A Multi-Period View on Actuarial and Financial Pricing for Guaranteed Minimum Death Benefits in Unit-Linked Life Insurance

S. Desmedt[*‡], X. Chenut[*], J.F. Walhin[*§]

* Secura, 9-31 Avenue des Nerviens, B-1040 Brussels, Belgium
‡ Corresponding author, Tel: +32 2 5048228, e-mail: stijn.desmedt@secura-re.com
§ Institut de Sciences Actuarielles, Université Catholique de Louvain, 6 Rue des Wallons, B-1348 Louvain-la-Neuve, Belgium

Abstract

We compare the actuarial and the financial approach for reserving and pricing for guaranteed minimum death benefits in unit-linked life insurance. In the first approach, no hedging strategy is applied, in the second, a dynamic hedging strategy based upon the Black and Scholes Model is used. In the financial approach, we incorporate hedging errors and transaction costs. Special attention is paid to the multi-period character of the guarantee. For both approaches, we propose a pricing method based upon a cash-flow model incorporating future information. We compare the prices obtained under several conditions.

Keywords

Unit-Linked Life Insurance, GMDB, Multi-Period Capital Allocation, Actuarial and Financial Pricing and Reserving

1 Introduction

In this paper, we compare actuarial and financial reserving and pricing for a guaranteed minimum death benefit (GMDB) in unit-linked life insurance. As pointed out in Brennan and Schwartz (1976), under the assumptions of the Black and Scholes market model and assuming the mortality risk can be completely eliminated, a riskless investment strategy for a GMDB exists. Financial reserving is based upon these results.

It is well known that in practice, there are limits to the assumptions underlying the Black and Scholes model. As explained in Hardy (2003), there are several sources of unhedged liability and additional costs when applying a dynamic hedging strategy: transaction costs (for rebalancing the hedge), hedging errors arising from the fact that discrete hedging intervals have to be maintained and additional costs arising from the fact that the log-normal equity model is subject to some imperfections. Furthermore, it may not be possible to eliminate mortality risk completely. In this paper we take into account each of these risks, which is necessary to be able to determine a price when applying the financial reserving strategy.

In practice, GMDB's are often reserved for in an actuarial way. Instead of hedging the liability of the guarantee, capital is allocated to ensure for the necessary security. When future information about the underlying asset and the mortality becomes available, capitals and reserves may be adapted to make sure the security level remains sufficiently high or to avoid unnecessary capital costs. We refer to Desmedt et al. (2004), where multi-period capital allocation strategies are discussed and methods are presented to calculate distribution functions of the future capitals and reserves when future information is taken into account. These provide useful information about the risks of a GMDB when actuarial reserving is performed and allow to determine an actuarial price using a cash-flow model.

The price a company should ask for a GMDB may depend on the reserving strategy it follows. We explain why, also within a given reserving strategy, prices may be different. On the other hand, prices are influenced by the market. GMDB's and portfolio's of GMDB's are of course not as liquidly traded as other common options, but nonetheless clients and insurers will still look for the lowest price. We believe that the price a company should ask for a GMDB may not be unique although the market can of course impose some constraints. We therefore compare the prices within the actuarial and financial framework under a set of conditions.

The remainder of this paper is organized as follows. In section 2, we compare the basic costs in the actuarial and financial reserving strategies. In section 3, we go deeper into the concept of multi-period capital allocation. We then explain a general cash-flow model for pricing multi-period risks in section 4. We explain the methodology and assumptions which are used for modelling a GMDB in section 5. In section 6, we explain how relevant future information can be accounted for when performing actuarial and financial reserving for a GMDB. After comparing several aspects of the actuarial and financial approach under a number of conditions in section 7, we end with a conclusion in section 8.

2 Actuarial Versus Financial Reserving

Consider a group of N_0 insured aged x years and 0 months in which everyone invests S_0 into a risky asset $(S_t)_{t \geq 0}$. Let $(N_t)_{t \geq 0}$ denote the survival process. Hence, N_t is equal to the number of survivors at time t. The insurer provides a guarantee of K in case the insured dies before retirement, which is assumed to be at the age of x years and T months. For convenience, we assume there is no surrender. Assume that when an insured dies in month $(t, t + 1]$ the guarantee leads to a payment of $\max\{0, K - S_{t+1}\} = (K - S_{t+1})_+$, which is equal to the payoff function of a European put option with strike price K and maturity date $t + 1$ on the underlying asset $(S_i)_{i \in [0, t+1]}$, to be paid at $t + 1$.

We now first focus on the financial approach. Let $_{s|}q_{x,t}$ denote the probability that an insured aged x years and t months, lives for another s months and then dies within month $(t + s, t + s + 1]$. Then, as seen from time 0, on average $N_0 \ _{t|}q_{x,0}$ put options with strike price K and maturity date $t + 1$ are needed at time $t + 1$.

Now assume the average log-return of the underlying asset is equal to μ and the volatility of the log-returns is equal to σ. Furthermore, assume that along with the stock process $(S_t)_{t \geq 0}$, there is a possibility to invest in a bond process $(B_t)_{t \geq 0}$, where for all t, $B_t = B_0 e^{rt}$. Assume monthly rebalancing of the hedging portfolio is performed. Then for all $t \in \{0, \ldots, T - 1\}$, the

Black and Scholes price of the hedge at time t of the GMDB described above is equal to:

$$BSP_t = N_t \sum_{s=t}^{T-1} {}_{s-t|}q_{x,t}P(t, s+1), \tag{1}$$

where $P(t, s+1)$ denotes the Black and Scholes price of a put option with strike K on the underlying asset $(S_i)_{i \in [t, s+1]}$, i.e.:

$$P(t, s+1) = -S_t\Phi(-d_1(t, s+1)) + Ke^{-r(s+1)}\Phi(-d_2(t, s+1)), \tag{2}$$

where

$$d_1(t, s+1) = \frac{\ln(S_t/K) + (r + \sigma^2/2)(s+1-t)}{\sigma\sqrt{s+1-t}} \tag{3}$$

$$d_2(t, s+1) = d_1(t, s+1) - \sigma\sqrt{s+1-t} \tag{4}$$

and Φ denotes the cumulative distribution function of the standard normal distribution.

To hedge this GMDB, a portfolio with

$$\xi_t = N_t \sum_{s=t}^{T-1} {}_{s-t|}q_{x,t}\left[N\left(\frac{\ln(S_t/K) + (r + \sigma^2/2)(s+1-t)}{\sigma\sqrt{s+1-t}}\right) - 1\right] \tag{5}$$

stocks and

$$\beta_t = (BSP_t - S_t\xi_t)/B_t \tag{6}$$

bonds is held at time t.

At time $t \in \{1, \dots, T\}$, the value of the hedging portfolio from the previous period $[t-1, t)$ is equal to:

$$V_t^- = N_{t-1}(\xi_{t-1}S_t + \beta_{t-1}B_t) \tag{7}$$

With V_t^-, we need to pay the new hedging portfolio and the eventual costs of the guarantee at time t. This means that the hedging error at time $t \in \{1, \dots, T-1\}$ is equal to:

$$HE_t = BSP_t + (N_{t-1} - N_t)(K - S_t)_+ - V_t^-. \tag{8}$$

At time T, the hedging error is equal to:

$$HE_T = (N_{T-1} - N_T)(K - S_T)_+ - V_T^-. \tag{9}$$

If $HE_t < 0$, then the hedging strategy lead to a benefit at time t, if $HE_t \geq 0$, then additional funding of the hedging portfolio is needed.

In order to incorporate the additional costs due to the imperfections of the log-normal model for describing the risky asset, it suffices to replace the log-normal model with a more advanced model such as the Regime-Switching Log-Normal model which is described in Hardy (2001). The underlying asset should be modelled under the real-world probability measure. The risk-neutral probability measure is only used for pricing and hedging purposes.

To incorporate the transaction costs, we use a similar approach as in Hardy (2003). We assume transaction costs on bonds are very small compared to those for the stocks and that

the transaction costs are a fixed percentage τ of the absolute change in the amount of stocks which is held. Hence the transaction costs at time $t \in \{1, \ldots, T-1\}$ are equal to:

$$TC_t = \tau S_t |\xi_t - \xi_{t-1}|. \tag{10}$$

At $t = 0$ and $t = T$, the transaction costs are equal to

$$TC_0 = \tau S_0 |\xi_0| \text{ and } TC_T = \tau S_T |\xi_T| \tag{11}$$

respectively.

In the financial approach, the discounted future costs at time 0 are equal to:

$$D_0^{(F)} = BSP_0 + \sum_{s=1}^{T} HE_s e^{-rs} + \sum_{s=0}^{T} TC_s e^{-rs}, \tag{12}$$

where (F) denotes that the financial approach is followed. At $t > 0$, the discounted future costs are equal to:

$$D_t^{(F)} = \sum_{s=t+1}^{T} HE_s e^{-r(s-t)} + \sum_{s=t+1}^{T} TC_s e^{-r(s-t)}. \tag{13}$$

In the actuarial approach, no hedging strategy is applied. The provider of the guarantee simply pays the value of the guarantee if this value is positive. This means the costs at time $t \in \{1, \ldots, T\}$ are equal to:

$$C_t = (N_{t-1} - N_t)(K - S_t)_+. \tag{14}$$

The discounted future costs at $t \in \{0, \ldots, T-1\}$ are then equal to

$$D_t^{(A)} = \sum_{s=t+1}^{T} C_s e^{-r(s-t)}, \tag{15}$$

where (A) denotes that the actuarial approach is followed.

3 Multi-Period Reserving and Capital Allocation

Future information about the underlying asset and the mortality will have an important impact both in the actuarial and the financial approach.

In the financial approach, the position of the underlying asset and the number of survivors will on the one hand of course have an impact on the hedging strategy which is applied. If the market were complete, a self-financing trading strategy replicating the GMDB would exist and no loading of the price would be necessary. As pointed out in section 2, in practice, there are both hedging errors and transaction costs. As we will see in section 7, both of these are uncertain and their level may depend on the future information which becomes available.

In the actuarial approach, the position of the underlying asset and the mortality will have a direct impact on the estimated future costs. If the underlying asset performs well, future costs become less probable and vice versa. If the mortality has been high, future costs become less probable and vice versa.

In IAA (2004), it is recommended to take available information into account on a regular basis when assessing the required solvency margin for the future. There are several reasons why it is useful to analyze the possible impact of future information. At time zero, it is important to assess the distribution functions of the future solvency margins. If increases of the required solvency margins are probable and can be large, it should be made sure it is possible to diversify this risk with other risks. Indeed, large increases of the total solvency margin of the company as a whole should be avoided since it may be hard to find shareholders willing to invest in something which will consume capital with a large probability. Future information may also be important in pricing. Indeed, since both in the actuarial and the financial approach future information may lead to reviewing the reserves and the capitals which are held, this may have an important impact on the price. For more details, we refer to section 4.

For calculating solvency requirements, we use risk measures. More details about these instruments and their applications can be found in Artzner et al. (1999), Dhaene et al. (2003), Dhaene et al. (2004a), Dhaene et al. (2004b) and Goovaerts et al. (2004).

3.1 Strategy Not Using Future Information

Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ denote the filtration describing the future information. In the strategy not using future information, the reserve at time t is defined as:

$$P_t^{(N)} = E[D_t | \mathcal{F}_0], \tag{16}$$

where (N) denotes that no future information is used in the evaluation of future reserves.

The total solvency level at t is defined as:

$$TSL_t^{(N)} = \rho[D_t | \mathcal{F}_0], \tag{17}$$

where ρ is a certain risk measure. The capital at time t is then equal to

$$K_t^{(N)} = TSL_t^{(N)} - P_t^{(N)}. \tag{18}$$

This approach may not be appropriate in practice. When the information is such that future costs are becoming more likely, then ruin becomes more likely if this is not taken into account. On the other hand, if future information is positive and this is not taken into account, we keep a level of capital which may be too high and which could be used for other business.

The usefulness of this method should rather be seen as a fast (but under certain conditions rough) computational alternative to the method using future information. For more details, we refer to section 4 and section 7.

3.2 Strategy Using Future Information

In the strategy using future information, the reserve at time t is defined as:

$$P_t^{(U)} = E[D_t | \mathcal{F}_t], \tag{19}$$

where (U) denotes that future information is used in the evaluation of the future reserves.

The total solvency level at t is defined as:

$$TSL_t^{(U)} = \rho[D_t | \mathcal{F}_t], \tag{20}$$

where ρ is a certain risk measure. The capital at time t is then equal to

$$K_t^{(U)} = TSL_t^{(U)} - P_t^{(U)}. \tag{21}$$

3.3 Calculating Total Solvency Requirements

For calculating total solvency requirements, we use the Tail Value-at-risk. This risk measure is defined as the average of all quantiles above a certain p-level. The quantile or Value-at-Risk is defined as follows.

Definition 3.1 (Value-at-Risk or Quantile) *For any $p \in (0,1)$, the VaR at level p of a risk X is defined and denoted by*

$$Q_p[X] = \inf\{x \in \mathbb{R}| F_X(x) \geq p\}, \tag{22}$$

where $F_X(x)$ denotes the distribution function of X.

The Tail Value-at-Risk is then defined as:

Definition 3.2 (Tail Value-at-Risk) *For any $p \in (0,1)$, the TVaR at level p of a risk X is defined and denoted by*

$$TVaR_p[X] = \frac{1}{1-p} \int_p^1 Q_q[X]dq. \tag{23}$$

When calculating the TVaR at level p of a set N of simulations, we just need to take the average of the $(1-p)N$ largest simulations.

4 Cash Flow Model in a Multi-Period Setting

For pricing, we model the average in-and outflows, taking the point of view of the shareholders. Assume the corporate tax rate is equal to γ and the average return on the invested capital (after taxation) is equal to $e^\delta - 1$. Assume the cost of capital is constant and equal to COC. Assume reserves are entirely invested in bonds (with return equal to $e^r - 1$) and a part of the capital is invested in bonds and another part is invested in shares. Therefore, δ is assumed to be larger than r. If we assume the return on the capital is independent of the total solvency levels which need to be held, we can work with the average return for pricing. It is good to be aware of the fact that this assumption may be a simplification of reality. If some of the capital is invested risky, the return on this part may be correlated with the performance of the GMDB.

We then have the following average outflows:

1. Net mean claim payments: $c_t(1 - \gamma)$, where $t \in \{1, \dots, T\}$.

2. Net mean change in the reserves: $\Delta p_t(1 - \gamma)$, where $t \in \{0, \dots, T\}$

3. Mean change in the allocated capital: Δk_t, where $t \in \{0, \dots, T\}$

We have the following inflows:

1. Net mean return on the reserves: $R_t(p)(1 - \gamma)$, where

$$R_t(p) = p_{t-1}(e^r - 1), t \in \{1, \dots, T\}. \tag{24}$$

2. Net mean return on the capital (after taxation): $R_t(k)$, where

$$R_t(k) = k_{t-1}(e^\delta - 1) \text{ and } t \in \{1, \dots, T\}. \tag{25}$$

3. Net premium income: $TFP(1 - \gamma)$, where TFP denotes the technico-financial premium which has to be determined.

At this point, we can understand some potential usefulness of the strategy described in section 3.1. To calculate the average in-and outflows, we need to determine the average reserves r_t and the average capitals k_t for all $t \in \{0, \ldots, T\}$. For the reserves, due to the iterativity property of the expectation, we have

$$p_t = \mathrm{E}[P_t] = \mathrm{E}[\mathrm{E}[D_t|\mathcal{F}_t]] = \mathrm{E}[D_t|\mathcal{F}_0], \text{ for all } t \in \{0, \ldots, T-1\}. \tag{26}$$

Hence, with respect to the reserves, there is no difference in the cash-flow model between the approach described in section 3.1 and section 3.2. For the capitals, we in general do not have the equality of

$$k_t = \mathrm{E}[K_t] = \mathrm{E}[\rho[D_t|\mathcal{F}_t] - E[D_t|\mathcal{F}_t]] \tag{27}$$

and $\rho[D_t|\mathcal{F}_0] - E[D_t|\mathcal{F}_0]$, for all $t \in \{0, \ldots, T-1\}$. We could use $\rho[D_t|\mathcal{F}_0] - E[D_t|\mathcal{F}_0]$ as an approximation of k_t but as shown in Desmedt et al. (2004), in the actuarial approach, this approximation may be very bad. Nonetheless, in circumstances where this approximation is acceptable and capital costs are relatively small, it may outweigh the additional complexity of incorporating future information. We refer to section 7 for some examples.

Now assume we discount all future cash-flows at the cost of capital. The technico-financial premium is then the value which makes the sum of the discounted inflows equal to the sum of the discounted outflows. Hence, the technico-financial premium is the value which solves equation (28)

$$\sum_{t=0}^{T} e^{-tCOC} \left[\Delta p_t - R_t(p) + c_t\right](1-\gamma) = \sum_{t=0}^{T} e^{-tCOC} \left[R_t(k) - \Delta k_t\right] + TFP(1-\gamma), \tag{28}$$

with the convention that $R_0(p) = R_0(k) = 0$ and $c_0 = 0$. From 26, it follows that both in the approaches using and not using future information,

$$p_t = \sum_{s=t+1}^{T} e^{-rs} c_s. \tag{29}$$

Using (29) it can be verified that the term $\Delta p_t - R_t(p) + c_t$ in (28) is equal to

$$P_0 \quad \text{if} \quad t = 0, \tag{30}$$
$$0 \quad \text{if} \quad t \in \{1, \ldots, T\}. \tag{31}$$

Using (30) and (31), we can write the technico-financial premium as

$$TFP = P_0 + \sum_{t=0}^{T} \frac{e^{-tCOC}[\Delta k_t - R_t(k)]}{1-\gamma}. \tag{32}$$

This means the technico-financial premium consists of two parts:

1. The reserve P_0 taken at time 0.

2. A loading for the average amounts of capital k_t which are allocated at the start of each year $t \in \{1, \ldots, T\}$.

To calculate expression (32), we need the average capitals at the start of the different periods. In the approach not using future information the future capitals are fixed at time 0. When using future information, the future capitals are unknown at time 0.

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In the approach using future information, we assume the future reserves and capitals are reviewed on a yearly basis. In the approach not using future information, reserves and capitals are assumed to be adapted on a monthly basis. This should lead to a small correction in the cash-flow model presented above since 31 is then not valid for all $t \in \{1, \ldots, T\}$. To avoid this correction when using future information, we assume that the payments in a given year are only made at the end of each year, i.e. we assume that at the end of year t', an average payment of

$$c'_{t'} = \sum_{s=1}^{12} e^{(12-s)r} c_{12t'-12+s} \tag{33}$$

has to be made, for all $t' \in \{1, \ldots, T/12\}$. $c'_{t'}$ denotes the average payment at the end of year t'. The reader may verify that in this yearly setting, an expression for the technico-financial premium with the same structure as 32 can be derived.

5 Modelling Methodology and Assumptions

5.1 Product Description

We suppose there is a group of $N_0 = 1000$ insured aged $x = 50$ which all invest $C = 1$ into the S&P 500 index. The insurer provides a guarantee of $K = 1$ in case the insured dies before retirement, which is assumed to be at the age of 65. For convenience, we assume there is no surrender.

5.2 Mortality Process

To model mortalities, we use a Gompertz-Makeham approach. The survival probability of a person aged x is then described as

$$_tp_x = \exp\left(-\alpha t - \frac{\beta e^{\gamma x}(e^{\gamma t} - 1)}{\gamma}\right), \tag{34}$$

for some $\alpha > 0$, $\beta > 0$ and γ. We use the first set of Gompertz-Makeham parameters in table 6 up to age 65 and the second set for ages higher than 65. These parameters are based upon table 197 of Assuralia, the Belgian Union of Insurers. We model mortalities on a monthly basis using random draws from binomial distributions.

5.3 Risky Asset

We use the regime-switching log-normal (RSLN) model with 2 regimes as described in Hardy (2001). This model provides us with monthly log-returns. We denote the regime applying to the interval $[t, t+1)$ as κ_t. Hence $\kappa_t \in \{1, 2\}$. In a certain regime κ_t we assume the log-return Y_t satisfies:

$$Y_t = \log(\frac{S_{t+1}}{S_t}|\kappa_t) \sim N(\mu_{\kappa_t}, \sigma_{\kappa_t}). \tag{35}$$

Furthermore, the transitions between the regimes are assumed to follow a Markov Process characterized by the matrix P of transition probabilities

$$p_{ij} = Pr[\kappa_{t+1} = j|\kappa_t = i], \text{for } i, j \in \{1, 2\}. \tag{36}$$

As parameters, we use the parameters estimated using the maximum log-likelihood techniques explained in Hardy (2001). We use the S&P 500 data (total returns) from January 1960 to December 2003. The maximum log-likelihood (MLE) estimates are given in table 6.

In the financial approach, an estimation of the standard deviation of the underlying asset is necessary to determine the Black and Scholes prices and the hedging strategy for the put options which are necessary for the GMDB. We use the standard deviation of the observed monthly log-returns of the S&P 500 index for the period 1960-2003 ($\sigma = 0.0432$). Note that the average monthly standard deviation of the log-returns found by making 10000 simulations with the RSLN model with the parameters as specified in table 6 is equal to 0.0428.

6 Future Information for a GMDB

6.1 Future Information about Mortality

In this paper, we only incorporate future information about the underlying asset. We first explain our motivation to do so. Therefore, we analyze the distribution functions obtained with 10000 simulations for the survivors in a group of 1000 insured aged 50 at the end of each year (until 65), which are given in figure 1. We summarize the first four moments in table 1.
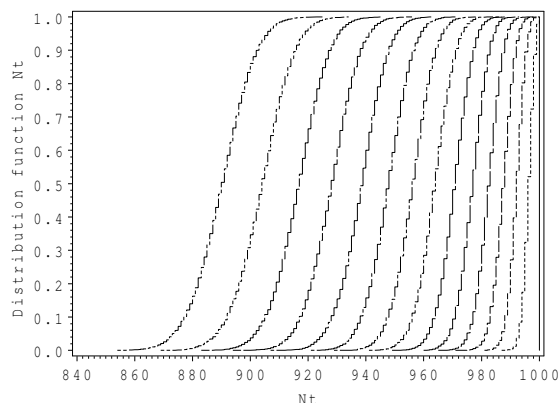

Figure 1: Distribution functions of survivors.

In figure 1, we see that the volatility of the distribution functions increases with time. Furthermore, the distribution functions become more and more symmetric if we move to the end. As we can see both in figure 1 and table 1, there is a negative skewness of the distributions mainly during the first years. The kurtosis is always close to the kurtosis of the normal distribution and gets closer if we move further into time.

As explained above, considering the impact of future information is important both for pricing purposes and for assessing the possible required solvency levels in the future. The impact of the number of survivors on the costs, the reserves and the total solvency levels is linear. Indeed, consider a situation A where there are 1000 survivors and a situation B where there are 900 survivors. Then the costs and the reserves in situation B are 0.9 times those in situation B. For the total solvency levels, we could claim that there could be some credit for a situation with more survivors, due to possible diversification effects. The level of the relative differences between the possible outcomes for the numbers of survivors is however such that these possible diversification effects may be neglected. Therefore, considering future information for pricing purposes will not have any considerable influence.

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Year	Mean	StDev	Skewness	Kurtosis
1	996	1.91	-0.47	3.14
2	992	2.79	-0.32	3.05
3	987	3.53	-0.24	3.05
4	982	4.14	-0.17	3.02
5	976	4.73	-0.17	2.92
6	970	5.31	-0.17	2.99
7	963	5.91	-0.13	2.99
8	956	6.49	-0.13	2.99
9	947	7.10	-0.12	2.98
10	938	7.66	-0.14	3.01
11	928	8.19	-0.11	2.99
12	916	8.75	-0.12	3.02
13	903	9.24	-0.11	3.01
14	890	9.81	-0.10	3.00

Table 1: First four moments of the numbers of survivors.

Considering future information for the mortality will not really contribute to assessing the future required total solvency levels as well. During the first years, the distribution functions for the numbers of survivors are very centered. Furthermore, deviations from the center, if any, are more pronounced at the side of the distribution where the number of survivors is low. Hence, increasing the total solvency level due to information about the mortality will during the first years almost never be necessary, and if it is, effects will be small. If we move further into the future, the volatility of the number of survivors increases. As we will see in section 7, after some years, the largest estimations of the required total solvency levels start decreasing if future information about the underlying asset is incorporated. When mortality information has a negative effect on the estimated future costs, it will only make required total solvency levels decrease slower.

6.2 Future Information about Underlying Asset

In Desmedt et al. (2004), a simulation strategy is described to incorporate future information about the underlying asset when the actuarial approach is followed. This simulation strategy is summarized in the following five steps.

1. Make N_S simulations for the underlying asset S_s from 0 to T and denote them as

$$\Omega = \{S_t^{(i)} | i \in \{1, \ldots, N_S\} \text{ and } t \in \{0, 1, \ldots, T\}\}. \tag{37}$$

   Since we only review the capitals and the reserves on a yearly basis, we define the value of the underlying asset at the end of year t' in the ith simulation as $S_{t'}^{(i)}$, where $i \in \{1, \ldots, N_S\}$ and $t' \in \{1, \ldots, T/12 - 1\}$.

2. For all $t' \in \{1, \ldots, T/12 - 1\}$, order the set $\{S_{t'}^{(i)} | i \in \{1, \ldots, N_S\}\}$. Assume now rank numbers are the same as the numbers in the superscript. This means $S_{t'}^{(1)}$ is the smallest simulation at t', $S_{t'}^{(2)}$ the second smallest and so on.

3. For all $t' \in \{1, \ldots, T/12 - 1\}$, divide $\{S_{t'}^{(i)} | i \in \{1, \ldots, N_S\}\}$ into N_A classes as follows

$$\{\{S_{t'}^{(1)}, \ldots, S_{t'}^{(N_S/N_A)}\}, \ldots, \{S_{t'}^{((N_A-1)N_S/N_A+1)}, \ldots, S_{t'}^{N_S}\}\}.$$

4. For all $t' \in \{1, \ldots, T/12 - 1\}$, calculate the average value of every class and retain it as a representing value for its class. For all $j \in \{1, \ldots, N_A\}$ and $t' \in \{1, \ldots, T/12 - 1\}$, denote the retained value for class j at the end of year t' as $S_{t'}^{(a,j)}$.

5. For all $j \in \{1, \ldots, N_A\}$ and $t' \in \{1, \ldots, T/12 - 1\}$, rescale the subset

$$\{S_t^{(i)} | i \in \{1, \ldots, N_S\} \text{ and } t \in \{12t', \ldots, T\}\} \tag{38}$$

of Ω such that $S_{12t'}^{(i)} = S_{t'}^{(a,j)}$ for all $i \in \{1, \ldots, N_S\}$. Hence for all $j \in \{1, \ldots, N_A\}$ and $t' \in \{1, \ldots, T/12 - 1\}$, we obtain a set $\Omega_{t'}^{(a,j)}$ of N_S simulations starting at $S_{t'}^{(a,j)}$ without resimulating.

When using future information in the actuarial approach, the simulation strategy as described above suffices to make the calculation of approximate distribution functions of the future reserves, capitals and total solvency levels possible in a reasonable amount of time. In the financial approach however, this alone is not sufficient. This can easily be understood by comparing 15 with 13. To calculate the distribution of $D_t^{(F)} | S_t = S_{t'}^{(a,i)}$, we need a set of simulations of future hedging errors and transaction costs. Therefore, we need to evaluate a set of put options starting at s and maturing at all $u \in \{s+1, \ldots, T\}$ for each $s \in \{t, \ldots, T\}$ for each of the simulations in $\Omega_{t'}^{(a,i)}$. Furthermore, for each of these put options, we need to calculate the portion of stocks which is necessary for the hedging strategy. The rest of the calculations are less time-consuming. In the actuarial strategy we only need to calculate $C_s | S_t = S_{t'}^{(a,i)}$ for all $s \in \{t+1, \ldots, T\}$ for all simulations to obtain a distribution function of $D_t^{(A)} | S_t = S_{t'}^{(a,i)}$, which is clearly quicker.

The speed of the calculations in the financial approach can be increased importantly using the following method:

1. Determine a set $\overline{\Omega}$ of 1000 values for the underlying asset as follows:

$$\overline{\Omega} = \{0.004, 0.008, \ldots, 1.996, 2, 2.012, 2.024, \ldots, 4.988, 5, 5.06, 5.012, \ldots, 20\} \tag{39}$$

2. For each $S \in \overline{\Omega}$ and each $t \in \{1, \ldots, T\}$, calculate the Black and Scholes price of one GMDB-contract on the underlying asset $(S_s)_{s \in [t,T]}$ with a guarantee of K for a person aged x years and t months with maturity date T, given that $S_t = S$. We then obtain:

$$\Pi = \left\{ \sum_{s=t}^{T-1} {}_{s-t|}q_{x,t} P(t, s+1) | S_t = S \right\}_{S \in \overline{\Omega} \text{ and } t \in \{1, \ldots, T\}} \tag{40}$$

3. For each of the prices $\pi \in \Pi$, calculate the number of stocks which needs to be bought to hedge the corresponding GMDB-contract. We then obtain:

$$\Xi = \left\{ \sum_{s=t}^{T-1} {}_{s-t|}q_{x,t} \left[N\left(\frac{\ln(S_t/K) + (r + \sigma^2/2)(s+1-t)}{\sigma\sqrt{s+1-t}} \right) - 1 \right] | S_t = S \right\}_{S \in \overline{\Omega} \text{ and } t \in \{1, \ldots, T\}} \tag{41}$$

4. Due to the known structure of $\overline{\Omega}$, for all $j \in \{1, \ldots, N_A\}$ and $t' \in \{1, \ldots, T/12 - 1\}$ very good approximations of the prices of the required GMDB-contracts and their corresponding stock part in the hedge can be quickly extracted from Π and Ξ for all of the N_S simulations in $\Omega_{t'}^{(a,j)}$. The distribution function of $D_t^{(F)} | S_t = S_{t'}^{(a,j)}$ can then be evaluated using the formulae in section 2. When a simulation in $\Omega_{t'}^{(a,j)}$ reaches a value below 0.004 or above 20, we use the values out of Π and Ξ corresponding to 0.004 or 20 respectively.

7 Actuarial Versus Financial Approach

7.1 Distribution of the Discounted Future Costs at Time 0

We first assess the differences between the distribution functions of the discounted future costs at time 0 in the actuarial and the financial approach. In figure 2, we compare the distribution functions and in table 2, we compare some risk measures of $D_0^{(A)}$ and $D_0^{(F)}$ for the set of parameters as specified in table 6. For all the results which follow, use the same set of simulations for the mortality pattern and underlying asset in the actuarial and financial approach to avoid differences due to statistical fluctuations.
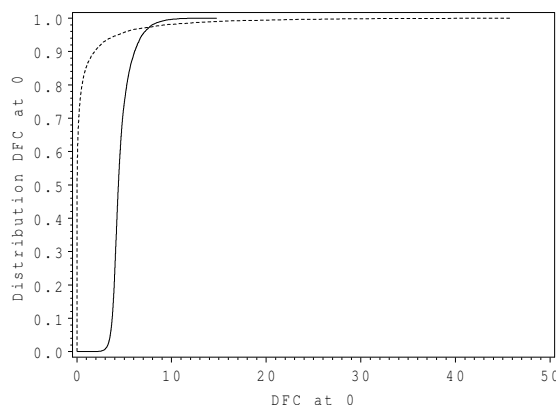
Figure 2: Comparison of $D_0^{(A)}$ (dashed line) and $D_0^{(F)}$ (full line) for $S_0 = 1$

With a probability of 97,3%, actuarial reserving leads to less costs than financial reserving. Given that the guarantee leads to any costs, the probability actuarial reserving is cheaper than financial reserving is equal to 95,2%. In the actuarial approach, the probability of having any costs is equal to 56%. Of course, the capital costs are not yet taken into account here.

Approach	BSP_0	Mean	StDev	Skewness	$\text{TVaR}_{0.95}$	$\text{TVaR}_{0.99}$
$D_0^{(F)}$	4.24	4.68	1.11	1.98	8.07	9.88
$D_0^{(A)}$	-	0.83	2.91	7.07	10.79	23.30

Table 2: Risk measures of D_0

In table 2, we see the average costs in the actuarial approach are substantially lower than in the financial approach. The standard deviation and the skewness are importantly larger in the actuarial approach. The TVaR's in the actuarial approach are importantly larger too. This means more capital will be required in the actuarial approach. As one may expect, the differences between the TVaR's increase importantly with increasing p-level.

The results mentioned above may of course depend on the probability any costs occur. In figure 3 and 4, we therefore compare the distribution functions of $D_0^{(A)}$ and $D_0^{(F)}$ for $S_0 \in \{0.5, 1.5\}$.

When $S_0 = 0.5$, the probability the actuarial approach leads to less costs than the financial approach is equal to 95.0%, when $S_0 = 1.5$, this probability is 99.1%.
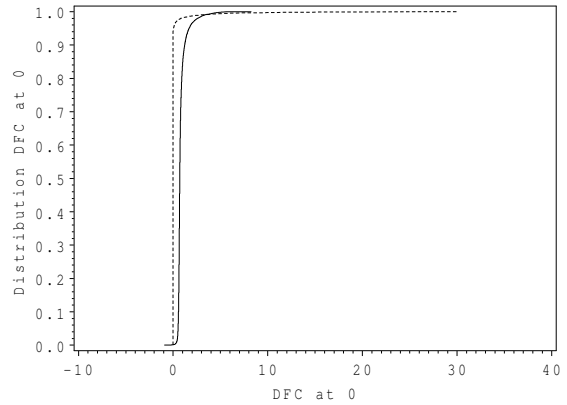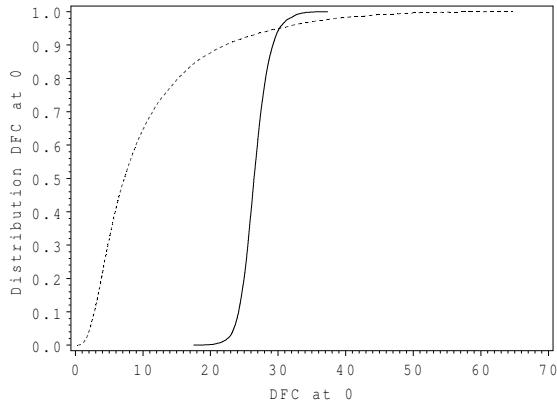
Figure 3: Comparison of $D_0^{(A)}$ (dashed line) and $D_0^{(F)}$ (full line) for $S_0 = 0.5$

Figure 4: Comparison of $D_0^{(A)}$ (dashed line) and $D_0^{(F)}$ (full line) for $S_0 = 1.5$

Approach	BSP_0	Mean	StDev	Skewness	TVaR$_{0.95}$	TVaR$_{0.99}$
$D_0^{(F)}$ for $S_0 = 0.5$	26.14	26.58	2.09	0.41	31.64	33.63
$D_0^{(A)}$ for $S_0 = 0.5$	-	10.33	9.06	2.00	38.70	50.29
$D_0^{(F)}$ for $S_0 = 1.5$	0.68	0.84	0.51	4.56	2.66	4.28
$D_0^{(A)}$ for $S_0 = 1.5$	-	0.11	1.09	17.00	2.22	8.62

Table 3: Risk measures of D_0 for $S_0 \in \{0.5, 1.5\}$

In table 3, we compare some risk measures of the discounted future costs for $S_0 \in \{0.5, 1.5\}$. Similar conclusions as for the case where S_0 is equal to 1 are valid for the mean, the standard deviation and the skewness. For the TVaR's when $S_0 = 0.5$, we again see that more capital will be required when performing actuarial pricing. When $S_0 = 1.5$, the required p-level will determine which strategy is the most capital consuming (when we use the TVaR as a risk measure). For a p-level of 0.95, the financial approach requires more capital. For a p-level of 0.99, the actuarial approach requires more capital.

7.2   Distributions Functions of Future Total Solvency Levels

We compare the distribution functions of the future total solvency levels in figure 5 and 6 when a TVaR at level 99% is used. The vertical line in these figures is the initial total solvency level. To characterize the other curves, the following rule can be used: the higher the line is at the left, the further it is in time.

In figure 5 and 6, we see that in the financial approach the relative amount at which future total solvency levels may need to be increased is importantly smaller than in the actuarial approach. This means the hedging strategy effectively reduces an important portion of the risk. The distribution functions are, for the comparable years, in general less skewed in the financial approach. We also see that, relative to the initial amount of capital, the actuarial strategy tends to move faster to situations with low capital amounts or capital amounts equal to zero. From these observations, we can conclude that the financial approach is less sensitive to future information than the actuarial approach.
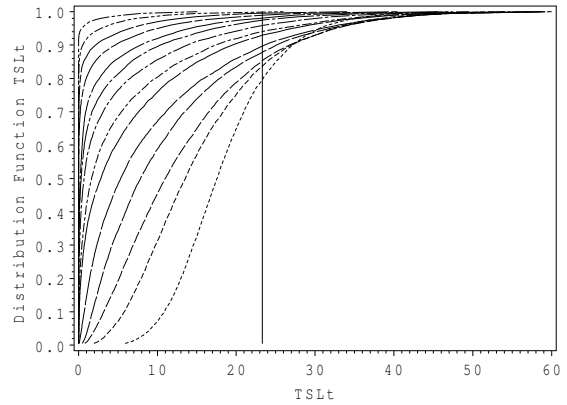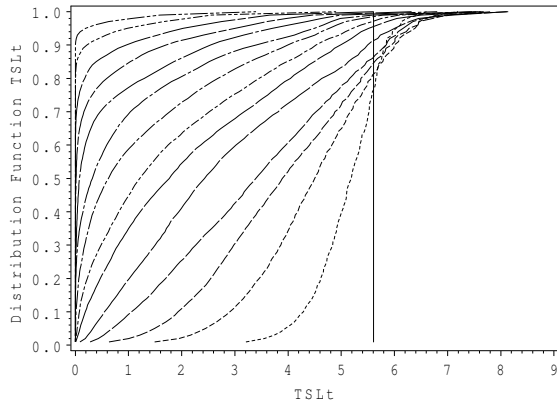
13

Figure 5: Distribution functions of $TSL_t^{(F)}$ Figure 6: Distribution functions of $TSL_t^{(A)}$

7.3   Pricing

From 32, it follows that for pricing, we need the reserve P_0 and the average capitals. Here, we will look at the average total solvency levels but if the average reserves are known, the average capitals can easily be obtained. As a test for the simulation strategy using future information, we also compare the average reserves in the strategy using future information with the reserves in the strategy not using future information.

For the parameters summarized in table 6, we compare the average reserves in the financial and actuarial approach in figure 7 and 8. We use 200 classes for the underlying asset in the actuarial approach and 100 in the financial approach.
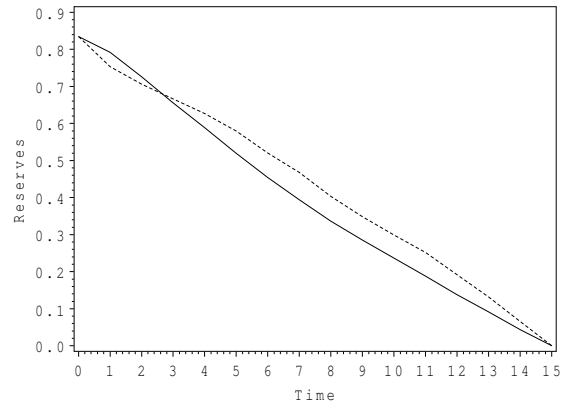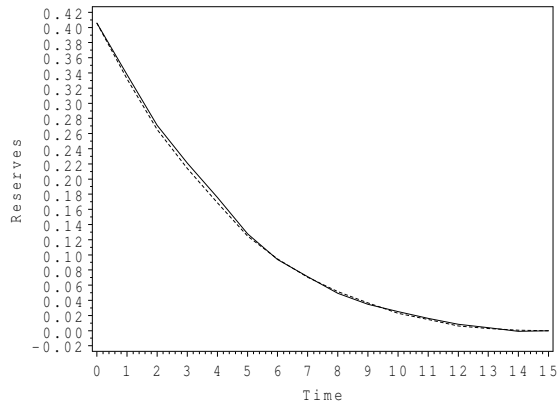

Figure 7: $\mathrm{E}[P_t^{(F,U)}]$ versus $P_t^{(F,N)}$ Figure 8: $\mathrm{E}[P_t^{(A,U)}]$ versus $P_t^{(A,N)}$

We see that, the iterativity property of the expectation is better satisfied in the financial approach than in the actuarial approach. In order to have better results in the actuarial approach, we could increase the number of simulations (N_S) and the number of classes for using future information (N_A).

In figure 9 and 10, we compare the average total solvency levels in the financial and actuarial approach using a TVaR at a level of 99%.
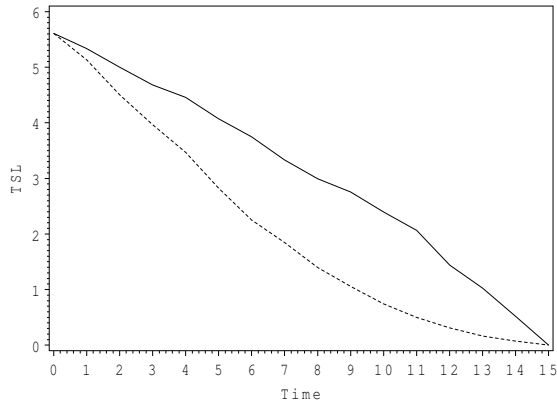
14

Figure 9: $\text{TVaR}_{0.99}[D_t^{(F)}|S_0]$ (full line) vs. $\text{E}[\text{TVaR}_{0.99}[D_t^{(F)}|S_t]]$ (dashed line)
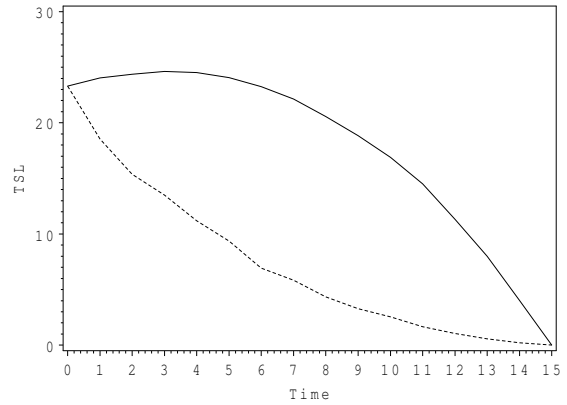

Figure 10: $\text{TVaR}_{0.99}[D_t^{(A)}|S_0]$ (full line) vs. $\text{E}[\text{TVaR}_{0.99}[D_t^{(A)}|S_t]]$ (dashed line)

We see that the relative differences between the average total solvency levels in the strategies where future information is and is not used are importantly larger in the actuarial approach than in the financial approach. This means that on average, the total solvency levels in the financial approach are less sensitive to future information than in the actuarial approach. Secondly, the average capital in the financial approach is substantially smaller than in the actuarial approach. The major part of the premium in the financial approach is determined by the Black and Scholes price of the GMDB. The influence of both these elements can be seen in table 4, where we summarize the technico-financial premia in the different approaches and in the strategies using and not using future information. This also means that the approximation of the price in the strategy using future information with the price in the strategy not using future information performs considerably better for the financial approach.

In figure 11 and 12, we compare the average total solvency levels in the two approaches for at TVaR at level 95%.

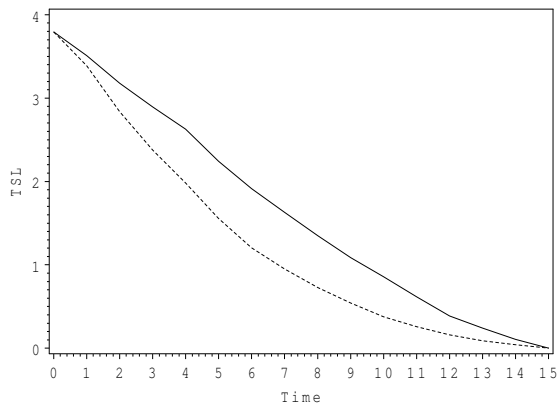Figure 11: $\text{TVaR}_{0.95}[D_t^{(F)}|S_0]$ (full line) vs. $\text{E}[\text{TVaR}_{0.95}[D_t^{(F)}|S_t]]$ (dashed line)
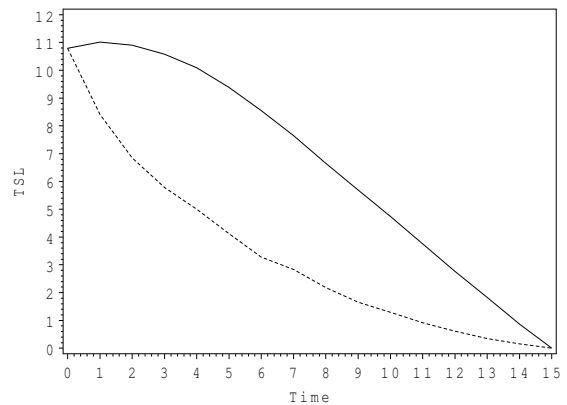

Figure 12: $\text{TVaR}_{0.95}[D_t^{(A)}|S_0]$ (full line) vs. $\text{E}[\text{TVaR}_{0.95}[D_t^{(A)}|S_t]]$ (dashed line)

15

We see in figure 11 and 12 the relative differences between the average total solvency levels in the strategies using and not using future information are considerably smaller than when a TVaR at level 99% is used. In the financial approach, the relative differences are, mainly during the first years, not very large. As we see in table 4, this clearly has an impact on the differences in the prices using and not using future information. By comparing table 2 and table 4, we see that in the financial approach, the most important part of the premium is the cost of the hedging strategy which is independent of the solvency level which is required, whereas in the actuarial approach the capital costs form the major part of the premium.

Technico-financial	$TVaR_{0.99}$		$TVaR_{0.95}$	
Premium	Act	Fin	Act	Fin
(N)	14.74	6.83	5.86	5.85
(U)	7.01	6.42	3.48	5.71

Table 4: Technico-financial premia

Note that for the results in table 4, we assume that the cost of capital is the same both in the actuarial and the financial approach.

When a high security level is used, the financial approach is a little more advantageous than the actuarial approach. If however a lower security level of 95% is sufficient, the actuarial approach is substantially less expensive.

7.4 Influence of Ratio of Guarantee and Initial Investment

In table 5, we compare the technico-financial premia for the actuarial and financial approaches using future information when a TVaR at level 0.95 and 0.99. We do this for $S_0 \in \{0.5, 1, 1.5\}$ and $K = 1$.

Approach	BSP_0	$E[D_0]$	$TVaR_{0.95}[D_0]$	$TVaR_{0.99}[D_0]$	$TFP_{0.95}^{(U)}$	$TFP_{0.99}^{(U)}$
(F) with $S_0 = 0.5$	26.14	26.58	31.64	33.63	28.80	29.89
(A) with $S_0 = 0.5$	-	10.33	38.70	50.29	20.57	25.48
(F) with $S_0 = 1$	4.24	4.68	8.07	9.88	5.71	6.42
(A) with $S_0 = 1$	-	0.83	10.79	23.30	3.48	7.01
(F) with $S_0 = 1.5$	0.68	0.84	2.66	4.28	1.28	1.75
(A) with $S_0 = 1.5$	-	0.11	2.22	8.62	0.73	2.24

Table 5: Influence of ratio of guarantee and initial investment

We see that in a situation where costs are more probable, the actuarial approach performs better under a wider range of conditions for the required total solvency level, when looking at the technico-financial premium. In a situation, where costs are less probable, the financial approach performs better under a wider range of conditions for the required total solvency level. We also see that both in the actuarial and the financial approach, the prices are more sensitive to the level of security when the probability of having costs is low. When costs are more probable, the price is less determined by capital costs since the distribution functions of the discounted future costs are less skewed.

7.5   Discussion

In general, situations where (large) augmentations of the required solvency margin may be necessary should be avoided. For a company writing different types of risks similar to the risk of a GMDB but which are possibly not correlated to it, it may be possible that some of the risk inherent to the GMDB can be diversified with other risks. Due to the potentially large necessary increases of the total solvency levels in the actuarial approach, this seems a necessary condition for being able to perform actuarial reserving. Reinsurers are typically exposed to low-frequency and high-severity risks which may be of a similar nature as the risk of a GMDB which is reserved for actuarially.

In the pricing strategy presented here, we do not take potential diversification benefits into account. We price each contract as if it were the only risk of the company. In reality, diversification benefits should be incorporated in pricing. Due to the way the prices in the actuarial and financial approach are composed, we may expect that, if diversification benefits are indeed present, the actuarial price will be affected more importantly. Furthermore, due to differences in diversification opportunities, prices may vary as well. To allow for reductions of this type, it may be required to quantify potential diversification benefits.

8   Conclusion

We have compared the distribution functions of the costs inherent to the actuarial and the financial approach for reserving for GMDB's. When capitals costs are not taken into account, actuarial reserving is on average and for the set of conditions analyzed always cheaper than financial reserving. Since the volatility and skewness of the costs in the actuarial approach are substantially larger than in the financial approach, more capital needs to be held when performing actuarial reserving.

We propose methods to incorporate future information about the underlying asset both in the actuarial and the financial approach. These methods allow to calculate approximate distribution functions of the future reserves, capitals and total solvency levels. This provides useful information for assessing the risks an insurer is exposed to when following a given approach. The impact of future information has a larger effect when performing actuarial reserving. We also find important differences between the average total solvency levels when using and not using future information in the actuarial approach. In the financial approach, these differences are considerably smaller.

Using the multi-period reserving and capital allocation strategies, we are able to develop a cash-flow model for pricing multi-period risks (on a stand alone basis). As we might have expected, we find that under a wide range of conditions, the actuarial price is mostly determined by capital costs whereas the major part of the financial price is the price of the hedge. In the financial approach, capital costs are more important for lower guarantees. For quite an important range of parameters, the actuarial price is lower than the financial price. When (very) high solvency requirements need to be met, financial reserving becomes more advantageous.

A Used Parameters and Notations: Summary

Simulations		
Number of simulations	N_S	10000
Number of classes for using future information for the underlying asset	N_A	200 for (A), 100 for (F)
Contractual Parameters		
Portfolio composition	$\{N_0, x, C\}$	$\{1000, 50, 1\}$
Initial value underlying asset	S_0	1
Guarantee at death	K	1
Age of retirement	x_R	65
Mortality Parameters		
	α	0.000591
Gompertz-	β_{0-65}	0.00000738
Makeham	γ_{0-65}	0.118
parameters	β_{65-99}	0.000619
	γ_{65-99}	0.0532
Financial Parameters		
Risk free rate	r	0.0035 (= 0.0425 yearly)
Volatility underlying asset	σ	0.0432 (= 0.150 yearly)
Tax rate	γ	0.4
Average return on invested capital	δ	0.00458 (= 0.055 yearly)
Cost of capital	COC	0.0083 (= 0.10 yearly)
Transaction costs	τ	0.2%
Parameters RSLN model (monthly)		
Average log-return in regime 1	μ_1	0.0135
Average log-return in regime 2	μ_2	-0.0109
Volatility in regime 1	σ_1	0.0344
Volatility in regime 2	σ_2	0.0645
Probability to move from regime 1 to 1	p_{11}	0.0483
Probability to move from regime 2 to 1	p_{21}	0.1985

Table 6: Used parameters and notations: summary

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