An Analysis of Default Correlations and Multiple Defaults

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Evaluating default correlations or the probabilities of default by more than one firm is an important task in credit analysis, derivatives pricing, and risk management. However, default correlations cannot be measured directly, multiple-default modeling is technically difficult, and most existing credit models cannot be applied to analyze multiple defaults. This article develops a first-passage-time model, providing an analytical formula for calculating default correlations that is easily implemented and conveniently used for a variety of financial applications. The model also provides a theoretical justification for several empirical regularities in the credit risk literature.

The fortunes of individual companies are linked together via industry-specific and/or general economic conditions. As a result, the default events of companies are often correlated. This correlation is very important in credit analysis. However, default correlations have not been satisfactorily modeled.

Currently, default correlations are estimated in two ways. One method uses historical data [e.g., Lucas (1995)]. The problems with this approach are well known. First, because defaults for bonds are rare—especially highly rated ones—there are not enough time-series data available to accurately estimate default correlations. Second, the historical approach as it has been implemented does not use firm-specific information and, therefore, cannot recognize that the default correlation between Exxon and Chevron could be very different from that between Exxon and Wal-Mart. Third, default correlations are time-varying, so past history may not reflect the current reality.

The second approach to estimating default correlations utilizes a particular theoretical structure of the default process. The most popular structure in practice is based on Merton's (1974) framework. In Merton's (1974) model, default can only occur at the maturity of a bond. The assumption that a bond may only default at its maturity is restrictive and unrealistic. For example, coupon bonds that may default at more than one date require a structure that accounts for the sample path of the firm's assets.[1] For this reason, the first-

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[1] According to Jones et al. (1984), the default risk implied by the Merton model is so low that its pricing ability for investment-grade bonds is no better than a naive model that does not consider default risk at all.

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passage-time model of credit evaluation is now widely used in the academic literature [Black and Cox (1976), Longstaff and Schwartsz (1995), Leland and Toft (1996), Leland (1998), Zhou (2001a), and many others].

This article develops a first-passage-time model, providing an analytical formula for calculating default correlations and joint default probability, which is conveniently used for a variety of financial applications. The article focuses on the joint default probability of multiple firms, while the existing first-passage-time models study the default probability of a single firm only. The contribution of this article is twofold. First, it presents a readily implementable approach to default correlations that makes use of various firm-specific information. Second, it provides a theoretical justification for various empirical regularities in the literature.

The article is organized as follows: Section 1 provides an overview of default correlation. Section 2 presents the model and investigates its properties. Section 3 discusses the empirical applications of the model. Section 4 concludes.

1. Default Correlation: Definition and Applications

1.1 Default correlation overview

Consider two random variables $D_1(t)$ and $D_2(t)$ that describe the default status of two firms, firm 1 and firm 2, over a given horizon t:

$$D_i(t) \equiv \begin{cases} 1 & \text{if firm } i \text{ defaults by } t \\ 0 & \text{otherwise} \end{cases}.$$

Assuming the independence of default events, the joint default probability of the two firms is

$$P(D_1(t) = 1 \text{ and } D_2(t) = 1) = P(D_1(t) = 1) \cdot P(D_2(t) = 1).$$

When examining the joint probability, however, it is reasonable to assume that when one entity defaults, the other entity may have a higher likelihood of defaulting. Perhaps both firms are experiencing pressures from the general economy, their industry, or their region. Thus, the two entities may have a positive default correlation.

We define the default correlation $\mathrm{Corr}(D_1(t), D_2(t))$ as

$$\mathrm{Corr}[D_1(t), D_2(t)] \equiv \frac{\mathrm{E}[D_1(t) \cdot D_2(t)] - \mathrm{E}[D_1(t)] \cdot \mathrm{E}[D_2(t)]}{\sqrt{\mathrm{Var}[D_1(t)] \cdot \mathrm{Var}[D_2(t)]}}. \quad (1)$$

Because $D_1(t)$ and $D_2(t)$ are Bernoulli binomial random variables, we have

$$\mathrm{E}[D_i(t)] = P(D_i(t) = 1),$$
$$\mathrm{Var}[D_i(t)] = P(D_i(t) = 1) \cdot [1 - P(D_i(t) = 1)].$$

Default correlation analysis plays a critical role in determining joint default probability—the probability of multiple defaults. From Equation (1), we have

$$\begin{aligned} P(D_1(t) = 1 \text{ and } D_2(t) = 1) &= \mathrm{E}[D_1(t) \cdot D_2(t)] \\ &= \mathrm{E}[D_1(t)] \cdot \mathrm{E}[D_2(t)] + \mathrm{Corr}[D_1(t), \\ &\quad D_2(t)].\sqrt{\mathrm{Var}[D_1(t)] \cdot \mathrm{Var}[D_2(t)]}. \quad (2) \end{aligned}$$

If $P(D_1(t) = 1) = \mathrm{E}[D_1(t)] = 5\%$ (i.e., firm 1 has a 5% probability of default), and $P(D_2(t) = 1) = E[D_2(t)] = 1\%$, the joint default probability of both firms, assuming their independence, is $5\% \times 1\% = 0.05\%$. If the default correlation equals 0.2, in this example, the joint probability of default would equal 0.48%. The latter is almost 10 times as large as the former. Thus, default correlation can have a large impact on the probability of joint default events.

The default correlation is also useful in evaluating the probability that either firm defaults:

$$\begin{aligned} &P(D_1(t) = 1 \text{ or } D_2(t) = 1) \\ &\quad = P(D_1(t) = 1) + P(D_2(t) = 1) - P(D_1(t) = 1 \text{ and } D_2(t) = 1) \\ &\quad = \mathrm{E}[D_1(t)] + \mathrm{E}[D_2(t)] - \mathrm{E}[D_1(t) \cdot D_2(t)]. \quad (3) \end{aligned}$$

Because the example shows that $\mathrm{E}[D_1(t) \cdot D_2(t)]$ can be very different from $\mathrm{E}[D_1(t)] \cdot \mathrm{E}[D_2(t)]$, it implies that estimates of the probability in Equation (3) can also be very sensitive to the default correlation.

1.2 Applications of default correlation

Default correlation analysis has many applications in asset pricing and risk management. Due to the rapid growth in the credit derivatives market and the increasing importance of measuring and controlling default risks in portfolios of loans, derivatives, and other securities, the importance of default correlation analysis has been widely recognized by the financial industry in recent years [J. P. Morgan (1995)]. This section briefly introduces some applications of default correlation analysis.

1.2.1 Asset pricing. A straightforward application of default correlation is the evaluation of a letter of credit (LOC)–backed debt. This contract transfers the risk of default from the debtholder to the LOC bank. Current rating agency practice is to rate LOC-backed debt with the same credit quality as the LOC bank. However, this overlooks the protection the debt holder receives directly from the debt obligor. Essentially, two failures have to occur before the debtholder experiences a financial loss: Both the LOC bank and the debt obligor have to default. Assume that the LOC bank has a 0.5% probability of default, and the debt obligor has a 2.0% probability of default. Using the same default probability as the LOC bank, current financial practice would assess a 0.5% default probability for the LOC-backed debt. However, if the default correlation between the LOC bank and the debt obligor is 0.20, the true default probability of the LOC-backed debt is about 0.2%. If the default correlation is 0.05, the default probability will be only 0.06%.

The default correlation is also important in pricing credit derivatives. The market for credit derivatives is relatively new but has grown rapidly in recent years [Rai et al. (1997)]. As an example, let us consider a simple credit default swap. This is an agreement between two counterparties in which a periodic fixed payment or up-front fee is exchanged for the promise of some specified payment(s) to be made at the maturity of the swap only if a prespecified reference party has defaulted by that time. In this example, the default protection buyer will experience a loss before the maturity of the default swap only if both the protection seller and the reference party have defaulted by that time. The default correlation must be known to determine the joint probability of default of the protection seller and the reference party.

1.2.2 Risk management. The problem of portfolio analysis with credit risk has been examined in the recent literature. A portfolio manager is concerned with not only the default of any single party but also the probability of multiple defaults in the portfolio. An effective measurement of credit risk in a portfolio involves three critical quantities: the probability of default for each individual position over various investment horizons, the *joint probability of default* between every pair of counterparties over various investment horizons, and the magnitude of financial loss in the event of each possible default. The most crucial and the most difficult part of credit aggregation analysis, as noted in the literature, is estimating the default correlations.

Suppose that a hypothetical loan portfolio consists of two loans; the annual default probability for each loan is 1%, and the annual default correlation is 10%. The probability that both loans default in a year is approximately 0.11%. Assume that the credit standings of the two loans subsequently deteriorate so that the annual default probability for each loan rises to 2%. If the default correlation remains constant, the probability that both loans default in a year becomes 0.24%, about twice the original joint default probability. However, as we shall see, a decline in the credit quality will typically lead to a rise in the default correlation. If the default correlation increases from 0.10 to 0.25, the joint probability of default becomes 0.53%, 5 times as large as the original probability. Thus, accurate measures of default correlations are crucial in modelling portfolios with credit risk.

2. The First-Passage-Time Model of Default Correlation

This section provides a basic theoretical model for default correlations based on first-passage-times. A firm defaults when its value first hits a default boundary. Determining the default correlation between two firms amounts to calculating the probability of a two-dimensional stochastic process passing a boundary.

Consider the default correlation between two arbitrary firms, firm 1 and firm 2. Our main assumptions are as follows.

Assumption 1. *Let V_1 and V_2 denote the total asset values of firm 1 and firm 2. The dynamics of V_1 and V_2 are given by the following vector stochastic process*

$$\begin{bmatrix} d\ln(V_1) \\ d\ln(V_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \Omega \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{4}$$

where μ_1 and μ_2 are constant drift terms, z_1 and z_2 are two independent standard Brownian motions, and Ω is a constant 2×2 matrix such that

$$\Omega \cdot \Omega' = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

The coefficient, $\rho \equiv Corr(d\ln(V_1), d\ln(V_2))$, reflects the correlation between the movements in the asset values of the two firms. This correlation coefficient plays a critical role in determining the default correlation between the firms.

Assumption 2. *The default of a firm is triggered by a decline in its asset value. For each firm i, there exists a time-dependent value $C_i(t)$ such that the firm continues to operate and meets its contractual obligations as long as $V_i(t) > C_i(t)$. However, if $V_i(t)$ falls to the threshold level $C_i(t)$, the firm defaults on all of its obligations immediately, and some form of corporate restructuring takes place. Following Black and Cox (1976), we assume that the time dependence of $C_i(t)$ takes an exponential form: $C_i(t) = e^{\lambda_i t} K_i$.*

There are many interpretations of the default boundary $C_i(t)$. Black and Cox (1976) interpret $C_i(t)$ as the minimum firm value required by the safety covenant of a debt contract. If the value of the firm falls to $C_i(t)$, its bondholders are entitled to force the firm into bankruptcy and obtain ownership of the firm's assets. According to Black and Cox, $C_i(t)$ takes an exponential form in t because the expected debt value usually takes this form. In many practical applications, $C_i(t)$ is set to a weighted average of the firm's long-term and short-term liabilities, and λ_i can be interpreted as the growth rate of firm i's liabilities.

2.1 Solution: a simplified version

We will first consider a simple case in which $\lambda_i = \mu_i$. Economically, $\lambda_i = \mu_i$ means that the debt value of a firm and the firm value have the same expected growth rate. Therefore, the assumption implies a constant leverage ratio in the steady state. A solution to the general model in which μ_i and λ_i are arbitrary constants will be provided later.

Denote $\tau_i \equiv \min_{t\geq 0}\{t|e^{-\lambda_i t}V_i(t) \leq K_i\}$ as the first time that firm i's value reaches its default threshold level. Then $P(D_i(t)=1) = P(\tau_i \leq t)$. Using the result of Harrison (1990), we have

$$P(D_i(t)=1) = 2\cdot N\left(-\frac{\ln(V_{i,0}/K_i)}{\sigma_i\sqrt{t}}\right) \tag{5}$$

$$= 2\cdot N\left(-\frac{Z_i}{\sqrt{t}}\right). \tag{6}$$

where

$$Z_i \equiv \frac{\ln(V_{i,0}/K_i)}{\sigma_i}$$

is the standardized distance of firm i to its default point and $N(.)$ denotes the cumulative probability distribution function for a standard normal variable. Based on Equations (1), (3), and (6), to determine the default correlation, the only remaining unknown we need to find is

$$P(D_1(t)=1 \text{ or } D_2(t)=1),$$

that is, the probability that at least one default has occurred by time t. We can prove the following result.

Main Result 1. *Assume that $\lambda_i = \mu_i$. We have*

$$\begin{aligned} &P(D_1(t)=1 \text{ or } D_2(t)=1)\\ &= 1 - \frac{2r_0}{\sqrt{2\pi t}}\cdot e^{-\frac{r_0^2}{4t}}\cdot \sum_{n=1,3,\cdots}\frac{1}{n}\cdot\sin\left(\frac{n\pi\theta_0}{\alpha}\right)\\ &\cdot\left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right)\right] \end{aligned} \tag{7}$$

where $I_\nu(z)$ is the modified Bessel function I with order ν and

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0\\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise.} \end{cases}$$

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{if } (.) > 0\\ \pi + \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{otherwise,} \end{cases}$$

$$r_0 = Z_2/\sin(\theta_0).$$

Proof. A thorough technical treatment of this result may be found in Rebholz (1994). A brief proof is provided in the Appendix to this paper.

2.2 Solution: a general version

This subsection briefly discusses a more general model that allows $\lambda_i \neq \mu_i$. Based on the results of Harrison (1990), when λ_i is not equal to μ_i, the default rate of an individual firm is

$$P(D_i(t) = 1) = N\left(-\frac{Z_i}{\sqrt{t}} - \frac{\mu_i - \lambda_i}{\sigma_i}\sqrt{t}\right) + e^{\frac{2(\lambda_i - \mu_i)Z_i}{\sigma_i}} N\left(-\frac{Z_i}{\sqrt{t}} + \frac{\mu_i - \lambda_i}{\sigma_i}\sqrt{t}\right). \tag{8}$$

Proposition 1. *Let λ_i and μ_i be any given constants. We can prove*

$$P(D_1(t) = 1 \text{ or } D_2(t) = 1) = 1 - \frac{2}{\alpha t} e^{a_1 x_1 + a_2 x_2 + a_t t} \cdot \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \cdot e^{-\frac{r_0^2}{2t}} \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta$$

where θ_0, r_0, and α are defined as in Main Result 1,

$$g_n(\theta) = \int_0^{\infty} r \cdot e^{-\frac{r^2}{2t}} \cdot e^{d_1 r \sin(\theta - \alpha) - d_2 r \cos(\theta - \alpha)} \cdot I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right) dr,$$

$$a_1 = \frac{(\lambda_1 - \mu_1)\sigma_2 - (\lambda_2 - \mu_2)\rho\sigma_1}{(1 - \rho^2)\sigma_1^2\sigma_2},$$

$$a_2 = \frac{(\lambda_2 - \mu_2)\sigma_1 - (\lambda_1 - \mu_1)\rho\sigma_2}{(1 - \rho^2)\sigma_2^2\sigma_1},$$

$$a_t = \frac{a_1^2\sigma_1^2}{2} + \rho a_1 a_2 \sigma_1 \sigma_2 + \frac{a_2^2\sigma_2^2}{2} - a_1(\lambda_1 - \mu_1) - a_2(\lambda_2 - \mu_2),$$

$$d_1 = a_1\sigma_1 + \rho a_2 \sigma_2,$$

$$d_2 = a_2\sigma_2\sqrt{1 - \rho^2}.$$

Proof. See Rebholz (1994).

As the result depends on the double integral of Bessel functions, its implementation becomes very computationally intensive. Table 1 shows three numerical examples of the impact of the drift terms λ_i and μ_i on the default correlations. The table suggests that the difference between μ and λ has little effect on default correlations with one-year or two-year horizons. The drift terms may have some impact on default correlations over long horizons (5 to

Table 1
The impact of drift terms in firm value processes and default boundaries on default correlations (%)

Horizon (years)	1	2	3	4	5	10
$\mu = \lambda$	0.04	1.2	3.7	6.5	9.2	17.1
$\mu = .05, \lambda = .03$	0.04	1.1	3.5	6.2	8.9	16.5
$\mu = .05, \lambda = .00$	0.04	1.0	3.3	6.0	8.5	15.7

Parameters in this table are: $V_1(0)/K_1(0) = V_2(0)/K_2(0) = 5$, $\sigma_1 = \sigma_2 = .30$, and $\rho = .40$.

10 years), but the impact is still relatively small. This result builds confidence in the assumption used in Main Result 1. The subsequent analyses are based on the simplified model with $\mu = \lambda$.

2.3 Implications of the model

2.3.1 Theoretical implications. Based on Main Result 1, Figure 1 plots the relation between the default correlations and the investment horizon t for different values of the asset-level correlation ρ. It illustrates the following results:

1. *The default correlation and the asset level correlation ρ have the same sign. The higher ρ is, ceteris paribus, the higher the default correlation.* This result is intuitive. For instance, if the asset level correlation ρ is positive, when one firm defaults, it is likely that the value of the other firm has also declined and moved closer to its default boundary. The result explains why firms in the same industry (region) often have higher default correlations than the firms in different industries (regions) [Lucas (1995)].
2. *Default correlations are generally very small over short horizons. They first increase and then slowly decrease with time.* Over a short horizon, default correlations are low because quick defaults are rare and nearly idiosyncratic. Default correlations eventually decrease with time because over a sufficiently long time horizon, the default of a firm is virtually inevitable in the model, and the nondefault events become rare and idiosyncratic. This result is consistent with an important phenomenon of historical default correlations reported in Lucas (1995). Lucas conjectures that the phenomenon is caused by business cycle fluctuations. Our result suggests that business cycle fluctuations are not necessary to explain this phenomenon.

Figure 2 illustrates the relation between default correlations and time, for various levels of the credit quality of the firms, proxied by the initial value of V/K. This figure illustrates some interesting results.

1. *High credit quality implies a low default correlation over typical horizons.* For the higher credit quality firms, the conditional default

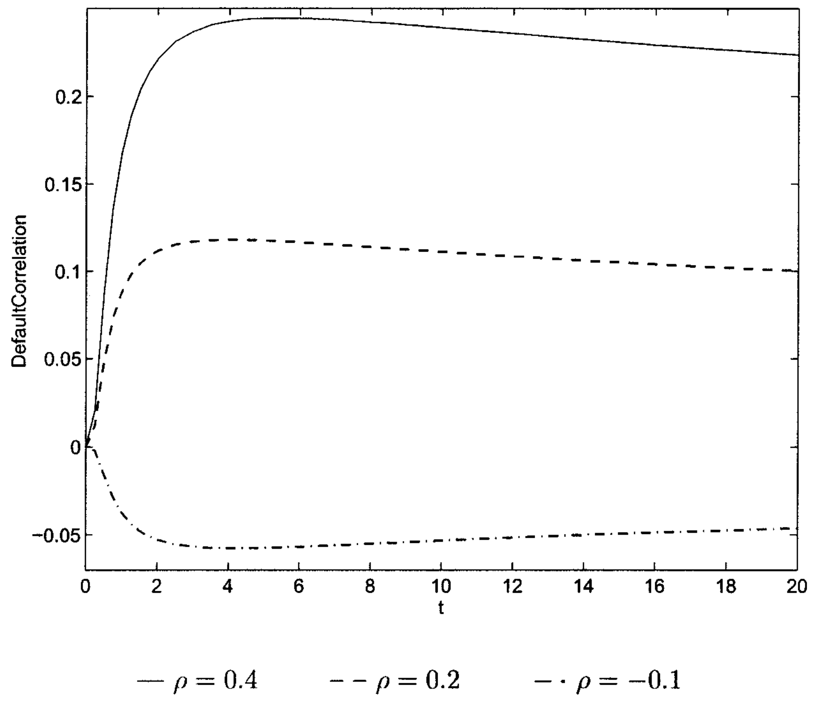
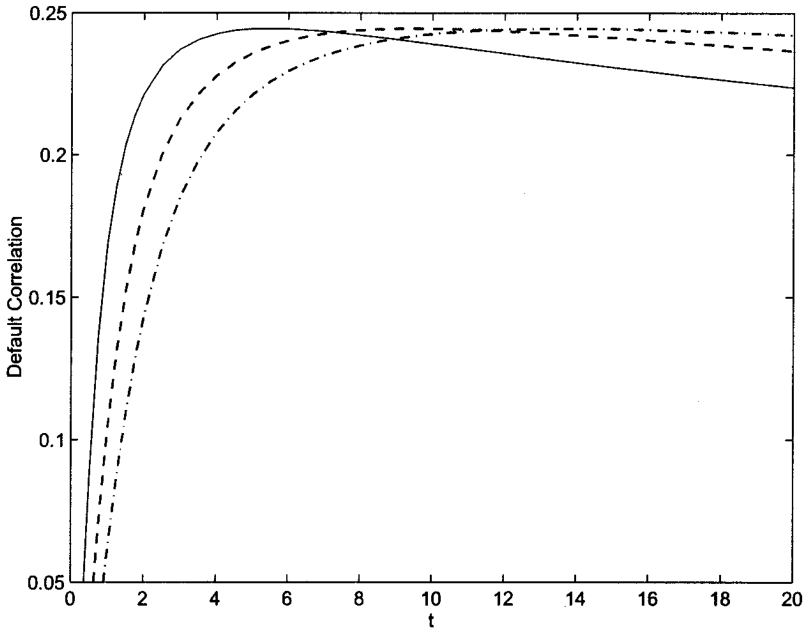


Figure 1
The relation between default correlation and time for given asset return correlations
Here ρ represents the asset level correlation between firms and t is the length of time horizon. The parameter values used here are $V_1/K_1 = V_2/K_2 = 1.8$ and $\sigma_1 = \sigma_2 = 0.4$.

probability $P(D_2(t) = 1|D_1(t) = 1)$ is small. Although the default of firm 1 signals that the value of firm 2, V_2, may have declined, because the original ratio V_2/K_2 is high, the probability that V_2 falls below K_2 is still very small. This result is consistent with the well-known empirical feature regarding the relation between default correlation and credit ratings.

2. *The time of peak default correlation depends on the credit quality of the underlying firms.* The higher quality firms take a longer time to peak. Lucas (1995) finds this result puzzling. Our model clarifies the source of this finding. The intuition here is similar to that in (1). The short-term defaults of the higher credit quality firms are idiosyncratic events and the joint defaults of the higher quality firms are rare. It takes a long time for the default correlation of higher quality firms to reach a high level.
3. *Because the credit quality of firms is time-varying, the default correlation is dynamic.* A rise in the credit quality leads to a substantial drop in the default correlation, and a decline in the credit quality leads



— $\frac{V_1}{K_1} = \frac{V_2}{K_2} = 2.0$ - - $\frac{V_1}{K_1} = \frac{V_2}{K_2} = 2.5$ – · $\frac{V_1}{K_1} = \frac{V_2}{K_2} = 3.0$

Figure 2
The relation between default correlation and time for various credit qualities
Here ρ represents the asset level correlation between firms and t is the length of time horizon. The parameter values used here are $\rho = 0.4$ and $\sigma_1 = \sigma_2 = 0.4$.

to a rise in the default correlation. This dynamic behavior arises even when the underlying assets and liabilities of the firm have constant expected returns and risks.

2.3.2 Risk management implications. The above results have many useful implications for credit analysis and risk management. Some examples are listed here.

1. Because the default correlation over short horizons is small, portfolio diversification should substantially reduce default risk over short horizons.
2. For long-term investments (e.g., 5 to 10 years), the default correlation can be quite a significant factor if the underlying firm values are highly correlated. In this case, concentration in one industry or one region, where defaults are highly correlated, could be very risky. Diversification across different regions or different industries may be desirable.

3. The dynamic nature of default correlations requires active management of the portfolio. This is true even if the expected returns and risks of the underlying assets and liabilities are constant over time.
4. The dynamic nature of default correlations has implications for capital requirements. Because the change in individual default risks may substantially affect the credit risk of a portfolio, the capital requirements must be adjusted accordingly. For example, if the default probability of each loan doubles, the probability of multiple defaults in a portfolio may be significantly more than doubled.

3. Applications of the Model

To apply the model in practice, we must estimate the following parameters: V_i, σ_i, K_i, ρ. We offer two approaches to estimate these parameters. One approach uses firm-specific information, such as the firm's stock returns, book value of liabilities, and so on. The other approach employs the firm's credit rating information and the default rates of similar firms.

3.1 An option approach to estimating parameters

3.1.1 Estimating V_i, σ_i, and ρ. Typically, the total value of a firm's underlying assets is not observable because the market value of the firm's liabilities is not known. In practice, this problem can be circumvented by an option theoretic model of the firm, which treats the firm's equity as a call option on the firm's underlying assets [Black and Scholes (1973)]. Denote S_i as the equity value of the firm, and assume that the debt structure (i.e., the principal or face value, the coupon arrangement, and the maturity of all debts) of a firm is observable; we have

$$\ln(S_i) = G(\ln(V_i), \sigma_i; \text{other values}), \tag{9}$$

where other values include the book value and the maturity of liabilities as well as the interest rate. These "other" values are generally observable.

Ito's lemma implies that the standard deviation of $d\ln(S_i)/dt$ satisfies

$$\begin{aligned} \sigma_{s,i} &= \frac{\partial G}{\partial \ln(V_i)} \cdot \sigma_i \\ &= H(V_i, \sigma_i; \text{other values}), \end{aligned} \tag{10}$$

Assume that we can observe stock price S_i and its volatility $\sigma_{s,i}$. Solving the joint equation system (9) and (10), we obtain V_i and σ_i. Using Ito's lemma, we also have $\rho = \mathrm{Corr}_t[d\ln(S_1), d\ln(S_2)]$.

Generally, firm value correlations estimated over finite horizons do not satisfy the above equation. However, the equation provides a good approximation for highly rated firms. This is because for a highly rated firm, the firm value is close to the equity value (leverage ratio is low). The dynamics of the firm value are therefore similar to the dynamics of the stock price. Whether the equation is also a good approximation for highly leveraged (lowly rated) firms is less certain.

3.1.2 Estimating K_i. The default point K_i captures or measures the liability structure of the firm. In practice, K_i is often defined as a firm's short-term liability plus 50% of the firm's long-term liability.

3.2 Statistical approach to estimating parameters

One may also use a statistical approach to estimate Z from Equation (6). If historical data provide cumulative default rates $\widetilde{A}(t)$ for similar firms at various investment horizons t, the parameter Z can be chosen to fit the theoretical default probabilities $P(Z, t)$ to the historical default rates $\widetilde{A}(t)$. One way to accomplish this is via a least-squares approach:

$$Z = \arg\min_{Z} \sum_{t} \left(\frac{P(Z,t)}{t} - \frac{\widetilde{A}(t)}{t} \right)^2.$$

In the above expression, cumulative default rates $P(Z, t)$ and $\widetilde{A}(t)$ are divided by time the horizon t so that they are transformed to average default rates per unit of time.

Compared with the option approach, the statistical approach is easier to use. However, because this approach is based solely on credit ratings, it does not effectively use all firm-specific information. In addition, because the default probability corresponding to a rating category is time-varying [Zhou (2001b)], historical default rates for firms in a given rating category may not reflect the true default probability of that rating category at any particular time.

3.3 Empirical analysis

To empirically examine the main result presented in Section 1, one may calculate default correlations for any two firms by using firm-specific information (Z, ρ, etc.) and then compare the calculated default correlations with the observed default correlations. Unfortunately, to date, there are no readily available statistics on observed firm-specific default correlations. For this reason, we can only use pooled data to see if the model can match some general empirical characteristics of default correlations.

The default data set used here is obtained from Moody's default studies as reported in Table 2. Using this data set, default correlations for various rating categories are estimated and are compared with the empirical results of Lucas (1995).

Table 3 reports z-values derived from default data. As expected, a high credit rating generally implies a large z. The only exception is for Aaa- and

Table 2
Historical cumulative default rates (%), 1970–93

Year	Aaa	Aa	A	Baa	Ba	B
1	0.00	0.02	0.01	0.16	1.79	8.31
2	0.00	0.04	0.09	0.51	4.38	14.85
3	0.00	0.08	0.28	0.91	6.92	20.38
4	0.04	0.20	0.46	1.46	9.41	24.78
5	0.12	0.32	0.62	1.97	11.85	28.38
6	0.22	0.43	0.83	2.46	13.78	31.88
7	0.33	0.52	1.06	3.09	15.33	34.32
8	0.45	0.64	1.31	3.75	16.75	36.71
9	0.58	0.76	1.61	4.39	18.14	38.38
10	0.73	0.91	1.96	4.96	19.48	39.96
11	0.90	1.09	2.30	5.56	20.84	41.08
12	1.09	1.29	2.65	6.19	22.22	41.74
13	1.30	1.51	2.99	6.77	23.54	42.45
14	1.55	1.76	3.29	7.44	24.52	43.04
15	1.84	1.76	3.62	8.16	25.46	43.70
16	2.18	1.76	3.95	8.91	26.43	44.43
17	2.38	1.89	4.26	9.69	27.29	45.27
18	2.63	2.05	4.58	10.45	28.06	45.58
19	2.63	2.24	4.96	11.07	28.88	45.58
20	2.63	2.48	5.23	11.70	29.76	45.58

Source: Fons (1994).

Table 3
Z-values implied by historical default rates

Aaa	Aa	A	Baa	Ba	B
9.28	9.38	8.06	6.46	3.73	2.10

Aa-rating categories. Table 3 shows that z is 9.28 for Aaa-rated firms and is 9.38 for Aa-rated firms. This abnormal finding is mainly due to statistical errors in the default rate data. As shown in Table 2, the statistical default rates for Aaa-rated firms are constantly higher than those for Aa-rated firms after 15 years. Because of this anomaly, we combine the two rating categories in the following default correlation analysis. We use Aa to represent this combined rating category and use 9.30 as its z-value.

The default correlations calculated from z-scores in Table 3 are reported in Tables 4 through 7. For ease of comparison, some results from Lucas (1995) are also reproduced. Generally speaking, the model matches the results of Lucas (1995) and displays the same patterns in the default correlations. The default correlations for high-rated firms are virtually zero at the short to middle investment horizons, but default correlations are rather high for low-rated firms, even at short time horizons.

The model's correlations over long horizons for high-rated firms are higher than those estimated by Lucas. There are several potential explanations for these differences. One is estimation errors. Rai et al. (1997) point out that the estimates of default correlations from a small sample of historical default

Table 4
One year default correlations (%)

	Model results					Lucas (1995) estimates				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.00					0.0				
A	0.00	0.00				0.0	0.0			
Baa	0.00	0.00	0.00			0.0	0.0	0.0		
Ba	0.00	0.00	0.01	1.32		0.0	0.0	0.0	2.0	
B	0.00	0.00	0.00	2.47	12.46	1.0	0.0	1.0	4.0	7.0

Asset level correlation $\rho = 0.4$.

Table 5
Two-year default correlations (%)

	Model results					Lucas (1995) estimates				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.00					0.0				
A	0.00	0.02				0.0	0.0			
Baa	0.01	0.05	0.25			0.0	0.0	0.0		
Ba	0.00	0.05	0.63	6.96		1.0	1.0	1.0	6.0	
B	0.00	0.02	0.41	9.24	19.61	1.0	1.0	2.0	10.0	16.0

Asset level correlation $\rho = 0.4$.

rates are generally downwardly biased. As noted by Lucas, the 15 overlapping time periods used in his study are possibly too short to represent the values of the true correlations. Another explanation is that the Z and/or ρ used in the calibrations contain errors and that these errors have a larger effect on default correlations for longer horizons. For instance, given $\rho = 0.4$, at a one-year horizon, the default correlation between two firms with $z = 8$ is 0%, and the default correlation between two firms with $z = 9.4$ is also 0%. At a five-year horizon, however, the default correlation between two firms with $z = 8$ is 1.65%, and the default correlation between two firms with $z = 9.4$ is only 0.6%. We can similarly show that ρ has a larger impact on default correlations over longer horizons. Of course, it is also possible that the model itself is misspecified so that it cannot precisely estimate default correlations for certain kinds of firms over long horizons.

Table 6
Five-year default correlations (%)

	Model results					Lucas (1995) estimates				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.59					0.0				
A	0.92	1.65				1.0	1.0			
Baa	1.24	2.60	5.01			1.0	1.0	0.0		
Ba	1.05	2.74	7.20	17.56		3.0	4.0	3.0	15.0	
B	0.65	1.88	5.67	18.43	24.01	4.0	6.0	7.0	25.0	29.0

Asset level correlation $\rho = 0.4$.

Table 7
Ten-year default correlations (%)

	Model results					Lucas (1995) estimates				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	4.66					1.0				
A	5.84	7.75				2.0	2.0			
Baa	6.76	9.63	13.12			1.0	1.0	0.0		
Ba	5.97	9.48	14.98	22.51		3.0	4.0	2.0	8.0	
B	4.32	7.21	12.28	21.80	24.37	8.0	9.0	6.0	17.0	38.0

Asset Level Correlation $\rho = 0.4$.

The model's default correlations between B and Ba are typically higher than the correlations within the category Ba.[2] This result is consistent with the estimates of Lucas (1995). We should understand that two companies with the same credit rating are not the same company and do not have a higher asset level correlation ρ. Also, default correlation tends to increase with the default probabilities of individual companies. The default correlation between two companies in the same rating category can be lower than the correlation between one of these companies and a lower-grade company because the lower-grade company has a larger default probability. If the difference in credit quality between two companies is too large, the default event of the lower quality company will provide little information on a higher quality company's default likelihood. As a result, the default correlation between two high-quality companies may be larger than the correlation between a high-quality company and a company with a much lower credit quality.

3.4 A comparison with the Merton-type model

Because of its simplicity, Merton's default model has been used widely by practitioners in credit risk analysis, yet Merton assumes that a firm has only one bond issue and can only default at the maturity of the bond. This assumption makes it hard to determine the default probability of a bond over a time horizon shorter than its remaining maturity. To overcome this limitation, practitioners simply assume that a bond can default at the end of whatever time horizon that they consider, but not at any other time. We can obtain this standard model as a special case for our analysis, if Assumption 2 in Section 2 is modified to the following: *At the end of the specified time horizon t, firm i defaults if its value $V_i(t)$ is less than the default boundary $C_i(t)$.*

For ease of discussion, we still assume that $\lambda_i = \mu_i$. It is straightforward to verify that under the new assumption:

$$P[D_i(t) = 1] = P[V_i(t) \leq C_i(t)] = N\left(-\frac{Z_i}{\sqrt{t}}\right) \tag{11}$$

[2] Similarly, we also find that the model's correlations within Baa are higher than those between Baa and Baa, the correlations between Ba and A are higher than those within the category A, and so on.

Table 8
Default correlations (%) implied by different models

		Time horizon (years)					
(Z_1, Z_2)	Model	1	2	3	4	5	10
(8,8)	Merton	0.00	0.01	0.17	0.60	1.30	6.10
(8,8)	Current	0.00	0.02	0.23	0.80	1.72	7.93
(3,3)	Merton	3.25	9.61	13.6	16.2	17.9	21.7
(3,3)	Current	4.29	12.2	16.8	19.5	21.1	24.0

Asset level correlation $\rho = 0.4$.

and

$$\begin{aligned} P[D_1(t) = 1 \text{ and } D_2(t) = 1] &= P[V_1(t) \le C_1(t) \text{ and } V_2(t) \le C_2(t)] \\ &= \int_{-\infty}^{Z_1/\sqrt{t}} \int_{-\infty}^{Z_2/\sqrt{t}} \frac{1}{2\pi\sqrt{1-\rho^2}} \\ &\quad \times \exp\left[\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right] dx_1 \cdot dx_2. \end{aligned} \tag{12}$$

Knowing the default probability of each single party and the joint default probability of both parties, we can obtain the default correlation as defined in Equation (1).

Equations (11) and (6) show that the default probability given by the Merton-type model is only half of the default probability implied by the first-passage-time model. This is consistent with the empirical evidence that Merton's model understates the default risk [Jones et al. (1984)].

Table 8 provides a comparison between default correlations implied by the Merton model and the first-passage-time model with given z-scores. According to Table 3, $z = 8$ roughly corresponds to A-rated firms and $z = 3$ corresponds to certain low-grade firms. Table 8 shows that the Merton approach typically generates lower default correlations. With z-values being low or time horizons being long, default correlations implied by the Merton approach are typically 20–30 % lower than those implied by the first-passage-time model. The two models generate similar and very low default correlations for high-quality firms over short time horizons.

Due to the lack of precise estimates of true default correlations, it is hard to judge how much better a first-passage-time model is than the Merton-type model in estimating default correlations. However, given that the Merton-type model substantially underestimates the default probability of a single party, it cannot provide accurate estimates of joint default probabilities, even if the true default correlations are known. In summary, the first-passage-time model has the following advantages over the Merton-type approach:

1. By ignoring the possibility of early default, the Merton approach underestimates both the probability of default of a single party and the probability of joint default. The first-passage-time model is based on a more realistic assumption and avoids this problem.
2. The Merton-type approach as used by the financial industry is inconsistent for multiple horizons. For example, to calculate two-year default correlations, the approach does not allow firms to default in the first year. The first-passage-time model avoids this inconsistency.
3. The likelihood of early default increases rapidly with the time horizon. For this reason, the Merton approach is mainly used by the financial industry to estimate default probabilities or default correlations over a one-year horizon. The first-passage-time model can be used to estimate default probabilities and default correlations over any horizon.
4. Our numerical simulations show that the first-passage approach is computationally more efficient than the Merton approach in estimating default correlations.

4. Conclusion

This article develops a first-passage-time model of default correlations and multiple defaults, providing an analytical formula for calculating default correlations and joint default probabilities. Because the formula is easily implemented, it provides a convenient tool for credit evaluation, risk management, and capital requirement allocation. The article offers a concrete illustration of the relation among asset-return correlations, default correlations, and time horizons and provides a theoretical justification for various empirical results in the credit risk literature.

Appendix

The Appendix provides a proof for the main result. By definition of D_1 and D_2, we have

$$\begin{aligned} P(D_1 = 1 \text{ or } D_2 = 1) &= P(\tau_1 \leq t \text{ or } \tau_2 \leq t) \\ &= P(\tau \leq t), \end{aligned} \tag{13}$$

where $\tau_i \equiv \min_{t \geq 0}\{t | e^{-\lambda_i t} V_i(t) \leq K_i\}$ and $\tau \equiv \min(\tau_1, \tau_2)$.

Define

$$X_i(t) = -\ln[e^{-\lambda_i t} V_i(t) / V_i(0)] \tag{14}$$

and

$$b_i = -\ln[K_i / V_i(0)]. \tag{15}$$

It is straightforward to verify that $[X_1(t), X_2(t)]$ follows a two-dimensional Brownian motion:

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = -\Omega \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{16}$$

with the initial condition

$$\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{17}$$

After this transformation, finding $P(\tau \leq t)$ is equivalent to finding the first passage time of $[X_1, X_2]'$ to a boundary consisting of two intersecting lines $X_1 = b_1$ and $X_2 = b_2$, where b_1 and b_2 are as defined in Equation (15). For notational convenience, this boundary will be denoted henceforth as $\partial(b_1, b_2)$.

Suppose that $[X_1(t), X_2(t)]'$ represents the position of a particle at time t and that $\partial(b_1, b_2)$ is an absorbing barrier. Let $f(x_1, x_2, t)$ be the transition probability density of the particle in the region $\{(x_1, x_2)|x_i s < b_1$ and $x_2 < b_2\}$, that is, the probability density that $[X_1(t), X_2(t)]' = [x_1, x_2]'$ and that the particle does not reach the barrier $\partial(b_1, b_2)$ in the time interval $(0, t)$. Of course, $f(x_1, x_2, t)$ depends on b_1 and b_2, but because b_1 and b_2 are fixed parameters, this dependence is suppressed in the function for notational convenience. We have

$$\begin{aligned} &P(X_1(t) < y_1 \text{and } X_2(t) < y_2 | X_1(s) < b_1 \\ &\quad \text{and } X_2(s) < b_2, \ \text{for} 0 < s < t) \\ &\quad = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(x_1, x_2, t) dx_1 dx_2 \equiv F(y_1, y_2, t). \end{aligned} \tag{18}$$

Thus, $F(b_1, b_2, t)$ is the probability that absorption has not yet occurred by time t, that is,

$$F(b_1, b_2, t) = P(\tau > t) = 1 - P(\tau \leq t) = 1 - P(D_1 = 1 \text{ or } D_2 = 1).$$

According to Cox and Miller (1972) and Karatzas and Shreve (1991), the transition probability density $f(x_1, x_2, t)$ satisfies the following Kolmogorov forward equation,

$$\begin{aligned} &\frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial f}{\partial t} \\ &\quad (x_1 < b_1, \quad x_2 < b_2), \end{aligned} \tag{19}$$

subject to the following boundary conditions:

$$\begin{aligned} &f(-\infty, x_2, t) = f(x_1, -\infty, t) = 0, \\ &f(x_1, x_2, 0) = \delta(x_1)\delta(x_2), \\ &\int_{-\infty}^{b_1} \int_{-\infty}^{b_2} f(x_1, x_2, t) dx_1 dx_2 \leq 1, t > 0, \\ &f(b_1, x_2, t) = f(x_1, b_2, t) = 0. \end{aligned} \tag{20}$$

Here, $\delta(x)$ is a Dirac's Delta function whose value is infinity at $x = 0$ and zero elsewhere. The function satisfies the condition $\int_{-\infty}^{\infty} \delta(x) \cdot dx = 1$ so that it can be a probability density function. The equation $f(x_1, x_2, 0) = \delta(x_1)\delta(x_2)$ describes the initial condition in Equation (17) that at time zero, $X_1(0) = 0$ and $X_2(0) = 0$.

Theorem 1. *The solution to partial differential equation (PDE) (19) subject to conditions (20) is given by*

$$f(x_1, x_2, t) = \frac{2}{\sigma_1\sigma_2\sqrt{1-\rho^2}\alpha t} e^{-\frac{r^2+r_0^2}{2t}}$$
$$\times \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right), \tag{21}$$

where

$$x_1 = b_1 - \sigma_1\left[\left(\sqrt{1-\rho^2}\right) r\cos(\theta) + \rho r\sin(\theta)\right],$$
$$x_2 = b_2 - \sigma_2 r\sin(\theta).$$

Proof of Theorem 1. To solve PDE (19), we define

$$u_1 = \frac{x_1}{\sigma_1}$$

and

$$u_2 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_2}{\sigma_2} - \rho\frac{x_1}{\sigma_1}\right).$$

Accordingly, we have

$$\frac{1}{2}\frac{\partial^2 f}{\partial u_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial u_2^2} = \frac{\partial f}{\partial t}. \tag{22}$$

The absorbing barriers are now the lines

$$u_1 = \frac{b_1}{\sigma_1}, u_2 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{b_2}{\sigma_2} - \rho\frac{u_1}{\sigma_1}\right),$$

which are generally not at right angles. The following transformations will put the intersection point of the barriers to the origin

$$v_1 = u_1 - \frac{b_1}{\sigma_1}$$
$$v_2 = u_2 - \frac{1}{\sqrt{1-\rho^2}}\left(\frac{b_2}{\sigma_2} - \rho\frac{u_1}{\sigma_1}\right).$$

The rotation through angle

$$\beta = \pi + \tan^{-1}\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)$$

gives

$$w_1 = -\left(\sqrt{1-\rho^2}\right)v_1 + \rho v_2,$$

and

$$_2 = -\rho v_1 - \left(\sqrt{1-\rho^2}\right)v_2.$$

Under these rigid transformations, the PDE (22) does not change form. The particle now starts from some point $[w_1(0), w_2(0)]'$ away from the origin and is absorbed at the boundaries

$$w_1 = 0, w_2 = -\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)w_1.$$

Based on above transformations, we obtain

$$x_1 = b_1 - \sigma_1\left[\left(\sqrt{1-\rho^2}\right)w_1 + \rho w_2\right],$$
$$x_2 = b_2 - \sigma_2 w_2. \tag{23}$$

Letting

$$w_1 = r\cos(\theta)$$
$$w_2 = r\sin(\theta)$$

P.D.E. (22) becomes

$$\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = 2\frac{\partial f}{\partial t} \tag{24}$$

subject to boundary conditions

$$f(r,0,t) = f(r,\alpha,t) = f(\infty,\theta,t) = 0,$$
$$f(r,\theta,0) = \delta(r-r_0)\delta(\theta-\theta_0),$$
$$\int_{\theta=0}^{\alpha}\int_{r=0}^{\infty} J\cdot f(r,\theta,t)drd\theta \le 1, t > 0, \tag{25}$$

where

$$x_1(r,\theta) = b_1 - \sigma_1\left[\left(\sqrt{1-\rho^2}\right)r\cos(\theta) + \rho r\sin(\theta)\right],$$
$$x_2(r,\theta) = b_2 - \sigma_2 r\sin(\theta),$$
$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise,} \end{cases} \tag{26}$$

and

$$J = r\sigma_1\sigma_2\sqrt{1-\rho^2}$$

is the Jacobian of the overall transformation from (x_1, x_2) to (r, θ).

Solving the PDE (24), we obtain:[3]

$$f = \frac{2r}{J\cdot\alpha\cdot t}e^{-\frac{r^2+r_0^2}{2t}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\theta}{\alpha}\right)\sin\left(\frac{n\pi\theta_0}{\alpha}\right)I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right). \tag{27}$$

Theorem 1 then follows immediately. ■

[3] The process for solving PDE (24) is available on request to the author.

Proof of the Main Result. According to Theorem 1,

$$
\begin{aligned}
F(b_1, b_2, t) &= \int_{-\infty}^{b_1}\int_{-\infty}^{b_2} f(x_1, x_2, t)dx_1 dx_2 \\
&= \int_{\theta=0}^{\alpha}\int_{r=0}^{\infty} J \cdot f(r, \theta, t)d\theta dr \\
&= \frac{2}{\alpha \cdot t} e^{-\frac{r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
&\quad \times \int_{\theta=0}^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta \int_{r=0}^{\infty} r e^{-\frac{r^2}{2t}} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) dr,
\end{aligned}
$$

where $J = r\sigma_1\sigma_2\sqrt{1-\rho^2}$ is the Jacobian of the transformation as defined above.

Using identities

$$\int_{\theta=0}^{\alpha} \frac{n\pi}{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta = 1 - (-1)^n,$$

and

$$\int_0^{\infty} re^{-c_1 r^2} I_v(c_2 r)dr = \frac{c_2}{8c_1}\sqrt{\frac{c_2^2}{8c_1}} e^{\frac{c_2^2}{8c_1}} \left[I_{\frac{1}{2}(v+1)}\left(\frac{c_2^2}{8c_1}\right) + I_{\frac{1}{2}(v-1)}\left(\frac{c_2^2}{8c_1}\right)\right],$$

one obtains the main result immediately. ■

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