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OPTIONS: A MONTE CARLO APPROACH

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This paper develops a Monte Carlo simulation method for solving option valuation problems. The method simulates the process generating the returns on the underlying asset and invokes the risk neutrality assumption to derive the value of the option. Techniques for improving the efficiency of the method are introduced. Some numerical examples are given to illustrate the procedure and additional applications are suggested.

1. Introduction

Option valuation models are very important in the theory of finance since many corporate liabilities can be expressed in terms of options or combinations of options. Smith (1976) has recently summarized the major advances in the theory of option pricing under conditions of general equilibrium. The seminal paper by Black and Scholes (1973) yields the option values that would obtain in conditions of market equilibrium. Black and Scholes assume that the stock price follows a lognormal distribution and show how a hedge position can be formed with a portfolio of one unit (long) of the stock and a short position of a number of options. Arbitrage arguments lead to a second-order linear partial differential equation governing the value of the option. A simple closed form solution of this equation exists in the case of a non-dividend paying stock and also in the case of a stock which pays a continuous dividend proportional to the stock price. For other dividend policies numerical methods must be used to solve the differential equation. Schwartz (1977) has recently developed a method which involves replacing the differential equation by a series of difference equations.

Cox and Ross (1976) have analyzed the structure of option valuation models and presented an alternative approach to their solution. Essentially they show that as long as a hedge position can be constructed the value of a European call

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option can be obtained by discounting the expected maturity value of the option at the risk free rate. The distribution of the maturity value of the option can be obtained from the distribution of the terminal stock value. Thus if the distribution of the terminal stock value is known the value of the option can be obtained by integration. In general the integrals involved will not have analytic solutions and they must be evaluated by numerical methods. Chen (1969) used a numerical integration method in his thesis on warrant valuation and more recently Parkinson (1976) has used this approach to obtain values of American put options.

The purpose of the present paper is to show that Monte Carlo simulation provides a third method of obtaining numerical solutions to option valuation problems. The technique proposed is simple and flexible in the sense that it can easily be modified to accommodate different processes governing the underlying stock returns. This method should provide a useful supplement to the two approaches mentioned above. Furthermore it has distinct advantages in some specialised situations – e.g. when the underlying stock returns involve jump processes. Essentially the method uses the fact that the distribution of terminal stock prices is determined by the process generating future stock price movements. This process can be simulated on a computer thus generating a series of stock price trajectories. This series determines a set of terminal stock values which can be used to obtain an estimate of the option value. Furthermore the standard deviation of the estimate can be obtained at the same time so that the accuracy of the results can be established.

Section 2 of the paper provides a brief description of the Monte Carlo method. One potential drawback of the method arises from the fact that the standard error of the estimate is inversely proportional to the square root of the number of simulation trials. Although any desired degree of accuracy can be obtained by performing enough simulation trials there are usually more efficient ways of reducing the error. One such method known as the control variate approach has proved effective in dealing with some option valuation problems and it is described in section 2. Another technique for increasing the precision of the estimates uses so called antithetic variates, and section 2 also includes a short account of this approach.

In section 3 the simulation method is used to obtain numerical estimates of a European call option on a stock which pays discrete dividends. In this case it is assumed that the returns on the underlying stock follow a lognormal distribution. The introduction of an appropriate control variate considerably improves the accuracy of the estimates. The true answers are obtained using numerical integration and it is verified that they lie as predicted within the calculated confidence limits.

Section 4 discusses further possible applications of the method and suggests when the Monte Carlo method should be used either in preference to or in conjunction with the alternative approaches available.

2. The Monte Carlo method

In this section the Monte Carlo method is described and two techniques for improving the efficiency of the method are discussed.[1] An excellent exposition of the Monte Carlo method is given by Hammersley and Handscomb (1964). Shreider (1966), Fishman (1973) and Meyer (1956) provide additional useful references.

It is convenient to couch the discussion in terms of the evaluation of a definite integral although Monte Carlo methods have a much wider range of applicability. Consider the integral

$$\int_A g(y) f(y)\, \mathrm{d}y = \bar{g}, \tag{1}$$

where $g(y)$ is an arbitrary function and $f(y)$ is a probability density function with

$$\int_A f(y)\, \mathrm{d}y = 1.$$

A denotes the range of integration and is omitted for convenience in the sequel.

To obtain an estimate of $\bar{g}$ a number (n) of sample values (y_i) is picked (at random) from the probability density function $f(y)$. The estimate of $\bar{g}$ is given by

$$\hat{g} = \frac{1}{n} \sum_{i=1}^{n} g(y_i). \tag{2}$$

The standard deviation of the estimate is given by $\hat{s}$ where

$$\hat{s}^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (g(y_i) - \hat{g})^2. \tag{3}$$

For large n the error involved in replacing $(n-1)$ by n in this formula is of little consequence. The distribution

$$\frac{\hat{g} - \bar{g}}{\sqrt{\dfrac{\hat{s}^2}{n}}} \tag{4}$$

tends to a standardized normal distribution with increasing values of n. For the values of n considered in this paper ($n > 1000$) the distribution can be regarded

[1] The material presented in this section is available elsewhere but it is repeated here for the sake of completeness.

as normal and confidence limits on the estimate of $\hat{g}$ can be obtained on this basis.

Since the standard deviation of $\hat{g}$ is equal to $\hat{s}/\sqrt{n}$ the confidence limits can be reduced by increasing n. To reduce the standard deviation by a factor of ten the number of simulations trials has to be increased one hundredfold. An alternative approach is to concentrate on reducing the size of $\hat{s}$. Such techniques are known as variance reduction techniques and a number of them are described in Chapter 5 of Hammersley and Handscomb (1964). The thrust of these methods is to modify or distort the original problem in such a way as to improve the accuracy of the results obtained by crude Monte Carlo methods. One such approach is known as the control variate method.

The basic idea underlying this method is to replace the problem under consideration by a similar but simpler problem which has an analytic solution. The solution of the simpler problem is used to increase the accuracy of the solution to the more complex problem. Suppose that the integral

$$\int g(y)h(y)\,\mathrm{d}y = G \tag{5}$$

can be evaluated analytically where h is a probability density function. From eqs. (1) and (5) it is clear that

$$\bar{g} = G+\int g(y)[f(y)-h(y)]\,\mathrm{d}y. \tag{6}$$

A revised estimate of $\bar{g}$, g^* can be obtained by evaluating the integral on the right-hand side of (6) by crude Monte Carlo methods. The function h is called the control variate. The gain in efficiency will be measured by the reduction in the variance of g^* as compared to the variance of $\hat{g}$. The increase in efficiency will depend on the degree to which h mimics the behaviour of f. Thus in selecting an appropriate control variate there are two (usually conflicting) requirements. First h must give rise to an integral that is easy to evaluate. Second h must model the behaviour of f.

In evaluating the integral in eq. (6) by crude Monte Carlo the same random number is used in the ith simulation trial to generate a value y_i from $g(y)$ and a value Y_i from $h(y)$. For a given value of n let

$$\hat{g} = \int g(y)f(y)\,\mathrm{d}y \quad \text{and} \quad \hat{G} = \int g(y)\,h(y)\,\mathrm{d}y$$

be the estimates obtained by crude Monte Carlo under these conditions. Then g^* is given by

$$g^* = G+(\hat{g}-\hat{G}). \tag{7}$$

This is an unbiased estimate and its variance is

$$\operatorname{var}(\hat{g})+\operatorname{var}(\hat{G})-2\operatorname{cov}(\hat{g}, \hat{G}).$$

This will be less than the variance of $\hat{g}$ as long as

$$\operatorname{cov}(\hat{g}, \hat{G}) > \operatorname{var}(\hat{G})/2,$$

or

$$\operatorname{corr}(\hat{g}, \hat{G}) > \tfrac{1}{2}\sqrt{(\operatorname{var}(\hat{G})/\operatorname{var}(\hat{g}))}. \tag{8}$$

This confirms the earlier observation that the efficiency gain is a function of the relationship between f and h. Furthermore eq. (8) furnishes a prescription for testing if a particular control variate will result in an efficiency gain.

Whereas the control variate approach used a second estimate of the integral with a high positive correlation with the estimate of interest the antithetic variate approach exploits the existence of negative correlation between two estimates. In a given problem there may be different methods of introducing an antithetic variate. One possible method is the following. Suppose that the series $y_1, y_2, \ldots, y_n$ has been generated using the random number sequence $u_1, u_2, \ldots, u_n$, where the u's are selected at random from the interval (0, 1). The sequence $(1-u_1), (1-u_2), \ldots, (1-u_n)$ is used to generate a second set of variates from the distribution $f(y)$. Suppose the estimate of (1) using the first set is $g(u)$ and the estimate of (1) using the second set is $g(1-u)$. Then

$$\tfrac{1}{2}[g(u)+g(1-u)]$$

will be unbiased estimate of (1) with variance equal to

$$\tfrac{1}{4}[\operatorname{var}(g(u))+\operatorname{var}(g(1-u))]+\tfrac{1}{2}[\operatorname{cov}(g(u), g(1-u))]. \tag{9}$$

If the covariance between $g(u)$ and $g(1-u)$ is negative this will yield a smaller estimate of the variance than an independent estimate. A more detailed discussion of this point is given by Fishman (1973).

3. Valuation of European call options on dividend paying stocks

In this section the Monte Carlo method is used to obtain estimates of option values on dividend paying stocks in the case of European call options. It is demonstrated that the introduction of an appropriate control variate significantly increases the accuracy of the results. The accuracy and reliability of the Monte Carlo method has been verified for a range of parameter values using the accurate results obtained by numerical integration and a representative sample of the results is given here. The effect of using antithetic variates is also examined and a modest gain in efficiency is observed.

Let

S_t = the current stock price at time t,
r = the risk-free rate per quarter compounded continuously,
σ^2 = the (assumed constant) variance rate per quarter on the underlying stock,
D_t = the dividend payable at time t; dividends payable quarterly,
E = the exercise price of the option,
T = expiration date of the option.

Cox and Ross (1976) have shown that the assumption of risk neutrality can be used to obtain solutions of option valuation problems.[2] This assumption implies that the equilibrium rate of return on all assets and in particular the common stock is equal to the risk-free rate. Hence the expected return on the stock is given by

$$E(S_T/S_t) = \exp(r[T-t]).$$

In addition it is assumed that the ratio of successive stock values follows a lognormal distribution and so the ratio S_{t+1}/S_t has a lognormal distribution with mean equal to $\exp(r)$. To relate r to the parameters of the normal distribution followed by $\log[S_{t+1}/S_t]$ it is convenient to write

$$E(S_{t+1}/S_t) = \exp(r) = \exp[(r-\sigma^2/2)+\sigma^2/2]. \tag{10}$$

Hence [see Aitchison and Brown (1963)] the expected value of $\log[S_{t+1}/S_t]$ is $(r-\sigma^2/2)$.

Using the properties of the lognormal distribution this means that we can generate the distribution of stock prices one period hence by forming the random variables,

$$S_{t+1} = S_t \exp[r-\sigma^2/2+\sigma\tilde{x}],$$

where $\tilde{x}$ is a normally distributed random variable with zero mean and unit variance. It is assumed that time is measured in units of one quarter. Assume also that S_t represents the stock price just after the quarterly dividend D_t has been paid.

To set up the simulation method in this case a value of S_{t+1} is generated. If this value is greater than D_{t+1} then $(S_{t+1}-D_{t+1})$ is used as the initial value at the start of the second period and the procedure is continued until a value of

[2] See Smith (1976, p. 22) and Cox and Ross (1976) for a discussion of this solution technique.

S_T is obtained. If at some stage $S_{t+m}(m = 1, 2, \ldots (T-t-1)$ is less than or equal to the corresponding dividend payment D_{t+m} the process stops. In this case another simulation trial is started again from time t. A series of simulation trials is carried out in this way and the expected value of Max $[S_T - E, 0]$ is found. This quantity is then discounted at the risk-free rate to yield an estimate of the option value $\hat{g}$. Eq. (3) is used to obtain the variance of the estimate. Ninety-five percent confidence limits are given by $\hat{g} \pm 2\hat{s}/\sqrt{n}$.

Table 1 gives the results of the simulation method for selected values of the

Table 1
European call option values on dividend paying stock; 5000 trials per estimate.[a]

S/E	Number of periods to maturity	Option values by crude Monte Carlo	Standard deviation of crude Monte Carlo estimates	Accurate option values obtained by numerical integration
0.50	2	0.004	0.002	0.003
	4	0.087	0.014	0.075
	6	0.273	0.031	0.266
	8	0.539	0.047	0.541
	10	0.879	0.066	0.867
	12	1.202	0.081	1.220
	14	1.547	0.094	1.587
	16	1.897	0.108	1.957
	18	2.281	0.130	2.327
	20	2.783	0.157	2.691
1.00	2	5.121	0.114	5.028
	4	7.427	0.170	7.251
	6	9.247	0.215	9.001
	8	10.793	0.258	10.493
	10	11.923	0.297	11.811
	12	13.135	0.330	13.000
	14	14.281	0.361	14.087
	16	15.079	0.389	15.090
	18	15.945	0.428	16.022
	20	17.190	0.479	16.891
1.50	2	26.445	0.240	26.369
	4	28.250	0.333	27.819
	6	29.832	0.405	29.300
	8	31.257	0.470	30.713
	10	32.319	0.528	32.042
	12	33.491	0.578	32.289
	14	34.835	0.624	34.459
	16	35.503	0.666	35.558
	18	36.382	0.722	36.594
	20	37.989	0.798	37.571

[a] S = 25, 50, 75 – current stock prices
E = 50
$$\left.\begin{aligned} r &= 0.015 \\ \sigma^2 &= 0.025 \\ D &= 0.25 \end{aligned}\right\} \text{per period}$$

underlying parameters. It is assumed that $r = 0.015$ per period, $\sigma^2 = 0.025$ per period, $D = 0.25$ per period and $E = 50$. The results for three different current stock prices are given corresponding to $S = 25$, 50 and 75. Crude Monte Carlo estimates of options with even maturities ranging from 2 to 20 are provided. Each estimate was obtained using 5000 trials where each trial corresponds to a possible stock path. For convenience it is assumed that level dividends are payable.

To provide a bench mark the accurate results obtained by numerical integration[3] are also given. All of the thirty crude Monte Carlo estimates lie within two standard deviations of the correct answers. However the 95 percent confidence limits are quite wide. For example when the current stock price is 50 and the option has 20 periods to maturity the crude Monte Carlo estimate is 17.190. The 95 percent confidence limits are 17.190 ± 0.958. To reduce the range of these confidence limits to $\pm$ 0.05 would require increasing the number of trials from 5000 to 1,835,500.[4]

This indicates that an alternative method of improving the accuracy of the Monte Carlo method should be sought. The fact that the Black–Scholes solution in the case of a non-dividend paying stock is exact permits the use of a suitable control variate. Assume the existence of another stock S_t^* which pays no dividends until the exercise date.[5] Thereafter S^* is identical with the dividend paying stock. An investor should be indifferent as between (European call) options on these two stocks (assuming both options have the same expiration date and the same exercise price). Since both stocks earn the same rate of return and will have the same value at the maturity date of the option, S_t^* will be less than S_t. In the appendix the first two moments of the return on S_t^* are derived in terms of D and the return parameters of S_t. The control variate is a stock having the same expected return and variance rate as S_t^* over the period until maturity and which in addition has returns that are lognormally distributed.[6]

The Black–Scholes solution is used to obtain the value of an option on the control variate. Then both problems are attacked together using crude Monte Carlo methods as described in section 2. In each simulation run the same

[3]To obtain very accurate option values it proved more convenient in this instance to use a numerical integration approach rather than a finite difference approach. The numerical integration method used here is based on a simple trapezoidal rule [cf. Chen (1969)]. By reducing the size of the interval, results accurate to three decimal places were obtained. This degree of accuracy was verified by showing that more refined intervals (using double precision) did not affect the results. As an additional check the numerical integration method was also used to obtain option values in the no dividend case. The results so obtained agreed exactly with the Black–Scholes results for the range of parameter values considered to at least three decimal places.

[4]To see this note that $\hat{s}/\sqrt{5000} = 0.479$. It is required to find a number n so that $\hat{s}/\sqrt{n} = 0.025$. Hence $n = (0.479/0.025)^2 \times 5000 = 1{,}835{,}528$.

[5]This technique of using an associated stock which pays no dividends has also been suggested by Rubenstein (1975).

[6]A more detailed discussion of this point is given in the appendix.

random normal deviates are used to generate both lognormal distributions. (These lognormal distributions correspond to the dividend paying stock and the control variate.) The use of the same set of normal deviates during corresponding periods in each simulation run induces high positive correlation. In the case of the control variate, D, the dividend payment is of course zero. The results are displayed in table 2 where for comparison purposes the same parameters are

Table 2
European call option values on dividend paying stocks; 5000 trials per estimate.[a]

S/E	Number of periods to maturity	Option values control variate method	Standard deviation of estimate using control variate method	Accurate option values using numerical integration
0.50	2	0.003	0.00004	0.003
	4	0.075	0.0002	0.075
	6	0.266	0.001	0.266
	8	0.542	0.001	0.541
	10	0.869	0.002	0.867
	12	1.222	0.003	1.220
	14	1.591	0.004	1.587
	16	1.964	0.006	1.957
	18	2.330	0.007	2.327
	20	2.689	0.010	2.691
1.00	2	5.028	0.0003	5.028
	4	7.251	0.001	7.251
	6	9.000	0.002	9.001
	8	10.494	0.003	10.493
	10	11.815	0.004	11.811
	12	13.006	0.006	13.000
	14	14.094	0.007	14.087
	16	15.106	0.009	15.090
	18	16.038	0.010	16.022
	20	16.903	0.013	16.891
1.50	2	26.369	0.0004	26.369
	4	27.818	0.001	27.819
	6	29.299	0.002	29.300
	8	30.715	0.003	30.713
	10	30.048	0.005	32.042
	12	33.298	0.006	33.289
	14	34.465	0.008	34.459
	16	35.578	0.009	35.558
	18	36.616	0.011	36.594
	20	37.582	0.014	37.571

[a] $S = 25, 50, 75$ – current stock prices
$E = 50$
$$\left.\begin{aligned} r &= 0.015 \\ r^2 &= 0.025 \\ D &= 0.25 \end{aligned}\right\} \text{per period}$$

used as in table 1. Observe that the introduction of the control variate greatly reduces the standard deviation of the estimates. For example the range of the 95 percent confidence limits in the case of the 20 period option with a current stock price of 50 have been reduced from $\pm$ 0.958 to $\pm$ 0.026. To achieve the same reduction by increasing the number of trials would require 6,788,000 trials instead of 5,000. In all 30 cases the 95 percent confidence limits contain the true answer. The confidence limits in table 2 would appear to be sufficiently accurate for most practical applications.

The introduction of an appropriate control variate provides a very efficient variance reduction technique in this problem. In some problems it may be difficult to find a suitable control variate. The alternative method discussed in section 2, the antithetic variate method, is often easier to apply since it concentrates on the procedure used for generating the random deviates. Essentially this technique relies on the introduction of negative correlation between two estimates. It is of interest to examine one method of using antithetic variates in the present problem.

Notice that if $\tilde{x}$ is a normally distributed random variable with zero mean and unit variance then so also is $-\tilde{x}$. Hence to apply the antithetic variate technique in this case use a set of random normal deviates to obtain the initial estimate. A second estimate is obtained by using the same set of random normal deviates with their signs reversed. The revised estimate will be the mean of these two estimates. The results of one such set of calculations are displayed in table 3

Table 3
Option values using crude Monte Carlo and antithetic variate method; 5000 trials per estimate.[a]

S/E	Number of periods to maturity	Crude Monte Carlo estimate	Standard deviation of of crude Monte Carlo estimates	Antithetic variate method estimate	Standard deviation of of antithetic variate estimate
1.00	2	5.121	0.114	5.093	0.060
	4	7.427	0.170	7.254	0.092
	6	9.247	0.215	9.101	0.126
	8	10.793	0.258	10.609	0.150
	10	11.923	0.297	11.716	0.170
	12	13.135	0.330	12.974	0.194
	14	14.281	0.361	14.050	0.215
	16	15.079	0.389	15.013	0.231
	18	15.945	0.428	16.018	0.260
	20	17.190	0.479	17.030	0.287

[a] $S = 50$
$E = 50$
$$\left.\begin{aligned} r &= 0.015 \\ \sigma^2 &= 0.025 \\ D &= 0.250 \end{aligned}\right\} \text{per period}$$

and comparison with the appropriate figures in table 1 shows that the standard deviations of the estimates have been reduced by approximately 50% – a very modest gain in efficiency. This low gain in efficiency may be explained as follows. While $\tilde{x}$ and $-\tilde{x}$ have perfect negative correlation this does not hold for the corresponding transformed lognormal variates. The variance of the revised estimate is given by an expression analogous to expression (9) in section 2 and while the covariance term is negative its magnitude is not large enough to effect a significant reduction in the variance of the revised estimate.

4. Discussion and additional applications

This section discusses some strengths and limitations of the Monte Carlo approach and identifies situations where its use would be advantageous. Some additional applications of the method are discussed. In particular the Monte Carlo method should prove useful in dealing with options where the underlying stock returns are generated by a mixture of both continuous and jump processes. (In general these mixed processes give rise to mixed partial differential – difference equations which are very difficult to solve.)

As was suggested in the last section the Monte Carlo method can be used to obtain numerical estimates of a European call option on a stock which pays discrete dividends. When an appropriate control variate was used the method gave very accurate results together with confidence limits on the estimates. This method provides the option value in respect of one particular current stock price. To obtain option values corresponding to different current stock prices a set of simulation trials has to be carried out for each starting stock price. When either a numerical integration or a finite difference approach is employed the complete range of stock prices is obtained as a by-product. This suggests that the Monte Carlo method should be used for one-off situations where only a few option values are required. Since the Monte Carlo method is very simple to set up it can play a useful role in checking the results obtained using one of the other approaches.

It is possible to modify the simulation technique to handle American call options on dividend paying stocks. Merton (1973) has shown that such options will only be exercised (by a rational investor) just before each dividend date if the stock price exceeds a certain critical value. These critical stock prices, one corresponding to each dividend date, must be determined as part of the solution. At each dividend date a series of option prices must be obtained to locate the critical stock price. This suggests that either the numerical integration approach or the finite difference method would be more effective. Notice however that if there is just one dividend payable during the lifetime of the contract the critical stock price can be found exactly by using the Black–Scholes solution and an iterative procedure. In such circumstances the Monte Carlo method should prove useful.

One advantage of the Monte Carlo method is that it is very flexible with regard to the distribution used to generate the returns on the underlying stock. Changing the underlying distribution merely involves using a different process for generating the random variates employed in the method. Furthermore the Monte Carlo method is perhaps unique in the sense that the distribution used to generate returns on the underlying stock need not have a closed form analytic expression. This opens the possibility of deriving option values using empirical distributions of stock returns.[7] The Monte Carlo method has the advantage that a distribution can be used for any of the parameters of the problem rather than a point estimate. For example it may be useful in some problems to regard the variance as a probability distribution since it is usually estimated from empirical data.

The simulation method will also have advantages in cases where the underlying stock returns are generated by a mixture of stochastic processes. As an example of this we consider the model proposed recently by Merton (1976). In this model the process generating stock returns is composed of a continuous part and a jump component. The continuous part is modelled by a Gauss–Weiner process and the jump component is modelled by an independent Poisson process. The jump corresponds to the arrival of important pieces of information which affect the stock price. The expected number of arrivals per unit time is λ. If the Poisson event occurs the stock price will jump from S to SY where Y is a random variable taking non-negative values. Under these assumptions the stock price dynamics become

$$\mathrm{d}S/S = (\alpha - \lambda k)\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{z} + \mathrm{d}q, \tag{11}$$

where α is the expected instantaneous return on the stock, σ^2 is the instantaneous variance of the return conditional on the Poisson event not occurring, $\mathrm{d}z$ is a Gauss–Weiner process, and $q(t)$ represents the Poisson process.

The expected value of Y is $(1+k)$ and it is assumed that the Y are independently and identically distributed.

Although it is not possible to create a riskless hedge against both the continuous changes and the discrete changes implied by eq. (11) Merton shows that an option pricing formula may be derived on the assumption that the risk associated with the jump is a diversifiable risk. The dynamics of the situation dictate that the distribution of stock prices at the end of a given time period will depend on

(i) the basic Gauss–Weiner process,
(ii) the number of jumps that occur,
(iii) the amplitude of these jumps.

[7]The author is indebted to A. Ananthanarayan for this observation.

Thus three different stochastic processes are involved. By generating random deviates from these three distributions a set of terminal stock values can be obtained and the simulation method used to obtain estimates of the option prices.

Merton has provided a solution to this problem in the form

$$\sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} [\varepsilon_n\{W(SX_n e^{-\lambda k t}, t, E, \sigma^2, r)\}],$$

where the random variable X_n is the product of n random variables each identically distributed to the random variable Y: $X_0 = 1$; and ε_n is the expectation operator over the distribution X_n. W represents the standard Black–Scholes solution.

To obtain an explicit solution Merton makes the additional assumption that the random variable Y also has the lognormal distribution. The Monte Carlo approach affords a method of providing numerical solutions in this case for different distributions of Y. In addition the Monte Carlo method can be used in this case when it is also assumed that the stock pays discrete dividends. Of course to improve the accuracy of the method without resorting to a large number of simulations it is necessary to incorporate a suitable variance reduction technique. While this point is not pursued here it is suggested that the use of either control variates or antithetic variates might prove useful.[8]

To conclude this section it will be useful to examine the role of Monte Carlo calculations in general and in particular their role in option valuation problems. The crude Monte Carlo approach provides a fast and flexible method of obtaining approximate answers together with confidence limits on the results. A number of effective techniques for reducing the variance of the estimates have been developed. However it is not possible to formulate general rules regarding the selection and implementation of the most effective technique. As Kahn[9] observes:

> "... the greatest gains in variance reduction are often made by exploiting specific details of the problem, rather than by the routine application of general principles."

In connection with option valuation problems it is suggested that the Monte Carlo method should prove most valuable in situations where it is difficult if not impossible to proceed using a more accurate approach. In particular we have in mind problems where the returns on the underlying stock are generated by a mixture of stochastic processes or else drawn from an empirical distribution. The Monte Carlo method provides a viable alternative in the case of European

[8]The author hopes to carry out further research on this point.

[9]See Meyer (1956, p. 146).

call options on dividend paying stocks because of the introduction of a very efficient control variate. In other situations it may be viewed as a useful supplement to existing methods. Indeed it should have some pedagogical value since it utilises the intuitive solution technique developed by Cox and Ross (1976) and shows how the process governing the returns on the underlying stock can be used to obtain option values. The present paper has drawn attention to some of the techniques available for reducing the variance of the crude Monte Carlo estimates. Since these techniques do not appear to be widely appreciated in the finance literature it is hoped this is a useful service.

Appendix

The determination of the parameters of the control variate is described below.

Let the current price of the dividend paying stock be S_t and assume that the maturity date of the option is m periods hence. Assume that a dividend has just been paid and that a dividend D will be payable at the end of the 1st, 2nd, 3rd $(m-1)$st period. It is also assumed that the stock returns in each period are independently and identically distributed. Thus the stock price evolution is generated by the series of random variables $x_1, x_2, \ldots, x_m$ where all the x_i are drawn from the same distribution. At the end of the first period the stock price is $S_t x_1$. Assuming that $S_t x_1$ is greater than D the stock price will be $(S_t x_1 - D)x_2$ at the end of the second period. After m periods under these assumptions the stock price will be

$$S_t \prod_{i=1}^{m} x_i - D \sum_{j=2}^{m} \prod_{i=j}^{m} x_i. \tag{A.1}$$

Since the x_i are independent the expected value of this quantity is

$$S_t \prod_{i=1}^{m} E(x_i) - D \sum_{j=2}^{m} \prod_{i=j}^{m} E(x_i). \tag{A.2}$$

The variance of (A.1) can be expressed in terms of $E(x_i)$ and $E(x_i^2)$ with somewhat more labour. It is convenient to write it first in the form

$$\begin{aligned} &S_t^2 \operatorname{var}\left[\prod_{i=1}^{m} x_i\right] + D^2 \operatorname{var}\left[\sum_{j=2}^{m} \prod_{i=j}^{m} x_i\right] \\ &-2S_t D \operatorname{cov}\left[\prod_{i=1}^{m} x_i, \sum_{j=2}^{m} \prod_{i=j}^{m} x_i\right], \end{aligned} \tag{A.3}$$

and then evaluate each term in turn. Since all the x_i have the same distribution it can be assumed that

$$E(x_i) = \rho \quad \text{and} \quad E(x_i^2) = \theta,$$

for all i and this simplifies the analysis.

With this notation it can be shown [Boyle (1976)] that

$$\operatorname{var}\left[\prod_{i=1}^{m} x_i\right] = \theta^m - \rho^{2m},$$

$$\operatorname{var}\left[\sum_{j=2}^{m}\prod_{i=j}^{m} x_i\right] = \frac{(\theta+\rho)}{(\theta-\rho)}\sum_{i=1}^{m-1}\theta^i - \frac{2\theta}{(\theta-\rho)}\sum_{i=1}^{m-1}\rho^i - \left[\sum_{i=1}^{m-1}\rho^i\right]^2,$$

$$\operatorname{cov}\left[\prod_{i=1}^{m} x_i, \sum_{j=2}^{m}\prod_{i=j}^{m} x_i\right] = \rho\theta\,\frac{(\theta^{m-1}-\rho^{m-1})}{(\theta-\rho)} - \rho^m\,\frac{(\rho-\rho^m)}{(1-\rho)}.$$

The control variate is derived using a stock S_t^* which pays no dividends until the option matures and thereafter is identical with the dividend paying stock. Hence *at maturity* the distribution of S^* is identical with that of S given by eq. (A.1) because from this point on both securities represent identical claims on the assets of the firm. Thus the first two moments of the distribution of S^* at the maturity date of the option are given by (A.2) and (A.3). Since S and S^* must earn the same return the current value of S^*, S_t^* is obtained by discounting (A.2) at the risk free rate. The control variate is a stock which follows a lognormal distribution and has the same first two moments as S^*. Notice that since S^* involves a linear combination of lognormal distributions it cannot follow a true lognormal distribution although this is a very good approximation for the range of parameters normally involved. Since the control variate follows a true lognormal distribution and its first two moments are known the mean and variance of the underlying normal distribution can be determined [cf. Aitchison and Brown (1963)]. Suppose the value of mean of the underlying normal distribution is M and that of the variance V. The normal distribution used to generate the one period returns on the control variate will have mean M/m and variance V/m. Because of the risk neutrality assumption,

$$rm = M + V/2,$$

and so M can be expressed in terms of r and V. These parameters are used in determining the crude Monte Carlo estimate of the option on the control variate. The same random numbers are used in these simulations as are used in obtaining the crude Monte Carlo estimates of the option on the dividend paying stock.

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