,

$$L(t) = \max\Big\{0, \min\{L(t-1) - 1 + N_t \times k_{pen}, s\}\Big\}.$$

- Let $L(\infty)$ be the level occupied by an “infinitely old" policy (stationary regime).
- Denoting as Θ the unknown (relative) expected claim frequency, Norberg Bayesian relativity attached to level ℓ is

$$r_\ell = \mathbb{E}[\Theta | L(\infty) = \ell].$$

Dependence in BM scales

- The random vector $\big(\Theta, L(1), \ldots, L(t)\big)$ is MTP$_2$ for any $t \geq 1$

  $\Rightarrow \big(\Theta, L(t)\big)$ and $\big(\Theta, L(\infty)\big)$ are both TP$_2$.

- The following stochastic inequalities hold true:

$$
\begin{aligned}
[\Theta|L(t)=\ell] \quad &\preceq_{\text{LR}} \quad [\Theta|L(t)=\ell'] \text{ for any } \ell \leq \ell', t \geq 1 \\
[\Theta|L(\infty)=\ell] \quad &\preceq_{\text{LR}} \quad [\Theta|L(\infty)=\ell'] \text{ for any } \ell \leq \ell' \\
&\Rightarrow \quad r_\ell \text{ is increasing with } \ell
\end{aligned}
$$

- Furthermore,

$$
[N_{t+1}|L(t)=\ell] \preceq_{\text{LR}} [N_{t+1}|L(t)=\ell'] \text{ for any } \ell \leq \ell'.
$$

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Stochastic bounds

- Often, actuaries act in a conservative way by basing the decision on the worst case compatible with the partial information at their disposal.
- In the univariate case, given the first few moments of the risk X, its support, mode, etc., two rv's X_- and X_+ are determined such that

$$X_- \preceq X \preceq X_+$$

  (here $\preceq$ can be $\preceq_{\text{ST}}$, $\preceq_{\text{ICX}}$ or $\preceq_{\text{CX}}$ for instance).
- This is closely related to the problem of maximizing/minimizing $\mathbb{E}[\phi(X)]$ for some function ϕ when X belongs to a given moment space.

Example with $\preceq_{ICX}$

$$\Pr[X_+ \le x] = \begin{cases} 0 & \text{if } x < 0, \\ \frac{\sigma^2}{\sigma^2+\mu^2} & \text{if } 0 \le x < \frac{\mu^2+\sigma^2}{2\mu}, \\ \frac{1}{2} + \frac{1}{2}\frac{x-\mu}{\sqrt{(x-\mu)^2+\sigma^2}} & \text{if } x \ge \frac{\mu^2+\sigma^2}{2\mu}. \end{cases}$$

$$\Pr[X_- \le x] = \begin{cases} 0 & \text{if } x < \mu - \frac{\sigma^2}{b-\mu}, \\ 1 - \frac{\mu}{b} & \text{if } \mu - \frac{\sigma^2}{b-\mu} \le x < \frac{\mu^2+\sigma^2}{\mu}, \\ 1 & \text{if } x \ge \frac{\mu^2+\sigma^2}{\mu}. \end{cases}$$

(see JANSEN ET AL. (1986) and DE VYLDER & GOOVAERTS (1982))

Stochastic bounds

- In the bivariate case, one could imagine that the marginal distributions are given but the underlying copula is only partially specified (it is PQD, for instance).
- Now, two random couples $\boldsymbol{X}_-$ and $\boldsymbol{X}_+$ are determined such that

$$\boldsymbol{X}_- \preceq \boldsymbol{X} \preceq \boldsymbol{X}_+$$

  (here $\preceq$ is a suitable bivariate order).
- Good candidates for $\preceq$ in the above stochastic inequality are the supermodular order and the directionally convex order.

Supermodular order

- A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is said to be supermodular when

$$\phi(b_1, b_2) - \phi(a_1, b_2) - \phi(b_1, a_2) + \phi(a_1, a_2) \geq 0$$

  for all $a_1 \leq b_1, a_2 \leq b_2$.

- Given two random couples $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$, $\boldsymbol{X} \preceq_{\text{SM}} \boldsymbol{Y}$ if $\mathbb{E}[\phi(\boldsymbol{X})] \leq \mathbb{E}[\phi(\boldsymbol{Y})]$ for all the (regular) supermodular functions ϕ for which the expectations exist.

- $\preceq_{\text{SM}}$ can only compare random vectors with identical marginals (it is a dependence order).

Extremal elements wrt $\preceq_{SM}$ with given marginals

- Any $\boldsymbol{X}$ satisfies $\boldsymbol{X}^- \preceq_{\text{SM}} \boldsymbol{X} \preceq_{\text{SM}} \boldsymbol{X}^+$, where $\boldsymbol{X}^-$ (resp. $\boldsymbol{X}^+$) has copula

$$C_L(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$$

(resp. $C_U(u_1, u_2) = \min\{u_1, u_2\}$)

and the same marginals as $\boldsymbol{X}$.

- If $\boldsymbol{X}$ is known to be PQD, that is if

$\Pr[X_1 > t_1, X_2 > t_2] \geq \Pr[X_1 > t_1] \Pr[X_2 > t_2]$ for all t_1, t_2,

then $\boldsymbol{X}^-$ can be taken with independent components.

- For any non-decreasing supermodular function Ψ, MÜLLER (1997) established that $\boldsymbol{X}^- \preceq_{\text{SM}} \boldsymbol{X} \preceq_{\text{SM}} \boldsymbol{X}^+$ implies

$$\Psi(X_1^-, X_2^-) \preceq_{\text{ICX}} \Psi(X_1, X_2) \preceq_{\text{ICX}} \Psi(X_1^+, X_2^+).$$

- True e.g. for

$$\Psi(x_1, x_2) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2,$$

with $\alpha_0 \in \mathbb{R}$, $\alpha_1 > 0$, $\alpha_2 > 0$, so that

$$X_1^- + X_2^- \preceq_{\text{CX}} X_1 + X_2 \preceq_{\text{CX}} X_1^- + X_2^-.$$

Directionally convex order

- A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is directionally convex, if it is supermodular, and in addition convex in each component, when the other component is held fixed.
- $\boldsymbol{X} \preceq_{\text{DIR-CX}} \boldsymbol{Y}$ if $\mathbb{E}[\phi(\boldsymbol{X})] \leq \mathbb{E}[\phi(\boldsymbol{Y})]$ for all the directionally convex functions ϕ for which the expectations exist.
- Directional convex order allows to compare random vectors with different marginals (and allows for shift in both the copula and the marginal cdf's).

A sufficient condition for $\preceq_{\text{DIR-CX}}$

- If $\boldsymbol{X}$ expresses less PQD than $\boldsymbol{Y}$, in the sense that

  $$\Pr[X_1 > t_1, X_2 > t_2] - \Pr[X_1 > t_1]\Pr[X_2 > t_2]$$

  $\leq \Pr[Y_1 > t_1, Y_2 > t_2] - \Pr[Y_1 > t_1]\Pr[Y_2 > t_2]$ for all t_1, t_2,

  then

  $$X_1 \preceq_{\text{CX}} Y_1 \text{ and } X_2 \preceq_{\text{CX}} Y_2 \Rightarrow \boldsymbol{X} \preceq_{\text{DIR-CX}} \boldsymbol{Y}.$$

- See RÜSCHENDORF (2004) for further results in that vein.

- Let $\boldsymbol{X}$, $\boldsymbol{X}^-$ and $\boldsymbol{X}^+$ have the same CI copula C, and $X_i^- \preceq_{\text{CX}} X_i \preceq_{\text{CX}} X_i^+$, $i = 1, 2$, Müller & Scarsini (2001) proved that

$$\boldsymbol{X}^- \preceq_{\text{DIR-CX}} \boldsymbol{X} \preceq_{\text{DIR-CX}} \boldsymbol{X}^+.$$

- Denuit, Genest & Mesfioui (2004) suggest to proceed in two steps:
  - first, the copula is replaced with a worse/better CI one (in the $\preceq_{\text{SM}}$-sense)
  - second, the marginals are replaced with worse/better ones (in the $\preceq_{\text{CX}}$-sense)

  giving bounds in the $\preceq_{\text{DIR-CX}}$-sense on $\boldsymbol{X}$.

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Loss-ALAE data set

- Data set provided by Insurance Services Office, Inc.
- ALAE's: expenses that are specifically attributable to the settlement of individual claims such as lawyers' fees and claims investigation expenses.
- The data consist of 1500 observed values of the pair (loss, ALAE), as well as a corresponding Policy Limit.

Losses and ALAE's in reinsurance

- Let us consider a reinsurance treaty on a policy with unlimited liability and insurer's retention R.
- Assuming a prorata sharing of expenses, the reinsurer's payment for a given realization of (LOSS,ALAE) is described by

$$g(\text{LOSS,ALAE}) = \begin{cases} 0 \text{ if LOSS} \leq R, \\ \text{LOSS} - R + \frac{\text{LOSS-}R}{\text{LOSS}}\text{ALAE} \\ \qquad \text{if LOSS} > R. \end{cases}$$

ISO Loss-ALAE data

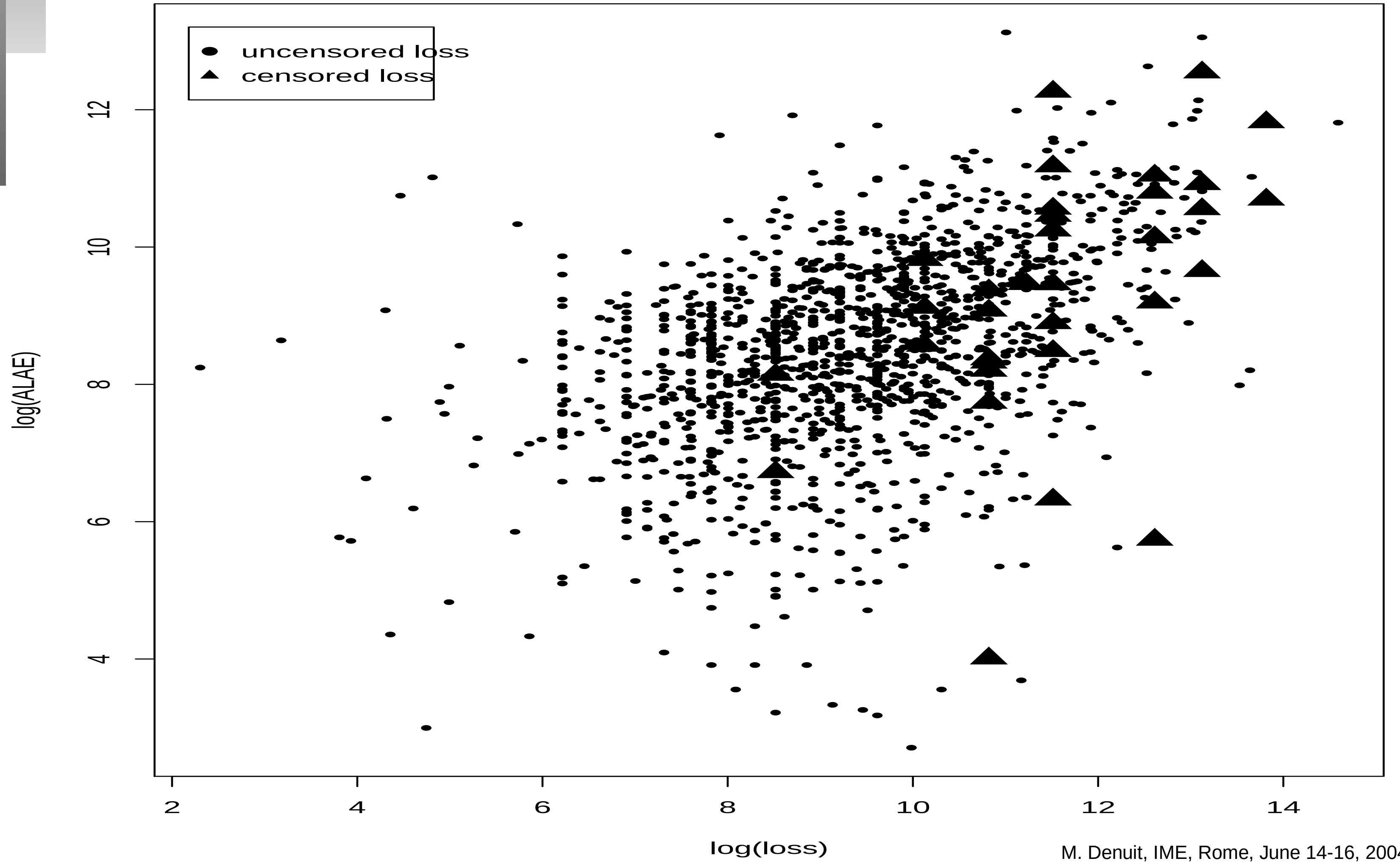
- Particularity of the data: some losses were <u>censored</u> because the claim amount cannot exceed the policy limit.
- Specifically,

$$\begin{cases} (T, ALAE_i), \quad i = 1, \ldots, n \quad \text{where} \quad T = \min(loss_i, \ell_i), \\ \\ \delta_i = \mathbb{I}[T = \ell_i] = \begin{cases} 1, & \text{if} \quad loss_i > \ell_i \Rightarrow \textit{censored claim} \\ 0, & \text{if} \quad loss_i \leq \ell_i \Rightarrow \textit{uncensored claim} \end{cases} \end{cases}$$

Summary statistics of the Loss-ALAE data

	Loss	ALAE	Loss (uncensored)	Loss (censored)
Total N	1,500	1,500	1,466	34
Min	10	15	10	5,000
1st Qu.	4,000	2,333	3,750	50,000
Mean	41,208	12,588	*37,110*	*217,941*
Median	12,000	5,471	11,049	100,000
3rd Qu.	35,000	12,577	32,000	300,000
Max	2,173,595	501,863	2,173,595	1,000,000
Std Dev.	102,748	28,146	92,513	258,205

Scatterplot of the Loss-ALAE data



Testing for PQD

- Empirical investigations carried out
  - by DENUIT & SCAILLET (2004), distance tests
  - by SCAILLET (2004), Kolmogorov-type tests

  strongly support PQD between Losses and their ALAE's.
- PQD means that large (resp. small) values of Loss and ALAE tend to occur simultaneously.
- Both methodologies only deal with complete data, and were thus applied to the 1466 uncensored pairs (loss,ALAE).

Archimedean copulas: definition

- Consider a function $\phi : [0,1] \to \overline{\mathbb{R}}^+$ satisfying $\phi(1) = 0$, $\phi^{(1)}(\tau) < 0$ and $\phi^{(2)}(\tau) > 0$ for all $\tau \in (0,1)$.
- Every such function ϕ generates a copula C_ϕ given by

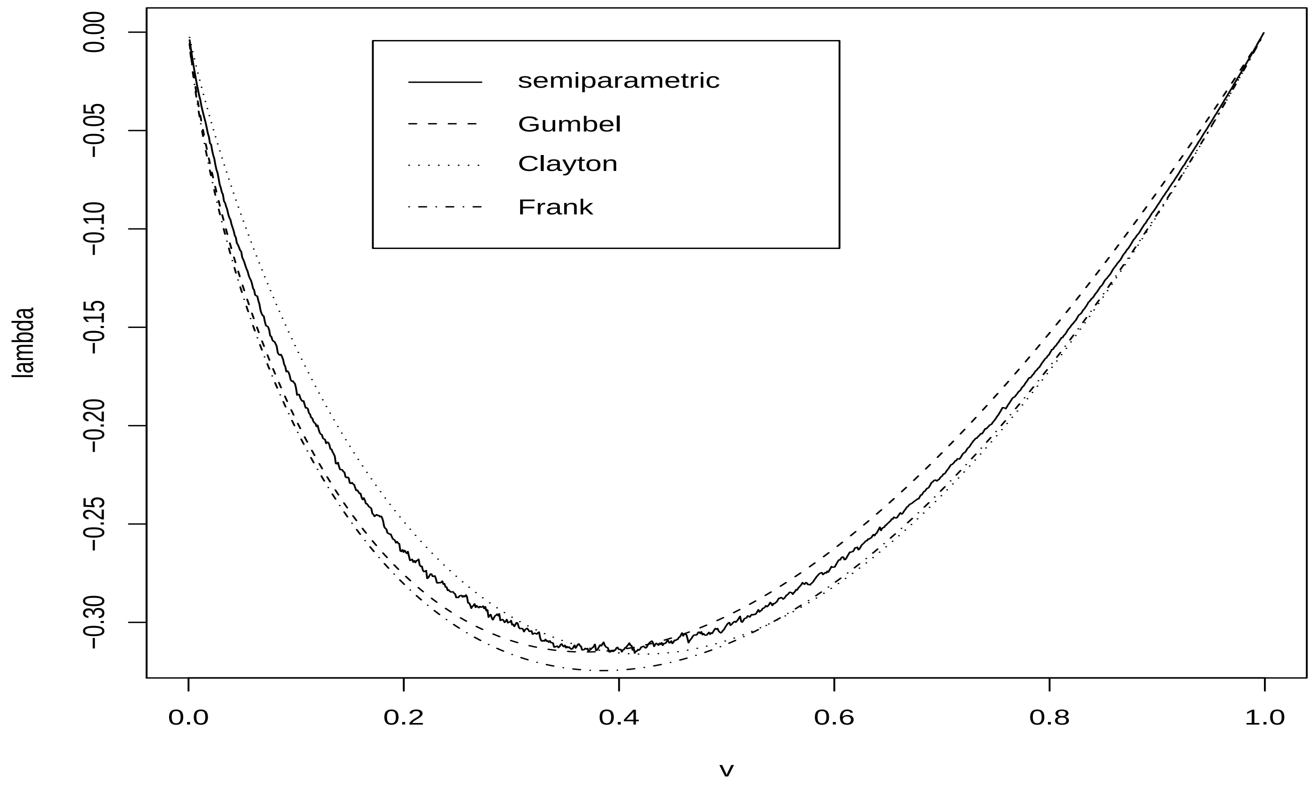
$$C_\phi(u_1, u_2) = \begin{cases} \phi^{-1}\left\{\phi(u_1) + \phi(u_2)\right\} \\ \qquad \text{if } \phi(u_1) + \phi(u_2) \leq \phi(0), \\ 0 \text{ otherwise;} \end{cases}$$

the copula C_ϕ is called an archimedean copula.

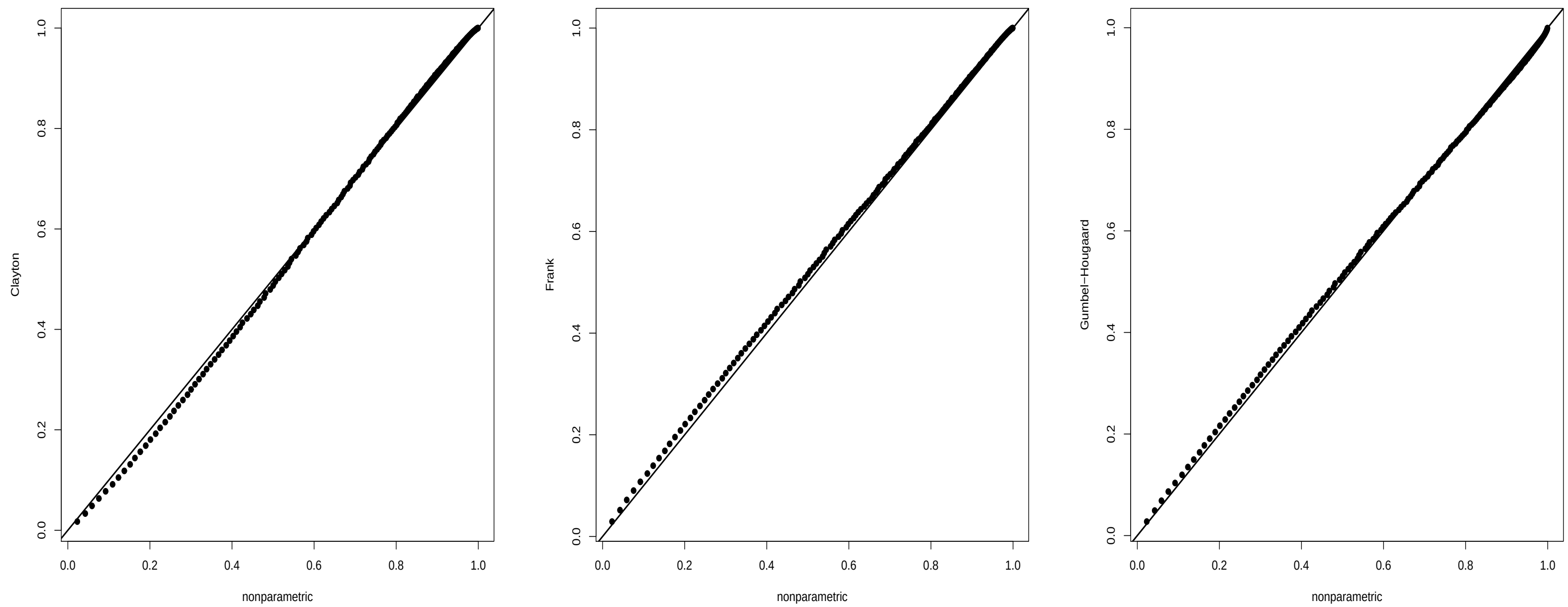
Nonparametric estimation of ϕ

- In the literature,
  1. GENEST & RIVEST (1993) for complete data, applied to the Loss-ALAE data by FREES & VALDEZ (1998)
  2. WANG & WELLS (2000) for doubly censored data
  3. DENUIT, PURCARU & VANKEILEGOM (2004) for Loss-ALAE data (truncation of loss).
- The nonparametric estimation of ϕ serves as a benchmark for selecting an appropriate parametric archimedean model.

Selection of the parametric generator on the basis of $\lambda = \phi/\phi^{(1)}$



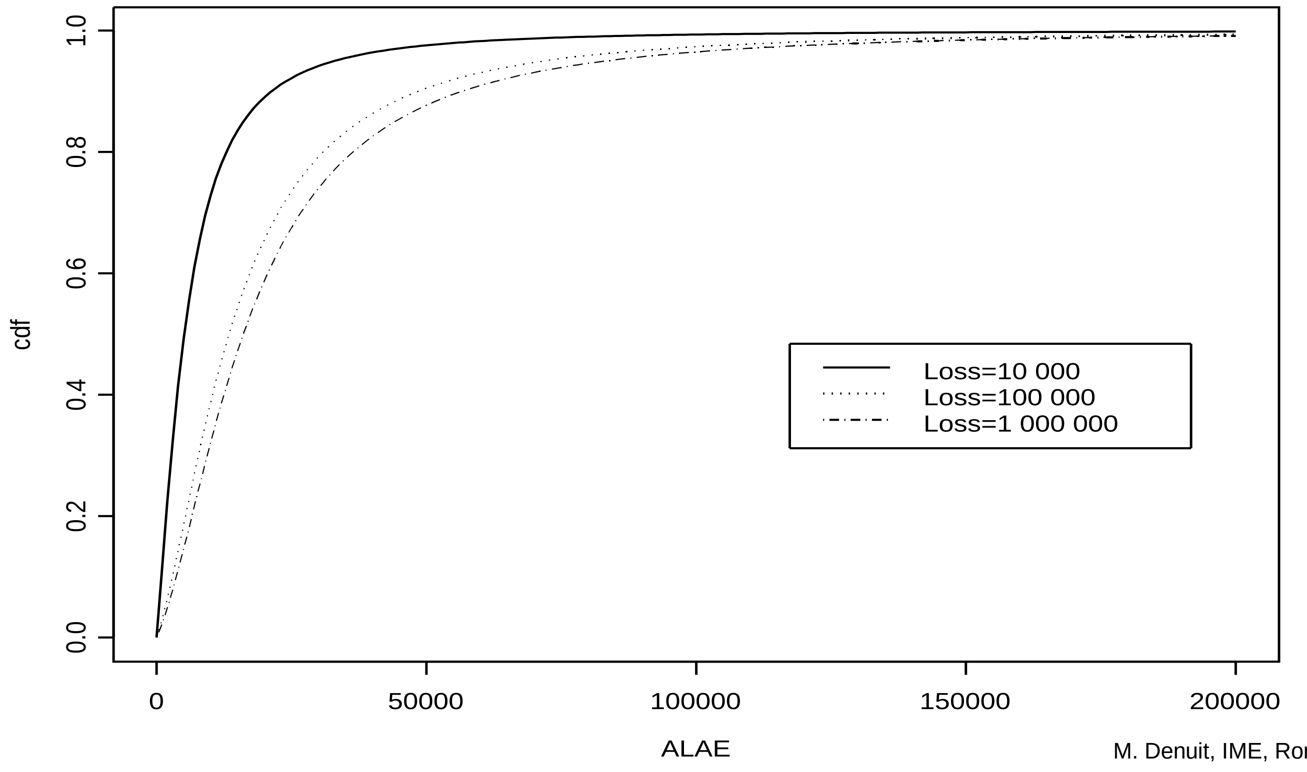
Selection of the parametric generator: QQ-plot of $K(z) = z - \lambda(z)$



	$Clayton$	$Frank$	$Gumbel$
$\widehat{\alpha}$ omnibus	0.517	3.077	1.444
$\int_0^1 (K_{\hat{\alpha}}(z) - \hat{K}(z))^2 dz$	0.0001123993	0.0001477749	0.00009302016

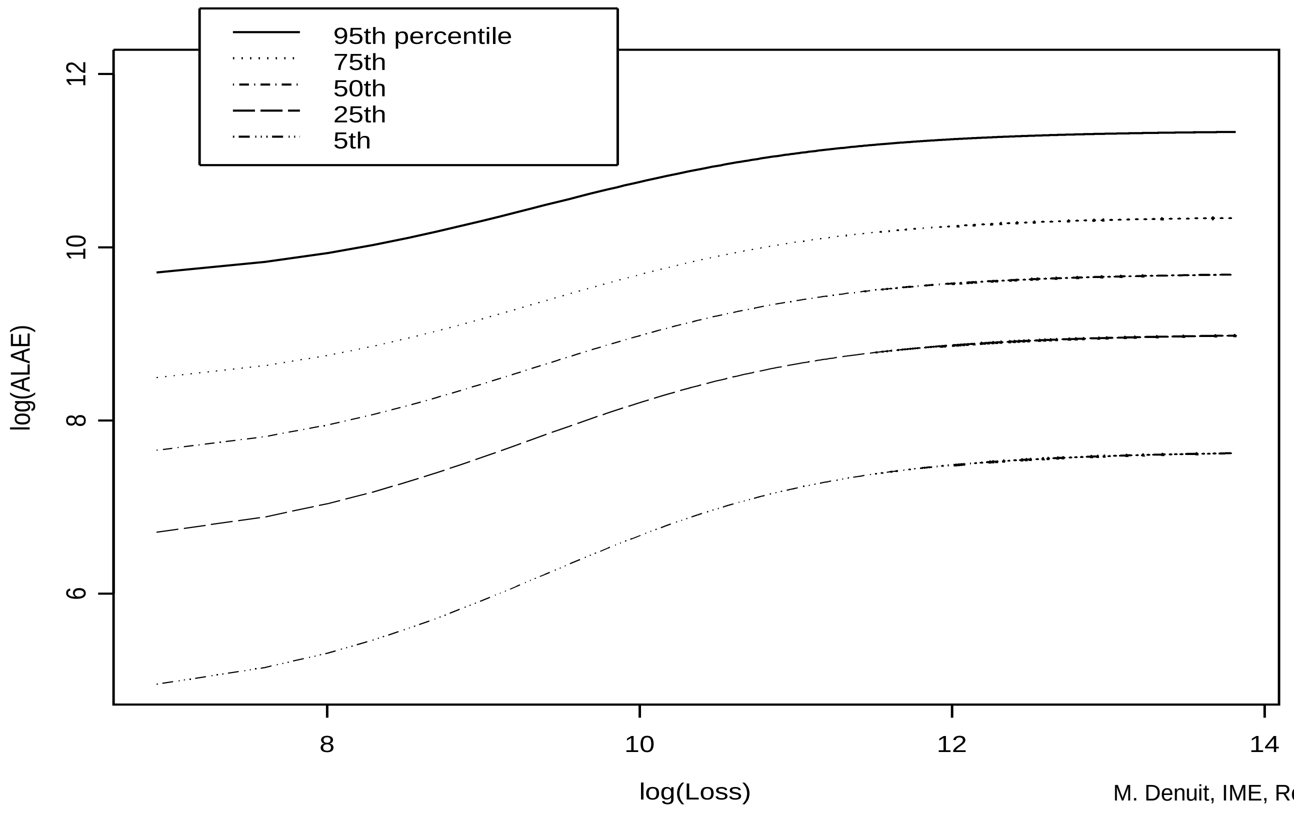
Application to Loss-ALAE

- To have an idea of the behavior of ALAE for some given Loss level, the next figure displays the graph of $x_2 \mapsto \Pr[\text{ALAE} \leq x_2|\text{Loss}]$:



Application to Loss-ALAE

- We also provide the quantile regression curves (i.e. the qth quantiles of ALAE for some given Loss level):



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