Asymptotics of ruin probabilities for risk processes under optimal reinsurance policies: the large claim case

Hanspeter Schmidli

Laboratory of Actuarial Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

Abstract

In a classical risk process reinsurance and investment can be chosen at any time. We find the Lundberg exponent and the Cramér-Lundberg approximation for the ruin probability under the optimal strategy in the case where no exponential moments for the claim size distribution exist. We also show that the optimal strategies converge.

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1. Introduction

Let $S_t = \sum_{i=1}^{N_t} Y_i$ be the aggregate claims process, where $\{N_t\}$ is a Poisson process with rate λ. The claim sizes $\{Y_i\}$ are iid, strictly positive and independent of the claim arrival process. We denote by Y a generic random variable, by $M_Y(r) = \mathbb{E}[\exp\{rY\}]$ its moment generating function and by $G(y)$ its distribution function. All stochastic quantities are defined on a large enough complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The insurer follows a strategy $(A(u), b(u))$ of feedback form, where $(A(u), b(u)) \in \mathcal{A} = [0, \infty) \times [0, 1]$. $A(u)$ denotes the amount invested into a risky asset, modelled as a geometric Brownian motion

$$\mathrm{d}Z_t = \mu Z_t\,\mathrm{d}t + \sigma Z_t\,\mathrm{d}W_t\,,$$

where $\{W_t\}$ is a standard Brownian motion independent of $\{S_t\}$. We assume here that all economic quantities are discounted. In particular, the claim sizes increase with inflation and the amount "not invested" is put on a bank account or invested in a riskless bond. It is even possible to borrow money at the same rate, as it will be the case under the optimal strategy. The latter can be interpreted that the portfolio under consideration has a debt to the capital resources of the company. The parameters fulfil $\mu, \sigma > 0$. $\sigma > 0$ is no loss of generality, $\mu > 0$ is necessary in order that investment reduces the ruin probability, but also makes sense from an economic point of view.

$b(u)$ is the retention level in proportional reinsurance, i.e. if a claim Y occurs at the time where the surplus is u (before the claim payment) then the insurer pays $b(u)Y$ and the reinsurer pays $(1-b(u))Y$. In order to get this reinsurance cover the insurer has to pay a continuous premium at rate $c(b(u))$. As in [7] we assume that $c(b)$ is strictly decreasing, $c(1) = 0$, and that $c < c(0) < \infty$, where c is the rate at which the insurer gets premiums. We have chosen here proportional reinsurance for simplicity. Other types of reinsurance can be treated similarly.

In this paper we work with the natural filtration $\{\mathcal{F}_t\}$ of $\{(S_t, W_t)\}$, i.e. the smallest right continuous filtration such that $\{(S_t, W_t)\}$ is adapted. The filtration has to be right continuous in order that the ruin time defined below is a stopping time.

Under the chosen strategy the surplus process is

$$\mathrm{d}X_t = (c - c(b(X_t)) + \mu A(X_t))\,\mathrm{d}t + \sigma A(X_t)\,\mathrm{d}W_t - b(X_{t-})\,\mathrm{d}S_t\,, \qquad X_0 = u\,. \quad (1)$$

The time of ruin is $\tau^{A,b} = \inf\{t \ge 0 : X_t < 0\}$ and the ruin probability $\psi^{A,b}(u) = \mathbb{P}[\tau^{A,b} < \infty]$. The control function is $\psi(u) = \inf_{\mathcal{A}} \psi^{A,b}(u)$. Note that $\psi(u) < 1$

even if no net profit condition is fulfilled. Positive safety loading can be achieved by investment.

In [7] it was proved that if there exists a twice continuously increasing function $\delta(u)$ solving the Hamilton-Jacobi-Bellman equation

$$\sup_{(A,b)\in\mathcal{A}} \tfrac{1}{2}\sigma^2 A^2\delta''(u) + (c - c(b) + \mu A)\delta'(u) + \lambda(\mathbb{E}[\delta(u - bY)] - \delta(u)) = 0 \qquad (2)$$

with $\delta(u) = 0$ for $u < 0$, then $\delta(u)$ is bounded and $\delta(u) = \delta(\infty)(1-\psi(u))$. Moreover, the arguments $(A(u), b(u))$ at which the supremum is taken in (2) define the optimal strategy $(A(X_t), b(X_{t-}))$. It is also shown that if $G(y)$ has a bounded density then there exists an increasing twice continuously differentiable solution to (2). Similar problems had been solved in [2] (no reinsurance possible) and [6] (no investment possible).

For the rest of this paper we suppose that $\psi(u)$ is twice continuously differentiable. Then $\psi(u)$ solves the Hamilton-Jacobi-Bellman equation

$$\inf_{(A,b)\in\mathcal{A}} \tfrac{1}{2}\sigma^2 A^2\psi''(u) + (c - c(b) + \mu A)\psi'(u) + \lambda(\mathbb{E}[\psi(u - bY)] - \psi(u)) = 0 \qquad (3)$$

where we let $\psi(u) = 1$ for $u < 0$. The optimal strategy $(A(u), b(u))$ are then the values of A, b in (3) for which the infimum is taken. In the following sections we investigate the asymptotic behaviour of $\psi(u)$ as $u \to \infty$ as well as the asymptotic behaviour of the strategies $(A(u), b(u))$ in the large claim case.

Similar problems had been considered before. In [3] the case without reinsurance and small claims is considered. The small claim case with reinsurance, with and without investment, is considered in [8]. The main step in the small claim case is to use exponential change of measure. This demands that exponential moments exist. Therefore we cannot use the same method here. However, it turns out that in our case we can conclude the results directly from the equation (3).

2. The Lundberg exponent and convergence of $\psi(u)\mathrm{e}^{Ru}$

In this paper we assume that no exponential moments exist, i.e. $M_Y(r) = \infty$ for all $r > 0$. We start by defining the Lundberg exponent.

Suppose we want to use a constant strategy (A, b). The corresponding Lundberg exponent $R(A, b)$ is the positive solution to

$$\lambda(M_Y(br) - 1) - (c - c(b) + \mu A)r + \tfrac{1}{2}\sigma^2 A^2 r^2 = 0\,. \tag{4}$$

Obviously, $b = 0$, otherwise the Lundberg exponent does not exist. In order that $R(A, 0) > 0$ it is needed that $A > \mu^{-1}(c(0) - c)$, i.e. that the process gets a positive drift. In this case

$$R(A, 0) = \frac{2(\mu A - (c(0) - c))}{\sigma^2 A^2}\,.$$

Thus $R(A, 0)$ has a unique maximum at $A^* = 2(c(0) - c)/\sigma^2$. Let $R = R(A^*, 0) = \mu^2/(2\sigma^2(c(0) - c))$. Note that $A^* = \mu/(\sigma^2 R)$. Observe, that A^* minimises the left hand side of (4) at $r = R$. We call R the Lundberg exponent.

We now can find an upper bound to the ruin probability.

Proposition 1. *The ruin probability is bounded by $\psi(u) \le \psi(0)\mathrm{e}^{-Ru}$. If $u > 0$ then the strict inequality holds*

Proof. Using the constant strategy $(A^*, 0)$ yields the ruin probability $\psi^{A^*,0}(u) = \mathrm{e}^{-Ru}$, see [4, p.427]. In [7] it is proved that $\psi(0) < 1$. Suppose we follow the following strategy. First the strategy $(A^*, 0)$ is used. The first time the process reaches 0 the optimal strategy is applied. Thus if $u > 0$, $\psi(u) < \psi(0)\mathrm{e}^{-Ru}$ because the strategy used is not optimal. □

Taking the infimum over A in (3), the Hamilton-Jacobi-Bellman equation reads

$$\inf_b -\frac{\mu^2}{2\sigma^2}\frac{\psi'(u)^2}{\psi''(u)} + (c - c(b))\psi'(u) + \lambda\Bigl(\int_0^{u/b} \psi(u - by)\,\mathrm{d}G(y) + 1 - G(u/b) - \psi(u)\Bigr) = 0\,. \tag{5}$$

b can be replaced by $b(u)$. We want to find the limit of $\psi(u)\mathrm{e}^{Ru}$. Therefore let $f(u) = \psi(u)\mathrm{e}^{Ru}$. Then

$$\begin{aligned}
&-\frac{\mu^2}{2\sigma^2}\frac{(Rf(u)-f'(u))^2}{R^2 f(u) - 2Rf'(u) + f''(u)} - (c - c(b(u)))(Rf(u) - f'(u)) \\
&\qquad + \lambda\Bigl(\int_0^{u/b(u)} f(u-b(u)y)\mathrm{e}^{Rb(u)y}\,\mathrm{d}G(y) + (1-G(u/b(u)))\mathrm{e}^{Ru} - f(u)\Bigr) = 0\,. \qquad (6)
\end{aligned}$$

Note that $Rf(u) - f'(u) > 0$ and $R^2 f(u) - 2Rf'(u) + f''(u) > 0$ by the corresponding properties of $\psi(u)$.

Let $g(u) = Rf(u) - f'(u) = -\psi'(u)\mathrm{e}^{Ru}$. Note that $g(u) > 0$ and $g'(u) < Rg(u)$. Equation (6) reads then

$$\begin{aligned}
&-\frac{\mu^2}{2\sigma^2}\frac{g^2(u)}{Rg(u) - g'(u)} - (c - c(b(u)))g(u) + \lambda\int_0^u g(u-y)\mathrm{e}^{Ry}(1 - G(y/b(u)))\,\mathrm{d}y \\
&\qquad + \lambda(1-\psi(0))\mathrm{e}^{Ru}(1 - G(u/b(u))) = 0\,. \qquad (7)
\end{aligned}$$

We are now ready to prove our main theorem.

Theorem 1. *There exists a $\zeta \in [0, \psi(0))$ such that*

$$\lim_{u\to\infty} \psi(u)\mathrm{e}^{Ru} = \zeta\,.$$

Moreover, the functions $f(u)$ and $g(u)$ are monotonically decreasing.

Proof. Replacing $b(u)$ by zero in (7) yields

$$-\frac{\mu^2}{2\sigma^2}\frac{g(u)^2}{Rg(u) - g'(u)} + (c(0) - c)g(u) \ge 0\,.$$

Because $g(u) > 0$ it is possible to divide by $g(u)/R$, giving

$$0 \le (c(0) - c)R - \frac{\mu^2}{2\sigma^2}\frac{Rg(u)}{Rg(u) - g'(u)} = -\frac{\mu^2}{2\sigma^2}\frac{g'(u)}{Rg(u) - g'(u)}\,,$$

using the definition of R. Because the denominator is strictly positive, the function $g'(u)$ is decreasing. If $f'(u) > 0$ at some point u then $f(u)$ is strictly increasing, implying that also $f'(u)$ must be strictly increasing because $g(u) = Rf(u) - f'(u)$ is decreasing. Because $f(u)$ is bounded this is not possible, yielding $f'(u) \le 0$ and $f(u)$ is decreasing. In particular, ζ exists and is smaller than $f(0) = \psi(0)$. From Proposition 1 we can conclude that $\zeta < \psi(0)$. $\square$

3. Convergence of the strategies

We first consider the strategy $b(u)$.

Theorem 2. *The strategy $b(u)$ converges to zero.*

Proof. Because $g(u)$ is decreasing we get for the terms in (7) involving $b(u)$

$$\begin{aligned}&(c(b(u)-c))g(u)+\lambda\int_0^u g(u-y)\mathrm{e}^{Ry}(1-G(y/b(u)))\,\mathrm{d}y\\&\qquad+\lambda(1-\psi(0))\mathrm{e}^{Ru}(1-G(u/b(u)))\\&\geq\Bigl(c(b(u)-c+\lambda\int_0^u \mathrm{e}^{Ry}(1-G(y/b(u)))\,\mathrm{d}y\Bigr)g(u)\,.\end{aligned}$$

Replacing $b(u)$ by 0 in (7) yields $(c(0)-c)g(u)$ for the terms involving $b(u)$. Because $b(u)$ is the argument for which the infimum is taken it follows that

$$\lambda\int_0^u \mathrm{e}^{Ry}(1-G(y/b(u)))\,\mathrm{d}y\leq c(0)-c\,.$$

Because $\int_0^\infty \mathrm{e}^{Ry}(1-G(y/b))\,\mathrm{d}y=\infty$ for all $b>0$ it follows that $b(u)\to 0$. ☐

The strategies $A(u)$ also converge.

Theorem 3. *The strategy $A(u)$ converges to A^* in the large claim case.*

Proof. Using the definition of the Lundberg exponent equation (7) can be written as

$$\begin{aligned}&-\frac{\mu^2}{2R\sigma^2}\frac{g'(u)g(u)}{Rg(u)-g'(u)}-(c(0)-c(b(u)))g(u)\\&\quad+\lambda\Bigl(\int_0^u g(u-y)\mathrm{e}^{Ry}(1-G(y/b(u)))\,\mathrm{d}y+(1-\psi(0))\mathrm{e}^{Ru}(1-G(u/b(u)))\Bigr)=0\,.\end{aligned}$$

The only negative term is $-(c(0)-c(b(u)))g(u)$. Dividing this term by $g(u)$ it converges to zero. Thus also the positive terms divided by $g(u)$ have to converge to zero. In particular,

$$\lim_{u\to\infty}\frac{g'(u)}{Rg(u)-g'(u)}=0\,.$$

This is only possible if $g'(u)/g(u)$ converges to zero. Thus

$$\lim_{u\to\infty} A(u) = \lim_{u\to\infty} \frac{\mu}{\sigma^2} \frac{g(u)}{Rg(u) - g'(u)} = A^* .$$

□

4. Correct exponent and positivity of ζ

Because we have not shown that $\lim_{u\to\infty} f(u) > 0$ we need to proof that R is in fact the Lundberg coefficient.

Proposition 2. *For any $\varepsilon > 0$,*

$$\lim_{u\to\infty} \psi(u)\mathrm{e}^{(R+\varepsilon)u} = \infty .$$

Remark. Because we have shown convergence of the strategies we could alternatively use the Gärtner-Ellis theorem to prove that $-\lim_{u\to\infty} u^{-1}\log\psi(u) = R$. ■

Proof. We have seen that $b(u)$ converges to zero. Choose $b > 0$ such that $c(b) > c$ and choose u_0 such that $b(u) \le b$ for all $u \ge u_0$. Consider now the following process $\{X_t^*\}$. If $X_t^* \le u_0$, the process follows the same law as $\{X_t\}$. For $u > u_0$ the full claims are reinsured but only the premium rate $c(b)$ has to be paid for reinsurance. For $u > u_0$ the optimal investment strategy for $\{X_t^*\}$ is chosen. The ruin probability of $\{X_t^*\}$ is denoted by $\psi^*(u)$. Clearly, $\psi(u) \ge \psi^*(u)$. Ruin for $\{X_t^*\}$ for an initial capital $u > u_0$ can only occur by reaching the level u_0. Thus for $u \ge u_0$

$$\psi^*(u) = \psi^*(u_0)\exp\Bigl\{-\frac{\mu^2}{2\sigma^2(c(b)-c)}(u-u_0)\Bigr\} .$$

This shows that

$$\lim_{u\to\infty} \psi(u)\exp\Bigl\{\Bigl(\frac{\mu^2}{2\sigma^2(c(b)-c)} + \frac{\varepsilon}{2}\Bigr)u\Bigr\} = \infty .$$

Because b is arbitrary this proves the result. □

An open question had been the following. We know that $b(u)$ converges to zero. Is it possible that $b(u) = 0$ for a finite u? The question is partially answered in the following result.

Lemma 1. *Consider the large claim case and suppose that $x(1-G(x))$ converges to zero as $x \to \infty$. If $\varliminf_{b \downarrow 0} b^{-1}(c(0) - c(b)) > \lambda \mathbb{E}[Y]$ then $b(u) > 0$ for all u.*

Remarks.

i) The condition $\lim_{x\to\infty} x(1 - G(x)) = 0$ is quite weak. Because the claim sizes have to be integrable, $x(1 - G(x)) > a$ for some $a > 0$ and all $x > x_1$ is not possible.

ii) If the expected value principle is used for the calculation of the reinsurance premium then $c'(b) = -(1 + \theta)\lambda\mathbb{E}[Y]$, i.e. $b(u) > 0$ for all u.

iii) In [7] the optimal strategy for Pareto distributed claim sizes was calculated. From the graph one gets the impression that $b(u) = 0$ for u large enough. This is not the case by the lemma above. Therefore the jump to zero in the graph is either due to numerical errors, or $b(u)$ is so small that it becomes zero by the discretisation. ■

Proof. Take the difference of the expression to be minimised with the expression at zero and divide it by $\lambda b g(u)$. This gives for $u > 1$,

$$-\frac{c(0) - c(b)}{\lambda b} \\ + \int_0^u \frac{g(u-y)}{g(u)} \frac{\mathrm{e}^{Ry}}{y} \frac{y}{b}(1 - G(y/b)) \,\mathrm{d}y + (1 - \psi(0)) \frac{\mathrm{e}^{Ru}}{ug(u)} \frac{u}{b}(1 - G(u/b))\,.$$

The last term goes to zero as $b \downarrow 0$. The integral part $\int_1^u$ also goes to zero as $b \downarrow 0$. The integral part $\int_0^1$ can be expressed as

$$\int_0^{1/b} \frac{g(u-by)}{g(u)} \,\mathrm{e}^{Rby}(1 - G(y)) \,\mathrm{d}y\,.$$

By bounded convergence this tends to $\mathbb{E}[Y]$. The expression to be minimised is strictly decreasing in b close to zero and therefore $b(u) > 0$. $\square$

To finish the paper we give a sufficient criterion under which $\zeta > 0$.

Theorem 4. *Suppose that there is a constant $K > 0$ such that $c(0) - c(b) \le Kb$ for all $b > 0$. Suppose also that there are constants $\alpha > 0$ and $0 < \gamma < \frac{1}{2}$ such that*

$$1 - G(u) \ge \alpha \exp\{-u^\gamma\}\,.$$

Then $\zeta = \lim_{u\to\infty} \psi(u)\mathrm{e}^{Ru} > 0$.

Proof. From the proof of Theorem 3 we know that $h(u) = -g'(u)/g(u)$ converges to zero. Thus we can express

$$g(u) = g(0)\exp\Bigl\{-\int_0^u h(v)\,\mathrm{d}v\Bigr\}$$

and $\zeta = 0$ is equivalent to $\int_0^\infty h(u)\,\mathrm{d}u = \infty$. Again from the proof of Theorem 3 we conclude that for u large enough

$$\frac{\mu^2}{4R^2\sigma^2}h(u) \le \frac{\mu^2}{2R\sigma^2}\frac{h(u)}{R + h(u)} = -\frac{\mu^2}{2R\sigma^2}\frac{g'(u)}{Rg(u) - g'(u)} < c(0) - c(b(u)) \le Kb(u)\,.$$

Thus $\int_0^\infty b(u)\,\mathrm{d}u < \infty$ implies $\zeta > 0$. Because $\mathrm{e}^{Ru}(1 - G(u/b(u)))$ converges to zero we have for u large enough

$$\alpha\mathrm{e}^{Ru-(u/b(u))^\gamma} \le \mathrm{e}^{Ru}(1 - G(u/b(u))) \le 1\,.$$

This is equivalent to

$$b(u) \le u(Ru + \log\alpha)^{-1/\gamma}\,.$$

This implies that $b(u)$ is integrable for $\gamma < \frac{1}{2}$. $\square$

The author conjectures that if $1 - G(u) \le \alpha\exp\{-u^\gamma\}$ for some $\gamma > \frac{1}{2}$, also called moderately heavy tailed case, then $\zeta = 0$. This conjecture is based upon the observation that often the behaviour in the moderately heavy tailed case is different from what one would expect, see [5] or [1]. Future research has to solve this open problem.

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