RISKMETRICS

The RiskMetrics Group
Working Paper Number 99-07

On Default Correlation: A Copula Function Approach

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This draft: April 2000
First draft: September 1999

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April 2000

Abstract

This paper studies the problem of default correlation. We first introduce a random variable called "time-until-default" to denote the survival time of each defaultable entity or financial instrument, and define the default correlation between two credit risks as the correlation coefficient between their survival times. Then we argue why a copula function approach should be used to specify the joint distribution of survival times after marginal distributions of survival times are derived from market information, such as risky bond prices or asset swap spreads. The definition and some basic properties of copula functions are given. We show that the current CreditMetrics approach to default correlation through asset correlation is equivalent to using a normal copula function. Finally, we give some numerical examples to illustrate the use of copula functions in the valuation of some credit derivatives, such as credit default swaps and first-to-default contracts.

1 Introduction

The rapidly growing credit derivative market has created a new set of financial instruments which can be used to manage the most important dimension of financial risk - credit risk. In addition to the standard credit derivative products, such as credit default swaps and total return swaps based upon a single underlying credit risk, many new products are now associated with a portfolio of credit risks. A typical example is the product with payment contingent upon the time and identity of the first or second-to-default in a given credit risk portfolio. Variations include instruments with payment contingent upon the cumulative loss before a given time in the future. The equity tranche of a collateralized bond obligation (CBO) or a collateralized loan obligation (CLO) is yet another variation, where the holder of the equity tranche incurs the first loss. Deductible and stop-loss in insurance products could also be incorporated into the basket credit derivatives structure. As more financial firms try to manage their credit risk at the portfolio level and the CBO/CLO market continues to expand, the demand for basket credit derivative products will most likely continue to grow.

Central to the valuation of the credit derivatives written on a credit portfolio is the problem of default correlation. The problem of default correlation even arises in the valuation of a simple credit default swap with one underlying reference asset if we do not assume the independence of default between the reference asset and the default swap seller. Surprising though it may seem, the default correlation has not been well defined and understood in finance. Existing literature tends to define default correlation based on discrete events which dichotomize according to survival or nonsurvival at a critical period such as one year. For example, if we denote

$$q_A = \Pr[E_A], \quad q_B = \Pr[E_B], \quad q_{AB} = \Pr[E_A E_B]$$

where E_A, E_B are defined as the default events of two securities A and B over 1 year. Then the default correlation ρ between two default events E_A and E_B, based on the standard definition of correlation of two random variables, are defined as follows

$$\rho = \frac{q_{AB} - q_A \cdot q_B}{\sqrt{q_A(1-q_A)q_B(1-q_B)}}. \tag{1}$$

This discrete event approach has been taken by Lucas [1995]. Hereafter we simply call this definition of default correlation the **discrete default correlation**.

However the choice of a specific period like one year is more or less arbitrary. It may correspond with many empirical studies of default rate over one year period. But the dependence of default correlation on a specific time interval has its disadvantages. First, default is a time dependent event, and so is default correlation. Let us take the survival time of a human being as an example. The probability of dying within one year for a person aged 50 years today is about 0.6%, but the probability of dying for the same person within 50 years is almost a sure event. Similarly default correlation is a time dependent quantity. Let us now take the survival times of a couple, both aged 50 years today. The correlation between the two discrete events that each dies within one year is very small. But the correlation between the two discrete events that each dies within 100 years is 1. Second, concentration on a single period of one year wastes important information. There are empirical studies which show that the default tendency of corporate bonds is linked to their age since issue. Also there are strong links between the economic cycle and defaults. Arbitrarily focusing on a one year period neglects this important information. Third, in the majority of credit derivative valuations, what we need is not the default correlation of two entities over the next year. We may need to have a joint distribution of survival times for the next 10 years. Fourth, the calculation of default rates as simple proportions is possible only when no samples are censored during the one year period[1].

This paper introduces a few techniques used in survival analysis. These techniques have been widely applied to other areas, such as life contingencies in actuarial science and industry life testing in reliability studies, which are similar to the credit problems we encounter here. We first introduce a random variable called

[1]A company who is observed, default free, by Moody's for 5-years and then withdrawn from the Moody's study must have a survival time exceeding 5 years. Another company may enter into Moody's study in the middle of a year, which implies that Moody's observes the company for only half of the one year observation period. In the survival analysis of statistics, such incomplete observation of default time is called *censoring*. According to Moody's studies, such incomplete observation does occur in Moody's credit default samples.

“time-until-default” to denote the survival time of each defaultable entity or financial instrument. Then, we define the default correlation of two entities as the correlation between their survival times. In credit derivative valuation we need first to construct a credit curve for each credit risk. A credit curve gives all marginal conditional default probabilities over a number of years. This curve is usually derived from the risky bond spread curve or asset swap spreads observed currently from the market. Spread curves and asset swap spreads contain information on default probabilities, recovery rate and liquidity factors etc. Assuming an exogenous recovery rate and a default treatment, we can extract a credit curve from the spread curve or asset swap spread curve. For two credit risks, we would obtain two credit curves from market observable information. Then, we need to specify a joint distribution for the survival times such that the marginal distributions are the credit curves. Obviously, this problem has no unique solution. Copula functions used in multivariate statistics provide a convenient way to specify the joint distribution of survival times with given marginal distributions. The concept of copula functions, their basic properties, and some commonly used copula functions are introduced. Finally, we give a few numerical examples of credit derivative valuation to demonstrate the use of copula functions and the impact of default correlation.

2 Characterization of Default by Time-Until-Default

In the study of default, interest centers on a group of individual companies for each of which there is defined a point event, often called default, (or survival) occurring after a length of time. We introduce a random variable called the **time-until-default**, or simply survival time, for a security, to denote this length of time. This random variable is the basic building block for the valuation of cash flows subject to default.

To precisely determine time-until-default, we need: an unambiguously defined time origin, a time scale for measuring the passage of time, and a clear definition of default.

We choose the current time as the time origin to allow use of current market information to build credit curves. The time scale is defined in terms of years for continuous models, or number of periods for discrete models. The meaning of default is defined by some rating agencies, such as Moody’s.

2.1 Survival Function

Let us consider an existing security A. This security's time-until-default, T_A, is a continuous random variable which measures the length of time from today to the time when default occurs. For simplicity we just use T which should be understood as the time-until-default for a specific security A. Let $F(t)$ denote the distribution function of T,

$$F(t) = \Pr(T \leq t), \qquad t \geq 0 \tag{2}$$

and set

$$S(t) = 1 - F(t) = \Pr(T > t), \qquad t \geq 0. \tag{3}$$

We also assume that $F(0) = 0$, which implies $S(0) = 1$. The function $S(t)$ is called the **survival function**. It gives the probability that a security will attain age t. The distribution of T_A can be defined by specifying either the distribution function $F(t)$ or the survival function $S(t)$. We can also define a probability density function as follows

$$f(t) = F'(t) = -S'(t) = \lim_{\Delta \to 0^+} \frac{\Pr[t \leq T < t + \Delta]}{\Delta}.$$

To make probability statements about a security which has survived x years, the future life time for this security is $T - x | T > x$. We introduce two more notations

$$\begin{aligned} {}_tq_x &= \Pr[T - x \leq t | T > x], \qquad t \geq 0 \\ {}_tp_x &= 1 - {}_tq_x = \Pr[T - x > t | T > x], \qquad t \geq 0. \end{aligned} \tag{4}$$

The symbol ${}_tq_x$ can be interpreted as the conditional probability that the security A will default within the next t years conditional on its survival for x years. In the special case of $X = 0$, we have

$${}_tp_0 = S(t) \quad x \geq 0.$$

If $t = 1$, we use the actuarial convention to omit the prefix 1 in the symbols ${}_tq_x$ and ${}_tp_x$, and we have

$$
\begin{aligned}
p_x &= \Pr[T - x > 1 | T > x] \\
q_x &= \Pr[T - x \le 1 | T > x].
\end{aligned}
$$

The symbol q_x is usually called the *marginal default probability*, which represents the probability of default in the next year conditional on the survival until the beginning of the year. A credit curve is then simply defined as the sequence of $q_0, q_1, \cdots, q_n$ in discrete models.

2.2 Hazard Rate Function

The distribution function $F(t)$ and the survival function $S(t)$ provide two mathematically equivalent ways of specifying the distribution of the random variable time-until-default, and there are many other equivalent functions. The one used most frequently by statisticians is the hazard rate function which gives the instantaneous default probability for a security that has attained age x.

$$
\begin{aligned}
\Pr[x < T \le x + \Delta x | T > x] &= \frac{F(x + \Delta x) - F(x)}{1 - F(x)} \\
&\approx \frac{f(x)\Delta x}{1 - F(x)}.
\end{aligned}
$$

The function

$$
\frac{f(x)}{1 - F(x)}
$$

has a conditional probability density interpretation: it gives the value of the conditional probability density function of T at exact age x, given survival to that time. Let's denote it as $h(x)$, which is usually called the **hazard rate function**. The relationship of the hazard rate function with the distribution function and survival function is as follows

$$h(x) = \frac{f(x)}{1 - F(x)} = -\frac{S'(x)}{S(x)}. \tag{5}$$

Then, the survival function can be expressed in terms of the hazard rate function,

$$S(t) = e^{-\int_0^t h(s)ds}.$$

Now, we can express ${}_tq_x$ and ${}_tp_x$ in terms of the hazard rate function as follows

$$\begin{aligned} {}_tp_x &= e^{-\int_0^t h(s+x)ds}, \\ {}_tq_x &= 1 - e^{-\int_0^t h(s+x)ds}. \end{aligned} \tag{6}$$

In addition,

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t h(s)ds},$$

and

$$f(t) = S(t) \cdot h(t). \tag{7}$$

which is the density function for T.

A typical assumption is that the hazard rate is a constant, h, over certain period, such as $[x, x+1]$. In this case, the density function is

$$f(t) = he^{-ht}$$

which shows that the survival time follows an exponential distribution with parameter h. Under this assumption, the survival probability over the time interval $[x, x+t]$ for $0 < t \leq 1$ is

$$
{}_t p_x = 1 - {}_t q_x = e^{-\int_0^t h(s)ds} = e^{-ht} = (p_x)^t
$$

where p_x is the probability of survival over one year period. This assumption can be used to scale down the default probability over one year to a default probability over a time interval less than one year.

Modelling a default process is equivalent to modelling a hazard function. There are a number of reasons why modelling the hazard rate function may be a good idea. First, it provides us information on the immediate default risk of each entity known to be alive at exact age t. Second, the comparisons of groups of individuals are most incisively made via the hazard rate function. Third, the hazard rate function based model can be easily adapted to more complicated situations, such as where there is censoring or there are several types of default or where we would like to consider stochastic default fluctuations. Fourth, there are a lot of similarities between the hazard rate function and the short rate. Many modeling techniques for the short rate processes can be readily borrowed to model the hazard rate.

Finally, we can define the joint survival function for two entities A and B based on their survival times T_A and T_B,

$$
S_{T_A T_B}(s, t) = \Pr[T_A > s, T_B > t].
$$

The joint distributional function is

$$
\begin{aligned}
F(s, t) &= \Pr[T_A \leq s, T_B \leq t] \\
&= 1 - S_{T_A}(s) - S_{T_B}(t) + S_{T_A T_B}(s, t).
\end{aligned}
$$

The aforementioned concepts and results can be found in survival analysis books, such as Bowers et al. [1997], Cox and Oakes [1984].

3 Definition of Default Correlations

The default correlation of two entities A and B can then be defined with respect to their survival times T_A and T_B as follows

$$\begin{aligned}
\rho_{AB} &= \frac{Cov(T_A, T_B)}{\sqrt{Var(T_A)Var(T_B)}} \\
&= \frac{E(T_A T_B) - E(T_A)E(T_B)}{\sqrt{Var(T_A)Var(T_B)}}.
\end{aligned} \tag{8}$$

Hereafter we simply call this definition of default correlation **the survival time correlation**. The survival time correlation is a much more general concept than that of the discrete default correlation based on a one period. If we have the joint distribution $f(s, t)$ of two survival times T_A, T_B, we can calculate the discrete default correlation. For example, if we define

$$\begin{aligned}
E_1 &= [T_A < 1], \\
E_2 &= [T_B < 1],
\end{aligned}$$

then the discrete default correlation can be calculated using equation (1) with the following calculation

$$\begin{aligned}
q_{12} &= \Pr[E_1 E_2] = \int_0^1 \int_0^1 f(s, t)dsdt \\
q_1 &= \int_0^1 f_A(s)ds \\
q_2 &= \int_0^1 f_B(t)dt.
\end{aligned}$$

However, knowing the discrete default correlation over one year period does not allow us to specify the survival time correlation.

4 The Construction of the Credit Curve

The distribution of survival time or time-until-default can be characterized by the distribution function, survival function or hazard rate function. It is shown in Section 2 that all default probabilities can be calculated once a characterization is given. The hazard rate function used to characterize the distribution of survival time can also be called a credit curve due to its similarity to a yield curve. But the basic question is: how do we obtain the credit curve or the distribution of survival time for a given credit?

There exist three methods to obtain the term structure of default rates:

(i) Obtaining historical default information from rating agencies;

(ii) Taking the Merton option theoretical approach;

(iii) Taking the implied approach using market prices of defaultable bonds or asset swap spreads.

Rating agencies like Moody's publish historical default rate studies regularly. In addition to the commonly cited one-year default rates, they also present multi-year default rates. From these rates we can obtain the hazard rate function. For example, Moody's (see Carty and Lieberman [1997]) publishes weighted average cumulative default rates from 1 to 20 years. For the B rating, the first 5 years cumulative default rates in percentage are 7.27, 13.87, 19.94, 25.03 and 29.45. From these rates we can obtain the marginal conditional default probabilities. The first marginal conditional default probability in year one is simply the one-year default probability, 7.27%. The other marginal conditional default probabilities can be obtained using the following formula:

$$ {}_{n+1}q_x = {}_nq_x + {}_np_x \cdot q_{x+n}, \tag{9} $$

which simply states that the probability of default over time interval $[0, n+1]$ is the sum of the probability of default over the time interval $[0, n]$, plus the probability of survival to the end of nth year and default in the following year. Using equation (9) we have the marginal conditional default probability:

$$ q_{x+n} = \frac{{}_{n+1}q_x - {}_nq_x}{1 - {}_nq_x} $$

which results in the marginal conditional default probabilities in year 2, 3, 4, 5 as 7.12%, 7.05%, 6.36% and 5.90%. If we assume a piecewise constant hazard rate function over each year, then we can obtain the hazard rate function using equation (6). The hazard rate function obtained is given in Figure (1).

Using diffusion processes to describe changes in the value of the firm, Merton [1974] demonstrated that a firm's default could be modeled with the Black and Scholes methodology. He showed that stock could be considered as a call option on the firm with strike price equal to the face value of a single payment debt. Using this framework we can obtain the default probability for the firm over one period, from which we can translate this default probability into a hazard rate function. Geske [1977] and Delianedis and Geske [1998] extended Merton's analysis to produce a term structure of default probabilities. Using the relationship between the hazard rate and the default probabilities we can obtain a credit curve.

Alternatively, we can take the implicit approach by using market observable information, such as asset swap spreads or risky corporate bond prices. This is the approach used by most credit derivative trading desks. The extracted default probabilities reflect the market-agreed perception today about the future default tendency of the underlying credit. Li [1998] presents one approach to building the credit curve from market information based on the Duffie and Singleton [1996] default treatment. In that paper the author assumes that there exists a series of bonds with maturity 1, 2, .., n years, which are issued by the same company and have the same seniority. All of those bonds have observable market prices. From the market price of these bonds we can calculate their yields to maturity. Using the yield to maturity of corresponding treasury bonds we obtain a yield spread curve over treasury (or asset swap spreads for a yield spread curve over LIBOR). The credit curve construction is based on this yield spread curve and an exogenous assumption about the recovery rate based on the seniority and the rating of the bonds, and the industry of the corporation.

The suggested approach is contrary to the use of historical default experience information provided by rating agencies such as Moody's. We intend to use market information rather than historical information for the following reasons:

- The calculation of profit and loss for a trading desk can only be based on current market information. This current market information reflects the market agreed perception about the evolution of the market in the future, on which the actual profit and loss depend. The default rate derived from current market information may be much different than historical default rates.
- Rating agencies use classification variables in the hope that homogeneous risks will be obtained

after classification. This technique has been used elsewhere like in pricing automobile insurance. Unfortunately, classification techniques omit often some firm specific information. Constructing a credit curve for each credit allows us to use more firm specific information.

- Rating agencies reacts much slower than the market in anticipation of future credit quality. A typical example is the rating agencies reaction to the recent Asian crisis.

- Ratings are primarily used to calculate default frequency instead of default severity. However, much of credit derivative value depends on both default frequency and severity.

- The information available from a rating agency is usually the one year default probability for each rating group and the rating migration matrix. Neither the transition matrixes, nor the default probabilities are necessarily stable over long periods of time. In addition, many credit derivative products have maturities well beyond one year, which requires the use of long term marginal default probability.

It is shown under the Duffie and Singleton approach that a defaultable instrument can be valued as if it is a default free instrument by discounting the defaultable cash flow at a *credit risk adjusted discount factor*. The credit risk adjusted discount factor or the total discount factor is the product of risk-free discount factor and the pure credit discount factor if the underlying factors affecting default and those affecting the interest rate are independent. Under this framework and the assumption of a piecewise constant hazard rate function, we can derive a credit curve or specify the distribution of the survival time.

5 Dependent Models - Copula Functions

Let us study some problems of an n credit portfolio. Using either the historical approach or the market implicit approach, we can construct the marginal distribution of survival time for each of the credit risks in the portfolio. If we assume mutual independence among the credit risks, we can study any problem associated with the portfolio. However, the independence assumption of the credit risks is obviously not realistic; in reality, the default rate for a group of credits tends to be higher in a recession and lower when the economy is booming. This implies that each credit is subject to the same set of macroeconomic environment, and that there exists some form of positive dependence among the credits. To introduce a correlation structure into the portfolio, we must determine how to specify a joint distribution of survival times, with given marginal distributions.

Obviously, this problem has no unique solution. Generally speaking, knowing the joint distribution of random variables allows us to derive the marginal distributions and the correlation structure among the random variables, but not vice versa. There are many different techniques in statistics which allow us to specify a joint distribution function with given marginal distributions and a correlation structure. Among them, copula function is a simple and convenient approach. We give a brief introduction to the concept of copula function in the next section.

5.1 Definition and Basic Properties of Copula Function

A copula function is a function that links or marries univariate marginals to their full multivariate distribution. For m uniform random variables, U_1, U_2, $\cdots$,U_m, the joint distribution function C, defined as

$$C(u_1, u_2, \cdots, u_m, \rho) = \Pr[U_1 \leq u_1, U_2 \leq u_2, \cdots, U_m \leq u_m]$$

can also be called a *copula function*.

Copula functions can be used to link marginal distributions with a joint distribution. For given univariate marginal distribution functions $F_1(x_1)$, $F_2(x_2)$,$\cdots$, $F_m(x_m)$, the function

$$C(F_1(x_1), F_2(x_2), \cdots, F_m(x_m)) = F(x_1, x_2, \cdots x_m),$$

which is defined using a copula function C, results in a multivariate distribution function with univariate marginal distributions as specified $F_1(x_1)$, $F_2(x_2)$,$\cdots$, $F_m(x_m)$.

This property can be easily shown as follows:

$$\begin{aligned} C(F_1(x_1), F_2(x_2), \cdots, F_m(x_m), \rho) &= \Pr[U_1 \leq F_1(x_1), U_2 \leq F_2(x_2), \cdots, U_m \leq F_m(x_m)] \\ &= \Pr\left[F_1^{-1}(U_1) \leq x_1, F_2^{-1}(U_2) \leq x_2, \cdots, F_m^{-1}(U_m) \leq x_m\right] \\ &= \Pr[X_1 \leq x_1, X_2 \leq x_2, \cdots, X_m \leq x_m] \\ &= F(x_1, x_2, \cdots x_m). \end{aligned}$$

The marginal distribution of X_i is

$$\begin{aligned} & C(F_1(+\infty), F_2(+\infty), \cdots F_i(x_i), \cdots, F_m(+\infty), \rho) \\ =\; & \Pr[X_1 \leq +\infty, X_2 \leq +\infty, \cdots, X_i \leq x_i, X_m \leq +\infty] \\ =\; & \Pr[X_i \leq x_i] \\ =\; & F_i(x_i). \end{aligned}$$

Sklar [1959] established the converse. He showed that any multivariate distribution function F can be written in the form of a copula function. He proved the following: If $F(x_1, x_2, \cdots x_m)$ is a joint multivariate distribution function with univariate marginal distribution functions $F_1(x_1)$, $F_2(x_2)$,$\cdots$, $F_m(x_m)$, then there exists a copula function $C(u_1, u_2, \cdots, u_m)$ such that

$$F(x_1, x_2, \cdots x_m) = C(F_1(x_1), F_2(x_2), \cdots, F_m(x_m)).$$

If each F_i is continuous then C is unique. Thus, copula functions provide a unifying and flexible way to study multivariate distributions.

For simplicity's sake, we discuss only the properties of bivariate copula functions $C(u, v, \rho)$ for uniform random variables U and V, defined over the area $\{(u, v)|0 < u \leq 1, 0 < v \leq 1\}$, where ρ is a correlation parameter. We call ρ simply a correlation parameter since it does not necessarily equal the usual correlation coefficient defined by Pearson, nor Spearman's Rho, nor Kendall's Tau. The bivariate copula function has the following properties:

(i) Since U and V are positive random variables, $C(0, v, \rho) = C(u, 0, \rho) = 0$.

(ii) Since U and V are bounded above by 1, the marginal distributions can be obtained by $C(1, v, \rho) = v$, $C(u, 1, \rho) = u$.

(iii) For independent random variables U and V, $C(u, v, \rho) = uv$.

Frechet [1951] showed there exist upper and lower bounds for a copula function

$$\max(0, u + v - 1) \leq C(u, v) \leq \min(u, v).$$

The multivariate extension of Frechet bounds is given by Dall'Aglio [1972].

5.2 Some Common Copula Functions

We present a few copula functions commonly used in biostatistics and actuarial science.

Frank Copula The Frank copula function is defined as

$$C(u, v) = \frac{1}{\alpha} \ln \left[1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1} \right], \quad -\infty < \alpha < \infty.$$

Bivariate Normal

$$C(u, v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v), \rho), \quad -1 \leq \rho \leq 1, \tag{10}$$

where Φ_2 is the bivariate normal distribution function with correlation coefficient ρ, and Φ^{-1} is the inverse of a univariate normal distribution function. As we shall see later, this is the copula function used in CreditMetrics.

Bivariate Mixture Copula Function We can form new copula function using existing copula functions. If the two uniform random variables u and v are independent, we have a copula function $C(u, v) = uv$. If the two random variables are perfect correlated we have the copula function $C(u, v) = \min(u, v)$. Mixing the two copula functions by a mixing coefficient ($\rho > 0$) we obtain a new copula function as follows

$$C(u,v) = (1-\rho)uv + \rho \min(u,v), \qquad \text{if } \rho > 0.$$

If $\rho \leq 0$ we have

$$C(u,v) = (1+\rho)uv - \rho(u-1+v)\Theta(u-1+v), \quad \text{if } \rho \leq 0,$$

where

$$\begin{aligned} \Theta(x) &= 1, \quad \text{if } x \geq 0 \\ &= 0, \quad \text{if } x < 0. \end{aligned}$$

5.3 Copula Function and Correlation Measurement

To compare different copula functions, we need to have a correlation measurement independent of marginal distributions. The usual Pearson's correlation coefficient, however, depends on the marginal distributions (See Lehmann [1966]). Both Spearman's Rho and Kendall's Tau can be defined using a copula function only as follows

$$\rho_s = 12 \iint [C(u,v) - uv] du dv,$$

$$\tau = 4 \iint C(u,v) dC(u,v) - 1.$$

Comparisons between results using different copula functions should be based on either a common Spearman's Rho or a Kendall's Tau.

Further examination of copula functions can be found in a survey paper by Frees and Valdez [1988] and a recent book by Nelsen [1999].

5.4 The Calibration of Default Correlation in Copula Function

Having chosen a copula function, we need to compute the pairwise correlation of survival times. Using the CreditMetrics (Gupton et al. [1997]) asset correlation approach, we can obtain the default correlation of two discrete events over one year period. As it happens, CreditMetrics uses the normal copula function in its default correlation formula even though it does not use the concept of copula function explicitly.

First let us summarize how CreditMetrics calculates joint default probability of two credits A and B. Suppose the one year default probabilities for A and B are q_A and q_B. CreditMetrics would use the following steps

- Obtain Z_A and Z_B such that

$$\begin{aligned} q_A &= \Pr[Z < Z_A] \\ q_B &= \Pr[Z < Z_B] \end{aligned}$$

  where Z is a standard normal random variable

- If ρ is the asset correlation, the joint default probability for credit A and B is calculated as follows,

$$\Pr[Z < Z_A, Z < Z_B] = \int_{-\infty}^{Z_A} \int_{-\infty}^{Z_B} \phi_2(x, y|\rho) dx dy = \Phi_2(Z_A, Z_B, \rho) \tag{11}$$

  where $\phi_2(x, y|\rho)$ is the standard bivariate normal density function with a correlation coefficient ρ, and Φ_2 is the bivariate accumulative normal distribution function.

If we use a bivariate normal copula function with a correlation parameter γ, and denote the survival times for A and B as T_A and T_B, the joint default probability can be calculated as follows

$$\Pr[T_A < 1, T_B < 1] = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1), \gamma) \tag{12}$$

where F_A and F_B are the distribution functions for the survival times T_A and T_B. If we notice that

$$q_i = \Pr[T_i < 1] = F_j(1) \text{ and } Z_i = \Phi^{-1}(q_i) \text{ for } i = A, B,$$

then we see that equation (12) and equation (11) give the same joint default probability over one year period if $\rho = \gamma$.

We can conclude that CreditMetrics uses a bivariate normal copula function with the asset correlation as the correlation parameter in the copula function. Thus, to generate survival times of two credit risks, we use a bivariate normal copula function with correlation parameter equal to the CreditMetrics asset correlation. We note that this correlation parameter is not the correlation coefficient between the two survival times. The correlation coefficient between the survival times is much smaller than the asset correlation. Conveniently, the marginal distribution of any subset of an n dimensional normal distribution is still a normal distribution. Using asset correlations, we can construct high dimensional normal copula functions to model the credit portfolio of any size.

6 Numerical Illustrations

This section gives some numerical examples to illustrate many of the points discussed above. Assume that we have two credit risks, A and B, which have flat spread curves of 300 bps and 500 bps over LIBOR. These spreads are usually given in the market as asset swap spreads. Using these spreads and a constant recovery assumption of 50% we build two credit curves for the two credit risks. For details, see Li [1998]. The two credit curves are given in Figures (2) and (3). These two curves will be used in the following numerical illustrations.

6.1 Illustration 1. Default Correlation v.s. Length of Time Period

In this example, we study the relationship between the discrete default correlation (1) and the survival time correlation (8). The survival time correlation is a much more general concept than the discrete default correlation defined for two discrete default events at an arbitrary period of time, such as one year. Knowing the former allows us to calculate the latter over any time interval in the future, but not vice versa.

Using two credit curves we can calculate all marginal default probabilities up to anytime t in the future, i.e.

$$ {}_tq_0 \quad = \quad \Pr[\tau < t] = 1 - e^{-\int_0^t h(s)ds}, $$

where $h(s)$ is the instantaneous default probability given by a credit curve. If we have the marginal default probabilities ${}_tq_0^A$ and ${}_tq_0^B$ for both A and B, we can also obtain the joint probability of default over the time interval $[0, t]$ by a copula function $C(u, v)$,

$$ \Pr[T_A < t, T_B < t] = C({}_tq_0^A, {}_tq_0^B). $$

Of course we need to specify a correlation parameter ρ in the copula function. We emphasize that knowing ρ would allow us to calculate the survival time correlation between T_A and T_B.

We can now obtain the discrete default correlation coefficient ρ_t between the two discrete events that A and B default over the time interval $[0, t]$ based on the formula (1). Intuitively, the discrete default correlation ρ_t should be an increasing function of t since the two underlying credits should have a higher tendency of joint default over longer periods. Using the bivariate normal copula function (10) and $\rho = 0.1$ as an example we obtain Figure (4).

From this graph we see explicitly that the discrete default correlation over time interval $[0, t]$ is a function of t. For example, this default correlation coefficient goes from 0.021 to 0.038 when t goes from six months to twelve months. The increase slows down as t becomes large.

6.2 Illustration 2. Default Correlation and Credit Swap Valuation

The second example shows the impact of default correlation on credit swap pricing. Suppose that credit A is the credit swap seller and credit B is the underlying reference asset. If we buy a default swap of 3 years with a reference asset of credit B from a risk-free counterparty we should pay 500 bps since holding the underlying asset and having a long position on the credit swap would create a riskless portfolio. But if we buy the default swap from a risky counterparty how much we should pay depends on the credit quality of the counterparty and the default correlation between the underlying reference asset and the counterparty.

Knowing only the discrete default correlation over one year we cannot value any credit swaps with a maturity longer than one year. Figure (5) shows the impact of asset correlation (or implicitly default correlation) on the credit swap premium. From the graph we see that the annualized premium decreases as the asset correlation between the counterparty and the underlying reference asset increases. Even at zero default correlation the credit swap has a value less than 500 bps since the counterparty is risky.

6.3 Illustration 3. Default Correlation and First-to-Default Valuation

The third example shows how to value a first-to-default contract. We assume we have a portfolio of n credits. Let us assume that for each credit i in the portfolio we have constructed a credit curve or a hazard rate function for its survival time T_i. The distribution function of T_i is $F_i(t)$. Using a copula function C we also obtain the joint distribution of the survival times as follows

$$F(t_1, t_2, \cdots, t_n) = C(F_1(t_1), F_2(t_2), \cdots, F_n(t_n)).$$

If we use normal copula function we have

$$F(t_1, t_2, \cdots, t_n) = \Phi_n(\Phi^{-1}(F_1(t_1)), \Phi^{-1}(F_2(t_2)), \cdots, \Phi^{-1}(F_n(t_n)))$$

where Φ_n is the n dimensional normal cumulative distribution function with correlation coefficient matrix Σ.

To simulate correlated survival times we introduce another series of random variables $Y_1, Y_2, \cdots Y_n$, such that

$$Y_1 = \Phi^{-1}(F_1(T_1)), Y_2 = \Phi^{-1}(F_2(T_2)), \cdots, Y_n = \Phi^{-1}(F_n(T_n)). \tag{13}$$

Then there is a one-to-one mapping between Y and T. Simulating $\{T_i | i = 1, 2, ..., n\}$ is equivalent to simulating $\{Y_i | i = 1, 2, ..., n\}$. As shown in the previous section the correlation between the Y's is the asset correlation of the underlying credits. Using CreditManager from RiskMetrics Group we can obtain the asset correlation matrix Σ. We have the following simulation scheme

- Simulate $Y_1, Y_2, \cdots Y_n$ from an n-dimension normal distribution with correlation coefficient matrix Σ.
- Obtain $T_1, T_2, \cdots T_n$ using $T_i = F_i^{-1}(N(Y_i)), i = 1, 2, \cdots, n$.

With each simulation run we generate the survival times for all the credits in the portfolio. With this information we can value any credit derivative structure written on the portfolio. We use a simple structure for illustration. The contract is a two-year transaction which pays one dollar if the first default occurs during the first two years.

We assume each credit has a constant hazard rate of $h = 0.1$ for $0 < t < +\infty$. From equation (7) we know the density function for the survival time T is he^{-ht}. This shows that the survival time is exponentially distributed with mean $1/h$. We also assume that every pair of credits in the portfolio has a constant asset correlation σ[2].

Suppose we have a constant interest rate $r = 0.1$. If all the credits in the portfolio are independent, the hazard rate of the minimum survival time $T = \min(T_1, T_2, \cdots, T_n)$ is easily shown to be

$$h_T = h_1 + h_2 + \cdots + h_n = nh.$$

If $T < 2$, the present value of the contract is $1 \cdot e^{-r \cdot T}$. The survival time for the first-to-default has a density function $f(t) = h_T \cdot e^{-h_T t}$, so the value of the contract is given by

[2]To have a positive definite correlation matrix, the constant correlation coefficient has to satisfy the condition $\sigma > -\frac{1}{n-1}$.

$$
\begin{aligned}
V &= \int_0^2 1 \cdot e^{-rt} f(t) dt \\
&= \int_0^2 1 \cdot e^{-r\,t} h_T \cdot e^{-h_T\,t} dt \\
&= \frac{h_T}{r + h_T} \left(1 - e^{-2.0 \cdot (r + h_T)}\right).
\end{aligned} \tag{14}
$$

In the general case we use the Monte Carlo simulation approach and the normal copula function to obtain the distribution of T. For each simulation run we have one scenario of default times $t_1, t_2, \cdots t_n$, from which we have the first-to-default time simply as $t = \min(t_1, t_2, \cdots t_n)$.

Let us examine the impact of the asset correlation on the value of the first-to-default contract of 5-assets. If $\sigma = 0$, the expected payoff function, based on equation (14), should give a value of 0.5823. Our simulation of 50,000 runs gives a value of 0.5830. If all 5 assets are perfectly correlated, then the first-to-default of 5-assets should be the same as the first-to-default of 1-asset since any one default induces all others to default. In this case the contract should worth 0.1648. Our simulation of 50,000 runs produces a result of 0.1638. Figure (6) shows the relationship between the value of the contract and the constant asset correlation coefficient. We see that the value of the contract decreases as the correlation increases. We also examine the impact of correlation on the value of the first-to-default of 20 assets in Figure (6). As expected, the first-to-default of 5 assets has the same value of the first-to-default of 20 assets when the asset correlation approaches to 1.

7 Conclusion

This paper introduces a few standard technique used in survival analysis to study the problem of default correlation. We first introduce a random variable called “the time-until-default” to characterize the default. Then the default correlation between two credit risks is defined as the correlation coefficient between their survival times. In practice we usually use market spread information to derive the distribution of survival times. When it comes to credit portfolio studies we need to specify a joint distribution with given marginal distributions. The problem cannot be solved uniquely. The copula function approach provides one way of specifying a joint distribution with known marginals. The concept of copula functions, their basic properties and some commonly used copula functions are introduced. The calibration of the correlation parameter used in copula functions against some popular credit models is also studied. We have shown that CreditMetrics essentially uses the normal copula function in its default correlation formula even though CreditMetrics does not use the concept of copula functions explicitly. Finally we show some numerical examples to illustrate the use of copula functions in the valuation of credit derivatives, such as credit default swaps and first-to-default contracts.

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Figure 1: Hazard Rate Function of B Grade Based on Moody's Study (1997)

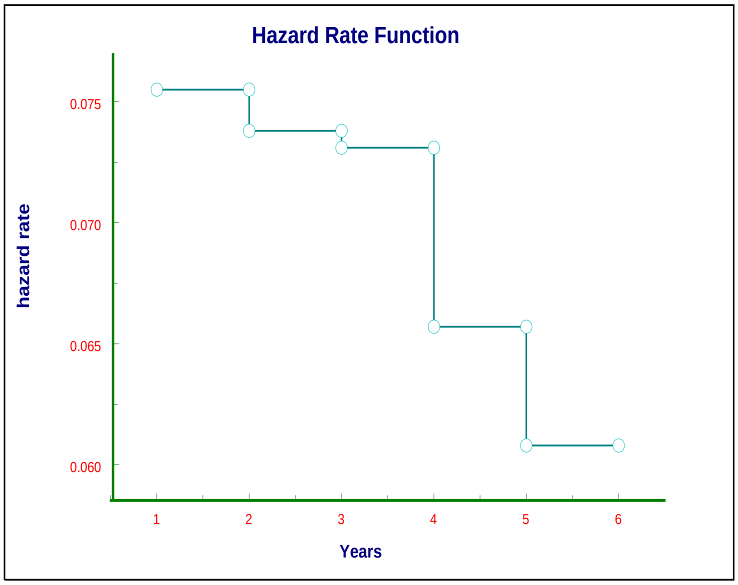

Figure 2: Credit Curve A

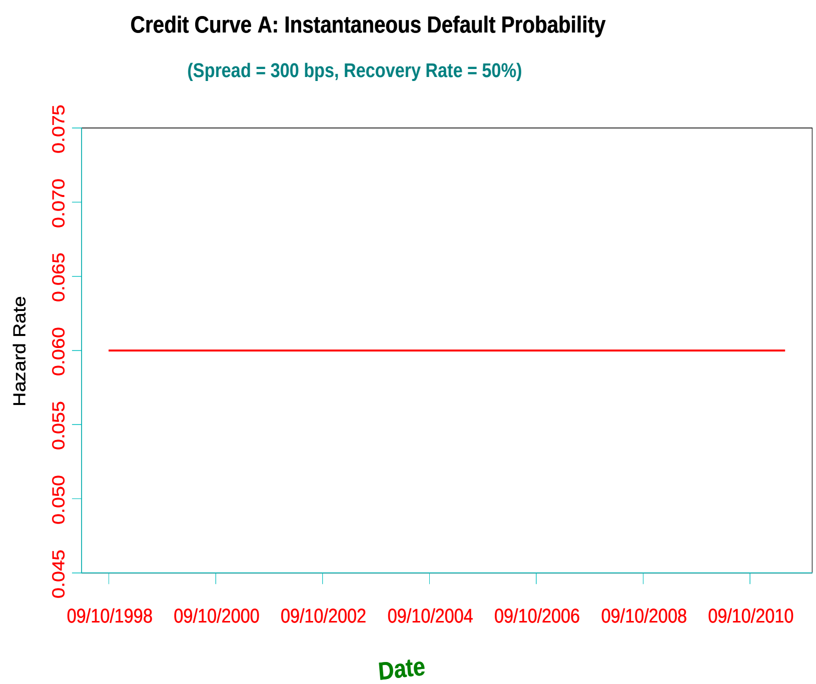

Figure 3: Credit Curve B

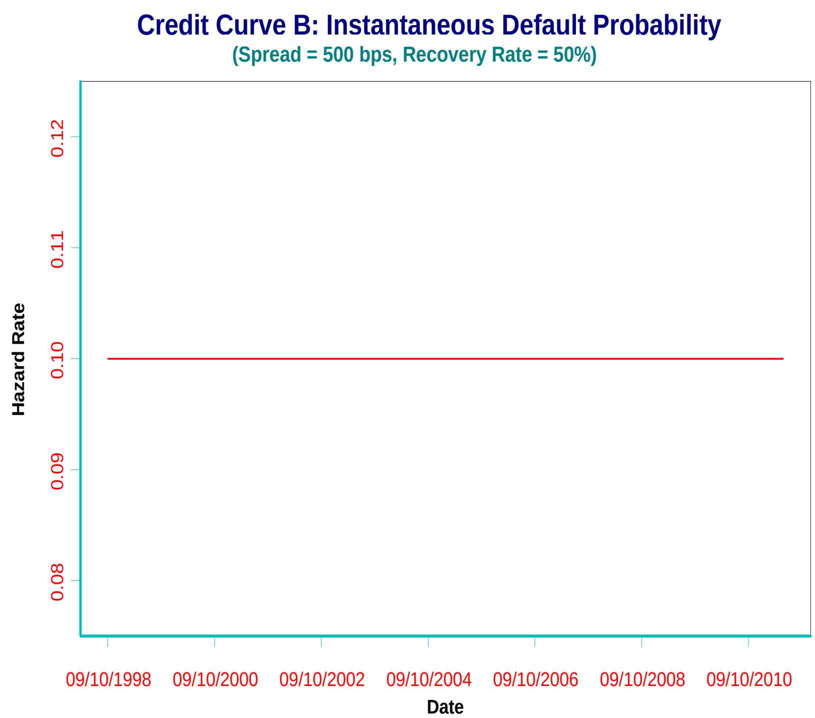

Figure 4: The Discrete Default Correlation v.s. the Length of Time Interval

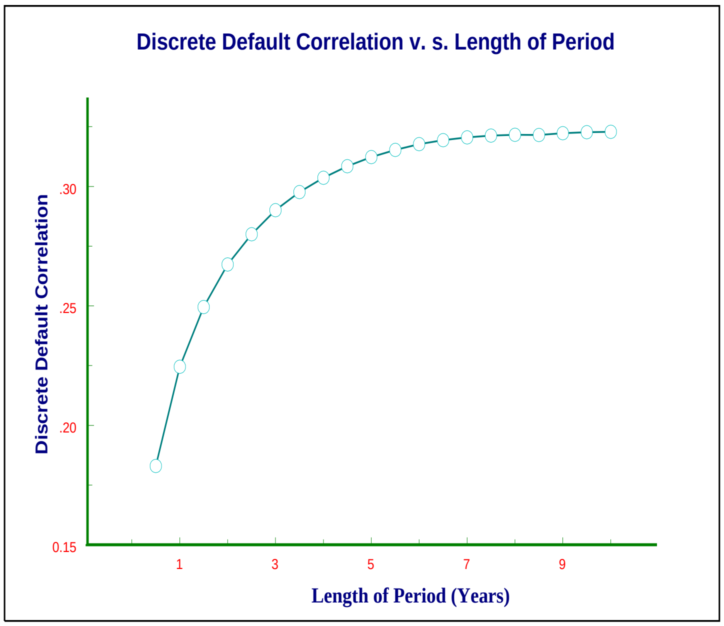

Figure 5: Impact of Asset Correlation on the Value of Credit Swap

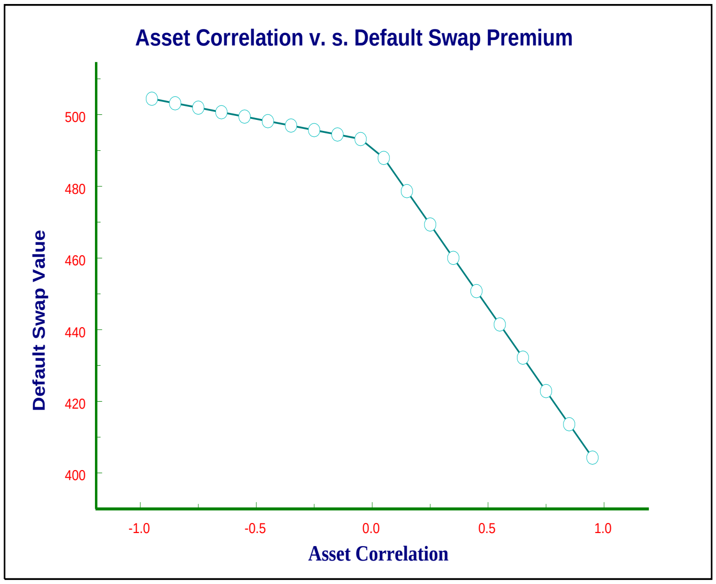

Figure 6: The Value of First-to-Default v. s. Asset Correlation