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A STOCHASTIC INVESTMENT MODEL FOR ACTUARIAL USE

by

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1. INTRODUCTION

1.1. The purpose of this paper is to present to the actuarial profession a stochastic investment model which can be used for simulations of " possible futures " extending for many years ahead. The ideas were first developed for the Maturity Guarantees Working Party (MGWP) whose report was published in 1980. The ideas were further developed in my own paper " Indexing Long Term Financial Contracts " (1981). However, these two papers restricted themselves to a consideration of ordinary shares and of inflation respectively, whereas in this paper I shall present what seems to me to be the minimum model that might be used to describe the total investments of a life office or pension fund.

1.2. After further introductory remarks in Section 1, I go on in Section 2 to discuss the general reasons for choosing the style of model that I have developed. In Section 3 I describe the model in detail. In Section 4 I explain how to use it, and in Section 5 I quote some results. In Section 6 I discuss briefly the sensitivity of the model to some of the assumptions made. In Section 7 I describe some of the possible applications of the model in the investment and actuarial fields. Many readers may prefer to go straight to this final section, and return to the model itself later. The investigations that led to the development of the model, and further details of how it behaves with varying parameters are contained in a separate note " Steps Towards a Stochastic Investment Model for Actuarial Use ", copies of which have been deposited in the Institute and Faculty libraries, and are available from the author on request.

1.3. The actuary's usual horizon is many years ahead, and he is usually content to progress there by annual steps. It is therefore desirable for him to have a stochastic model to describe the way in which appropriate investment variables have moved over the *long* *term*, without being too concerned with very short term fluctuations. It is also desirable to have a model that, while still being an adequate representation of past history, is based on plausible economic and investment assumptions, and produces simulated futures that might be considered generally realistic. It is satisfactory for the actuary to use the simplest model consistent with these objectives, so that features that may be statistically significant but that do not affect the long term structure of the model may be omitted. Thus the actuary's desiderata for a stochastic model may be different from those of short term forecasters, whose objective may be accurate forecasting of the values of the variables, or of a range of values within which the variables may be expected to fall, in the comparatively short term, and who may wish to use any statistically significant features of the model that might improve the accuracy of such short term forecasts. The model described in this paper is for the use of the actuary, and I do not pretend that it competes with other methods, either statistical or economic, of obtaining short term forecasts.

2. THE GENERAL FEATURES OF THE MODEL

2.1. A great deal of actuarial thought developed at a time when the main investments of insurance companies were fixed interest loans and securities, which provided relatively low yields at a time when long-term inflation was virtually nil. The conventional actuarial concept of a single fixed rate of interest was reasonably appropriate in these circumstances. Since the middle of the twentieth century life offices and pension funds have invested to a much greater extent than previously in ordinary shares, inflation of retail prices has been a continuing feature, and fixed interest rates have risen substantially. It has become unreasonable to consider the investments of a life office without including the possibility of both ordinary shares and fixed interest stocks, and it is difficult to forecast the long-term future of a life office without taking into account the inflation of management expenses. A minimum investment model, therefore, requires us to consider inflation, ordinary shares and fixed interest securities.

2.2. The substantial fluctuations that have been observed in the rate of inflation, the prices of ordinary shares and the rate of interest on fixed interest securities lead one to wish to consider more carefully likely possible future fluctuations in these variables. The actuary should not only be interested in the average return that may be achieved on investments, but in the range of possible returns.

Unless he does this, he cannot know to what extent any single figure he chooses is sufficiently much " on the safe side ". A consideration also of the possible fluctuations in investment experience may be of value in considering alternative strategies for investment policy, bonus declaration, etc, which are discussed more fully in Section 7.

2.3. The classical model used by financial economists to describe the stochastic movement of ordinary share prices has been that of a random walk. The MGWP showed that, over a long time period, a model based on dividends and dividend yields was more appropriate. In fact over the short term the MGWP model and the pure random walk model are sufficiently similar for many investigations, based on, say, daily share prices over a period of two years, to have been unable to distinguish between them. The model presented here continues the separation into two series, with prices being treated as a function of dividends and dividend yields.

2.4. The MGWP did not take into account the effect of inflation on company dividends, arguing that, if they were to do so, it would then be necessary to postulate a stochastic model for inflation. Their assumption was that the combination of an unknown model for inflation and an unknown influence of inflation upon dividends would lead to the same stochastic model for dividends as they actually chose. However, for many purposes one wishes to forecast both inflation and company dividends and share prices, and to do so in a self-consistent way. It is clear that dividends, which are measured in money terms, ought, other things being equal, to be related to the general level of money prices elsewhere in the economy. Both are measured by the same *numeraire* of current pounds. It is, therefore, appropriate to relate company dividends directly in some way to the index being used as a measure of general prices, which for my purpose is the Retail Prices Index, or its predecessors.

2.5. I postulate that inflation is the driving force for the other investment variables that have been investigated. Each is thus made dependent on inflation rather than vice versa. Some might argue that interest rates, or even the level of share prices themselves have an influence on future inflation, and that a two-way relationship should have been introduced throughout. Investigations showed that this would have been an unnecessary complication.

2.6. It is too great a simplification to talk about " the " rate of interest on fixed interest stock. There is in reality a complete structure of interest rates that vary by term to maturity, by the level of coupon and by the characteristics of the borrower. In order to make the problem manageable I have restricted myself to yields on long term Government securities, in fact represented by Consols, on the assumption that this one variable can be used to represent adequately the whole level of interest rates for varying terms. I have not included consideration of short term interest rates, though these would be a necessary pivot for any model that wished to represent the whole term structure of interest rates.

2.7. From about 1750 to 1950 fixed interest rates in Britain fluctuated at comparatively low levels; in the last 25 years yields have moved to considerably higher levels. The view of most economists and investment practitioners would be that these higher nominal rates of interest represent the sum of a real rate, which may fluctuate perhaps around 3 per cent, and an allowance for expected future inflation. As inflationary expectations have risen, so also have interest rates. It is not easy to measure what " the market " expects inflation to be. All that we can do is to assume that the market's expectations are influenced by the past history of inflation. It is plausible to assume that the market's estimate of inflation over a long time period does not change rapidly in response to short term price changes, but nevertheless does respond to a succession of rates of inflation at some different level from previously. I, therefore, hypothesise a model in which the yield on Consols responds with a considerable time lag to changes in the rate of inflation. This agrees in principle with the methods used by economists such as Sargent (1973) and Friedman and Schwartz (1982).

2.8. A number of North American investigations (for example Vasicek (1977), Boyle (1978), Cox, Ingersoll & Ross (1978), Brennan & Schwartz (1983), Nelson & Schaefer (1983)), have postulated stochastic models for interest rates and their term structure. So far as I know there have been no satisfactory investigations into stochastic models that introduce the level of inflation as a driving factor, nor any that investigate U.K. data in a stochastic way.

2.9. A fuller model of the investment markets open to U.K. insurance companies would include property rentals and property yields. Unfortunately, reliable series for these for a sufficiently long time period do not seem to exist.

2.10. A fully comprehensive model should also include overseas shares, particularly those in the United States. This would require also a study of exchange rates, in order to convert the results into sterling. Although good data series exist for consumer prices, company dividends and share dividend yields in the United States, and also for the pound/dollar exchange rates, to extend my present investigation in this way would have made it impossibly large. I leave it to later studies to make this extension.

2.11. The preliminary discussions above show that four variables are the minimum necessary to describe the investments of a fund for actuarial purposes. These four variables were also those for which an adequate data series existed. I have used data for the period from 1919 to 1982 at annual intervals for all the series. The variables are:

$Q(t)$ The Retail Prices Index, or its predecessors.

$D(t)$ An index of share dividends, based on a succession of share indices, the last being the Financial Times-Actuaries All-Share Index.

$Y(t)$ The dividend yield on these same share indices, that is, the dividend index at the specified date divided by the price index at that date.

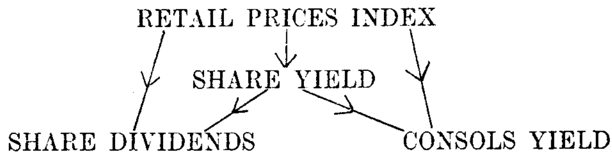
$C(t)$ The yield on 2·5% Consols, which is taken as a measure of the general level of fixed interest yields in the market.

Details of the sources and the values of the variables are given in the separate note referred to in Section 1.2.

2.12. The model that will be presented in Section 3 was derived after careful consideration of these four series, and the parameters chosen are those which appear to me to be the most suitable for actuarial use. The particular models have been chosen after consideration of a great variety of alternatives, and the particular values of the parameters are those which appear both to fit the past data satisfactorily, and to be appropriate for the long-term future. In general they are very close to the " best estimates " of the parameters, in a least squares or maximum likelihood sense. Full details of the investigations leading to the choice of model and of parameters are also given in the separate note.

3. THE STOCHASTIC MODEL

3.1. Although four separate variables are involved in the model, I did not find it necessary to use a full multivariate structure, in which each variable could affect each of the others. Instead I chose to use a "cascade" one, which can be shown diagrammatically below, where the arrows indicate the direction of influence.



3.2. Thus the Retail Prices Index series, $Q(t)$, is described first, entirely in terms of its own previous values, and the values of a random "white noise" series. White noise is the name given by electrical engineers to a sequence of independent identically distributed random variables, which thus have no single dominant frequency, and so bear the same relation to sound as white light does to light.

3.3. The model for $Q(t)$ is

$$\nabla\, ln Q(t) = QMU + QA(\nabla\, ln Q(t-1) - QMU) + QSD.QZ(t),$$

where the backwards difference operator ∇ is defined by

$$\nabla\, X(t) = X(t) - X(t-1).$$

and $QZ(t)$ is a sequence of independent identically distributed unit normal variates.

3.4. This model says that the annual rate of inflation follows a first order autoregressive process, with a fixed mean QMU, and a parameter QA such that the expected rate of inflation each year is equal to the mean plus QA times last year's deviation from the mean. Appropriate values for the parameters are:

$$QMU = 0{\cdot}05,\ QA = 0{\cdot}6,\ QSD = 0{\cdot}05.$$

There is fairly little uncertainty about the appropriate values for QA and QSD, but considerable uncertainty about the value to use for QMU, where anything between 0·04 and 0·10 might be justifiable, depending on the past period of observation one wishes to consider.

3.5. The dividend yield depends both on the current level of inflation and on previous values of itself and on a white noise series. The model is:

$$ln Y(t) = YW.\nabla\, ln Q(t) + YN(t),$$

where $$YN(t) = ln\, YMU + YA(YN(t-1) - ln YMU) + YE(t),$$

$$YE(t) = YSD.YZ(t),$$

and $YZ(t)$ is a sequence of independent identically distributed unit normal variates.

3.6. This model says that the natural logarithm of the yield consists of two parts: the first is directly dependent on the current rate of inflation (a high rate of inflation implying a high share yield and vice versa), and the second, YN(t), follows a first order autoregressive model, similar to that of the rate of inflation itself. Appropriate values for the parameters are:

$$\text{YMU} = 0{\cdot}04,\ \text{YA} = 0{\cdot}6,\ \text{YW} = 1{\cdot}35,\ \text{YSD} = 0{\cdot}175.$$

3.7. The index of share dividends is made to depend on inflation, with both an exponentially lagged effect and an additional direct effect, and on the residual, YE(t), from the yield model, plus a white noise series, which has both a simultaneous and a lagged effect. The parameters are such that a given percentage increase in the Retail Prices Index ultimately results in the same percentage increase in the dividend index, so the model is said to have unit gain. The model is:

$$\nabla\, ln\text{D}(t) = \text{DW}\left(\frac{\text{DD}}{1-(1-\text{DD})\text{B}}\right)\nabla\, ln\text{Q}(t) + \text{DX}\ \nabla\, ln\text{Q}(t)$$
$$+\text{DMU}+\text{DY}.\text{YE}(t-1)+\text{DE}(t)+\text{DB}.\text{DE}(t-1),$$

where the backwards step operator is defined by

$$\text{BX}(t) = \text{X}(t-1),$$
$$\text{DE}(t) = \text{DSD}.\text{DZ}(t),$$

and DZ(t) is a sequence of independent identically distributed unit normal variates.

3.8. The term in parentheses above involving DD represents an infinite series of lag effects, with exponentially declining coefficients:

$$\text{DD},$$
$$\text{DD}\ (1-\text{DD}),$$
$$\text{DD}\ (1-\text{DD})^2,\ \text{etc.}$$

The sum of these coefficients is unity, so this part of the formula represents the lagged effect of inflation, with unit gain. This means that if retail prices rise by 1 per cent this term will also, eventually, rise by 1 per cent. We can alternatively describe it as the " carried forward " effect of inflation, DM(t), where

$$\text{DM}(t) = \text{DD}.\ \nabla\, ln\ \text{Q}(t)+(1-\text{DD})\ \text{DM}(t-1),$$

from which we see that the amount that enters the dividend model each year is DD times the current inflation rate, plus (1 − DD) times the amount brought forward from the previous year, and that this total is then carried forward to the next year.

3.9. Appropriate values for the parameters are:

DW = 0·8, DD = 0·2, DX = 0·2, DMU = 0·0, DY = −0·2, DB = 0·375, DSD = 0·075.

Investigations showed that for practical purposes in the long term very similar results could be obtained from a slightly simpler model that omits the term in DB, and alters two other parameters to give:

DW = 0·8, DD = 0·2, DX = 0·2, DMU = 0·0, DY = −0·3, DB = 0·0, DSD = 0·10.

3.10. The model makes the dividend index appear to depend on the residual of the share yield. In fact share prices to some extent correctly anticipate changes in dividends. For example, an unusual rise in dividends may be correctly forecast by investment analysts, so that share prices take account of this and so rise. The yield is calculated on the previous year's dividend, and so falls. Although this is the *causal* sequence, it is convenient in the model to reflect the *temporal* sequence, so that an unexpected fall in yields results in an upwards change in the dividend index in the following period.

3.11. Although the parameter DMU is set to zero, it is retained in the model, since one may wish to investigate the results of assuming a small positive or negative value for it, implying a positive or negative long-term change in real dividends.

3.12. The Consols yield is assumed to consist of a real part, CN(t), plus an allowance for expected future inflation. The latter is based on the actual values of present and past inflation. The real part is defined by a third order autoregressive model, together with an influence from the residual of the yield series, YE(t), and a residual white noise series. The model is:

$$\mathrm{C}(t) = \mathrm{CW}\left(\frac{\mathrm{CD}}{1-(1-\mathrm{CD})\mathrm{B}}\right)\nabla\, ln\mathrm{Q}(t) + \mathrm{CN}(t),$$

$$\text{where } ln\mathrm{CN}(t) = ln\mathrm{CMU} + (\mathrm{CA1}.\mathrm{B} + \mathrm{CA2}.\mathrm{B}^2 + \mathrm{CA3}.\mathrm{B}^3)\,(ln\mathrm{CN}(t) - ln\mathrm{CMU}) + \mathrm{CY}.\mathrm{YE}(t) + \mathrm{CSD}.\mathrm{CZ}(t),$$

where CZ(t) is a sequence of independent identically distributed unit normal variates.

3.13. The term in parentheses in CD has a similar form to the DD term in the dividend model, though the parameter value is different. It represents the current value of expected future inflation as an exponentially weighted moving average of past rates of inflation.

3.14. Appropriate values for the parameters are:

$$CW = 1{\cdot}0,\ CD = 0{\cdot}045,\ CMU = 0{\cdot}035,\ CA1 = 1{\cdot}20,$$
$$CA2 = -0{\cdot}48,\ CA3 = 0{\cdot}20,\ CY = 0{\cdot}06,\ CSD = 0{\cdot}14.$$

The value of CW is 1·0, and it might appear that this term could be omitted; however, it is of interest to investigate variations in this parameter.

3.15. Just as for the dividend model, investigations showed that very similar long-term forecasts were obtained by setting three of these parameters equal to zero, and changing the values of three of the others, to give:

$$CW = 1{\cdot}0,\ CD = 0{\cdot}05,\ CMU = 0{\cdot}035,\ CA1 = 0{\cdot}91,$$
$$CA2 = 0{\cdot}0,\ CA3 = 0{\cdot}0,\ CY = 0{\cdot}0,\ CSD = 0{\cdot}165.$$

This form of the model says that the influence of inflation on the Consols yield is reflected by using as expected inflation an exponentially weighted moving average of past inflation, with a parameter of 0·05. The real rate of return has a mean of 3·5%, and follows a first order autoregressive series with a parameter of 0·91, so that it tends back towards its mean rather slowly.

3.16. It will be seen that the complete model is wholly self-contained. The only inputs are the four separate white noise series, and no exogenous variables are included. In my view, whatever may be the case for short-term forecasting, such a self-contained model is better for long-term simulations. The rate of inflation, the amount of company dividends, the level of interest rates, and the prices at which shares trade may well depend on such extraneous factors as government policy, business conditions and the political, military, economic and climatic condition of the world. Wars, famines and natural disasters may or may not occur. But they are not forecastable in the long run and their influence is subsumed in the white noise series.

4. HOW TO USE THE MODEL

4.1. It would be possible to derive analytically the joint probability distribution of the unknown values of certain of the variables in successive future years, given a suitable set of data to represent the past history and current state at some particular starting time. However, it seems to me particularly complicated to do this for any realistic actuarial purpose, whereas a simulation method facilitates many more possible investigations. The method of simulation that is appropriate for this model is similar to that used by the MGWP. On the basis of a starting position at time $t = 0$, one can generate values for the four series, $Q(t)$, $Y(t)$, $D(t)$ and $C(t)$, for $t = 1$ to N, where N is, for example, 100. It is necessary to simulate independent unit normal pseudo-random variables for each of the white noise series, QZ, YZ, DZ and CZ, using, for example, Marsaglia's Polar method, as described in Appendix E of the MGWP Report.

4.2. It is necessary to choose certain initial values to represent the present state, and to start the indices. One can set Q(0) arbitrarily as 1. The model for the Retail Prices Index requires us to postulate a value for $\nabla\, ln Q(0)$, the rate of inflation " last year ", i.e. in the year just preceding the beginning of the simulation period. I denote this by QI. A neutral value for QI is QMU, the average force of inflation. However, one may wish to investigate the effect of a different starting value, or to insert the actual current real value.

4.3. The model for dividend yield requires us to choose a value for the yield at the start of the simulation period. This is Y(0) or YI. A neutral value for this is given by YMU.exp (YW.QMU). As with inflation, it may be of interest to investigate the effect of choosing different values for the starting yield, such as the actual current value. The model for yields requires also a value for $\nabla\, ln Q(0)$, which has already been given by QI.

4.4. To start the dividend series one needs to chose an arbitrary value for D(0). It is of no importance whether one uses a value of 1, or a value equal to Y(0), which would then imply a starting share price, P(0), of 1; either may be used. One then needs to choose a value for the carried forward exponentially lagged effect of inflation, viz:

$$DM(0) = \left(\frac{DD}{1-(1-DD)B}\right) \nabla\, ln Q(0),$$

which I denote just as DM. The neutral value for this is also QMU, but one may wish to use an estimate of the current carry forward. One also needs a value of $\nabla\, ln Q(0)$, given as before by QI. One then needs a value for YEI = YE(0), the random residual that took the share yield to its present level. This could either be stated explicitly, or be calculated given also values for $Y(-1)$ and $\nabla\, ln Q(-1)$. The neutral value is zero.

4.5. The starting values required for the Consols yield series include a carry forward from past inflation, similar to that required for dividends, though based on a different parameter, viz:

$$\left(\frac{CD}{1-(1-CD)B}\right)\nabla\, ln Q(0),$$

which I denote CM. The neutral value for this is QMU. One also needs to select a value for the starting Consols yield, C(0), and, for the full model, for the two past years, $C(-1)$ and $C(-2)$ also. The neutral value for these is QMU + CMU, but the actual current values could be used. The model for $C(t)$ would allow the possibility of negative values of the yield if inflation were negative for long enough. To avoid these occurring I postulate a minimum value for $C(t)$ of CMIN, set equal to 0·5%.

4.6. Besides calculating values for the four basic series it is also convenient to calculate values for three derived series. The first of these is the share price, $P(t)$, which is easily derived from the formula:

$$P(t) = D(t)/Y(t).$$

4.7. One can next calculate a " rolled-up " share index being the value of a share index where dividends, net of tax, are reinvested in shares. This is denoted $PR(t)$, where

$$PR(t) = PR(t-1)\left(\frac{P(t)+D(t)\,(1-\text{tax A})}{P(t-1)}\right),$$

and tax A is the rate of tax on share dividends, assumed constant. In fact I have taken this throughout as zero, so I have assumed a gross roll-up. An arbitrary starting value of PR(0) = 1 is appropriate.

4.8. The third additional series is a corresponding rolled-up index for Consols, denoted $CR(t)$, where

$$CR(t) = CR(t-1)\left(\frac{1}{C(t)}+(1-\text{tax B})\right)C(t-1),$$

and tax B is the rate of tax on " unfranked " income. I take this also as zero throughout. An arbitrary starting value of CR(0) = 1 is appropriate. This formula assumes that " Consols " are truly irredeemable stocks, and would not be repaid if interest rates fell below the coupon rate, possibly being then refinanced at a lower coupon rate. This complication could easily be allowed for in the calculation of CR(t) if desired.

5. PRELIMINARY RESULTS

5.1. In order to comprehend the results of simulating these seven connected series for, say, 100 years for, say, 1,000 simulations it is necessary to summarise the results in some convenient way. One can plot the results for a small number of simulations in order to get an impression of the variety of experiences that might result from the model. It is particularly illuminating to study in detail some of the extreme cases that result from a large number of simulations; but of course one must then remember that it is in fact an extreme case that one has chosen to study. However, it is more convenient for presentation to record the means and standard deviations of selected statistics, and the correlation coefficients between them. But there is a considerable choice of statistics to use.

5.2. It is convenient to restrict ourselves to the results after selected t years. I have chosen to take t = 1, 5, 10, 15, 20, 30, 50, 75, and 100. The first set of statistics to consider are the " final values " of the appropriate series at time t, i.e.

$$\mathrm{F}x(t) = x(t)/x(0),$$

where x = Q, D, P, PR and CR respectively. All these are the series that would be measured in money terms, and they represent the consolidated results up to time t. The values of Y(t) and C(t) are of comparatively less interest, and I have not considered them further.

5.3. However, these final values have very different sizes for different values of t. It is therefore convenient to calculate the equivalent annual uniform rates of growth, Gx(t), where again x = Q, D, P, PR and CR, and

$$\mathrm{G}x(t) = 100\ (\mathrm{F}x(t)^{1/t} - 1).$$

It should be noted, however, that Gx(t) is not related to Fx(t) by a linear transformation, so that in general the mean of Gx(t) is not equal to the transformed values of the mean of Fx(t). Thus

$$\mathrm{E}(\mathrm{G}x(t)) \neq 100\ (\mathrm{E}(\mathrm{F}x(t))^{1/t} - 1).$$

However, quantiles, such as the median value, or the kth highest value, do correspond.

5.4. Besides the money values it is of interest also to look at the "real" returns for the series other than the Retail Prices Series. Thus we can calculate $Hx(t)$ for x = D, P, PR and CR, where

$$Hx(t) = Fx(t)/FQ(t).$$

Similarly, we can calculate the equivalent uniform real rate of return over the period, $Jx(t)$ where

$$Jx(t) = 100\ (Hx(t)^{1/t} - 1).$$

5.5. This gives us a considerable number of different statistics. I shall therefore restrict myself to considering the rate of growth of the price index and of the two rolled-up indices, both in money and in real terms. These are the values: GQ(t), GPR(t), GCR(t), JPR(t) and JCR(t). I shall quote the means and variances of these statistics, and selected correlation coefficients between pairs of them.

5.6. I start by considering the model that best represents the data, that with the largest number of parameters outlined above, which I call the "Full Standard Basis".

5.7. The first column of Table 1 lists the complete set of parameters for the Full Standard Basis, and Table 2 shows the results on this Basis. It is worth studying these in some detail, in order to get a feel for the pattern of results. Later I consider the effect of varying the parameters, and the results will be compared with those shown in such a table.

5.8. *Mean rate of inflation (GQ)*: The observed average of this value is a little over 5% for low terms, dropping to very close to 5% for longer terms. This of course is consistent with a value for QMU of 0·05. The standard deviation is over 5% for term 1, but drops steadily as the term increases, so that there is greater relative certainty about inflation in the longer run than in the shorter.

5.9. *Mean rate of money return on shares (GPR)*: The average value of this starts high at over 12% for term 1, dropping quickly to a little over 10% by term 5, and declining further to under 10% thereafter. The standard deviation is very high at term 1, over 20%, dropping sharply and then more slowly to about 2% for very long terms. However, the distribution of GPR is positively skewed, like the distribution of all these observed statistics, and with a positively skewed distribution the mean is likely to be somewhat higher than the median. The higher the standard deviation, the greater the divergence between mean and median. The median value of

TABLE 1

Values of Parameters in Standard Bases

		Full Standard Basis	Reduced Standard Basis
Inflation			
	QMU	0·05	0·05
	QA	0·6	0·6
	QSD	0·05	0·05
	QI	0·05	0·05
Share Yield			
	YW	1·35	1·35
	YMU %	4·0	4·0
	YA	0·6	0·6
	YSD	0·175	0·175
	YI %	4·27932	4·27932
	YEI	0·0	0·0
Share Dividend			
	DW	0·8	0·8
	DD	0·2	0·2
	DX	0·2	0·2
	DY	−0·2	−0·3
	DMU	0·0	0·0
	DB	0·375	0·0
	DSD	0·075	0·1
	DEI	0·0	0·0
	DM	0·05	0·05
Consols Yield			
	CW	1·0	1·0
	CD	0·045	0·05
	CMU %	3·5	3·5
	CY	0·06	0·0
	CA1	1·20	0·91
	CA2	−0·48	0·0
	CA3	0·20	0·0
	CSD	0·14	0·165
	CI %	8·5	8·5
	CM	0·05	0·05

GPR for term 1 is 10·2, for term 5 is 9·85, and thereafter is fairly close to the mean value.

5.10. The correlation coefficient between GPR and GQ is negative at term 1, −0·26: this is caused by the term in YW; an increase in inflation causes an increase in share yield, a reduction in share price, and therefore a low rate of money return. However, a high rate of inflation ultimately affects dividends, which in turn affect share prices and increase the money return in later years. Thus the correlation coefficient become positive (+0·24) by term 5, increasing to nearly 0·8 by term 100.

TABLE 2

Results on Full Standard Basis

Term (years):	1	5	10	15	20	30	50	75	100
Mean Rate of Inflation (GQ):									
E(GQ)	5·37	5·37	5·14	5·16	5·07	5·03	4·99	5·06	5·08
SD(GQ)	5·34	4·48	3·61	3·13	2·83	2·38	1·80	1·54	1·34
Mean Rate of Money Return on Shares (GPR):									
E(GPR)	12·17	10·17	10·07	9·85	9·80	9·69	9·66	9·72	9·71
SD(GPR)	21·72	7·27	5·23	4·28	3·78	3·29	2·60	2·14	1·94
Correlation Coefficient:									
C(GPR,GQ)	−0·26	0·24	0·46	0·57	0·64	0·71	0·76	0·77	0·78
Mean Rate of Money Return on Consols (GCR):									
E(GCR)	8·05	8·53	8·69	8·68	8·73	8·74	8·71	8·66	8·68
SD(GCR)	6·27	2·81	1·55	1·12	1·00	1·08	1·19	1·15	1·13
Correlation Coefficients:									
C(GCR,GQ)	−0·44	−0·62	−0·61	−0·37	−0·13	0·27	0·58	0·67	0·73
C(GCR,GPR)	0·18	−0·09	−0·17	−0·09	0·02	0·29	0·49	0·54	0·60
Mean Rate of Real Return on Shares (JPR):									
E(JPR)	6·99	4·68	4·73	4·48	4·52	4·44	4·44	4·44	4·41
SD(JPR)	22·75	7·29	4·59	3·46	2·82	2·20	1·61	1·29	1·17
Correlation Coefficient:									
C(JPR,GQ)	−0·47	−0·39	−0·28	−0·23	−0·18	−0·06	0·05	0·04	0·09
Mean Rate of Real Return on Consols (JCR):									
E(JCR)	2·94	3·26	3·53	3·45	3·56	3·58	3·56	3·44	3·43
SD(JCR)	9·53	6·41	4·62	3·63	3·07	2·31	1·46	1·13	0·91
Correlation Coefficients:									
C(JCR,GQ)	−0·82	−0·94	−0·96	−0·96	−0·95	−0·90	−0·77	−0·69	−0·59
C(JCR,JPR)	0·43	0·39	0·31	0·27	0·21	0·12	0·02	0·01	0·01

5·11. *Mean rate of money return on Consols* (*GCR*): The average value of GCR lies between 8% and 9% for all terms, which is consistent with the initial yield assumed on Consols of 8·5%. The standard deviation over one year is highish, at 6%, but drops rapidly with term, to reach a minimum of 1% at term 20, increasing slightly thereafter. This interesting pattern can be explained as follows: in the short term there is uncertainty about the price of Consols, which influences the 1-year return; as time goes by the influence of the current price becomes less, and the interest received on the initial investment comes to dominate; but as the duration extends far into the future, the uncertain yield at which future interest will be re- nvested attains greater importance.

5.12. The correlation coefficient between the money return on Consols, GCR, and inflation, GQ, is negative and quite large at lower terms, changing to become positive and quite large at the highest terms. On this Basis the change from negative to positive correlation coefficients occurs between terms 20 and 30. Again, this can be explained: in the short run a rise in inflation causes a rise in interest rates, a fall in prices and poor money returns; in the long run interest rates rise to compensate for higher inflation. But it is interesting to see how long it takes for this compensation to occur. The correlation coefficient between the money returns on Consols, GCR, and on shares, GPR, starts rather small and positive (+0·18 at term 1) then becomes negative up to almost term 20, rising thereafter to quite large and positive by term 100. Several factors influence this: a rise in inflation causes a drop in the prices of both shares and Consols in the short term; high inflation pushes share prices up, but harms Consols returns in the medium term; high inflation then has a positive effect on both over the longer term; in addition, yields on Consols and shares are positively connected through the parameter CY.

5.13. *Mean rate of real return on shares (JPR)*: The average values are roughly equivalent to the difference between the money return on shares and the rate of inflation, starting at almost 7% for term 1 and falling to just under 4·5% for longer terms. As with the money return on shares the skew distribution with a high standard deviation pushes up the average value; the median value of JPR for term 1 is just over 5%. The standard deviation of JPR is very high at term 1, nearly 23%, but drops to a little over 1% at the longest term. It slightly exceeds the standard deviation of the money return at shorter terms, but falls below it at longer terms. Thus the real return on shares is more certain than the money return.

5.14. The correlation coefficient between the real return on shares, JPR, and inflation, GQ, is quoted. This is negative at term 1 (−0·47) approaching virtually zero for long durations. The explanation for this can be easily seen: in the short term high inflation adversely affects share prices, as discussed above, and in the medium term dividends do not respond immediately to higher inflation; in the long run they are independent, but it takes up to thirty years before the correlation coefficient becomes negligible.

5.15. *Mean rate of real return on Consols (JCR)*: The average value of the mean rate of real return on Consols is just below 3% for term 1, rising to about 3·5% for longer terms. This is consistent with the value of CMU of 3·5% and with the difference between the mean rate of money return on Consols and inflation. The standard deviation is high at term 1, dropping as usual with increasing term, and reducing to less than 1% at the longest term. From term 10 onwards these standard deviations are similar to those of the mean rate of real return on shares.

5.16. The correlation coefficient between the real rate of return on Consols and inflation is negative and large throughout, exceeding −0·9 for all the medium terms. There is a positive correlation of 0·43 between the real rate of return on Consols and that on shares at term 1, which drops as the term rises, falling to virtually zero for longer terms.

5.17. In my description of the model in Section 3 I stated alternative sets of parameters for the dividend series and for Consols. Investigations showed that, among a variety of different simplifications that might have been chosen, the use of these two alternative sets of parameters produced results which, at least for longer terms, were very similar to those produced by the Full Standard Basis. This rather simpler model I call the Reduced Standard Basis. The parameters are listed in the second column of Table 1, and the results are shown in Table 3, for the same statistics as in Table 2.

5.18. The parameters for the inflation model are unchanged, so the results for inflation, GQ, are identical. The model for share yields is unchanged, and the model for share dividends is changed in such a way that one would expect a slightly higher standard deviation for dividend changes, and hence both money and real returns, for short terms, with very little difference between the results for longer terms. This is indeed the case. The results for both money and real return on Consols also show very similar results on the two Bases. The Reduced Standard Basis shows a slightly higher standard deviation of mean return at shorter terms. Also the correlation coefficient between Consols return and share return at term 1 is reduced somewhat because the CY term has been omitted.

5.19. These alternative results have both started from a neutral initial position. If the current and recent past values of the variables were very different from their mean values the more complicated models for dividend yields and for Consols would reflect this more in the short term. The Basis that is appropriate to use in any particular case will depend on the particular circumstances. If the short term carries great weight, and present conditions are thought to be unusual, then a more elaborate model should be used. In general,

TABLE 3

Results on Reduced Standard Basis

Term (years):	1	5	10	15	20	30	50	75	100
Mean Rate of Inflation (GQ):									
E(GQ)	5·37	5·37	5·14	5·16	5·07	5·03	4·99	5·06	5·08
SD(GQ)	5·34	4·48	3·61	3·13	2·83	2·38	1·80	1·54	1·34
Mean Rate of Money Return on Shares (GPR):									
E(GPR)	12·51	10·21	10·10	9·87	9·81	9·69	9·66	9·72	9·71
SD(GPR)	22·91	7·64	5·41	4·37	3·84	3·32	2·62	2·15	1·95
Correlation Coefficient:									
C(GPR,GQ)	−0·24	0·22	0·45	0·55	0·62	0·70	0·75	0·77	0·77
Mean Rate of Money Return on Consols (GCR):									
E(GCR)	7·97	8·54	8·72	8·71	8·77	8·76	8·73	8·69	8·70
SD(GCR)	7·27	3·00	1·66	1·21	1·10	1·17	1·27	1·20	1·17
Correlation Coefficients:									
C(GCR,GQ)	−0·42	−0·64	−0·62	−0·36	−0·12	0·28	0·58	0·67	0·73
C(GCR,GPR)	0·11	−0·13	−0·18	−0·11	0·01	0·28	0·48	0·53	0·60
Mean Rate of Real Return on Shares (JPR):									
E(JPR)	7·31	4·72	4·76	4·50	4·53	4·44	4·45	4·44	4·40
SD(JPR)	23·79	7·65	4·79	3·59	2·91	2·27	1·65	1·31	1·18
Correlation Coefficient:									
C(JPR,GQ)	−0·45	−0·37	−0·27	−0·23	−0·19	−0·07	0·04	0·03	0·09
Mean Rate of Real Return on Consols (JCR):									
E(JCR)	2·87	3·27	3·56	3·48	3·60	3·60	3·58	3·47	3·46
SD (JCR)	10·31	6·61	4·72	3·67	3·11	2·32	1·47	1·14	0·92
Correlation Coefficients:									
C(JCR,GQ)	−0·79	−0·94	−0·96	−0·95	−0·94	−0·88	−0·74	−0·66	−0·56
C(JCR,JPR)	0·35	0·36	0·30	0·26	0·21	0·12	0·02	0·01	0·02

however, I would think that it is sufficient to use the Reduced Standard Basis. Fewer parameters need to be considered, and fewer past values of the variables are required to set the initial conditions.

6. SENSITIVITY ANALYSIS

6.1. In order to get a feel for the effect of alternative parameter values on the results, it is desirable to explore the sensitivity of the results to changes in the value of each parameter in turn. I have generally used a variation for each parameter of roughly 1·5 standard errors in each direction, but in some cases I have also used more extreme values. A list of the 63 variations is shown in Table 4; 41 of these show variations in the basic long term parameters and 22 of them show variations in the initial conditions.

6.2. Because of the cascade structure of the model it is convenient to consider variations in each section of the model in turn. Variations in the parameters for the inflation model affect each of the series, whereas variations in the parameters for Consols affect only the Consols returns. It would, however, be laborious to discuss every variation in full, and an indication of the technique and results can be obtained by discussing only the variation in the inflation parameters. Table 5 shows results for variations A1 to C2, the variations in the basic inflation parameters, for a twenty-year term only. Where changes in the parameters have differential effects on different terms I shall comment on these below.

TABLE 4

Variations of Reduced Standard Basis

I. *Variations in basic parameters*

I.1 *Inflation*

A.	1. QA = 0·45		2. QA = 0·75
	3. QA = 0·95		4. QA = 1·0
B.	1. QMU = 0·03		2. QMU = 0·07
	QI = 0·03		QI = 0·07
	YI = 4·16533%		YI = 4·39644%
	DM = 0·03		DM = 0·07
	CI = 6·5%		CI = 10·5%
	CM = 0·03		CM = 0·07
	3. QMU = 0·10		
	QI = 0·10		
	YI = 4·57815%		
	DM = 0·10		
	CI = 13·5%		
	CM = 0·10		
C.	1. QSD = 0·4		2. QSD = 0·06

I.2 *Shares*

D.	1. YW = 0·85	2. YW = 1·85
E.	1. YA = 0·45	2. YA = 0·75
F.	1. YMU = 3·5%	2. YMU = 4·5%
	YI = 3·74441%	YI = 4·81424%
G.	1. YSD = 0·15	2. YSD = 0·20
H.	1. DD = 0·1	2. DD = 0·3
I.	1. DW = 0·6	2. DW = 1·0
J.	1. DX = 0·0	2. DX = 0·4
K.	1. DW = 1·0	2. DW = 0·6
	DX = 0·0	DX = 0·4
L.	1. DMU = −0·01	2. DMU = 0·01
M.	1. DY = −0·4	2. DY = −0·2
N.	1. DSD = 0·085	2. DSD = 0·115

I.3 *Consols*

O.	1. CW = 0·9	2. CW = 1·1
P.	1. CD = 0·04	2. CD = 0·06
Q.	1. CMU = 2·5%	2. CMU = 4·5%
	CI = 7·5%	CI = 9·5%
R.	1. CA1 = 0·86	2. CA1 = 0·96
S.	1. CSD = 0·145	2. CSD = 0·185

TABLE 4 (cont.)

II. *Variations in initial conditions*

II.1 *Inflation*

T.	1. QI	=	0·0	2. QI	=	0·02
	3. QI	=	0·07	4. QI	=	0·10
U.	1. QI	=	0·0	2. QI	=	0·02
	YI	=	4·0%	YI	=	4·10947%
	DM	=	0·0	DM	=	0·02
	CI	=	3·5%	CI	=	5·5%
	CM	=	0·0	CM	=	0·02
	3. QI	=	0·07	4. QI	=	0·10
	YI	=	4·39644%	YI	=	4·57815%
	DM	=	0·07	DM	=	0·10
	CI	=	10·5%	CI	=	13·5%
	CM	=	0·07	CM	=	0·10

II.2 *Shares*

V.	1. YI	=	2·27932%	2. YI	=	3·27932%
	3. YI	=	5·27932%	4. YI	=	6·27932%
W.	1. YEI	=	−0·175	2. YEI	=	0·175
X.	1. DM	=	0·02	2. DM	=	0·08

II.3 *Consols*

Y.	1. CI	=	5·5%	2. CI	=	6·5%
	3. CI	=	10·5%	4. CI	=	12·5%
Z.	1. CM	=	0·03	2. CM	=	0·07

TABLE 5

Results for variations of Reduced Standard Basis: Term 20

Variation	Reduced Standard	A1 QA=0·45	A2 QA=0·75	A3 QA=0·95	A4 QA=1·0	B1 QMU=0·03 etc.
Mean Rate of Inflation (GQ):						
E(GQ)	5·07	5·07	5·09	5·45	5·91	2·99
SD(GQ)	2·83	2·12	4·22	10·32	14·48	2·78
Mean Rate of Money Return on Shares (GPR):						
E(GPR)	9·81	9·80	9·84	10·14	10·44	7·52
SD(GPR)	3·84	3·46	4·71	8·90	11·57	3·76
Correlation Coefficient:						
C(GPR,GQ)	0·62	0·51	0·76	0·93	0·96	0·62
Mean Rate of Money Return on Consols (GCR):						
E(GCR)	8·77	8·80	8·98	10·91	11·84	6·89
SD(GCR)	1·10	0·96	1·52	4·18	4·87	1·50
Correlation Coefficients:						
C(GPR,GQ)	−0·12	−0·07	−0·25	−0·61	−0·64	−0·36
C(GCR,GPR)	0·01	0·02	−0·10	−0·51	−0·57	−0·16
Mean Rate of Real Return on Shares (JPR):						
E(JPR)	4·53	4·51	4·56	4·68	4·79	4·41
SD(JPR)	2·91	2·87	3·02	3·94	4·99	2·91
Correlation Coefficient:						
C(JPR,GQ)	−0·19	−0·15	−0·26	−0·60	−0·76	−0·19
Mean Rate of Real Return on Consols (JCR):						
E(JCR)	3·60	3·50	3·88	6·45	8·01	3·88
SD(JCR)	3·11	2·35	4·79	13·70	18·66	3·62
Correlation Coefficients:						
C(JCR,GQ)	−0·94	−0·92	−0·95	−0·94	−0·95	−0·92
C(JCR,JPR)	0·21	0·16	0·29	0·62	0·77	0·20

TABLE 5 (cont.)

Variation	Reduced Standard	B2 QMU=0·07 etc.	B3 QMU=0·10 etc.	C1 QSD=0·04	C2 QSD=0·06
Mean Rate of Inflation (GQ):					
E(GQ)	5·07	7·19	10·46	5·07	5·07
SD(GQ)	2·83	2·89	2·98	2·27	3·40
Mean Rate of Money Return on Shares (GPR):					
E(GPR)	9·81	12·16	15·78	9·80	9·82
SD(GPR)	3·84	3·92	4·05	3·52	4·20
Correlation Coefficient:					
C(GPR,GQ)	0·62	0·62	0·63	0·54	0·69
Mean Rate of Money Return on Consols (GCR):					
E(GCR)	8·77	10·72	13·69	8·71	8·84
SD(GCR)	1·10	0·99	0·98	0·97	1·27
Correlation Coefficients:					
C(GCR,GQ)	−0·12	0·10	0·30	−0·09	−0·15
C(GCR,GPR)	0·01	0·14	0·27	0·01	−0·01
Mean Rate of Real Return on Shares (JPR):					
E(JPR)	4·53	4·65	4·83	4·51	4·55
SD(JPR)	2·91	2·91	2·91	2·87	2·97
Correlation Coefficient:					
C(JPR,GQ)	−0·19	−0·18	−0·18	−0·16	−0·21
Mean Rate of Real Return on Consols (JCR):					
E(JCR)	3·60	3·36	3·00	3·51	3·70
SD(JCR)	3·11	2·86	2·66	2·50	3·76
Correlation Coefficients:					
C(JCR,GQ)	−0·94	−0·94	−0·95	−0·93	−0·94
C(JCR,JPR)	0·21	0·21	0·21	0·17	0·24

6.3. A reduction in QA implies that inflation returns more rapidly to its mean level. Results for Variation A1 (QA = 0·45) show almost no change in the average mean rates, at any term, but some reduction in the standard deviations, except at term 1. The correlation coefficients are also reduced in absolute value. Variation A2 (QA=0·75) shows the opposite, with an increase in the standard deviations for all terms other than 1, and an increase in the absolute value of all the correlation coefficients. Variations A3 (QA = 0·95) and A4 (QA = 1·0) take this further. In Variation A3 the standard deviation of the mean rate of inflation does not reduce with term; in Variation A4 it increases very considerably with term, so that at term 100 the average mean rate of inflation is 7·69% with a standard deviation of 32·39%. The range is enormous, and includes both "hyper-inflations" and "hyper-deflations". These latter are a consequence of the model, but, unlike the former, which have in fact occurred, they seem to me to be economically unrealistic. The mean rate of money return on shares also shows both a rising average and a substantially rising standard deviation; but the mean rate of real return on shares is not affected so much. The mean rate of money return on Consols also increases considerably with term, with a high standard deviation, and the mean rate of real return on Consols is also greatly increased (in part because of the hyper-deflations). One can conclude that this parameter is a critical one, in that if it is pushed to its extreme value it creates a very unstable model. No other parameter has this sort of effect.

6.4. Besides varying the mean inflation rate, QMU, I vary all those initial values that are dependent on it, QI, YI, DM, CI and CM. Thus each variation starts in a neutral position with respect to QMU. The results are much as might be expected. The mean rate of inflation reduces or increases correspondingly, with rather little change in the standard deviation. However, the average mean rate of money return on shares is altered to a rather greater extent, and the standard deviation reduces or increases with QMU. The mean rate of real return on shares also reduces or increases to some extent; a high value of QMU produces a high average mean rate of real return on shares. The reverse is true for Consols: the mean rate of money return reduces or increases rather less than proportionately, and the mean rate of real return changes in the opposite direction; that is, a reduction in the mean rate of inflation increases the mean rate of real return on Consols and vice versa; a reduction in QMU also increases the standard deviation of both returns on Consols. It is interesting to see that the view that a high mean rate of inflation is good for shares and bad for Consols is true even when starting from a neutral position.

6.5. Variation in QSD, the standard deviation in the inflation model, has very little effect on any of the average mean rates, but naturally it reduces or increases all the standard deviations to some extent.

6.6. Table 6 shows the results for Variations T1 to U4, the variations of the initial parameters for the inflation model. Variations in the initial rate of inflation can be looked at in two ways. Variations T only change the value of QI, but not any of the other initial values associated with QI. Variations U also change these other initial variables, viz. YI, DM, CI and CM. Variations T can be thought of as representing a state where inflation has been stable with a force of 0·05 for some time, and there is a single unexpected change in the rate of inflation one year, not reflected in investment variables. Variations U represent a world where inflation has been stable at a mean rate other than 0·05 for some time, and investment variables reflect this, but the mean rate is then changed to 0·05 for

TABLE 6

Results for variations in initial conditions of Reduced Standard Basis

Variation	T1: QI = 0·00			T2: QI = 0·02			T3: QI = 0·07		
Term	1	10	20	1	10	20	1	10	20
Mean Rate of Inflation (GQ):									
E(GQ)	2·26	4·36	4·68	3·49	4·67	4·83	6·65	5·46	5·23
SD(GQ)	5·18	3·58	3·82	5·24	3·59	2·83	5·40	3·62	2·84
Mean Rate of Money Return on Shares (GPR):									
E(GPR)	11·30	9·39	9·41	11·78	9·68	9·57	12·99	10·39	9·97
SD(GPR)	22·66	5·37	3·82	22·76	5·39	3·83	23·01	5·42	3·84
Correlation Coefficient:									
C(GPR,GQ)	−0·24	0·45	0·62	−0·24	0·45	0·62	−0·24	0·45	0·62
Mean Rate of Money Return on Consols (GCR):									
E(GCR)	9·75	8·80	8·64	9·03	8·77	8·69	7·27	8·69	8·82
SD(GCR)	7·53	1·74	1·11	7·43	1·71	1·11	7·17	1·63	1·09
Correlation Coefficients:									
C(GPR,GQ)	−0·42	−0·62	−0·13	−0·42	−0·62	−0·13	−0·42	−0·61	−0·11
C(GCR,GPR)	0·11	−0·18	−0·01	0·11	−0·18	−0·00	0·11	−0·18	0·01
Mean Rate of Real Return on Shares (JPR):									
E(JPR)	9·39	4·87	4·53	8·55	4·82	4·53	6·49	4·72	4·53
SD(JPR)	24·25	4·79	2·91	24·07	4·79	2·91	23·61	4·78	2·91
Correlation Coefficient:									
C(JPR,GQ)	−0·45	−0·27	−0·19	−0·45	−0·27	−0·19	−0·45	−0·27	−0·19
Mean Rate of Real Return on Consols (JCR):									
E(JCR)	7·76	4·41	3·87	5·77	4·07	3·76	0·99	3·22	3·49
SD(JCR)	10·92	4·83	3·14	10·67	4·79	3·13	10·07	4·68	3·09
Correlation Coefficients:									
C(JCR,GQ)	−0·78	−0·96	−0·94	−0·79	−0·96	−0·94	−0·79	−0·96	−0·94
C(JCR,JPR)	0·35	0·30	0·21	0·35	0·30	0·21	0·36	0·30	0·21

TABLE 6 (cont.)

Variation	T4: QI = 0·10			U1: QI = 0·00 etc			U2: QI = 0·02 etc		
Term	1	10	20	1	10	20	1	10	20
Mean Rate of Inflation (GQ):									
E(GQ)	8·58	5·93	5·46	2·26	4·36	4·68	3·49	4·67	4·83
SD(GQ)	5·50	3·63	2·84	5·18	3·58	2·82	5·24	3·59	2·83
Mean Rate of Money Return on Shares (GPR):									
E(GPR)	13·73	10·82	10·22	4·75	7·08	8·16	7·79	8·28	8·82
SD(GPR)	23·16	5·44	3·85	21·36	5·26	3·78	21·97	5·32	3·80
Correlation Coefficient:									
C(GPR,GQ)	−0·24	0·45	0·62	−0·24	0·45	0·62	−0·24	0·45	0·62
Mean Rate of Money Return on Consols (GCR):									
E(GCR)	6·24	8·65	8·89	1·18	0·89	2·34	4·04	4·59	5·33
SD(GCR)	7·02	1·59	1·08	17·27	3·50	1·42	11·02	2·44	1·26
Correlation Coefficients:									
C(GPR,GQ)	−0·42	−0·61	−0·10	−0·42	−0·66	−0·35	−0·42	−0·65	−0·26
C(GCR,GPR)	0·11	−0·17	−0·02	0·11	−0·22	−0·16	0·11	−0·21	−0·09
Mean Rate of Real Return on Shares (JPR):									
E(JPR)	5·27	4·66	4·52	2·96	2·64	3·34	4·68	3·49	3·81
SD(JPR)	23·34	4·78	2·91	22·85	4·69	2·88	23·22	4·73	2·89
Correlation Coefficient:									
C(JPR,GQ)	−0·45	−0·27	−0·19	−0·45	−0·27	−0·19	−0·45	−0·27	−0·19
Mean Rate of Real Return on Consols (JCR):									
E(JCR)	−1·77	2·72	3·33	−0·44	−3·13	−2·14	1·01	0·09	0·56
SD(JCR)	9·73	4·62	3·07	19·84	6·19	3·39	13·74	5·32	3·25
Correlation Coefficients:									
C(JCR,GQ)	−0·79	−0·96	−0·94	−0·61	−0·90	−0·92	−0·70	−0·93	−0·93
C(JCR,JPR)	0·36	0·30	0·21	0·28	0·29	0·20	0·32	0·30	0·20

TABLE 6 (cont.)

Variation	U3: QI = 0·07 etc			U4: QI = 0·10 etc		
Term	1	10	20	1	10	20
Mean Rate of Inflation (GQ):						
E(GQ)	6·65	5·46	5·23	8·58	5·93	5·46
SD(GQ)	5·40	3·62	2·84	5·50	3·63	2·84
Mean Rate of Money Return on Shares (GPR):						
E(GPR)	15·77	11·34	10·48	20·84	13·22	11·49
SD(GPR)	23·56	5·47	3·86	24·58	5·56	3·90
Correlation Coefficient:						
C(GPR,GQ)	−0·24	0·45	0·62	−0·24	0·45	0·62
Mean Rate of Money Return on Consols (GCR):						
E(GCR)	10·36	11·07	10·74	13·78	14·30	13·44
SD(GCR)	5·94	1·37	1·05	4·67	1·08	1·01
Correlation Coefficients:						
C(GPR,GQ)	−0·42	−0·58	−0·03	−0·42	−0·50	−0·08
C(GCR,GPR)	0·11	−0·15	0·06	0·11	−0·10	0·14
Mean Rate of Real Return on Shares (JPR):						
E(JPR)	9·10	5·62	5·01	11·85	6·93	5·73
SD(JPR)	24·18	4·83	2·92	24·77	4·88	2·94
Correlation Coefficient:						
C(JPR,GQ)	−0·45	−0·27	−0·19	−0·45	−0·27	−0·19
Mean Rate of Real Return on Consols (JCR):						
E(JCR)	3·87	5·48	5·31	5·15	8·04	7·64
SD(JCR)	9·17	4·51	3·05	8·16	4·32	2·99
Correlation Coefficients:						
C(JCR,GQ)	−0·83	−0·97	−0·94	−0·87	−0·98	−0·95
C(JCR,JPR)	0·37	0·30	0·21	0·39	0·30	0·21

the future. Both these sets of variations affect the results for short terms considerably, with the long term results tending towards those of the Standard Basis. Thus in Variation T1 (QI = 0·0) the average mean rate of inflation over one year is 2·26%, rising to 5% by term 100. The standard deviation is not much altered. The average mean money rate of return on shares is reduced at term 1, but rises to its normal level with increasing term; the average mean real rate of return on shares, however, is high at term 1, and falls to its normal level with increasing term. In contrast the average mean rates of return on Consols, both money and real, are high at term 1, and fall to their normal level with increasing term. All these statements are reversed when QI is high. Thus, starting from an unexpectedly low rate of inflation is good for both shares and Consols in real terms, and vice versa, but the effect on money returns is inconsistent.

6·7. Variations U show just the same picture as Variations T for the rate of inflation, but quite a different one for returns on shares and Consols. Variation U1 shows a low starting value of QI (QI = 0·0) and of the other initial variables. The average mean rate of return on shares, both money and real, is very low initially, rising slowly to a normal level. The mean rate of money return on Consols is only just positive in the early terms, and the average mean rate of real return is negative up to term 30. The position is entirely reversed with a high set of initial values. These variations show that an unexpected change in the level of the mean rate of inflation is poor for shares and extremely bad for fixed interest investments.

6.8. Variations in the remaining parameters do not affect the results for inflation, and indeed only affect their own variables. Thus the variations in the parameters for the yield and dividend models only affect the money and real returns on shares. Their effect can be summarised: the mean rate of return, money and real, on shares is increased by increases in YW, YMU, DW, DX and DMU; the standard deviation of return is increased by increases in YA, YSD, DD and DSD, and by a decrease in DY. DW and DX have a distorting effect on the standard deviations; an increase in either of them increases the standard deviation of money return and reduces the standard deviation of real return. They also alter the correlation coefficients between GPR and GQ, and between JPR and GQ, as do DD and DSD in a variety of ways.

6.9. Changes in the values of the initial parameters for shares have a very marked effect. When the initial yield, YI, is low the average mean money returns and real returns on ordinary shares are very much reduced, being substantially negative over the short term, and never quite catching up. The reverse is true when the initial yield is above average. This effect, however, should be considered jointly with the effect of YEI, which is in effect the unexpected change in yield in the preceding year. If the yield has fallen unexpectedly (YEI is negative) this presages a rise in dividends, and a good return on shares; the reverse is true if share yields have unexpectedly risen (YEI positive). A high carry forward of inflation, DM, also gives an increase in the average mean rates of return on shares.

6.10. Variations in the Consols parameters affect only the returns on Consols, and the correlation coefficients between them and inflation and the return on shares. The results are in some ways more complex to describe. The standard deviation of return is increased by increases in CA1 and CSD. An increase in CMU increases the mean returns correspondingly, and also distorts the standard deviation of return, increasing it at short terms and decreasing it at longer terms. An increase in CD increases both the mean and the standard deviations. An increase in CW distorts the pattern both of mean and of standard deviations, reducing them at short terms and increasing them at longer terms. Changes in all these parameters also have differential effects on the correlation coefficients, particularly those of GCR with GQ and with GPR.

6.11. Varying the initial parameters for Consols has some expected effects: an increase in the initial yield, CI, increases the mean returns, both money and real. But it also differentially affects the standard deviations and correlation coefficients. An increase in the carry forward effect of inflation, CM, also has differential effects, reducing the mean return at shorter terms and increasing it at longer terms.

6.12. It would be of interest at any particular time to investigate the results using the complete set of current values as initial values for these simulations. However, to report on such results for any one date would be only of historic interest by the time this paper is published, and it would invite comparison with the actual outcome since that date. This would be to misunderstand an " expected value ", which cannot be verified by a single observation. For verification of the model in this specific way one would need to find a series of occasions when current conditions were the same, and observe the outcomes over a reasonable number of non-overlapping periods.

7. APPLICATIONS

7.1. The stochastic model for investments described above can be used by actuaries in almost any circumstance where a rate of interest enters their calculations at present. But it opens up wider possibilities for investigation too. I shall describe some such applications below.

7.2. A stochastic model has, of course, already been used for estimation of the contingency reserves necessary for unit linked assurance policies with performance guarantees. The approach was fully documented in the Report of the Maturity Guarantees Working Party, and it has subsequently been used by a number of life offices, as is evidenced by their Returns to the Department of Trade. The model put forward in this paper is quite compatible with that proposed by the Maturity Guarantees Working Party for ordinary shares alone, provided that a suitable basis is chosen. The essential feature is the choice of the mean rate of inflation, QMU, of 0·04, rather than 0·05.

7.3. A second practical use has been described in a paper prepared for the 22nd International Congress of Actuaries (Wilkie, 1984), which uses a stochastic model for inflation to estimate the cost of providing minimum money guarantees for annuities where the benefit is linked to the price index, but for example does not decrease if the price index decreases.

7.4. The conventional " portfolio selection " model, first developed by Markowitz (1959) and described in many modern books on investments, assumes in its simplest form a single time period for investment and a set of possible investments whose expected returns, variances and covariances are known. If we assume that share dividends are always reinvested in ordinary shares and that interest on Consols is always reinvested in Consols, then the simulation returns already quoted give enough information to select optimum portfolios for a specific duration.

7.5. However, such a simplification is unnecessarily unrealistic. There is no commitment to reinvest income arising from an asset in further purchases of the same asset. Investors do not make a single investment decision and leave it for a large number of years without review. The stochastic model allows for the investigation of various investment strategies. One can try to devise simple (or complex) trading rules that would allow one to construct an optimal portfolio strategy over some suitable time period. The simplest such strategy might be to invest the interest received each year in new purchases of assets in predetermined proportions. This is the least " dynamic " strategy that one might adopt. The next step up is to postulate a decision rule which determines the proportions to be invested in each class of asset depending on the current yields, or on recent performance, or some other features of the experience that depend on the particular simulation, and can be assumed to be known at the time the decision is made. A further elaboration would be to admit the possibility of switching already purchased assets from one class to another, possibly allowing for transaction costs too.

7.6. Any such decision rule or set of rules, in which the investment decisions to be made depend in some way on the actual experience of each simulation, can be described as a " dynamic investment strategy ". As can be seen from the preceding description such strategies can be more or less dynamic in that they can allow for more or less flexibility in the assumed movement of investments, and in the amount of information in the experience taken account of.

7.7. One must beware, however, of relying too greatly on the successful outcomes of any such strategy. If such a profitable strategy can be found, then presumably any investor may be able to find it. The collective decisions of many investors may alter the parameters of the model in such a way that the profitable strategy disappears. It seems to me, without having fully investigated such strategies, that a likely such successful strategy would be to buy shares when dividend yields are high, and sell when dividend yields are low. If such a policy were widely followed the effect would be to stabilise share dividend yields much more closely around the mean level. This might result in either a lower value of QA, implying a more rapid return to a mean yield whenever a divergence occurred, or a lower value of QSD, implying less variation away from the mean position, or possibly both. There is some slight evidence that this is already occurring, in that dividend yields in the years from 1975 to 1983 have fluctuated within a much narrower band than over preceding years. This could be just by chance, but could be the beginning of a change in the parameters. A similar argument would apply to fixed interest yields, at least as far as the " real " part is concerned, where again a lower value of CA1 or of CSD would produce greater stability. However, interest rates are also influenced by many other economic factors besides the actions of investors (as indeed are share prices), and one must still expect some random fluctuation to continue, in such a way that mechanical investment strategies guaranteed to be profitable cannot be devised.

7.8. From investments I now turn to life insurance. The stochastic model can be of use in assessing the premium for any kind of without profits contract. One can make deterministic assumptions about mortality and expenses, possibly relating them to inflation, and then assume a reasonable, but not too optimistic, investment strategy (fixed or dynamic). On the basis of each simulated future experience one can calculate what premium would have been enough to provide the specified benefits. This gives an empirical frequency distribution for the premium.

7.9. What level of premium an office should then charge is worth further consideration. The mean of this empirical distribution would be too low, since in many cases the premium would be insufficient. Since all policies entering at the same date experience the same investment returns, there is no averaging over simultaneous policies, in contrast to the position with mortality, where a large number of lives can be assumed to die or to survive independently of one another. One approach to selecting the premium is to choose the premium that has a suitably high probability of being sufficient. But this means overcharging on most occasions. Another approach is to assume that shareholders (or some other capitalists) provide additional capital, both for initial expenses and for contingencies, on which an additional rate of return must be earned. Reserves can be set up on a very strong basis, and surplus released each year if the experience is satisfactory. The value of the policy to the capitalist is then the present value of surplus, discounted at a suitably high rate of return (which in turn may depend on the particular simulation). This then gives the amount of capital to be put up; the policyholder must provide the rest.

7.10. A similar technique can be used for determining premiums for with profit policies, but the possibility of profits raises a further complication. One could assume a predetermined bonus rate, thus turning the policy into a without profits policy with an increasing sum assured. But this is hardly realistic. In fact bonus rates depend substantially on investment performance. It is, therefore, necessary to devise a " dynamic bonus strategy ", in which the decisions that an actuary might take about the declaration of bonus are reproduced in some realistic but mechanical way, taking account of the actual experience so far within the simulation. I do not suggest that the actuary can be replaced by mechanical rules; rather, that the actuary can investigate what would happen if he were to apply mechanical rules himself, in order to see the consequences of any set of principles he may wish to establish, and to estimate the probability of his having to alter his principles.

7.11. Besides being used for the determination of premiums, the stochastic model can be used for valuation. At a minimum level it can be used to assess the solvency of a life office, with an existing portfolio of liabilities and of assets, and on the assumption both of a reasonable, but not too optimistic, dynamic investment strategy and of a sensible dynamic bonus strategy. The actuary wishes to ensure that the office will neither run out of assets, nor become technically insolvent before then by having insufficient assets to satisfy a reasonable minimum valuation basis, which could well be a statutory one, and whose interest basis could depend on the current yields on the porfolio of assets resulting from the particular simulation. But it is not enough to say that the office will be able to meet its liabilities at the very worst if the office declares no future bonus; one must bring in the realistic assumption that bonuses will continue to be declared in accordance with the dynamic bonus policy provided there is sufficient surplus to support them. A stochastic investment model allows investigation of solvency on realistic lines, conditional on the dynamic investment strategy and dynamic bonus strategy chosen. It also allows for the investigation of different dynamic investment and bonus strategies, in order to see whether a more desirable strategy can be found. The Faculty Solvency Working Party has made investigations along these lines, and this paper has been in part a contribution to those investigations, which were reported at last month's sessional meeting.

7.12. Besides the investigation of minimum solvency levels the stochastic model allows the investigation of desirable bonus strategies, and can therefore give guidance on an appropriate bonus to declare. The details of the model would be very similar to that used to investigate solvency, but with a different objective. An office presumably wishes to have a very low probability of becoming insolvent, and this puts a barrier on its possible actions. But for its bonus policy it may wish to employ an optimal approach, in the sense of trying to maximise the expected bonus subject to minimum risk, or vice versa. There are additional considerations of equity between generations, and the desirability of the office not accumulating too large an estate.

7.13. A stochastic investment model can also be used to investigate the cost of all sorts of investment options. The maturity guarantee on a unit linked policy is one such; so also are the index linked annuities with minimum guarantees discussed above. But flexible endowment policies, discussed by Ford and Masters (1979), guaranteed annuity options and guaranteed surrender values are other examples.

7.14. An office writing linked business can use the model to investigate many other aspects besides the maturity guarantee one. Where the sum at risk on death is in some way dependent on the value of the units, one can use a stochastic model for the unit prices to assess the mortality cost. There is usually a management charge which depends on the unit values, but actual expenses may be proportional to retail prices (or assumed to grow faster or slower than the Retail Prices Index). The use of a stochastic model allows investigation of the relationship between charges and expenses so as to determine the probability that the one is insufficient to cover the other.

7.15. Similar techniques can be used in pension fund work, to investigate the effect of alternative strategies of investment, or to investigate the degree of solvency of the fund, or the sufficiency of any chosen contribution rate. However, in order to complete the picture for a pension fund one would have to postulate a rate of increase of earnings in excess of prices. The traditional actuarial method, of using a constant percentage for this, could be followed. Alternatively, one could investigate the stochastic model for earnings as a function of prices, taking into account possible connections with growth of dividends. In considering the future of a pension fund one might also wish to consider the development of a fund open to new entrants, rather than, as is traditionally done in Britain, a closed population. This problem has received little discussion in British actuarial circles, though it is covered fully by Müller (1973).

7.16. The technique of matching assets to liabilities in a stochastic environment has been discussed by Wise (1984) in a recent paper presented to the Institute. It should be possible to use my model as an input to his technique. Indeed, since I have two types of asset, shares and Consols, each of which can be sold to " mature " in any year, my asset base is a particularly full one. However, at the time of writing I have not had an opportunity to combine my own model with that of Wise.

7.17. I hope that the suggestions above stimulate others to make use of what I consider to be an exciting new tool.

8. ACKNOWLEDGEMENTS

This model has been long in the gestation, and many others have contributed to its development. I should like to thank my colleagues on the Maturity Guarantees Working Party, who examined critically the first stages of development, and also Mr. E. J. Godolphin of Royal Holloway College, with whom I had many useful discussions at the time the MGWP was working. More recently my colleagues on the Solvency Working Party have been exposed to my explanations, and have commented critically and helpfully on the draft paper. Gordon McLeod of Gwilym Jenkins & Partners contributed by preparing his own report on the variables and, in the course of doing that, by teaching me much about transfer functions and multivariate models. Finally I should like to pay tribute to the late Gwilym Jenkins himself, whose work in all modern time series

analysis has been seminal, and whose conversations and correspondence shortly before his untimely death in 1982 were enormously stimulating to me.

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DISCUSSION

Professor A. D. Wilkie, introducing his paper, said:—It gives me great pleasure to present this paper on A Stochastic Investment Model for Actuarial Use to the scrutiny of the profession in this hall.

It has taken a long time to reach the model described in this paper. I think it started about 1972, when I had recently been appointed Economics Research Manager of a Scottish Life Office. I told my friend Sidney Benjamin, in passing, that I was supposed to produce some sort of economic and investment forecasts. He replied: "But it's all random anyway isn't it?" I respect Sidney's views sufficiently for me to take a seemingly facetious remark seriously, and I started to study the literature on random walks in the stock market.

At that time the response of many investment analysts to the words "random walk" was: "It can't be true, or we would all be out of a job". I approached the matter rather differently, thinking instead: "If it is true, how does this affect the actuarial management of a life office?" Actuaries, after all, ought to be used to dealing with random events; that is what our training is all about.

At the time one topical problem was maturity guarantees on linked life policies. A research paper, prepared by Sidney Benjamin, and presented to a closed meeting at the Institute (but later published in part in the proceedings of the Tokyo Congress), stimulated me to write a paper for that Congress which looked at maturity guarantees with a simple theoretical investment model.

Later, both Sidney and I were members of the Maturity Guarantees Working Party, which produced a very long report that was discussed at the Faculty in 1980. My contribution to that report was to derive the separate models for share dividend and share yield that seemed to fit the facts better than the previously suggested models that looked only at share prices.

One of the criticisms made of that model was that it took no account of inflation, which clearly ought to have some influence on share dividends. I therefore turned my attention to looking at inflation. The results of some of those investigations appeared in a paper on "Indexing of Long Term Financial Contracts", presented to the Faculty in 1982. This began to establish the third leg of tonight's model.

The next stimulus came through being asked to join the Working Party on Solvency which reported to you last month. In order to carry out the sorts of calculation that that Working Party wanted, it was clearly necessary to look at inflation, share dividends, share yields and fixed interest yields as a whole. We should have liked to include both long-term and short-term interest rates, but it eventually proved difficult to get a satisfactory long series for short-term interest rates, and this feature, which is rather like having an extra room in the house, remains to be built.

It seemed sensible to find out how others who were expert in time series analysis might approach the problem, so I got my office to commission a report from Gwilym Jenkins & Partners, the firm founded by the late Gwilym Jenkins, who, along with George Box, has possibly contributed more to time series analysis than anyone else. Their report was very helpful, and taught me a lot about the analysis of multiple time series. But I also learned that the objective of Box-Jenkins' methods, which is getting the best one period ahead forecast, is not the same as I was wanting. I wanted to find a satisfactory long-term structure for my variables, even if this did not give me the best one step ahead forecast.

So I started again, and produced a model that satisfies me, and I think also satisfied my colleagues on the Solvency Working Party. It resembles the model produced by Gordon McLeod of Gwilym Jenkins & Partners only in some respects, and is clearly different from it in others.

So much for the background. You will see how much Sidney Benjamin has influenced the progress of this study. I am only sorry that other commitments prevent him from being here this evening.

Now for the paper. I should perhaps explain to those who find even the title formidable that "stochastic" just means that something varies randomly with time. And by "randomly" I do not mean wholly and utterly unpredictably, but rather in accordance with some stated and known or estimated probability structure. Successive tosses of a coin are an example of a simple stochastic sequence; so are the numbers of deaths notified each week to a life office; so is the lapping of waves on a harbour wall, which bob up and down with a visible short-term periodicity driven by random wave movements, but which are also subject to the long-term periodicity of the tides whose exact amplitude on any occasion is also random, depending on wind and weather; so also is the movement of dust particles seen in a shaft of sunlight coming through the windows of a great church; so is the demand for electricity from minute to minute of the day; and so too are share prices and other investment variables.

My model is driven by four independent "white noise" series, which enter my model and produce as output the four economic series I discuss, the price index, the share dividend index, the share yield, and the yield on irredeemable fixed interest stock, that I describe as "Consols".

Those who know about electrical systems analysis and control theory (which I realise may be few in this hall) will be familiar with such systems, and may recognise my model as one that can be represented by a state space or state variable model. However, because of the logs and exponentials that I have included, it is not a wholly linear model, and non-linear models are somewhat harder to analyse.

We have to take the investment variables as given, but when we feed them in to the system that we call a life office or a pension fund, things are more under our control. The dynamic investment policy and dynamic bonus policy that I describe in Part 7 of the paper can be thought of as corresponding to control systems with dynamic feedback.

But this opens up yet another new subject. In order to get this far I have had to learn something about time series analysis, something about statistical estimation with non-linear models, and the numerical analysis that goes along with that, quite a lot about what I call the "classical" models of financial economists, and now something about stochastic control theory. I think this is an example of how actuaries can push forward the frontiers of their knowledge only by going outside their own discipline. Actuarial education is too inward-looking and actuarial practice too much concerned with the very necessary day-to-day management of institutions to achieve on their own such advances as I hope this is. But I hope that there are among the younger of you some, perhaps many, whose mathematical equipment is more powerful than mine, and who may be able to see how to use what I am sure you have already learned in applying what I can see is an enormously exciting model.

The applications of my model are legion. Another that was brought to my attention last week is to aid life offices to make bonus forecasts, both on a consistent basis between offices, and such as to produce a range of possible values, rather than a point forecast.

I said in the paper that it was inappropriate to produce simulations based on the initial conditions at any particular date. I have not taken my own advice. The table below gives the results of simulations using the parameters of the Full Standard Basis, with the initial conditions as at close of business on Friday, 16th November 1984, taking into account the just published Retail Prices Index for October. I can safely publish these at this meeting, since we have not had enough experience for me to be proved wrong. Even in a year's time all that you will find is that the observed values of the variables differ from my expected values by a certain number of standard deviations.

A lot of the statistical background, and a lot more tables and diagrams, are to be found in the supplementary note "Steps Towards a Comprehensive Stochastic Investment Model" referred to in the paper. Its 231 pages do not actually contain all that much text.

One of the sets of figures given in that note shows the results of ten simulations on the Reduced Standard Basis, starting at a neutral position. The corresponding figures on the Full Standard Basis using last Friday's initial conditions are shown below too. You will see how quickly you can get an impression of the possible spread of results from quite a small number of simulations. But if I added more and more simulations some of them would be well outside the broad band that you can see developing in the figures.

The future is uncertain. But as actuaries we have learned how to manage financial institutions in such a way that the uncertainty of mortality that affects each of us individually can be dealt with satisfactorily on a collective basis. We have not yet applied the same techniques to the management of the investment side of the businesses we advise. I hope that this paper gives us some of the tools with which to do this, and I await your response to it.

I am sorry to have taken so long to introduce the paper, but I felt that it was necessary. I hope it has been helpful.

TABLE 1

Results on Full Standard Basis as at close, 16th November 1984

Term	1	5	10	15	20	30	50	75	100
Mean Rate of Inflation — GQ..									
E(GQ)	5·59	5·45	5·18	5·19	5·09	5·04	5·00	5·06	5·08
SD(GQ)	6·66	4·83	3·75	3·21	2·88	2·40	1·81	1·54	1·35
Mean Rate of Money Return on Shares Rolled Up — GPR									
E(GPR)	13·03	10·55	10·26	9·97	9·89	9·75	9·69	9·74	9·73
SD(GPR)	24·66	9·45	6·19	4·93	4·21	3·53	2·71	2·21	1·99
Correlation Coefficient									
C(GPR,GQ)	−0·01	0·34	0·49	0·57	0·62	0·69	0·75	0·76	0·77
Mean Rate of Money Return on Consols — Rolled Up — GCR									
E(GCR)	8·45	8·85	8·90	8·83	8·84	8·81	8·75	8·69	8·70
SD(GCR)	8·28	4·51	3·41	2·92	2·64	2·23	1·81	1·51	1·36
Correlation Coefficients									
C(GCR,GQ)	−0·35	−0·33	−0·21	−0·08	−0·01	0·16	0·41	0·53	0·62
C(GCR,GPR)	0·10	0·09	0·07	0·13	0·15	0·26	0·40	0·48	0·54
Mean Rate of Real Return on Shares — Rolled Up — JPR									
E(JPR)	7·48	4·92	4·86	4·56	4·58	4·48	4·47	4·45	4·42
SD(JPR)	24·41	8·65	5·22	3·90	3·15	2·43	1·71	1·36	1·21
Correlation Coefficient									
C(JPR,GQ)	−0·29	−0·21	−0·17	−0·14	−0·11	−0·03	0·07	0·06	0·10
Mean Rate of Real Return on Consols — Rolled Up JCR									
E(JCR)	3·30	3·51	3·69	3·57	3·65	3·63	3·59	3·47	3·45
SD(JCR)	11·90	7·38	5·44	4·39	3·81	2·93	1·91	1·43	1·15
Correlation Coefficients									
C(JCR,GQ)	−0·78	−0·83	−0·81	−0·77	−0·75	−0·69	−0·57	−0·53	−0·46
C(JCR,JPR)	0·28	0·30	0·25	0·24	0·22	0·17	0·09	0·08	0·07

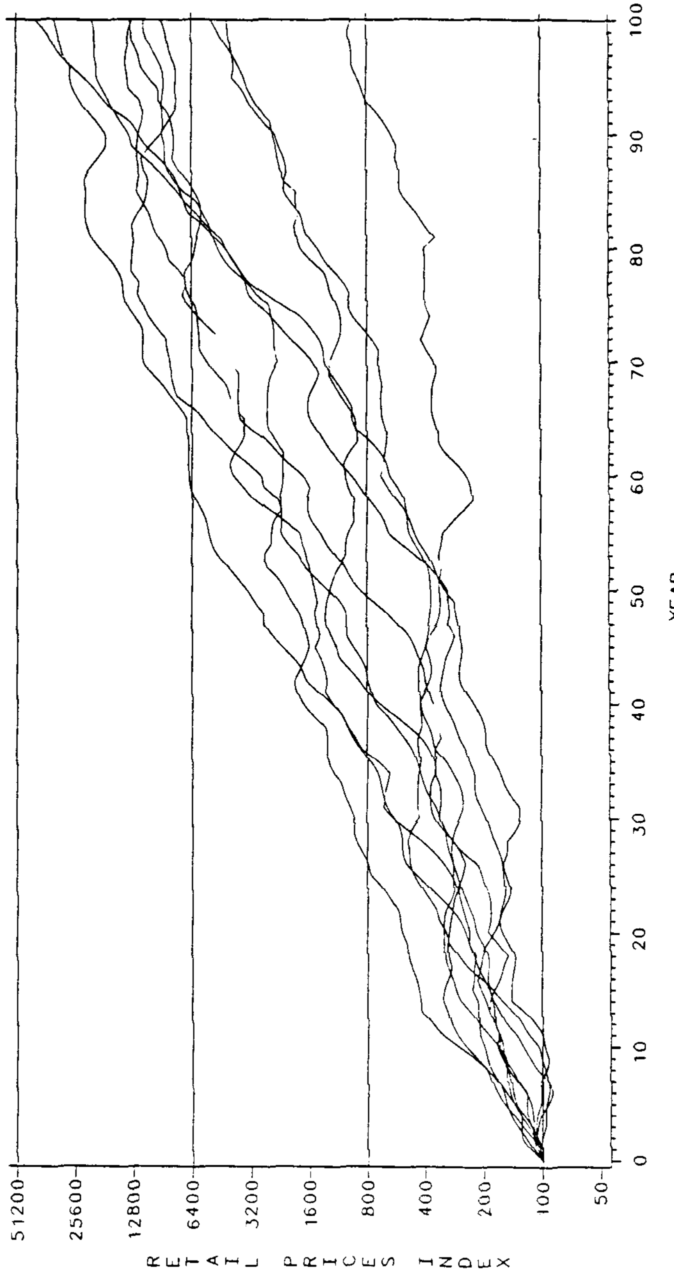


FIGURE A.1
Ten Simulations of Retail Prices Index for 100 Years using Full Standard Basis and Closing Prices of 16th November 1984.

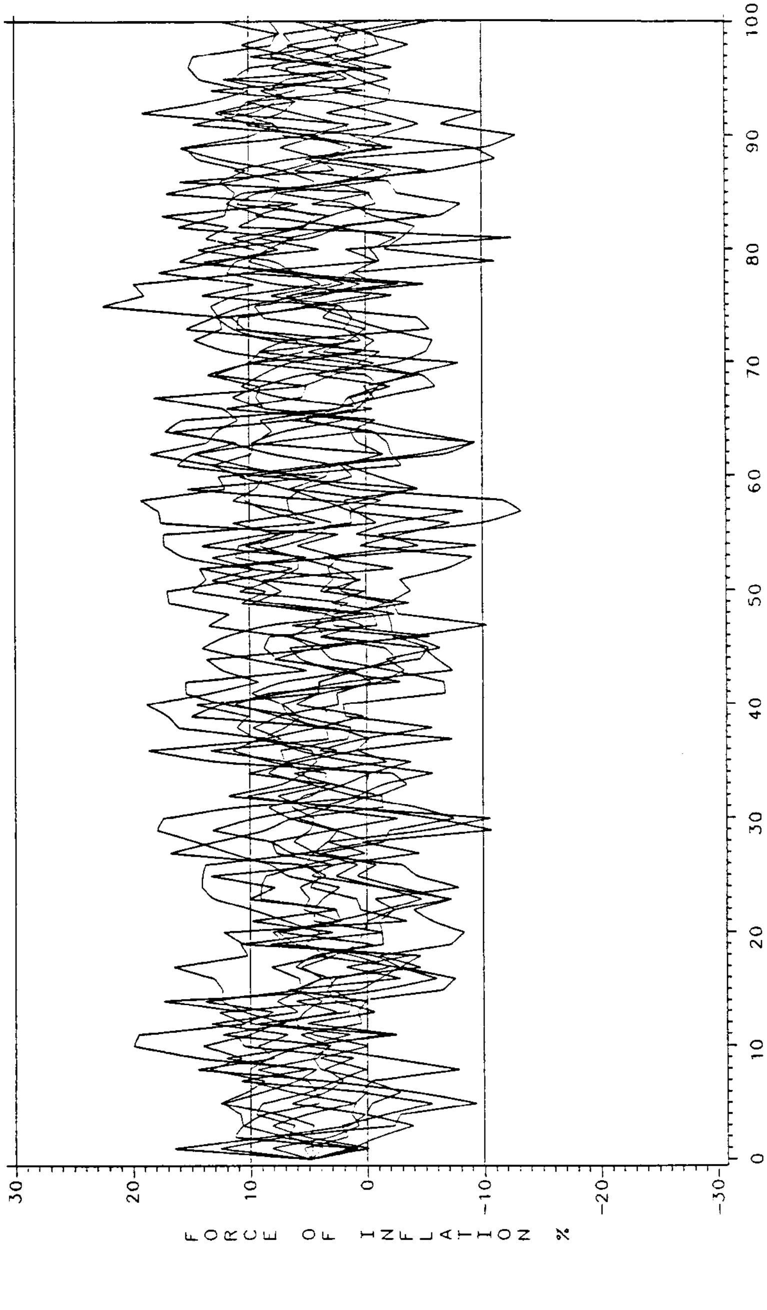


FIGURE A.2

Ten Simulations of Force of Inflation % for 100 Years using Full Standard Basis and Closing Prices of 16th November 1984.

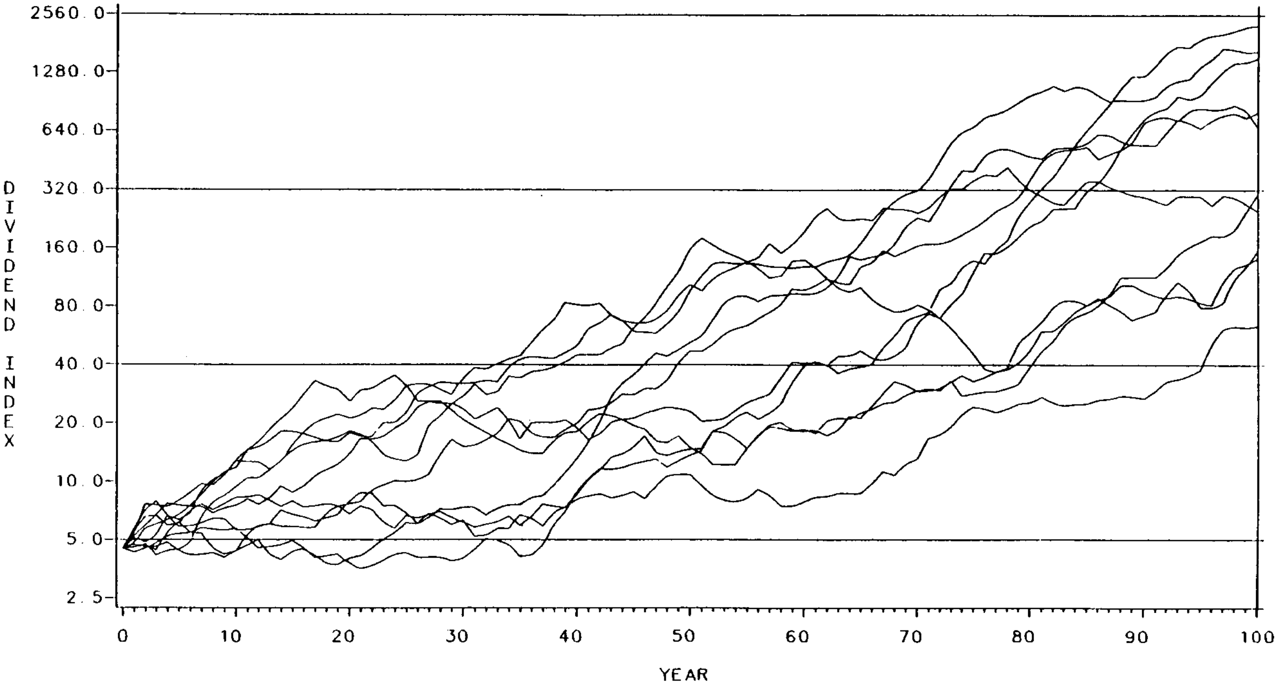


FIGURE A.3
Ten Simulations of Share Dividend Index for 100 Years using Full Standard Basis and Closing Prices of 16th November 1984.

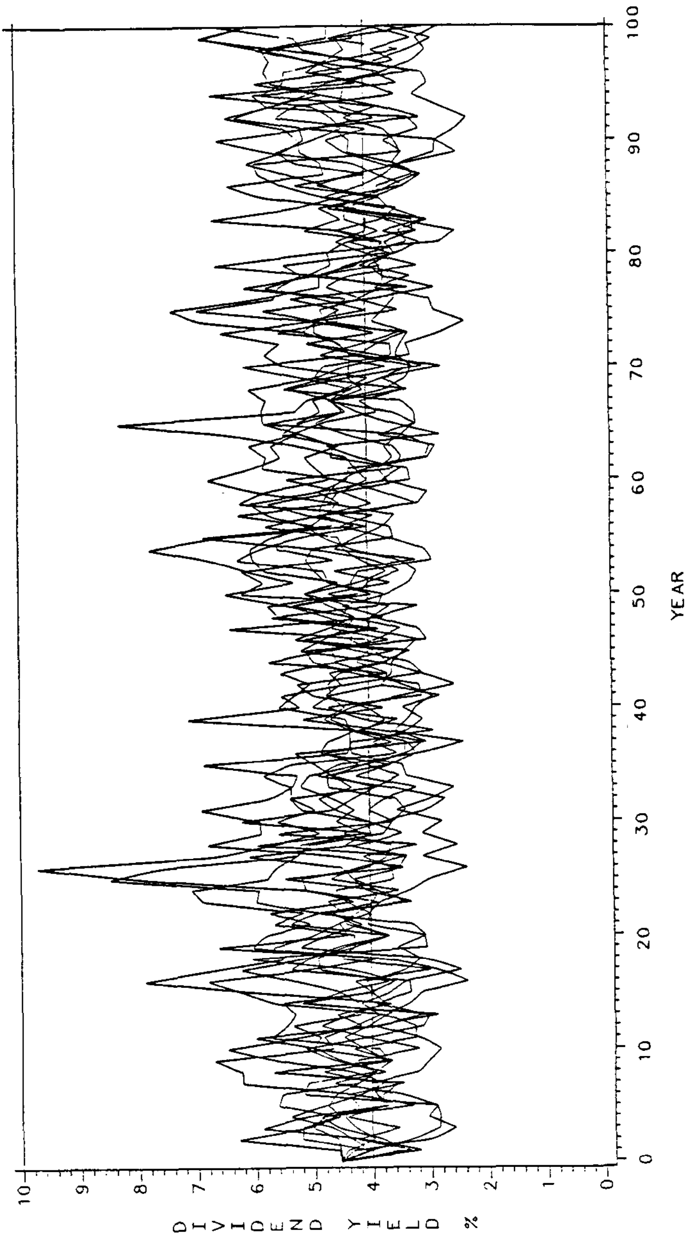


FIGURE A.4
Ten Simulations of Share Dividend Yield % for 100 Years using Full Standard Basis and Closing Prices of 16th November 1984.

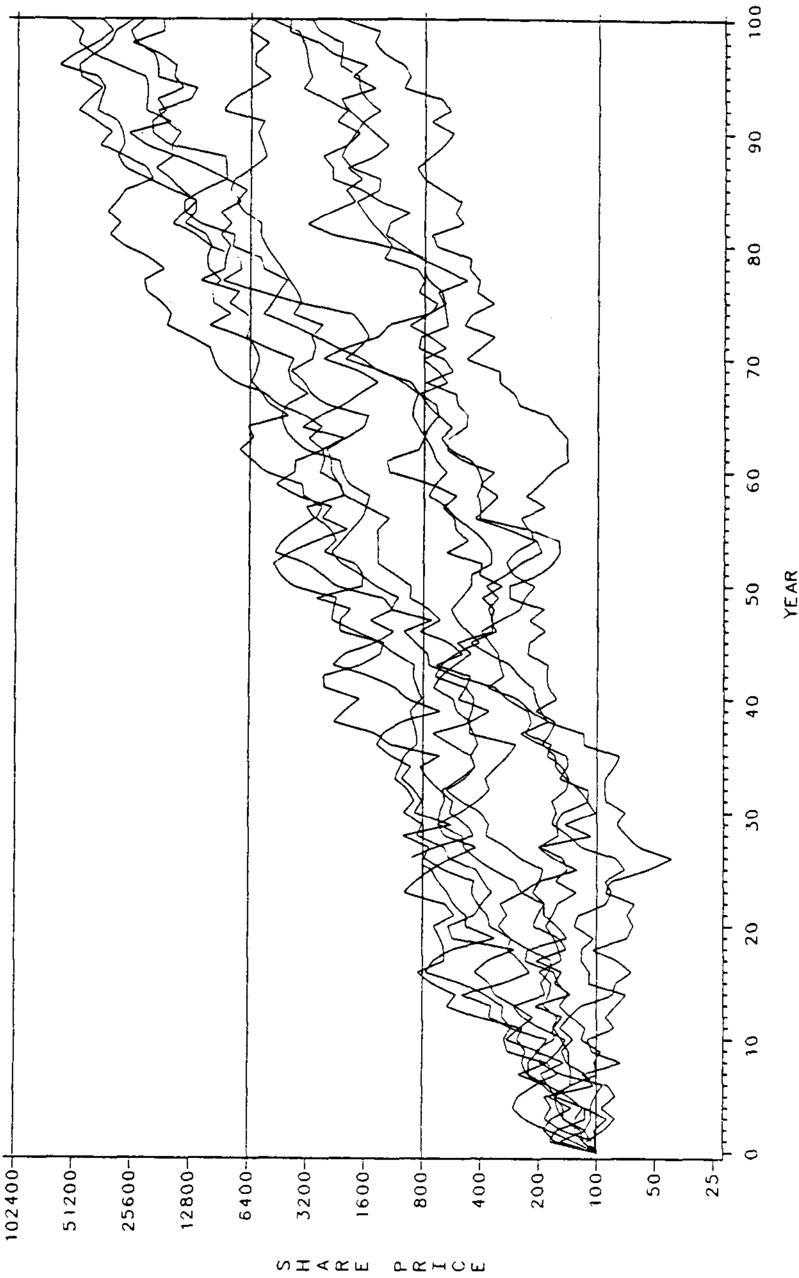


FIGURE A.5
Ten Simulations of Share Price for 100 Years using Full Standard Basis and Closing Prices of 16th November 1984.

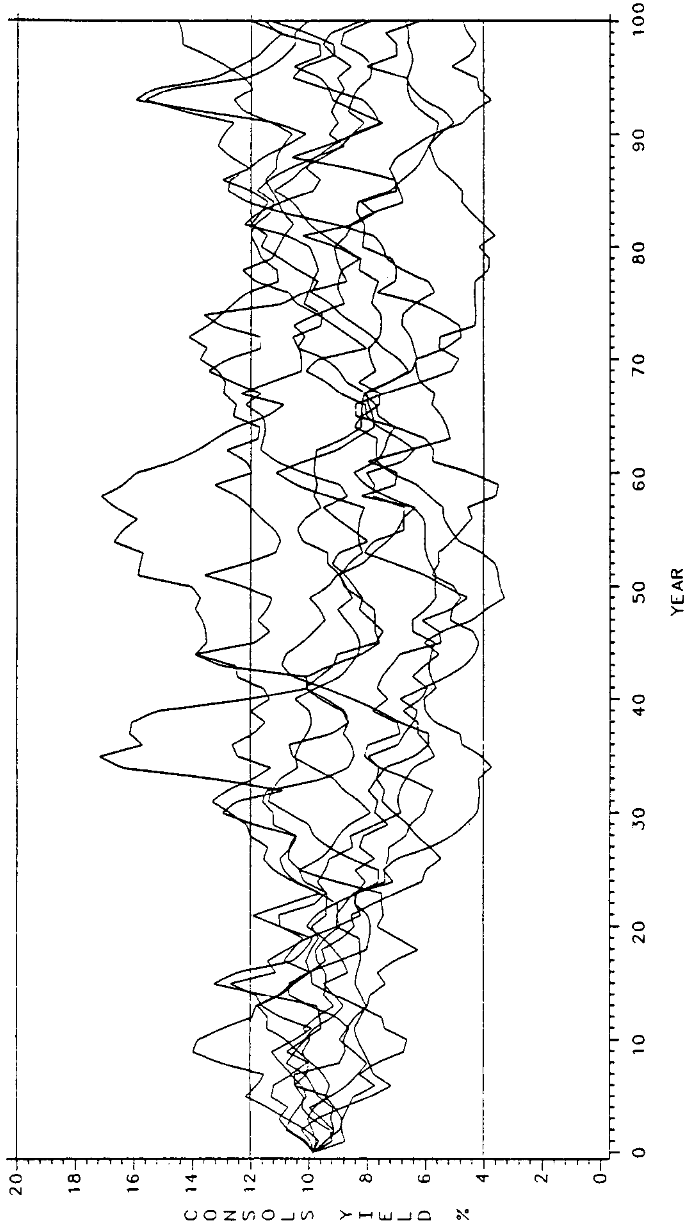


FIGURE A.6
Ten Simulations of Consols Yield % for 100 Years using Full Standard Basis and Closing Prices of 16th November 1984.

Mr. C. W. McLean, opening the discussion, said:—It is with pleasure that I open the discussion tonight on this, latest work of our most prolific researcher and author. I must congratulate Professor Wilkie on his paper which I have no doubt will in time be recognised as a landmark in actuarial thought. The need for this model was indicated in previous papers presented here and the background to this work was described in the Report of the Working Party on The Solvency of ~~Life Assurance~~ Companies, presented in this hall last month. Indeed, in the discussion on that Report it was suggested that we had been discussing Acts 1 and 3 of Hamlet, with tonight's paper as the missing Act 2. The play, however, is somewhat shorter.

While a fully independent stochastic investment model has been the long-sought 'holy grail' of our profession, its use when found will be so widespread that it is essential that all claims to provide such a model are carefully scrutinised. We must decide if we have been presented this evening with such a suitably firm foundation. Thus if I appear critical in a number of my remarks it is because of the importance of the paper and because my acceptance of the stochastic approach is implicit.

The paper can be analysed from two approaches: the rigour of the model itself and, secondly, the particular use the author has made of it. It is necessary therefore to appraise the assumptions and conclusions. Regarding the former, the principal aim (Section 1.3) is that the model should produce realistic results for the long term, based on plausible economic and investment assumptions. Specifically no claim is made with regard to short-term forecasts although I am sure many may still be uneasy with regard to the concept of describing a long-term result while knowing nothing about the series of short terms which it comprises. It is not stated precisely what length of time might be considered the long term but I shall return to this later.

What Professor Wilkie does claim at an early stage is that "features that may be statistically significant but that do not alter the long-term structure of the model may be omitted". Economic data is notoriously difficult to work with, with autocorrelation, leads and lags, multicollinearity, and endogenous variables. For this reason, econometrics is properly considered to be a separate branch of statistics, and correct specification of the economic relationships involved in any economic analysis is essential for statistical rigour. I am convinced that the stochastic approach avoids these problems.

I now turn to the assumptions, detailed in Section 2, which characterise the general features of the model. While I suspect that the results of the M.G.W.P. quoted in Section 2.3 have fitted recent experience less well, the first heroic assumption appears to be made in Section 2.4, where the author relates dividends to the retail prices index: 'other things being equal'. This assumes much more than the absence of dividend restraint, involving productivity of the different factors of production (known to exhibit secular change) and of course corporate gearing. Trends in these influences can certainly persist for decades. Similarly in Section 2.5 I am unconvinced by the justification of the postulated relationship between interest rates, share prices and inflation. The research is described in general in the supplementary "Steps Towards" note but few specific reasons are given for specifying the inflation:share price relationship as one-way.

Section 2.7 correctly cites the work of Sargent (1973) in support of the lagged relationship of inflation to yield on Consols. This comprehensive, though slightly dated, work by Sargent also draws less reassuring conclusions. In particular his empirical results, and I quote, "casts considerable doubt on the adequacy of the hypothesis that there is a single direction of influence, one flowing from inflation to interest". He goes on to describe an approach that accommodates the apparent feedback from interest to inflation.

While the use of a single long-term interest rate to represent the term structure and the exclusion of property are necessary simplifying assumptions the assumption, of Section 2.10, that overseas equities be excluded is more demanding. The fact is that our economy *is* an open one; while international capital flows and the exchange rate mechanism can be stabilising influences in the long run they could well disturb the model. In general, the explicit assumptions of Section 2 of the paper are reasonable in order to establish the minimum working model. It is, however, important that we remember that implicit assumptions are involved also, which move the model further from the real world. The selection of the parameters has involved a good deal of subjectivity, as described in Section 2.12 and the separate note, and it has not been demonstrated that they represent the empirical conclusion of a fully specified and internally consistent econometric model. Judgement of this can best be made on the long-term results.

The Retail Price Index itself contains an interest rate element, although this is of short-term, mortgage interest rates. It is, however, relevant (being 4·2% in 1981) and that suggests a degree of simultaneity. The model for inflation itself is a reasonable one, although it is possible that independent events could influence the persistence, or autoregression, of inflation rates. I would suggest that expectations could be strongly influenced by external or political factors leading, for example, to the term of collective bargaining agreements varying according to expectations of such of a sea-change. Experience in the U.K. and U.S.A. over the last decade supports this, which is one of a number of ratchet effects which destabilise inflation, particularly at higher levels. The specific values chosen for the parameters QMU, QA and to a lesser extent QSD should be treated with some caution. I do not agree with the statement in Section 3.4 that "there is fairly little uncertainty about the appropriate value for QA" for these reasons. It could hold different values if the inflation rate was moving sharply in one direction or if external influences convince the real "moving forces" in the economy that this was the case. Also, justifiable values of QMU could be found outwith the range 0·04 and 0·10. As the author himself suggested in his earlier work, "Indexing Long-Term Financial Contracts", an observer in 1914, looking at more than 250 years' history of prices, would have felt confident that QMU was zero. Indeed, given the lack of trend in prices over this long period and looking at the long-term social and economic trends in standard of living and industrial production, our observer in 1914 would have needed some convincing that inflation was the *driving force* for *anything*. It is with the benefit of more recent hindsight that QMU has been quantified and we should perhaps bear in mind the author's own previous admonition (in the same work), that it is a good *ad hoc* rule that one should not forecast more than *n* years ahead on the basis of 2*n* years of past history.

The use of a Retail Price Index rather than earnings, or some such index which reflected long-term productivity gains or quality improvements, has already been questioned. This factor undermines the unit gain assumption (Section 3.7) of the equity model (although it is probably intended that a small positive value for DMU would allow for this). Similar considerations to those already mentioned with respect to the stability of QA apply to the dividend model. Certainly dividends are slow to adjust to current real profitability but analysis of distribution patterns over the 1970's (by a number of authors in bank reviews) suggests that gearing and belated acceptance of the inflation accounting concept had a large role in recent experience. While interest in current cost accounting is now fading almost as rapidly as it did in the 1920's, it is likely that future dividend policy will be influenced by recent experience of overdistributions. Even apart from the fluctuation in dividend cover, corporate profitability in the face of accelerating inflation will depend on the speed at which price increases can be passed on. Rigidities in the system will place an upper limit on this in the short term (12-18 months), producing a dip in corporate profitability and dividend growth when the rate of inflation jumps. I would suggest therefore that DD (Section 3.7) is not constant and that furthermore it may be set at too low a level, producing an exaggerated persistence of the effects of single year's inflation and dividends into the future. The phrase repeated in Section 3.9, "appropriate values for the parameters are", could reflect anything from the statistically rigorous to the purely subjective. Having read the "Steps Towards" note referred to in Section 1.2 I consider the methodology to be somewhere in between, with an element of arbitrariness introduced to reconcile contradictions between the M.G.W.P. results, those of Gordon Pepper, and intuition. In any event the period analysed may be a poor guide to the future. Social changes produce persistent secular trends; dividend patterns cannot be extrapolated independently of them. The assumption of the permanence of the mixed market economy as we know it has been previously questioned in this hall, casting doubt on the long-term value of any model to which such an assumption is built in. Other secular trends include concern for the environment and distribution of wealth and the effects of new technology. The answer is not merely to state that all bets are off if such assumptions cannot be made, but to insert parameters which represent these factors when analysing the historical data. While the concept, in Section 3.15, of a first order autoregressive series with parameter 0·91 is intuitively appealing, and indeed broadly matches Mr. Pepper's analysis here earlier this year, I suspect that movement away from and return to the mean is at times more rapid than indicated by the model.

Finally, at the end of Section 3, I would invite others to comment on the assumption (Section 3.16) that non-forecastable long-run extraneous factors can be subsumed in the white noise series. It all depends what is meant by the long run — but clearly exogenous events such as the discovery of gold in the New World and, much later, the end of the gold standard are pivotal factors which could break any model with rigid parameters. While these events may not be forecastable, what we do know is that such shocks are likely in the long term and that while the system may continue to operate in a similar stochastic fashion after a major disturbance, it is likely to be with new values of several parameters. We cannot say that we are being conservative in using such a model.

Use of the model is well described in Section 4 and allows anyone to pick the starting point which interests him. Selection of a range of starting points tests the stability of the model and is something the author has attempted himself, as he describes in Sections 5 and 6.

It is at this point, in Section 5, that we are able to make our first assessment of the plausibility of the results from the model. We read, in Section 5.8, that on the Full Standard Basis the observed average of QMU (inflation) is close to 5% at longer terms, and that the standard deviation declines as the term increases. The suggested conclusion is 'that there is greater relative certainty about inflation in the longer run than in the short'. I think that we should reflect on this for a moment. Does this result support the model or is it merely a direct consequence of its specification? I would suggest the latter, as the plausibility itself is *ex-post* rationalisation. Empirical analysis of any historical data in terms of fluctuations about a mean implicitly assumes stability *ex-post*. One would of course arrive at a different mean value doing the analysis in 1984 than in 1914, but data can be found to support the premises of stability about a number of different levels. I would suggest that this historical perspective of *ex-post* stability in the long run is not the same thing as *ex-ante* confidence about the level of inflation in future. The standard deviation result is specious and the conclusion is a direct result of the initial assumptions. I do not wish to labour this point but the model for Q(t), inflation, is integral to the whole, and without stating the term for which the model is expected to be applicable and consequently rationalising the period from which the parameters should be derived there is a danger of subjectivity and a false perspective. The model has been empirically derived and if the same data has been used to provide the parameters, more is needed to validate it.

I shall leave others to comment on the plausibility and permanence of the results from the full standard basis. It is summarised as, a rise in inflation is bad for share and Consols prices in the short term, in the medium term high inflation pushes share prices up as dividends start to respond with a lag, while Consols still suffers, and in the long run it has a positive effect on the returns from both. I wonder whether the supply of real assets could keep up over a given period with investors increasing demand for them in the face of high inflation — if not real returns could fall or, if inflation rates were volatile, a higher risk premium would be required for Consols.

The detailed sensitivity analysis has explored the characteristics of the model fairly fully. It is particularly interesting to note that the model can simulate hyper-inflations with $QA = 1{\cdot}0$, although it also, less realistically, produces hyper-deflations. The author rightly states that "pushing QA to its extreme value creates a very unstable model" but as this accords with our perception of reality it is perhaps more appropriate. There are a number of economies today exhibiting such instability and only an extreme optimist would discount their relevance to the U.K. Limits to the instability could be produced by extending the model to that of an open economy, incorporating the stabilising force of an exchange rate.

Many of the conclusions from the sensitivity analysis in Section 6 will strike a chord, and I would be surprised if anyone agreed with all, but nevertheless the results are intuitively pleasing. This impression may be strongly influenced by recent history and present circumstances — we should not underestimate the capacity of the system to surprise and the possible permanence of new parameter values after such a shock, rather than return to the previous mean.

Section 7 certainly opens our eyes to the vast potential of a stochastic investment model. Before application to portfolio strategy, however, extension of the model to cover overseas markets and deal with tax might be appropriate. A potentially successful strategy is suggested in Section 7.7 but a better known yield relationship is that, since the FTA index began, gilts have outperformed equities in any calendar year at the start of which the reverse yield gap exceeds 6%. This broadly supports the relationship initially postulated, but suggests that dividend yields are not viewed in isolation.

The applications suggested in Sections 7.8 to 7.16 must have whetted the appetite of all, particularly if they followed the author's advice and went straight to this section of the paper. A comprehensive stochastic investment model (even a simplification of the 'holy grail' for practical purposes) would allow our profession to make major progress in premium calculations (with and without profits), valuation, solvency, matching, option valuation, and pension fund analysis. Those, however, who have carefully followed the derivation of the parameters may counsel caution at this stage — even on the relevance to solvency.

In conclusion, if I appear to have concentrated overly on the early parts of the paper — derivation and assumption — it is because they are integral to the soundness of the conclusions. Use of stochastic models has been discussed by the profession for some time — I think it is now generally accepted that actuarial work is essentially stochastic in its nature and that such models are appropriate for simulations of many of the factors which influence a life office. As such, the model itself is a useful contribution to actuarial thought and, subject to the limitations of four variables and 21 parameters, it is an excellent starting point for analysis of risk. What I consider *is* open to discussion is the choice of parameter values. We should question terms such as "other things being equal", "appropriate values" and "close to best estimates". Just what constitutes "the long term" might indeed also be discussed.

Finally, with my criticism confined to the actual parameter values and the claimed long-run stability, I have no hesitation in endorsing use of the model, with inputs of one's own choice. There is no doubt that tonight's work takes the profession a leap forward and I thank Professor Wilkie for putting this research before us.

Mr. R. S. Clarkson:—I should like to concentrate on the suitability of the time series chosen to represent inflation, since this is by far the most important element in the whole approach put forward for discussion by Professor Wilkie tonight. In the paper itself, the inflation time series is taken care of in Section 3 in a mere 23 lines. We are told that inflation follows a certain first order autoregressive process involving three parameters, that there is very little uncertainty regarding the appropriate values for two of the parameters, and that the third parameter could lie anywhere in the range 0·04 to 0·10 depending on the period of observation used. However, the Supplementary Note referred to in Section 1.2 of the paper contains no less than 30 pages of commentary, graphs and tables relating to the inflation time series, and — on reading this Note — it immediately becomes obvious that the situation is far from being as clear-cut as implied in Section 3 of the paper. My main criticism of the paper is that the treatment of the inflation time series is totally inadequate; it should not be necessary to consult another source — which will not appear in the "Transactions" — for details of the most important element of a model put forward for use by the actuarial profession.

In the Supplementary Note, the Price Index data go back to 1661 and I think it is instructive first of all to glance at the broad picture that we see from that data. Before the twentieth century, there were long periods of very high inflation. For instance, from 1777 to 1801, the Price Index almost exactly doubled over that period of 24 years, and again from 1789 to 1813 it doubled. After some of these peaks the Price Index declined quite sharply. These declines of around 40% sometimes took 9 years and sometimes took 30 years. If we now come to the twentieth century, where obviously the data are of more relevance in fitting the model, we have from 1914 a very sharp burst of inflation, with the Retail Price Index doubling in only 4 years. Again from 1973 to 1977 the Price Index very nearly doubled in 4 years. In the twentieth century we have also had a significant fall: from 1920 to 1933, a period of 13 years, the Price Index fell over 40%.

I conclude, therefore, that — not only in the dim and distant past, but also in the twentieth century — the price series has exhibited very pronounced cyclical movements, and the underlying time series should be capable of reproducing these very considerable cyclical swings.

Consider now Section 3.3 of the paper. The first order autoregressive time series is such that the natural logarithm of the Price Index oscillates around a straight line trend and in fact oscillates more in the short run than in the long run. The point is made in the paper that the time series gives more stability to inflation in the long run rather than in the short run.

The main problem in fitting a series such as this is in deciding the slope of this straight line trend in the logarithm of the price series, and this slope is represented in the formula by QMU. As I have just indicated, in the past we had very long wavelengths of changes in the price index and the average force of inflation (which QMU can be interpreted as being) depends on the period of observation chosen. This is confirmed very vividly in Table 5.2 of the Note where three possible time periods are shown for the price data — 1919 to 1982, 1933 to 1982, and 1946 to 1982. 1919 was more or less the peak of the post-First World War inflation. Because it was at a relatively high point in the wave, the average QMU comes out at about $3\frac{1}{2}\%$ per annum. The second possible starting point to give 50 years of data happens to be 1933, which was the bottom of the wave. As opposed to the previous $3\frac{1}{2}\%$, the average for that particular period is 6%. The third period considered gives an even higher value of 7%. We have a serious problem — at what point do we begin our measurement of the average force of inflation? I see this as an intrinsic difficulty of the time series chosen. The other point I would make about Table 5.2 in the Note is that the standard deviation of the force of inflation is very high compared to the average values. After a considerable amount of statistical testing, mainly looking at residual variances, Professor Wilkie concludes that, as opposed to the $3\frac{1}{2}\%$, 6% and 7% values that might be used for the force of inflation, 0·05 or 5% should be chosen and that is the value that is used in the paper.

In summary, my criticism of the time series is twofold. Firstly, it seems to attach too much certainty to what inflation will be in the future. It strikes me as far too tame a series to represent some of the wild and uncontrollable events that occur in the real world of economics and investment. Secondly, given the rather high residual variances there must be considerable doubt about the values of the fitted parameters found.

Professor Wilkie, in the Note, has a very long commentary on the suitability of the distribution and considers a very large number of possible tests such as whether the residuals are normally distributed. There are two problems here — we have only 64 points of data, and there were very sharp shocks to the system. There were, for example, very sharp downward movements in 1920 and 1921, a sharp upward movement in 1941 and another sharp upward movement in 1975.

We appear to have what I would call the Mandelbrot factor emerging. Many of the time series used in economic work tend to have rather large tails, and a few rogue values out in the tails of the distributions can cause innumerable problems.

After all his investigations, Professor Wilkie clearly had reservations about the suitability of the time series described in Section 3.3 but decided that given the constraints of having a Box-Jenkins approach this series was the best in the circumstances.

Three years ago I, too, became involved in some fairly extensive investigations regarding Box-Jenkins time series analysis. Having fitted a model to gilt-edged prices and obtained a residual which gave a measure of short-term dearness or cheapness, I used elementary control theory methods to try to find the turning points in the series and identify switching opportunities. However, it occurred to me that Box-Jenkins series could possibly be applied to the daily data of the gilt-edged market. The detailed work, including the writing of the very extensive suites of computer programs, was done by a colleague of mine, Dr. Mossaheb. Our work was based on over 800 daily observations. We carried out a large number of statistical tests, almost exactly as described by Professor Wilkie in the Note, and we concluded that four apparently quite different time series all gave an equally satisfactory representation of the data — ARIMA 400, ARIMA 410, ARIMA 012, and ARIMA 111.

Although this approach appeared to hold out considerable promise, we found that the residual variance was so high that the model had virtually no forecasting ability. The daily random shocks were very large relative to the movements predicted by the model, and it soon became apparent that the underlying price series was too "tame". In Box-Jenkins analysis, it is assumed that the white noise series does not become overwhelmed by periodic random shocks. With very great reluctance we decided that the entire Box-Jenkins approach had to be abandoned completely, and that a much more robust model was required.

The model that we adopted for the price residual Z_t was

$$Z_t = Z_{t-1} - \theta(Z_{t-1} - \bar{Z}_{t-1}) + a(t)$$

where $a(t)$ is the daily random shock and $0<\theta<1$. Because of the existence of some abnormally large values of $a(t)$ we did not attempt to find θ on the basis of conventional techniques but instead investigated the values of θ which maximised the function:

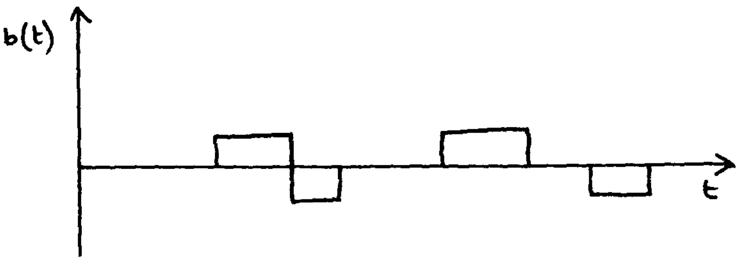
$$\{\text{frequency} \mid a(t) \mid < \varepsilon\}$$

For various small values of ε, the corresponding values of θ were stable for a particular stock and were generally about 0·1.

Returning now to the paper under discussion, I suggest that the inflation series should be what I would call a universal economic random shock model:

$$Z_t = Z_{t-1} - \theta(Z_{t-1} - \bar{Z}_{t-1}) + a(t) + b(t)$$

where $b(t)$ is the random shock component and θ and $a(t)$ are as before. The function $b(t)$ might be portrayed as below:



The first strongly positive shocks would represent the rapid build-up of inflationary pressures such as in 1915 and 1975, and the ensuing negative shocks would represent the subsequent period of deflation, such as in 1920 and 1921. The general "shape" of $b(t)$ could be estimated with the benefit of hindsight, and in simulations of future values of the price series we could either feed in specific values or use randomly generated shocks that follow certain boundary conditions.

It is an understatement to say that a lot more work would have to be done before a series like this could be used in practice, but I have one interesting suggestion to leave for Professor Wilkie. For a quite different purpose I was looking at the price movements of an index-linked gilt-edged stock for the third quarter of 1984. The daily price changes were generally nil or ⅛ of a point. However, there were two periods, each of five days duration, when first of all there were strongly negative movements, then later there were strongly positive movements. Most of the movement in the quarter was accounted for by five daily shocks downwards and then later five consecutive daily shocks upwards. That of course was the behaviour of only one gilt-edged stock over a short period, but by looking at similar proxies for inflation, we might find enough inflation data to avoid having to wait another 64 years to double the number of observations used in the paper.

My comments clearly have concentrated entirely on the inflation series, since I believe that this series is the key to the whole model. In opening the discussion, Mr. McLean said that he is happy with the general structure of the model, but is critical of some of the parameter values. For the reasons I have just given, I believe that the model in its present form is unsatisfactory. Others may well disagree with my conclusion, but I suggest that the relevant parts of the Supplementary Note should appear as an appendix to the paper, as otherwise the record in the "Transactions" will not contain sufficient of the technical detail to allow readers to assess the suitability of the inflation time series. I very much appreciate all the work Professor Wilkie has put in to take us this far along the road, and I am confident that a much more satisfactory time series for inflation can in fact be devised fairly easily. Then, and only then, I would accept that the actuarial profession has — at last — a satisfactory stochastic investment model.

Mr. J. Plymen:— I would like to congratulate Professor Wilkie on a most notable contribution to Faculty proceedings. His handling of the statistical work is masterly and is an example to us all.

Unfortunately the data, on which all this has been built, is subject to very considerable difficulties and limitations. I will start off with the data for dividends. In previous discussions on the subject I have pointed out the unsatisfactory nature of the dividend indices, the de Zoete and Bevan from 1919 to 1930, the old Actuaries' Index for the next 30 years, and the FTA for the last 20 years. These indices are very different animals, the first with 30 shares, the second with 170 and the last with 650. In particular, the old Actuaries' Index contains very suspect information from the point of view of dividends and dividend growth. It had by modern standards a curious industrial coverage, including all sorts of shares that do not exist now, such as rails and steel, together with much shipping and heavy industry. During the 1930's these "heavy industry" categories suffered severely from the depression. Altogether, I think the old Actuaries' Index which represents half the period of observations is really most unsatisfactory to work on and, of course, when you join together three different indices you have got the jump as you change over. Altogether, if one is trying to make sophisticated estimates of dividend changes relative to inflation and so on, I do not like the use of these three different indices.

Obviously a long period for this study is desirable. Nevertheless, as a guide to the future, surely it is better to base our material on later figures rather than earlier ones. I doubt whether the earlier experience, based on the financial and economic progress of equities during the 1920's, is of much relevance now, bearing in mind the different social conditions, the lower rates of taxation, the low level of welfare, and the strength of the currency.

I would much prefer to work on the last 22 years for which we have the FT Actuaries' Index with a complete consistency of dividend factors. The difficulty is when one is talking about the relationship to inflation. I confirm what Mr. Clarkson said that the question of the effect of inflation on dividends is absolutely critical to the whole thing. Fortunately, as regards high rates of inflation, we have got really very little statistics over the 60 years. We had a short burst of high inflation which really contributes quite inadequate data on, say, 15%+ inflation, for any relationship to be derived. I have not, unfortunately, been able to study Mr. Wilkie's detailed material which obviously appreciates all these points, but I do not feel at all happy about the assumptions that he makes regarding the relationships between dividends and inflation. For one thing, for all the examples quoted in the paper he assumes that there is no dividend growth without inflation. Admittedly of course he can put in a positive or negative factor for this. But I would regard this as surely an assumption somewhat difficult to justify in economic and financial terms. Surely the whole point of the concept of the equity method of financing is that dividends are distributed much below the earnings, every year there is a certain amount of plough-back which surely earns a reasonable rate of return and builds up the underlying strength of the business and makes for higher dividends in future. Surely we reckon that overall there is some real growth in the gross domestic product which should be reflected in the real growth of dividends. I will come back to this later.

Then there is the question of the responsiveness of dividends to inflation. I do not think that the assumption of a 100% response to inflation, admittedly with various time lags, makes sense. Looking at it in practical terms based on following company profits over many years, I would say that if you have inflation at a containable, moderate level of up to about 5%, probably that does not distort company finances too much and possibly you do get dividends responding to this inflation 100%. But surely when inflation rises as it did in the 1970's we then have to pay some regard to inflation accounting with subsequent dividends likely to be prejudiced. I do not think it is possible to establish this relationship from statistics, while the statistics are so poor and unsatisfactory.

One can make a crude comparison. I have just taken the figures for 1950–70 inflation, which, apart from one peak figure of 9·5%, goes between 0% and 6·8% with an average of 3·5%. Over this period the de Zoete and Bevan dividend index grows 3·6 times and 6·75% compound. The compound growth of inflation was something like 3·5% so that over this period of moderate inflation there was a degree of real growth of dividends of about 3%. On the other hand, let us look at the period from 1970 to 1980. The Actuaries' Index gained 11%, the inflation rate gained 13½% so that over this period with inflation much higher the dividends definitely lagged inflation.

I would suggest that probably the overall 3–4% gain was there all the time but that when inflation reached about 15% the dividends only responded by about half that, as you have to allow for considerable cut-back. Altogether, I do not think the way that dividends follow inflation can clearly be obtained from statistical studies. I think it should be obtained by what I call a proper capital and income model, that is to say, the sort of model that is used by industrialists now, whereby you project more or less the whole profit and loss account and balance sheet forward over the years allowing as one would for inflation, rates of growth and that sort of thing. With a comprehensive model like this, you could apply some sort of inflation accounting to it and could test it to see how the resulting profits would move with various degrees of assumed inflation.

This brings me back to my further point where Professor Wilkie's model could be improved. He only uses the four parameters because they are the four variables for which an adequate data series existed. That is to say they are the four variables that could be obtained over the 60 years. Why not confine one's information to the last 20 years, which I think makes sense? (After all we have thrown away the mortality tables of 1924–29 – why should we use financial tables from the same period?) If we confine ourselves to the last 20 years we then have from the FTA Index the earnings factor which could very well be incorporated in the model. I have been saying for years that I would like the FTA Index to be a real model incorporating further factors. There is no reason why the Index should not include the asset value of the equity and then, with the earnings, you would have the earnings on capital employed. If you had an index like that or used data banks with the same information the possibility would exist of producing a comprehensive model where the build-up of dividends would be rather more logical and consistent than the somewhat arbitrary assumptions that Professor Wilkie has had to make.

Despite having criticised the statistical underlying data, I would like to talk about the actual conclusions. I feel that, as Professor Wilkie has ignored, I think incorrectly, the likely long-term rate of growth of dividends due to plough-back, he has somewhat understated the prospects for the equity. If his calculations were redone, as they can very well be in a moment, with a 3% gain of real dividends he would of course get a tremendously different picture. As it stands, the projections suggest that there is no particular advantage in the long term in investing in equities. After all on most of the projections you get the equity return of, say, $9{\cdot}8 \pm 3{\cdot}78$ and the corresponding gilt rate $8{\cdot}73 \pm 1{\cdot}00$. On the usual principle of allowing for risk and return, on the whole an extra 1 point from the equity situation is not really attractive, bearing in mind you have to stand 3·8 times the fluctuation, 3·8 times the risk. At any rate I would certainly like to see some alternative figures worked out on the basis of a positive long-term upward trend of equity dividends.

Mr. G. B. Chaplin:—First of all, let me add my thanks to Professor Wilkie for an excellent paper. I should like to take this opportunity to describe briefly another application of a time series model of interest rates to investment problems.

Options on bonds are a common feature of today's markets. For example, split redemption dates on U.K., U.S. and Eurobonds; convertible bonds in the U.K.; options to buy or sell bonds at fixed prices in both U.S. and European markets and soon in the U.K.

A Black-Scholes option pricing model is a familiar tool in the equity markets but is not directly applicable to bond options. Of much greater practical significance is a time series model such as the one discussed in the paper. Given such a model, a simulation of future interest rates can be made to determine the profit (if any) on exercise of an option. Repeating the process many times, an expected value of an option can be calculated.

In using such a method one is often examining the tail of a distribution and in such circumstances sensitivity analysis such as is described in Professor Wilkie's paper is very important. One's uncertainty in the final value of the option is often quite high, but as a general rule, it seems these options both in favour of, and against, investors tend to be undervalued by the market.

In practice, the approach I have used tends to differ in detail from that of the author. I have taken as my guide the method used in calculating a gross redemption yield. One implicitly assumes that it will be possible to reinvest dividends at the gross redemption yield — i.e. one assumes that expected yields will equal the current yield. I have followed this basic approach in assuming that interest rates follow a time series model both starting at current levels and fluctuating about current levels. This is essentially a "neutral" view. No direction is assumed for future interest rate movements and no assumption is made that the correct level of interest rates is some value different from current values. This in a sense is a negative approach — we are throwing our hands up in the air and saying we do not know what the expected long-term rate of interest is. The basic model used, however, does allow one to assume an expected long-term value different from current values, if desired.

When very long time periods are concerned, the results are very much dependent on one's assumptions and again sensitivity analysis is very important.

Finally, I note that the author states in paragraph 3.15 that the real rate of return on Consols has averaged 3½%, and if real yields deviate from this level they only slowly return to the mean level. This is a particularly interesting point at a time when real yields on fixed interest securities appear to be between about 6% and 9% around the world, while long-dated index-linked stocks have remained in the 3% to 3½% range.

Mr. D. H. Loades:—I have been associated with David Wilkie for many years both in the Maturity Guarantees Working Party and in the Faculty Solvency Working Party and, of course, I am now wedded to the idea of stochastic processes.

I would like to start off with just a minor criticism of the paper for newcomers to this subject and that is notation. I had to struggle with the concepts and notation in the Maturity Guarantees Working Party and in the end managed to master the standard notation. If you look back into David Wilkie's deposited papers you will see that he starts with the standard notation defining all symbols. In this paper we have the expanded formulae which are so much easier to understand if they are put into their finite difference form, e.g. the formula in paragraph 3.5 becomes $(1-\text{YA.B})(\text{YN}(t)-\ln \text{YMU})=\text{YE}(t)$. Finite difference is not a subject which is studied today but basically if you can remember the subject you have the formula $E = 1 + \Delta$. You can simply separate out the symbols and manipulate them algebraically. The corresponding formulae the time series analysis is either $\nabla = 1 - B$ or $B = 1 - \nabla$. There is some confusion in Section 3.7 where the backwards operator (B) is defined but looks more like a variable rather than an operator with $X(t)$ being the variable. It looks as if $BX(t)$ is a new variable.

Referring again to Section 3.7 and later, DMU is the real rate of increase. We had some discussion on whether that should be zero or not. But if DMU represents the whole of the real increase in dividends, does that automatically imply that DW + DX must equal 1. It does in the list of parameters given in Section 3.9. If that is a necessary condition it should be made clear. I tried to find that out from the deposited papers but once again I could not really see whether it was so or not. It seems to me that in Section 3.9 the fact that DD = DX is fortuitous and not a necessary condition. The other point that I found difficult is the mixture of logarithmic and unlogged variables and I wonder whether it is possible to keep to logged variables all the way through.

Pension fund actuaries tend to think in terms of real rates of interest and use the sort of formula which is quite obvious, i.e. (1 + real rate of interest) = (1 + the gross rate of interest) ÷ (1 + the inflation rate) and this is effectively what is done in Section 5.4 to obtain average rates of interest over the period t. That is what I would call a multiplicative model. In the multi-variate analysis in the paper we appear to be getting an additive model mixed up with logarithmic transformations (which produce multiplicative models). The two can be consistent in the short term, i.e. if you take logs of the formula above and express the logs as a series, you get back to an additive model which is: the gross rate of interest = the sum of (the real rate of interest + the inflation rate). But I wonder whether you get the long term and the short term mixed up by combining a logarithmic model with a straightforward linear model.

As a member of the Working Party, I was privy to much of the work that has not been published plus various private papers. Reference has been made to Gordon McLeod's work which David commissioned and I have studied that. One thing that impressed me from that analysis is that when you took different time periods, the resultant models had the same structure. I regard that as very important. There is a difference between the structure of the model and the level of the parameters that you use.

When the Government Actuary's Department started to fit mathematical curves to population mortality, every time we graduated a new set of data, we had to change the model. It was not simply a matter of recalibrating the parameters; therefore, the model was unstable and people have probably even forgotten what it was. But in Gordon McLeod's models which were the starting point of much of David Wilkie's later work, the structure of the model was unaltered but certain of the key parameters, particularly the means, changed. You would expect the means to vary. If you calibrate a model during a period of high inflation and use it to predict the future, you predict a future based on high inflation and vice versa. I am not too worried about the level of the parameter. It seems to bother a lot of people whether the inflation parameter is 3% or 10%. I am a pension fund actuary. Liabilities are dependent on inflation, investment returns are dependent on inflation. If the structure of the model is the same irrespective of the level of inflation, do you get the same real rates of return over the long period, whether you use a model with a high level of parameters or one with a low level? I think that is the crux of the issue for pension fund actuaries. Those concerned solely with investment returns obviously are worried.

I am also not too worried about the problems of whether the model for inflation is robust or whatever expression one likes to use for it. If you study uni-variate models, that is, where a time series is defined in terms of its past history, you get one result. You can presuppose that the series you are studying is highly dependent on another series. If you project that series first and use it to predict the second series you get another result. The two will not be too different. You find that most of the variability of the second series is explained by the first series. You get similar results because the first series has influenced the second so much that you can also explain most of the variability of the second series in terms of its own history. If the correlation is not particularly strong, additional terms will be introduced.

Therefore, it seems to me that one could have a pretty loose fit for inflation (as long as it does not go off to extremes such as hyper-inflation or hyper-deflation) and still get the right relation between liabilities and investments. This once again is looking at it from the point of view of a pension fund actuary.

I was interested in Mr. Clarkson's discussion on adjustments to David Wilkie's model to make it more variable. Once again this seems to echo work done by Gordon McLeod in what is called intervention analysis. With hindsight you can see where the model has got extreme variation. You simply put in a new variable which is limited in time and examine the effect. This reduces the variability of the time series, making it easier to understand the underlying structure of the model. But it leaves you with a difficulty not in analysis but in forecasting. You have taken out a source of variation in the analysis, now you have got to put it back into the projection. I have not seen any rationale for doing this. For the long term which David Wilkie was looking at I do not suppose it makes a great deal of difference. He tends to increase the variance of the residuals. If you wish to put it back directly you can either suppose that it has a distribution or perhaps that the distribution of the time interval between occurrences of the intervention variable is known. I am not sure whether that is helpful in the sort of applications that I see for David Wilkie's model. What may be an interesting exercise is seeing the extreme variation you can get in investment returns.

Turning to applications. One of the problems that the pension fund actuary has is presenting the results of valuations where you have asset values on one side of the balance sheet with liabilities on the other side and you want consistency between the two. You need either a method of putting a value on the liabilities that flows from the value of the assets or a consistent method of valuing both liabilities and assets. Yet you have to explain the methodology in the report. This causes a great many problems. It seems to me that David Wilkie's models give one way of analysing the problem. You can start with assets at their market value; you can generate the future income flows, using a stochastic model related to the inflation series; you can generate the liabilities by the same method. This is simply looking at emerging costs. You can use the methods set out in the Faculty's Solvency Working Party for accumulating either a single sum of money or an annual sum of money. The simulations are continued to the end of the day which will disclose a surplus or a shortfall. From the calculations for the value of the accumulation of a single amount of money (or an annual amount), you can decide what that shortfall means in terms of cash today or contributions tomorrow. That simply means that the balance sheet is calculated by differencing. In other words, assets = market value, liabilities = market value +/ — the cash injection that is needed or the surplus. Alternatively one can express the shortfall in terms of additional contributions. I am not sure that makes it any easier to explain to people what you have done but conceptually I find it an interesting way of looking at the problem. As a bonus you will have a guide to the stringency of the valuation.

It is quite simple to calculate the slope of the central forecast derived from a uni-variate model. It is quite easy to calculate the funnel of doubt which in the terminology of time series is called a variance multiplier. That comes out of the algebraic manipulation of the formulae which can be done on a pocket calculator. It is much more difficult to determine where the central forecast should lie. It is different from a regression line whose position is calculated and fixed. The starting point for the central forecast varies with the recent history and that is one of the strengths of the Box-Jenkins approach. Now what interests me is, given multi-variate formulae, can you without doing a great many simulations determine the central forecast? Is it good enough simply to put all the random variables to zero for the future? Is there a simple method of calculating the funnel of doubt, i.e. the variance multipliers? I do not see how to do it. It would help me greatly if I were able to calculate these factors without going through the effort of large scale simulations.

Mr. J. G. Spain:—As a member of the Institute and a visitor to the Faculty, I am grateful to you for being able to put in my oar. I am not going to talk statistically, just a few random thoughts. The first point I would like to cover is one brought up by Mr. Plymen on the data base that is used, in that three indices were mixed up, being all the data that was available. It did occur to me that it should be possible for some interested person to go back and look, for a few specimen dates, to see what the FT-Actuaries All Share Index yield would have been if it had been calculated on the day. I am sure the information must be available. And secondly, Mr. Plymen sparked off in my mind the thought that perhaps is accepted by all of you already, but not in London yet, that the estate of a life assurance company, that is the hidden margins, is a concept that can also be applied to commercial companies in that not all the earnings are distributed, part being retained to fuel future growth.

I was very interested to see a paper such as this presented because the Americans have already gone very much over to statistics on this sort of thing. What we call A3 in the Institute, life contingencies, is now heavily statistical and very difficult to follow, but I think we are going to be stuck with it. So this is probably going to be a landmark, and we had better get used to it. The only thing is, though, that it is going to be very much an internal thing to the profession, because there is almost no way we can use this outside the profession because they will not understand it. They already do not understand what we do when we say we are making assumptions. They will understand it even less if we say we have built this particular model or we have changed it.

I would like to go on, though, to the use of a model such as this, for example, from the point of view of a Life Office actuary in assessing whether or not a premium rate will be sufficient for the job.

One can price an annuity or an endowment on traditional bases and come up with a rate. One can then do a thousand simulations, on varying different assumptions, and come out with a rate twice as much. If one really believes that the simulations are correct and that the assumptions are what one should have started with, the answer must be to stop writing business at the current rate and cease to write new business for quite a long while.

I just cannot see it happening, but that must be the implication of using statistical methods if those are the conclusions to which one comes. I do not speak from the point of view of a Life Office actuary; my interest is solely in Pension Funds where the outlook for planning is so long, far longer than for Life Offices excluding their pension business. One can try to simulate these things, in fact the Americans have done this for the long term for pension funds. Nobody so far has been able to convince other consulting actuaries that the results have been of any use yet but we all live in hope.

David Loades did say that the Pension Fund actuary is particularly interested in real rates of return and communicating this to clients. Very often, in fact, for U.K. private pension fund valuations the assets and liabilities are valued in tandem one with the other, the same assumptions being used on both sides of the valuation balance sheet, and the Trustees are persuaded that this is the right way to do things. Unfortunately, from one point of view, the Trustees have also been subscribing in many cases to monitoring services based on investment performance which suggest that the investment returns (which are based on market value) have been giving them fantastic returns of, say, 20% or 30% per annum over 5 or 10 years. They then turn round to the actuaries and say: "Why, if we have had such fantastic yields, are you giving us such a low starting point?" There is an answer to this but we will not go into that.

My final point on the use of stochastic processes applies to tonight's paper as it does to Andrew Wise's paper which was presented a couple of months ago to the Institute. If you start off making a certain set of assumptions, and assuming that you have got the model right more or less, you are hypothecating from time zero where you will be at time 3, at time 5, at time 10, and whenever you like. When you get to time 3 you have to start again. How do you know what your starting point should be? There are so many different values from each simulation that you end up, I think, with a conclusion that you know where you start from and you know where you think you are going to go, but I do not think you can really be certain in the interim that you know where you have been. I find that rather disturbing.

Mr. A. P. Limb:—I would like to add my congratulations to those expressed by others to Professor Wilkie for a most interesting and challenging paper. It is interesting for a number of reasons. First the use of stochastic methods to investigate actuarial problems is still, although we seem to have many aficionados here tonight, a relatively new area although an increasing number of papers have been written having a bearing on the technique as the bibliography of the paper clearly demonstrates. It is very interesting to see displayed at any rate a summary of the methods adopted by the author in fixing the models described in this paper and particularly I think the techniques used to investigate the sensitivity of these models to changes in parameters.

The paper is challenging in a number of ways also. Some of the mathematics involved is perhaps somewhat daunting; nevertheless, I think we have got to get to grips with it. If we do not we are, I think, to say the very least, in no position to judge the value of the techniques and approaches which the author illustrates. We may, of course, prejudge them from a position of comparative ignorance of the mathematical approach used but prejudice and judgement are two very different things. I think the paper is challenging because it brings us as a profession once again up against the unavoidable question of the attitude we should take towards this exciting but very complex new tool which Professor Wilkie has laid before us tonight and which has been edging into view over the actuarial horizon for some years. I suppose the challenge can be divided into two parts. The first is: "Should we seek to use stochastic methods if we are sure that we can devise a reliable stochastic model?" and the second, even more searching, is: "Are we sure that we can devise a reliable stochastic model?"

For my part I have little hesitation in saying that if we were sure we could devise a reliable stochastic model, there are most certainly a number of situations in which we should use it. It is my impression that the profession as a whole has already accepted this over the question of the setting of reserves for maturity guarantees under unit-linked policies although I suppose that having seen the consequences of granting such guarantees in terms of the substantial reserves required to meet them with a high degree of probability, most offices have now stopped giving them and so this application will be increasingly little used. The second application where I have little doubt that such a technique should be used is in the determination of solvency reserves. The author lists a number of other uses of varying degrees of general interest in the last section of his paper. Probably the most widespread possible such use would be in the fixing of premium bases and the author's comments that this would result in overcharging in most cases and his suggestions for a way of meeting this equitably are interesting. Of course, I suppose with-profits policies developed in this way long before stochastic models were even thought of but the level of bonus loading is susceptible to investigation using a stochastic approach.

One should not underestimate the complications involved in using the method; as the author himself states, these are substantial. Dynamic investment policies and dynamic bonus policies which must be employed if the model is to be in any sense realistic are not easy problems to resolve and the effort which is required is very considerable indeed. Nevertheless, with the general availability of computers it is now a possibility whereas until recently we would have had no chance whatever of using it.

The second aspect of current challenge is the question of whether or not we can develop a reliable stochastic model, and I emphasise the word "reliable", for use in illuminating possible futures and assessing the degree of probability of a particular range of outcomes. The bulk of the paper before us tonight addresses itself to this problem and the author has produced a practical answer. By so doing one might think that he has given the lie to anyone who might suggest that we cannot in fact produce such a model. There are those who believe that the future is unquantifiable and it is a waste of time to seek in any sense to know the unknowable. They would quote unexpected changes in the past which have altered the course of history, such as wars, diseases, advances in medicine and so on. Indeed, one might fear that at the present time we may be seeing just such a problem in the medical area or one might fear that the present military stance taken by the super powers is such as to make the future so uncertain that any attempt to quantify its possible fluctuations is a waste of time and we should simply stick to our old deterministic methods and be done with it. To them I would say that one should know the enemy and his resources as well as possible and that anything which adds to our store of awareness of the range of possible future experience and of its distribution is so obviously worthwhile that the question scarcely deserves further discussion.

There are some who would say that one can learn little of the future from studying the past and one should therefore postulate, let us say, two possible futures. Perhaps one might call one the optimistic future and the other the pessimistic future and explore these in a deterministic fashion ignoring any further refinements. I must confess that even now I have a sneaking sympathy with this approach myself but I think in my more enlightened moments that this is nothing but laziness. I will, however, return to the point. Surely it is better to know if we can the distribution of likely possible futures rather than simply guess at two variants, one of which we think is optimistic and the other of which we think is pessimistic, with no good evidence to support these thoughts?

There are those also who would argue that if you seek to construct a model of the future from the study of the past you ruin your scientific approach if, on the grounds that you dislike some features of the model, you alter them subjectively, and I think that the author has done this here and there. To them I would say that there is no virtue in slavish adherence to a model derived from the past if it offends good sense. What one ought to say is that a stochastic model for the future is better than a deterministic approach in many circumstances and we should do the best we can in setting up such a model. This may involve analysis of past experience and subjective modification dictated by the tenets of good sense.

There is, however, one particular point which worries me in the paper before us tonight. It is to be found in paragraph 3.4 where the author says that there is considerable uncertainty about the value of the QMU, where anything between

0·04 and 0·10 might be justifiable, depending on the past period of observation one wishes to consider, and this seems to me to be a central area of doubt. He returns to the point in his sensitivity analysis and he illustrates the effect of using QMU as 3% rather than the 5% he had earlier used, and the effect as you might expect is considerable. Furthermore, he again implicitly acknowledges this difficulty in paragraph 7.2 where he points out that when the Report of the Maturity Guarantees Working Party was submitted, a mean rate of inflation of 4% was used rather than the 5% he is using here. Finally, he illustrated himself extensively the truth of his statement about the difficulty of knowing what value of QMU to use in his paper, "Indexing Long-Term Financial Contracts".

If there is such evident uncertainty as to what long-term rate of mean future inflation to use, does this not seriously vitiate the value of the investment model set out in this paper or indeed any other such model which has to acknowledge the same doubt? If in fact there is such admitted difficulty and uncertainty in knowing the value of a fundamental parameter, the mean rate of inflation over the future, one is, I suggest, bound to ask if the model really does anything more than illustrate in a stochastic way what would happen if a particular level for this parameter (and others) were to be appropriate and perhaps one is back yet again in the unenviable position which I referred to a moment ago: that of illustrating two possible views of the future, albeit two possible stochastic views rather than deterministic ones, but still without any real confidence that either is more likely than the other and yet with an uncomfortable awareness that they are very different one from another.

Mr. I. C. Lumsden, *closing the discussion,* said:—Let me begin by adding my thanks to those of other speakers for Professor Wilkie's well researched and interesting paper.

The Professor, and other members of the Maturity Guarantees Working Party, will be pleased by the extent to which so many members of the profession have come to accept the use of random-walk models in recent years. His work is an important contribution to actuarial knowledge, but I feel sure he will agree that his model may be more suited to some applications than to others, and requires to be handled with care.

In considering the Professor's model, a first step should perhaps be to ask whether or not one can use past experience as a reasonable guide to the future. The use of models similar to the Professor's is well established in the natural sciences, particularly for short-term forecasting. It is perhaps not unreasonable to imagine that nature will evolve slowly and consistently. Is it, however, fair to assume that future economic conditions will be consistent with those of the past, when we know that the political and social environment will almost certainly be different from any we have seen before? The answer must surely be "not necessarily", and indeed a number of tonight's speakers have suggested that an econometric model designed to predict future experience would be preferable to one relying at all heavily on the past.

In this context it is perhaps fair to recall that the Professor's model is designed for general long-term actuarial use, rather than for short-term investment appraisal. Given the purpose of his model econometric complexity is perhaps out-of-place, and the Professor's approach of combining a careful study of the past with a sensible, if simplified, view of the future is perhaps a fair one.

Having decided to base his model at least in part on a study of past experience, Professor Wilkie had then to decide on the general form it should take — on the particular investment variables it should involve, and on what they should be assumed to depend. He has set out briefly his reasons for adopting the particular variables of Retail Price Index, Dividend Index, Share Yield Index and Consols yield in Section 2 of his paper. The Professor has described in his separate Note the extensive statistical analyses of the correlations between the past values of these four variables on which he then based the final structure of his model. In years to come either Professor Wilkie or others may be able to improve on his structure — no structure can be said with certainty to be the best one. Given the extent of the Professor's analyses, however, we are bound I think to treat his conclusions with respect.

There are just a few aspects of the structure of the Professor's model which give me some cause for concern. Firstly, the cross-correlation analyses originally carried out by Gwilym Jenkins and Partners, which are referred to in the Professor's special note, did not disclose much of a functional relationship between fixed interest yields and inflation. I agree with the Professor, of course, that intuitively such a result must be wrong. It is perhaps as well to recognise, however, that the Professor's final choice of model for fixed interest yields involves a large element of subjectivity, and owes relatively little to objective statistical analysis.

Whether or not as a result of that subjectivity, the Professor's fixed interest model involves a relatively loose relationship between share prices and fixed interest yields. Both are assumed to depend on inflation, of course, but in a very different way, whilst the only direct relationship between the two variables is a small assumed dependence of the fixed interest yield on the random residual of the share yield in the same year. In our work many of us will occasionally take comfort from the assumption that if share prices fall substantially, fixed interest yields will probably rise at the same time. If that comfort is misplaced it deserves to be taken away, but bearing in mind that particularly in the area of fixed interest yields the Professor's final choice of model has been in many respects subjective, I am not sure just how much reliance can be put on his assumed correlation between these two variables.

A second aspect of the Professor's model which gives me, and I note Mr. Limb also, some concern is the importance in long-term projections of the subjectively chosen mean future inflation rate. This may be illustrated using some of the figures in Table 3. The mean rate of money return on Consols of around 8·7% over 100 years is simply the result of an assumed real return of around 3·5%, combined with an assumed mean inflation rate of a little over 5% per annum. If the Professor had chosen an inflation assumption of only 4% per annum, the mean rate of money return would have been only 7·5%. It can be easily seen that over long periods the subjective assessment of the inflation rate may well assume a greater importance than any element of stochastic variation.

The third aspect of the Professor's model which concerns me is the central tendency assumed for all the variables — their tendency to return always towards predetermined mean values. Mr. McLean has already expressed a similar concern. Central tendencies are admittedly required within the model in order to limit the variability of the results. Nevertheless the idea that a variable will tend to a single predetermined mean implies an increasingly certain average value over time, which conflicts with our intuitive belief that future investment returns will be increasingly *un*certain as time goes by.

As an example of the effect of these central tendencies, the standard deviation of the modelled mean rate of money return reduces as the period of the simulation increases. According to Table 3 we can be about 95% certain to earn 5·4% per annum on Consols over the next 10 years, but we can be equally confident of earning more than 6·5% per annum over 20 years. Over longer and longer periods the model will show us as being increasingly likely to earn the assumed mean return. This pattern of reducing variability of returns over time does raise questions in my mind as to the model's direct suitability for use over long future periods.

This brings me to the last section of the Professor's paper, dealing with possible applications of his model. Whilst I find the prospects of such applications most interesting, it is I think only sensible that we recognise the potential limitations of the model in some circumstances. The model is, perhaps, best suited to the calculation of terms for inflation-linked contracts such as the Professor mentions in paragraph 7.3, where the specific level of the future inflation assumption will usually be of secondary importance. Again, for short future periods over which the fluctuations in share prices are likely to be of much greater importance than the assumed mean inflation rate, the model with suitably chosen parameters will form a useful basis for assessing the reserves to cover maturity guarantees backed by equity investments.

Over longer periods, however, the assumed mean future inflation rate assumes relatively a greater importance, and the central tendencies within the model produce smaller and smaller variability in the results. When it comes to assessing premium rates for long-term contracts, it is not immediately obvious to me that the model will be of much help. Not only must the inflation assumption be chosen independently, but in order to produce a credible pattern of assumed future earnings it will I think be necessary to introduce into the model an element of variability in the inflation assumption as the forecast period increases. I would have to express these same reservations over the usefulness of the model as it stands for valuation, added to a reiteration of my concern over the strength of its assumed correlation between share prices and fixed interest yields.

In conclusion, let me thank Professor Wilkie once more for his impressive paper. If some of us have expressed reservations about his model I am sure we have done so with a constructive intent. The Professor's work does not I am sure represent an end to actuarial research. It may perhaps mark a beginning to proving ourselves better equipped for our work than mere astrologers, and for that beginning we owe the Professor a great debt.

Professor A. D. Wilkie, replying to the discussion, said:—I am very pleased at the reception my paper has had tonight. The main detailed criticisms, put forward by Mr. McLean and Mr. Clarkson, were that the model was not complicated enough to reflect reality satisfactorily. If this is the case, I have no objection to others putting forward a more complicated model, if it makes any difference to the results. But my own technique has been to simplify as far as possible, without affecting the long-term results, which is what I have done in producing what I have called the Reduced Standard Basis.

The comments made by Mr. McLean and Mr. Clarkson were so detailed that I should prefer to study them first, before replying in writing. The fact that their comments were detailed indicate that in principle they accept the idea of using a stochastic model for investment variables, though they disagree about the details of the model they would like to use. One of the features of the model I have put forward is that the user can choose his own parameters. This is similar to the model of a life table, first put forward by Abraham de Moivre. He got the principle right; others have modified the parameters, i.e. the values of the mortality rates, in the light of current experience from time to time.

Mr. Plymen criticised the dividend series I have used. I agree with the criticisms, and I would willingly have used a better one if it had been available. I remember that in 1972, when I first became interested in the FT-Actuaries Index, I rang up Jack Plymen, who was then Chairman of the Joint Index Committee, asking whether the Committee could produce and publish an "ex-dividend adjustment" that would show the actual amount of dividends paid by companies on that day. He replied that nobody had ever suggested this before. When I joined the Committee myself, and became responsible for constructing the fixed interest indices, I included an ex-dividend adjustment on the lines I wanted. The corresponding adjustment for the equity indices is being prepared, and is due to start publication next year. In 60 years' time I hope that our successors will be able to make use of what is now a more accurate index.

Like Mr. Plymen, I should like to have used company earnings instead of, or as well as, company dividends. Unfortunately, earnings were not published at all until 1948, and in recent years have been so distorted by the lack of inflation adjustments that I do not think them reliable. In any case, they are not available for all companies, and hence not available for the All-Share Index.

Mr. Plymen suggests that the 1920's are not relevant to today. Does he think that today's experience is not relevant to the 2020's? If so, how does he think insurance companies or pension funds can conduct business that far ahead at all?

I shall thank the other contributors to the discussion, without replying to their points individually, and I shall answer any specific questions in writing later.

Professor A. D. Wilkie subsequently wrote:—Mr. McLean asked me to distinguish the short term and the long term. Clearly, since I only use annual values in my model, I can say nothing about short-term fluctuations within a year that may be superimposed on my model. Further, since econometric forecasters appear able to produce reasonable forecasts for one or two years ahead, using a great deal more information than I have attempted to use, I do not pretend that my model produces as good forecasts as theirs over one or two years. In particular, my model may well have higher forecast intervals over such periods. I suppose my long term begins about three years out, and continues for as long as you want.

While it might be nice to use more information, such as is done by econometricians, as I wrote in paragraph 3.16 of the paper it is not possible to forecast all the other exogenous factors, and they are subsumed in the white noise series.

Mr. McLean is unhappy about my relating share dividends to the retail prices index "other things being equal", in effect criticising my assumption that the gain is unity, and the mean real rate of dividend increase (DMU) is zero. Mr. Plymen also thinks that DMU should not be zero. If a positive value of DMU is chosen, then the effect of inflation on dividends needs to be less than unity, if past data is to be represented adequately, and vice versa if DMU is negative. There is nothing to prevent Messrs McLean and Plymen using different values for the parameters if they wish.

Mr. McLean quotes Sargent, and suggests that the influence of inflation on the other variables may not be all in one direction. I agree that it would have been possible to construct a fully multivariate time-series model, in which changes in all the variables affected all the others with suitable lags. This indeed was the form of model investigated by Gordon McLeod. However, he found that although the influence of inflation on dividends, etc. was strong, the counter influence was statistically weak. Further, tests showed that this made very little difference to the long-term results from the model. I therefore selected the simplest model that would both represent the past and be a reasonable model for simulating the future. Subsequent investigators may well prefer to use a more complicated model, especially after additional data becomes available.

Mr. McLean is doubtful about the appropriate value for QA. In the separate Note, I quote the standard errors of my parameter estimates. Those for QA are small; further, the values of QA found by fitting the model over different time periods are very similar. I therefore concluded that this was a reasonably constant parameter. By contrast, the standard errors for QMU are fairly large, and the fitted values vary considerably over different time periods. If Mr. McLean, or Mr. Clarkson, who makes much the same point, wish to use a more complicated model, in which QA and QMU vary with time, then they need to find some way of describing this, possibly including stochastic variation of these parameters. Again, I thought this was too complicated.

One way of using the model, but introducing greater uncertainty into it, as both Mr. McLean and Mr. Clarkson seem to desire, is to simulate as follows: at the start of the simulations we prescribe a mean and standard deviation for each of the parameters. Before each particular simulation we pick the parameter values to use during this simulation of, say, 100 years, by picking from a normal distribution with the given mean and standard deviation. We then fix the parameter values for that particular simulation. We pick new parameter values for the next simulation. Some results of this are shown in a paper presented on 29th January 1985 to the Institute of Actuaries Students' Society, "Some *Applications of Stochastic Investment Models*".

Mr. Clarkson is concerned about the distributions of my residuals. In some cases they indeed are fatter-tailed than normal. Nevertheless I have used a normal distribution in these simulations. An alternative way would be to choose a different distribution. One such alternative would be to choose a member of the stable Pareto distributions, which have infinite variance, not a very nice feature; apart from the Cauchy distribution, which has no mean either, they are not obviously easy to simulate. However, it may well be worth trying such a distribution. A second possibility is to use a fixed model where one picks first from a normal distribution, then from a Poisson distribution, with quite a low parameter, so that usually the number picked is zero, and only occasionally is 1, 2, . . . One then makes that number of pickings from another normal distribution with a larger standard deviation, and adds the whole lot up. This is fairly easy to simulate, but it gives rather a lot of parameters to estimate from the available data.

It is not clear that Mr. Clarkson's model with a wandering mean is not in fact another standard autoregressive time-series model, or perhaps a model with two series and a transfer function between them. Again, I have no objection to using such a model; it would not be too difficult to simulate; but it might be rather hard to estimate the parameters satisfactorily.

Mr. Clarkson refers to his investigation into gilt-edged prices and produced four apparently quite different ARIMA models. I wonder whether he found the "roots" of his models. It may well be that the principal roots of the models were similar, and only the secondary roots were different. This is the case with my third-order autoregressive model for the real yield on Consols. The term $(1 - CA1.B - CA2.B^2 - CA3.B^3)$ factorises, and the principal factor is similar to the *only factor* $(1 - CA1.B)$ *of the Reduced Standard model*.

Mr. Clarkson says that the retail price series has "exhibited very pronounced cyclical movements". If he means that there is any regular periodicity in the Retail Price Index, then I must flatly contradict this. I do not quote the results of investigating the Fourier transforms of the values, which would indicate any marked regular periodicity, but I have investigated them and they show no strong period at all. However, even pure "random walks" show considerable runs in one direction or the other that can easily be mistaken for secular trends; and more complicated autoregressive models may respond to a single shock by a damped periodic oscillation, so a single large shock or by coincidence a number of suitably positioned shocks may therefore generate something that looks like a regular periodicity until it dies away.

In reply to Mr. Loades, I have to say that the fact that $DW + DX = 1$ is necessary for dividends to respond to changes in prices with unit gain. However, the fact that $DD = DX = 0{\cdot}2$ is indeed coincidental. Mr. Loades asked what sort of results one got if one set all the standard deviations to zero, and calculated the future forecasts mechanically. Because of the logarithmic transforms, one does not get the mean of the unlogged series, but in some cases one certainly gets the median value. In other cases one gets a value that is just central, but I do not know whether it is necessarily the median or not. Partly this is because of the mixture of logged and unlogged terms. I am sorry about the mixture, but it seems necessary in order to avoid, for example, yields becoming negative.

For some of the series it is certainly possible to calculate the future central forecasts and forecast intervals by using the "variance multipliers" as Mr. Loades suggests. But it is difficult to do this for all of them, especially for the rolled-up indices. I therefore resorted to simulation everywhere even though some of my results could have been derived analytically.

Finally, Mr. Limb seemed to think the mathematics rather daunting. On the contrary, apart from the generation of pseudo-random normal variables, using the model involves no more than simple arithmetic. Fitting the model requires a fair knowledge of mathematical statistics, but even here the mathematics involved is very little more than the Faculty or the Institute require for entry, and much less than anybody with a mathematics degree would at one time have known.

Both Mr. Spain and Mr. Limb were concerned that the use of stochastic models might deter companies from writing a particular sort of business. I do not think that this is the right conclusion to draw. It behoves companies instead to design contracts that take proper account of the stochastic risks, and minimise the probability of "ruin" of a particular portfolio. I think that my stochastic model makes it easier to do this.