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Brownian sheet and related processes

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On quadratic functionals of the Brownian sheet and related processes

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Abstract

Motivated by asymptotic problems in the theory of empirical processes, and specifically by tests of independence, we study the law of quadratic functionals of the (weighted) Brownian sheet and of the bivariate Brownian bridge on $[0,1]^2$. In particular: (i) we use Fubini type techniques to establish identities in law with quadratic functionals of other Gaussian processes, (ii) we explicitly calculate the Laplace transform of such functionals by means of Karhunen-Loève expansions, (iii) we prove central and non-central limit theorems in the same spirit of Peccati and Yor (2004) and Nualart and Peccati (2004). Our results extend some classical computations due to P. Lévy (1950), as well as the formulae recently obtained by Deheuvels and Martynov (2003).
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Contents

1 Introduction, Notation and Preliminaries

1.1 Introduction

In this paper, we study quadratic functionals of Gaussian processes related to the multi-parameter Wiener process. In the one-parameter case, the study of such functionals goes back to [6] and [36], and has been further developed e.g. in [27] and [37] for purely mathematical purposes (see [54], as well as [23] and [7] for applications to polymer theory). For some early study in the multi parameter case, the reader is referred to [11], [20], [21] and [22]. One of the motivations of our study is the investigation of Cramér-von Mises-type independence tests, where such quadratic functionals turn out to play a crucial role. This problem is investigated in §1.2, where we concentrate on *bivariate* distributions. It will become obvious later on that our results can be written in the more general framework of $\mathbb{R}^d$-valued random vectors for an arbitrary $d \geq 2$, at the price of minor additional technicalities. The choice of $d = 2$ turns out, however, to be of particular interest, first, because of the specific tools available in this case (refer to [4, 5]), and second, because of the fact that it is the most useful for statistical applications. Our work is closely related to the study of *copula functions*, which has received a considerable interest in the recent literature (see, e.g., [38] and the references therein).

1.2 Preliminaries on Bivariate Tests of Independence

Let $\{(X_n, Y_n) : n \geq 1\}$ be independent replicæ of a random vector (X, Y) with distribution function [df] $F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$. We assume that the corresponding marginal df's $G(x) = \mathbb{P}(X \leq x)$ and $H(y) = \mathbb{P}(Y \leq y)$ are continuous. The quantile functions pertaining to $G(\cdot)$ and $H(\cdot)$ are denoted, respectively, by $G^{\text{inv}}(s) = \inf\{x : G(x) \geq s\}$ and $H^{\text{inv}}(t) = \inf\{y : H(y) \geq t\}$, for $0 < s, t < 1$. Throughout the sequel, we will set $U_n = G(X_n)$ and $V_n = H(Y_n)$, together with $U = G(X)$ and

$V = H(Y)$, and keep in mind that these random variables are uniformly distributed on $(0,1)$. The *copula function* (see, e.g., Sklar, [47], Schweizer, [44]) of $F(\cdot,\cdot)$ is defined as the distribution function $C(u,v) = \mathbb{P}(U \le u, V \le v)$ of the random vector $(U,V) = (G(X), H(Y))$. This function fulfills the identity

$$\begin{aligned} C(s,t) &= F\left(G^{\mathrm{inv}}(s), H^{\mathrm{inv}}(t)\right) \quad \text{for} \quad 0 < s,t < 1, \\ &= s \wedge t \quad \text{when either} \quad s \vee t = 1 \quad \text{or} \quad s \wedge t = 0 \quad \text{with} \quad 0 \le s,t \le 1. \end{aligned} \tag{1.1}$$

The empirical counterparts of $F(\cdot,\cdot)$, $G(\cdot)$ and $H(\cdot)$, based upon $(X_1,Y_1),\ldots,(X_n,Y_n)$, are given, respectively, for each $n \ge 1$ and $x,y \in \mathbb{R}$, by

$$F_n(x,y) = n^{-1}\sum_{i=1}^{n} \mathbb{I}_{\{X_i \le x, Y_i \le y\}},\ G_n(x) = n^{-1}\sum_{i=1}^{n} \mathbb{I}_{\{X_i \le x\}} \text{ and } H_n(y) = n^{-1}\sum_{i=1}^{n} \mathbb{I}_{\{Y_i \le y\}}. \tag{1.2}$$

Let $G_n^{\mathrm{inv}}(s) = \inf\{x : G_n(x) \ge s\}$ and $H_n^{\mathrm{inv}}(t) = \inf\{y : H_n(y) \ge t\}$, for $0 \le s,t \le 1$ and $n \ge 1$, denote the empirical quantile functions of F_n and G_n. By a straightforward analogue of (1.1), we define the *empirical copula function* of $F_n(\cdot,\cdot)$ (see, e.g., [13]) by

$$\begin{aligned} C_n(s,t) &= F_n\left(G_n^{\mathrm{inv}}(s), H_n^{\mathrm{inv}}(t)\right) \quad \text{for} \quad 0 < s,t < 1, \\ &= s \wedge t \quad \text{when either} \quad s \vee t = 1 \quad \text{or} \quad s \wedge t = 0 \quad \text{with} \quad 0 \le s,t \le 1. \end{aligned} \tag{1.3}$$

Remark 1.1 *(a) The* empirical copula function $C_n(\cdot,\cdot)$ *is* distribution-free*, in the sense that it is invariant with respect to changes of (X_n,Y_n), into $(\phi(X_n),\psi(Y_n))$, $n = 1,2,\ldots$, where $\phi(\cdot)$ and $\psi(\cdot)$ are arbitrary one-to-one nondecreasing mappings of $\mathbb{R}$ onto itself. This property entails that the empirical copula functions based, respectively, upon $(X_1,Y_1),\ldots,(X_n,Y_n)$ and $(U_1 = G(X_1), V_1 = H(Y_1)),\ldots,(U_n = G(X_n), V_n = H(Y_n))$, are identical.*

(b) As follows from (1.1), the copula function $C(s,t)$ is the df of a bivariate random vector with uniform $(0,1)$ marginals. This property is not shared by the empirical copula function $C_n(s,t)$ in (1.3). The latter is, conditionally upon the sample $(X_1,Y_1),\ldots,(X_n,Y_n)$, the df of a bivariate random vector with marginals uniformly distributed on the discrete set $\{0,1/n,\ldots,(n-1)/n\}$.

One may check (refer to Theorem 3.1 in [13]) that there exists a constant κ (depending upon $C(\cdot,\cdot)$ only) such that, with probability 1,

$$\limsup_{n\to\infty}\left\{\frac{n}{\operatorname{loglog} n}\right\}^{1/2} \sup_{0\le u,v\le 1}|C_n(u,v) - C(u,v)| = \kappa < \infty. \tag{1.4}$$

In particular, $\kappa = 1/4$ in the *independence case*, where $C(u,v) = uv$ (see, e.g., (2.41) in the sequel). By introducing the empirical copula process

$$\Gamma_n(u,v) = n^{1/2}\left(C_n(u,v) - C(u,v)\right) \quad \text{for} \quad 0 \le u,v \le 1, \tag{1.5}$$

one may show further (see, e.g., p. 389 in van der Vaart and Wellner [49]) that, for each specified pair of constants a and b with $0 < a < b < 1$, $\{\Gamma_n(u,v) : a \le u,v \le b\}$ converges weakly to a centered Gaussian process $\{\Gamma(u,v) : a \le u,v \le b\}$. For a general $C(\cdot,\cdot)$, this property is not necessarily true when $a = 0$ and $b = 1$, and some additional regularity assumptions on $C(\cdot,\cdot)$ are required for its validity. For example, Fermanian, Radulović and Wegkamp, [26], show that the weak convergence of $\Gamma_n(\cdot,\cdot)$ to $\Gamma(\cdot,\cdot)$ holds on $[0,1]^2$ when $C(\cdot,\cdot)$ has continuous partial derivatives on $[0,1]^2$. In particular, these conditions are satisfied under the *independence assumption* of X and Y, which holds iff

$$C(u,v) = uv \quad \text{for} \quad 0 \le u,v \le 1. \tag{1.6}$$

The fact that, under (1.6), $\{\Gamma_n(u,v) : 0 \le u,v \le 1\}$ converges weakly to a centered Gaussian process $\{\Gamma_n(u,v) : 0 \le u,v \le 1\}$ can be proved by specific arguments (see, e.g., [13], Theorem 1 in [14], and the forthcoming §2.1). In this case, the limiting Gaussian process $\Gamma(u,v)$ reduces to a tied-down two-parameter Brownian bridge $\mathbf{B}_*(\cdot,\cdot)$ (see, e.g., §2.1 below). This property motivates the idea of testing the independence assumption by either one of the following Cramér-von Mises-type test statistics

$$\Omega^2_{n;\mathrm{BKR}} = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F_n(x,y)-G_n(x)H_n(y)\}^2\,dF_n(x,y), \tag{1.7}$$

$$\Omega^2_{n;\mathrm{H}} = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F_n(x,y)-G_n(x)H_n(y)\}^2\,dG_n(x)dH_n(y), \tag{1.8}$$

$$\Omega^2_{n;\mathrm{C}} = n\int_0^1\int_0^1\{C_n(u,v)-uv\}^2\,dudv. \tag{1.9}$$

The best-known among the above three statistics, namely $\Omega^2_{n;\mathrm{BKR}}$, is due to Blum, Kiefer and Rosenblatt [3], and has been investigated by several authors among which we may cite Csörgő [12]. The statistic $\Omega^2_{n;\mathrm{H}}$, due to Hoeffding [29], is less popular than $\Omega^2_{n;\mathrm{BKR}}$, since it requires the summation of n^2 terms instead of n for $\Omega^2_{n;\mathrm{BKR}}$. The statistic $\Omega^2_{n;\mathrm{C}}$ is a variant of the preceding two statistics, and was introduced by Deheuvels (refer to (6.1) in [13]). A fourth related random variable is

$$\Omega^2_{n;\mathrm{T}} = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F_n(x,y)-G_n(x)H_n(y)\}^2\,dG(x)dH(y). \tag{1.10}$$

We note that, unlike $\Omega^2_{n;\mathrm{BKR}}$, $\Omega^2_{n;\mathrm{H}}$ and $\Omega^2_{n;\mathrm{C}}$, the random variable $\Omega^2_{n;\mathrm{T}}$, as defined in (1.10), is not a statistic in the strict sense, given that it depends upon the unknown marginal df's $G(\cdot)$ and $H(\cdot)$.

Under the independence assumption (1.6), each one of these random variables converges weakly to

$$\int_0^1\int_0^1 \mathbf{B}_*^2(s,t)dsdt, \tag{1.11}$$

where $\mathbf{B}_*$ is a bivariate tied-down Brownian bridge (see §2.1 below). On the other hand, for finite n, these statistics behave somewhat differently. It is beyond the scope of this paper to investigate this general problem, and we will rather, in the forthcoming §2, concentrate on the study of $\Omega^2_{n;\mathrm{C}}$. In particular, we will show in this section that $\Omega^2_{n;\mathrm{C}}$ is, in a certain sense, more natural that $\Omega^2_{n;\mathrm{BKR}}$ and $\Omega^2_{n;\mathrm{H}}$, since it remains asymptotically very close to the random variable $\Omega^2_{n;\mathrm{T}}$ (we denote this property by $\Omega^2_{n;\mathrm{C}} \simeq \Omega^2_{n;\mathrm{T}}$). Namely, we will show in the forthcoming Corollary 2.2 that, under (1.6), as $n\to\infty$,

$$\left|\Omega^2_{n;\mathrm{C}}-\Omega^2_{n;\mathrm{T}}\right| = O\left(n^{-1/4}(\log n)^{1/2}(\log\log n)^{3/4}\right)\quad \text{a.s.} \tag{1.12}$$

In spite of the above-mentioned fact that $\Omega^2_{n;\mathrm{T}}$ is not determined only by the sample observations, this random quantity should appear as a more natural discrepancy measure than $\Omega^2_{n;\mathrm{BKR}}$ and $\Omega^2_{n;\mathrm{H}}$, to test the independence assumption. This is due to the fact that $\Omega^2_{n;\mathrm{T}}$ does not weigh the square deviation $\{F_n(x,y)-G_n(x)H_n(y)\}^2$ with a random measure, such as $dF_n(x,y)$ or $dG_n(x)dH_n(y)$, but rather, with the "exact" deterministic distribution $dG(x)dF(y)$. For this reason, one should expect some advantages in the replacement of the previous statistics, $\Omega^2_{n;\mathrm{BKR}}$ and $\Omega^2_{n;\mathrm{H}}$ by $\Omega^2_{n;\mathrm{C}}$. This question will be investigated elsewhere.

Among other results, we will also establish in the sequel (see, e.g., (2.43))) that, on a suitable probability space, there exists a sequence $\mathbf{B}_{n;*}(\cdot,\cdot)$ of bivariate tied-down Brownian bridges such that, almost surely as $n\to\infty$,

$$\Omega^2_{n;\mathrm{C}} = \int_0^1\int_0^1 \mathbf{B}^2_{n;*}(u,v)dudv + O\left(n^{-1/4}(\log n)^{1/2}(\log\log n)^{3/4}\right). \tag{1.13}$$

This result will be extended below in a more general framework.

1.3 Weighted Bivariate Tests of Independence

Motivated by these preliminaries, we are led to introduce weighted bivariate tests of independence. Namely, for selected constants $\gamma, \delta \in \mathbb{R}$, we set

$$\Omega^2_{n;\mathrm{BKR};\gamma,\delta} = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_n(x)^{2\gamma}H_n(y)^{2\delta}\left\{F_n(x,y)-G_n(x)H_n(y)\right\}^2 dF_n(x,y), \tag{1.14}$$

$$\Omega^2_{n;\mathrm{H};\gamma,\delta} = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_n(x)^{2\gamma}H_n(y)^{2\delta}\left\{F_n(x,y)-G_n(x)H_n(y)\right\}^2 dG_n(x)dH_n(y), \tag{1.15}$$

$$\Omega^2_{n;\mathrm{C};\gamma,\delta} = n\int_0^1\int_0^1 u^{2\gamma}v^{2\delta}\left\{C_n(u,v)-uv\right\}^2 dudv. \tag{1.16}$$

$$\Omega^2_{n;\mathrm{T};\gamma,\delta} = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x)^{2\gamma}H(y)^{2\delta}\left\{F_n(x,y)-G_n(x)H_n(y)\right\}^2 dG(x)dH(y). \tag{1.17}$$

The investigation of $\Omega^2_{n;\mathrm{BKR};\gamma,\delta}$ and $\Omega^2_{n;\mathrm{H};\gamma,\delta}$ will be undertaken elsewhere, and here we will limit ourselves to the study of $\Omega^2_{n;\mathrm{C}}$ and $\Omega^2_{n;\mathrm{T}}$. In particular, we will show that, under (1.6) and appropriate conditions on $\gamma, \delta \in \mathbb{R}$, these two statistics converge in distribution to the limiting random variable (with $\mathbf{B}_*$ denoting again a bivariate tied-down Brownian bridge)

$$\int_0^1\int_0^1 u^{2\gamma}v^{2\delta}\mathbf{B}_*^2(u,v)dudv. \tag{1.18}$$

This leads us to a general study of quadratic functionals of bivariate Brownian bridges. Starting from Section 3, we will concentrate on this problem, by largely extending results of the kind obtained in the univariate framework by Lévy [36] and Deheuvels and Martynov [16]. In the next section, we provide some empirical process arguments to establish the above-mentioned limiting results.

1.4 Organization of the Paper

The rest of the paper is organized as follows. In § 2, the statistical discussion of § 1.1-§ 1.3 is developed and made precise. In § 3, we generalize to the multi-parameter case some Fubini-Wiener identities in law, formerly obtained (see, e.g., [17] and [55]) between quadratic functionals of one-parameter Gaussian processes. In § 4, we use the Karhunen-Loève expansion of the previously considered Gaussian processes to derive the Laplace transform of quadratic functionals such as (1.18). In § 5, we consider weak convergence results involving the same kind of quadratic functionals of bivariate Gaussian processes.

2 Some Empirical Process Arguments

2.1 Strong Approximation Results

In this section, we will work in the setup of § 1.2. Namely, we assume that X and Y are mutually independent with continuous distribution functions $G(\cdot)$ and $H(\cdot)$, so that $F(x,y) = G(x)H(y)$ for $x, y \in \mathbb{R}$ and $C(u,v) = uv$ for $0 \le u, v \le 1$. Thus, $U_1 = G(X_1), U_2 = G(X_2), \ldots$ and $V_1 = H(Y_1), V_2 = H(Y_2), \ldots$, are two independent sequences of independent and identically distributed uniform $(0,1)$ random variables. The following notation will be useful, in view of (1.2). For each $n \ge 1$ and $0 \le u, v \le 1$, set

$$\mathbb{T}_n(u,v) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{U_i\le u, V_i \le v\}} = F_n\left(G^{\mathrm{inv}}(u), H^{\mathrm{inv}}(v)\right), \tag{2.1}$$

$$\mathbb{U}_n(u) = \mathbb{T}_n(u,1) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{U_i\le u\}} = G_n\left(G^{\mathrm{inv}}(u)\right), \tag{2.2}$$

$$\mathbb{V}_n(v) = \mathbb{T}_n(1,v) = \frac{1}{n}\sum_{j=1}^n \mathbb{I}_{\{V_j\le v\}} = H_n\left(H^{\mathrm{inv}}(v)\right). \tag{2.3}$$

The empirical quantile functions of $\mathbb{U}_n(\cdot)$ and $\mathbb{V}_n(\cdot)$ are given, for $0 \leq u, v \leq 1$, by

$$\mathbb{U}_n^{\mathrm{inv}}(u) = \inf\{s \geq 0 : \mathbb{U}_n(s) \geq u\} = G\left(G_n^{\mathrm{inv}}(u)\right), \tag{2.4}$$
$$\mathbb{V}_n^{\mathrm{inv}}(v) = \inf\{t \geq 0 : \mathbb{V}_n(t) \geq v\} = H\left(H_n^{\mathrm{inv}}(v)\right). \tag{2.5}$$

Consider the empirical processes defined, respectively, for $n \geq 1$ and $0 \leq u, v \leq 1$, by

$$\alpha_n(u,v) = n^{1/2}\{\mathbb{T}_n(u,v) - uv\}, \tag{2.6}$$
$$\alpha_{n;\mathrm{U}}(u) = \alpha_n(u,1) = n^{1/2}\{\mathbb{U}_n(u) - u\} \tag{2.7}$$
$$\alpha_{n;\mathrm{V}}(v) = \alpha_n(1,v) = n^{1/2}\{\mathbb{V}_n(v) - v\}, \tag{2.8}$$
$$\beta_{n;\mathrm{U}}(u) = n^{1/2}\left\{\mathbb{U}_n^{\mathrm{inv}}(u) - u\right\}, \tag{2.9}$$
$$\beta_{n;\mathrm{V}}(v) = n^{1/2}\left\{\mathbb{V}_n^{\mathrm{inv}}(v) - v\right\}. \tag{2.10}$$

Set further, in view of (1.3)–(1.5) and (2.6)–(2.10),

$$\begin{aligned}
\alpha_{n;0}(u,v) &= n^{1/2}\{\mathbb{T}_n(u,v) - u\mathbb{V}_n(v) - v\mathbb{U}_n(u) + uv\} \\
&= \alpha_n(u,v) - u\alpha_n(1,v) - v\alpha_n(u,1) \\
&= \alpha_n(u,v) - u\alpha_{n;\mathrm{V}}(v) - v\alpha_{n;\mathrm{U}}(u), \qquad (2.11)\\
\alpha_{n;1}(u,v) &= n^{1/2}\{\mathbb{T}_n(u,v) - \mathbb{U}_n(u)\mathbb{V}_n(v)\} \\
&= \alpha_{n;0}(u,v) - n^{-1/2}\alpha_n(u,1)\alpha_n(1,v) \\
&= \alpha_{n;0}(u,v) - n^{-1/2}\alpha_{n;\mathrm{U}}(u)\alpha_{n;\mathrm{V}}(v) \qquad (2.12)\\
\alpha_{n;2}(u,v) &= n^{1/2}\left\{\mathbb{T}_n\left(\mathbb{U}_n^{\mathrm{inv}}(u), \mathbb{V}_n^{\mathrm{inv}}(v)\right) - uv\right\} \\
&= \Gamma_n(u,v) = n^{1/2}\{C_n(u,v) - uv\} \\
&= \alpha_n\left(u + n^{-1/2}\beta_{n;\mathrm{U}}(u), v + n^{-1/2}\beta_{n;\mathrm{V}}(v)\right) \\
&\quad + u\beta_{n;\mathrm{V}}(v) + v\beta_{n;\mathrm{U}}(u) + n^{-1/2}\beta_{n;\mathrm{U}}(u)\beta_{n;\mathrm{V}}(v). \qquad (2.13)
\end{aligned}$$

Below, $\{\mathbf{W}(s,t) : s \geq 0, t \geq 0\}$ will denote a (standard) *bivariate Wiener process* (or *Brownian sheet*), namely, a centered Gaussian process with continuous paths and covariance function given by

$$\mathbb{E}\left(\mathbf{W}(s',t')\mathbf{W}(s'',t'')\right) = (s' \wedge s'')(t' \wedge t'') \quad \text{for} \quad s', s'', t', t'' \geq 0. \tag{2.14}$$

A *bivariate Brownian bridge* is defined, in terms of $\mathbf{W}(\cdot,\cdot)$, via

$$\mathbf{B}(s,t) = \mathbf{W}(s,t) - st\mathbf{W}(1,1) \quad \text{for} \quad 0 \leq s, t \leq 1. \tag{2.15}$$

A *tied-down Brownian bridge* is, in turn, defined, in terms of $\mathbf{B}(\cdot,\cdot)$ and $\mathbf{W}(\cdot,\cdot)$, via

$$\begin{aligned}
\mathbf{B}_*(s,t) &= \mathbf{B}(s,t) - s\mathbf{B}(1,t) - t\mathbf{B}(s,1) \\
&= \mathbf{W}(s,t) - s\mathbf{W}(1,t) - t\mathbf{W}(s,1) + st\mathbf{W}(1,1) \quad \text{for} \quad 0 \leq s, t \leq 1. \qquad (2.16)
\end{aligned}$$

The processes $\mathbf{B}(\cdot,\cdot)$ and $\mathbf{B}_*(\cdot,\cdot)$ are both Gaussian, with continuous sample paths and covariance functions given by, for $0 \leq s', t', s'', t'' \leq 1$,

$$\mathbb{E}\left(\mathbf{B}(s',t')\mathbf{B}(s'',t'')\right) = (s' \wedge s'')(t' \wedge t'') - s's''t't'', \tag{2.17}$$
$$\mathbb{E}\left(\mathbf{B}_*(s',t')\mathbf{B}_*(s'',t'')\right) = (s' \wedge s'' - s's'')(t' \wedge t'' - t't''). \tag{2.18}$$

The following fact is of particular interest in the present framework (see, e.g., [15]). Consider the (univariate) Brownian bridges defined by

$$B_{\mathrm{U}}(u) = \mathbf{B}(u,1) \quad \text{and} \quad B_{\mathrm{V}}(v) = \mathbf{B}(1,v). \tag{2.19}$$

Fact 2.1 *The processes $\mathbf{B}_*(\cdot,\cdot)$, $B_{\mathrm{U}}(\cdot)$ and $B_{\mathrm{V}}(\cdot)$ are independent.*

For convenience, we will denote the *sup-norm* of a bounded function f, defined on $J=[0,1]$ or $J=[0,1]^2$, by $\|f\|=\sup_{x\in J}|f(x)|$, and set $\mathcal{I}(x)=x$ for the identity. The next fact, due to Castelle (see, e.g. [5]) and Bonvalot and Castelle (see, e.g. [4]) provides a strong approximation result appropriate for our needs.

Fact 2.2 *On a suitable probability space, it is possible to define $\{\alpha_n(u,v):0\le u,v\le 1\}$, jointly with a sequence of bivariate Brownian bridges $\{\mathbf{B}_n(u,v):0\le u,v\le 1\}$, $n=1,2,\ldots$, in such a way that, with probability 1 as $n\to\infty$,*

$$\|\alpha_n-\mathbf{B}_n\|=O\left(n^{-1/2}(\log n)^2\right). \tag{2.20}$$

Below, unless otherwise specified, we will work on the probability space of Fact 2.2, and define sequences of tied-down bivariate brownian bridges $\mathbf{B}_{n;*}(\cdot,\cdot)$ and univariate Brownian bridges $B_{n;\mathrm{U}}(\cdot)$, $B_{n;\mathrm{V}}(\cdot)$ (with $\mathbf{B}_{n;*}(\cdot,\cdot)$, $B_{n;\mathrm{U}}(\cdot)$, $B_{n;\mathrm{V}}(\cdot)$ independent for each $n\ge 1$), by setting, for $n=1,2,\ldots$,

$$\begin{aligned}\mathbf{B}_{n;*}(u,v) &= \mathbf{B}_n(u,v)-u\mathbf{B}_n(1,v)-v\mathbf{B}_n(u,1)\\ &= \mathbf{B}_n(u,v)-uB_{n;\mathrm{V}}(v)-vB_{n;\mathrm{U}}(u),\end{aligned} \tag{2.21}$$

where

$$B_{n;\mathrm{U}}(u) = \mathbf{B}_n(u,1) \quad\text{and}\quad B_{n;\mathrm{V}}(v) = \mathbf{B}_n(1,v) \quad\text{for}\quad 0\le u,v\le 1. \tag{2.22}$$

The next fact follows from (2.20) and a result of Chung (see, e.g., [8]).

Fact 2.3 *We have, as $n\to\infty$*

$$\|\alpha_n\|=O_{\mathbb{P}}(1) \quad \textit{and} \quad \|\alpha_{n;k}\|=O_{\mathbb{P}}(1) \quad \textit{for} \quad k=0,1,2, \tag{2.23}$$

$$\|\alpha_{n,\mathrm{U}}\|=\|\beta_{n,\mathrm{U}}\|=O_{\mathbb{P}}(1) \quad \textit{and} \quad \|\alpha_{n,\mathrm{V}}\|=\|\beta_{n,\mathrm{V}}\|=O_{\mathbb{P}}(1). \tag{2.24}$$

Moreover, with probability 1,

$$\limsup_{n\to\infty}(\operatorname{loglog} n)^{-1/2}\|\alpha_{n,\mathrm{U}}\|=\limsup_{n\to\infty}(\operatorname{loglog} n)^{-1/2}\|\beta_{n,\mathrm{U}}\|=2^{-1/2}, \tag{2.25}$$

$$\limsup_{n\to\infty}(\operatorname{loglog} n)^{-1/2}\|\alpha_{n,\mathrm{V}}\|=\limsup_{n\to\infty}(\operatorname{loglog} n)^{-1/2}\|\beta_{n,\mathrm{V}}\|=2^{-1/2}. \tag{2.26}$$

The next proposition collects some easy consequences of the above definitions and facts.

Proposition 2.1 *We have, with probability 1 as $n\to\infty$,*

$$\|\alpha_{n;0}-\mathbf{B}_{n;*}\| = O\left(n^{-1/2}(\log n)^2\right) \quad \textit{and} \quad \|\alpha_{n;1}-\mathbf{B}_{n;*}\| = O\left(n^{-1/2}(\log n)^2\right), \tag{2.27}$$

$$\|\alpha_{n;\mathrm{U}}-B_{n;\mathrm{U}}\| = O\left(n^{-1/2}(\log n)^2\right) \quad \textit{and} \quad \|\alpha_{n;\mathrm{V}}-B_{n;\mathrm{V}}\| = O\left(n^{-1/2}(\log n)^2\right). \tag{2.28}$$

Moreover, we have

$$\|\alpha_{n;0}-\alpha_{n;1}\| = O\left(n^{-1/2}(\operatorname{loglog} n)^2\right). \tag{2.29}$$

Proof. Combine (2.20) with (2.11)–(2.13).□

Proposition 2.2 *We have, with probability 1,*

$$\limsup_{n\to\infty}(2\operatorname{loglog} n)^{-1/2}\|\alpha_n\| = \frac{1}{2}, \tag{2.30}$$

$$\limsup_{n\to\infty}(2\operatorname{loglog} n)^{-1/2}\|\alpha_{n;0}\| = \frac{1}{4}. \tag{2.31}$$

Proof. This non-trivial result turns out to follow from classical arguments, based upon some well-known facts collected from the literature. We limit ourselves to establish (2.31), the proof of (2.30) being similar, and therefore, omitted. Recall (2.6) and (2.11). As follows from Révész [43], in combination with (2.11) and (2.16), there exists (on a suitable probability space) a sequence $\widetilde{\mathbf{B}}_{n;*}, n = 1, 2, ...$ of independent tied-down bivariate Brownian bridges such that, with probability 1 as $n \to \infty$,

$$\left\|\alpha_{n;0} - n^{-1/2}\sum_{j=1}^{n}\widetilde{\mathbf{B}}_{j;*}\right\| = O\left(n^{-1/6}(\log n)^{3/2}\right). \tag{2.32}$$

In spite that the rate in (2.32) is sub-optimal (see, e.g., [4, 5] and the references therein), it is sufficient for our needs. We next apply a result of Lai [34] to show that the sequence

$$\zeta_n = (2n \operatorname{loglog} n)^{-1/2}\sum_{j=1}^{n}\widetilde{\mathbf{B}}_{j;*}, \tag{2.33}$$

is almost surely relatively compact in the space $C\left([0,1]^2\right)$ of continuous functions on $[0,1]^2$, endowed with the uniform topology. The corresponding limit set is the unit ball $\mathbb{K}_0$ of the reproducing kernel Hilbert space pertaining to the tied-down Brownian bridge $\mathbf{B}_*(\cdot,\cdot)$ (as defined in (2.16)). This, when combined with (2.32), entails that the sequence $(2 \operatorname{loglog} n)^{-1/2}\alpha_{n;0}$ is almost surely relatively compact in the space $\mathcal{B}\left([0,1]^2\right)$ of bounded functions on $[0,1]^2$, with limit set $\mathbb{K}_0$. In view of (2.18), we combine this last result with an argument in Section 4 of Lai [34], to show that, almost surely,

$$\limsup_{n\to\infty}\,(2 \operatorname{loglog} n)^{-1/2}\|\alpha_{n;0}\| = \limsup_{n\to\infty}\,(2 \operatorname{loglog} n)^{-1/2}\|\zeta_n\|$$
$$= \sup_{h\in\mathbb{K}_0}\|h\| = \sup_{0\le u\le 1}\left\{\operatorname{Var}\left(\mathbf{B}_*(u,u)\right)\right\}^{1/2} = \sup_{0\le u\le 1}\{u-u^2\} = \frac{1}{4}.$$

We so obtain (2.31), as sought.□

The next fact is due to Kiefer [33]. Recall the notation (2.7)–(2.10).

Fact 2.4 *We have, almost surely,*

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{1/4}\,\|\alpha_{n;\mathrm{U}} + \beta_{n;\mathrm{U}}\| \;=\; 2^{-1/4}, \tag{2.34}$$
$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{1/4}\,\|\alpha_{n;\mathrm{V}} + \beta_{n;\mathrm{V}}\| \;=\; 2^{-1/4}. \tag{2.35}$$

For our needs, it will be convenient to denote, for each measurable subset R of $[0,1]^2$, the empirical measure of R by $\mathbb{T}_n(R) = n^{-1}\#\{(U_i,V_i)\in R : 1\le i\le n\}$. The corresponding set-indexed empirical process is given likewise by

$$\alpha_n(R) = n^{1/2}\left\{\mathbb{T}_n(R) - |R|\right\}, \tag{2.36}$$

where $|R|$ stands for the Lebesgue measure of R. Note also the relation with the notation introduced in (2.1): for every $0 \le u, v \le 1$, $\mathbb{T}_n(u,v) = \mathbb{T}_n([0,u]\times[0,v])$. We will consider especially the class $\overline{\mathcal{R}}$ of closed rectangles, of the form $[a,b]\times[c,d]$, for $0\le a\le b\le 1$ and $0\le c\le d\le 1$. Following the notation of Einmahl (see, e.g., p.67 in [24]), we set

$$\omega_n(a) = \sup_{R\in\overline{\mathcal{R}}:|R|\le a}|\mathbb{T}_n(R)|. \tag{2.37}$$

We will make use of the following fact due to Einmahl (see Theorem 5.3, p.75 in [24], and, e.g., [25]).

Fact 2.5 *Let* $\{a_n : n\ge 1\}$ *be a sequence of positive numbers such that* $a_n \downarrow 0$*,* $na_n \uparrow \infty$*,* $na_n/\log n \to \infty$ *and* $(\log(1/a_n))/\operatorname{loglog} n \to \infty$*. Then, with probability* 1*,*

$$\lim_{n\to\infty}\left(2a_n\log(1/a_n)\right)^{-1/2}\omega_n(a_n) = 1. \tag{2.38}$$

We are now equipped to prove the next proposition. Recall the definitions (2.11)-(2.13) of $\alpha_{n;0}$ and $\alpha_{n;2}$.

Proposition 2.3 *We have, almost surely,*

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\operatorname{loglog} n)^{-1/4} \|\alpha_{n,2}-\alpha_{n,0}\| \leq 5\times 2^{-1/4}. \tag{2.39}$$

Proof. Recalling (2.6), we will make use of the straightforward inequality, for $0 \leq u', v', u'', v'' \leq 1$.

$$\begin{aligned} |\alpha_n(u',v')-\alpha_n(u'',v'')| &\leq \omega_n\left(|u'-u''|\right)+\omega_n\left(|v'-v''|\right)+\omega_n\left(|u'-u''|\times|v'-v''|\right) \\ &\leq 3\omega_n\left(|u'-u''|\vee|v'-v''|\right). \end{aligned} \tag{2.40}$$

Fix any $\epsilon>0$, and set $a_n=(1+\epsilon)2^{-1/2}n^{-1/2}(\operatorname{loglog} n)^{1/2}$. By combining (2.25)–(2.26) with (2.40) and the triangle inequality, we get that, with probability 1 for all n sufficiently large,

$$\sup_{0\leq u,v\leq 1}\left|\alpha_n\big(u+n^{-1/2}\beta_{n;\mathrm{U}}(u),v+n^{-1/2}\beta_{n;\mathrm{V}}(v)\big)-\alpha_n(u,v)\right| \leq 3\omega_n(a_n),$$

whence, by (2.34), almost surely,

$$\begin{aligned} &\limsup_{n\to\infty}\ n^{1/4}(\log n)^{-1/2}(\operatorname{loglog} n)^{-1/4} \\ &\times \sup_{0\leq u,v\leq 1}\left|\alpha_n\big(u+n^{-1/2}\beta_{n;\mathrm{U}}(u),v+n^{-1/2}\beta_{n;\mathrm{V}}(v)\big)-\alpha_n(u,v)\right| \\ \leq\ & 3\limsup_{n\to\infty}\ n^{1/4}(\log n)^{-1/2}(\operatorname{loglog} n)^{-1/4}\omega_n(a_n) = 3\times 2^{-1/4}\sqrt{1+\epsilon}. \end{aligned}$$

Observe that $\epsilon>0$ may be chosen as small as desired in this last expression. Therefore, by combining this last result with (2.7)–(2.8), (2.11)–(2.13) and (2.25)–(2.26), we conclude that, almost surely,

$$\begin{aligned} &\limsup_{n\to\infty}\ n^{1/4}(\log n)^{-1/2}(\operatorname{loglog} n)^{-1/4} \|\alpha_{n;0}-\alpha_{n;2}\| \leq 3\times 2^{-1/4} \\ &+\limsup_{n\to\infty}\ n^{1/4}(\log n)^{-1/2}(\operatorname{loglog} n)^{-1/4}\left\{\|\alpha_{n;\mathrm{U}}+\beta_{n;\mathrm{U}}\|+\|\alpha_{n;\mathrm{V}}+\beta_{n;\mathrm{V}}\|\right\} \\ &\leq 2^{-1/4}\left\{3+2\right\} = 5\times 2^{-1/4}, \end{aligned}$$

where the last inequality follows from (2.34)–(2.35). We so obtain (2.39), as sought.□

The following corollary is a straightforward consequence of Propositions 2.1, 2.2 and 2.3.

Corollary 2.1 *We have, with probability* 1,

$$\limsup_{n\to\infty}(2\operatorname{loglog} n)^{-1/2}\|\alpha_{n;1}\| = \limsup_{n\to\infty}(2\operatorname{loglog} n)^{-1/2}\|\alpha_{n;2}\| = \frac{1}{4}. \tag{2.41}$$

Moreover, with probability 1,

$$\limsup_{n\to\infty}\ n^{1/4}(\log n)^{-1/2}(\log n)^{-1/2}(\operatorname{loglog} n)^{-1/4}\|\alpha_{n;2}-\mathbf{B}_{n;*}\| \leq 5\times 2^{-1/4}. \tag{2.42}$$

Proof. The proof of (2.41) is achieved by combining (2.29), (2.31) and (2.39). In the same spirit, we infer readily (2.42) from (2.27) and (2.39).□

2.2 Application to Tests of Independence

The following corollary is a natural consequence of Proposition 2.1 and Corollary 2.1, when combined with the definitions (1.9) and (1.10) of $\Omega^2_{n;\mathrm{C}}$ and $\Omega^2_{n;\mathrm{T}}$. Below, we assume, unless otherwise specified, that our random observations are defined on the probability space of Fact 2.2

Corollary 2.2 *We have, almost surely,*

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\operatorname{loglog} n)^{-3/4}\left|\Omega^2_{n;\mathrm{C}}-\int_0^1\int_0^1 \mathbf{B}^2_{n;*}(u,v)dudv\right|\le 5\times 2^{5/4}, \tag{2.43}$$

and

$$\limsup_{n\to\infty} n^{1/2}(\log n)^{-2}(\operatorname{loglog} n)^{-1/2}\left|\Omega^2_{n;\mathrm{T}}-\int_0^1\int_0^1 \mathbf{B}^2_{n;*}(u,v)dudv\right|<\infty, \tag{2.44}$$

Proof. In view of (1.9) and (2.13), we have

$$\Omega^2_{n;\mathrm{C}}=n\int_0^1\int_0^1\{C_n(u,v)-uv\}^2\,dudv=\int_0^1\int_0^1\alpha^2_{n;2}(u,v)dudv. \tag{2.45}$$

Therefore, by the triangle inequality,

$$\begin{aligned}&\left|\Omega^2_{n;\mathrm{C}}-\int_0^1\int_0^1 \mathbf{B}^2_{n;*}(u,v)dudv\right|\le\|\alpha_{n;2}-\mathbf{B}_{n;*}\|\times\|\alpha_{n;2}+\mathbf{B}_{n;*}\|\\&\le\|\alpha_{n;2}-\mathbf{B}_{n;*}\|\times\{2\,\|\alpha_{n;2}\|+\|\alpha_{n;2}-\mathbf{B}_{n;*}\|\}\,.\end{aligned}$$

This, when combined with (2.41) and (2.42) readily yields (2.43).□

Likewise, by (1.10) and (2.12), we get

$$\Omega^2_{n;\mathrm{T}}=\int_0^1\int_0^1\alpha^2_{n;1}(u,v)dudv. \tag{2.46}$$

whence, by the triangle inequality,

$$\begin{aligned}&\left|\Omega^2_{n;\mathrm{T}}-\int_0^1\int_0^1 \mathbf{B}^2_{n;*}(u,v)dudv\right|\le\|\alpha_{n;1}-\mathbf{B}_{n;*}\|\times\|\alpha_{n;1}+\mathbf{B}_{n;*}\|\\&\le\|\alpha_{n;1}-\mathbf{B}_{n;*}\|\times\{2\,\|\alpha_{n;1}\|+\|\alpha_{n;1}-\mathbf{B}_{n;*}\|\}\,.\end{aligned}$$

By combining this last inequality with (2.27) and (2.41), we conclude (2.44).□

We now turn to the study of weighted versions of the statistic $\Omega^2_{n;\mathrm{C}}$. We limit ourselves to the following more or the less straightforward result, given the arguments above.

Corollary 2.3 *For each choice of $\gamma>-1/2$ and $\delta>-1/2$, we have, almost surely,*

$$\left|\Omega^2_{n;\mathrm{C};\gamma,\delta}-\int_0^1\int_0^1 u^{2\gamma}v^{2\delta}\mathbf{B}^2_{n;*}(u,v)dudv\right|=O\left(n^{-1/4}(\log n)^{1/2}(\operatorname{loglog} n)^{3/4}\right). \tag{2.47}$$

Moreover, (2.47) *also holds with $\Omega^2_{n;\mathrm{T};\gamma,\delta}$ replacing $\Omega^2_{n;\mathrm{C};\gamma,\delta}$.*

Proof. The proof is essentially identical to the proof of Corollary 2.2, with the added simple observation that

$$\int_0^1\int_0^1 u^{2\gamma}v^{2\delta}dudv<\infty,$$

when $\gamma>-1/2$ and $\delta>-1/2$.□

Remark 2.1 *In spite of the fact that the random variable*

$$\Xi_* (\gamma, \delta) = \int_0^1 \int_0^1 u^{2\gamma} v^{2\delta} \mathbf{B}_{n;*}^2 (u, v) du dv, \tag{2.48}$$

is still defined for $\gamma > -1$ and $\delta > -1$, the conditions of Corollary 2.3 turn out to be sharp. This follows from the observation that $C_n(u, v) = 1/n$ for $0 < u, v \leq 1/n$. Therefore $\Omega^2_{n;\mathrm{C};\gamma,\delta}$ is not defined when either $-1 < \gamma \leq -1/2$ or $-1 < \delta \leq -1/2$. In Section 5 we will complete this discussion by investigating the asymptotic behavior of the r.v. $\Xi_* (\gamma, \delta)$, when $\gamma, \delta \downarrow -1$.

These results motivate a systematic study of the laws of weighted quadratic forms of the type (2.48). We will start by establishing some useful identity in laws between quadratic functionals of Gaussian processes. The results of Sections 2 and 3 will be further developed in § 4, where the study of such functionals as (2.48) is performed by means of Karhunen-Loève expansions.

3 Distributional Identities via Stochastic Fubini Theorems

3.1 Weighted Processes

Let $\{W(t) : t \geq 0\}$ denote a (standard) Wiener process, and let $B(t) = W(t) - tW(1)$ denote a (standard) Brownian bridge for $0 \leq t \leq 1$. For each $\gamma > -1$, consider the weighted processes

$$W_\gamma = \Big\{ W_\gamma(t) = t^\gamma W(t) : 0 < t \leq 1 \Big\}, \tag{3.1}$$

and

$$B_\gamma = \Big\{ B_\gamma(t) = t^\gamma B(t) = t^\gamma W(t) - t^{\gamma+1} W(1) : 0 < t \leq 1 \Big\}, \tag{3.2}$$

with $W_\gamma(0) = B_\gamma(0) := 0$. The Karhunen-Loève expansions of these processes have been given in [16], in terms of Bessel functions. One of the purposes of the present paper is to extend the results of [16] to the multivariate case. Towards this aim, we shall make use of the following notation.

For each $\gamma > -1$ and $\delta > -1$ we denote by $\mathbf{W}^{(\gamma,\delta)} = \big\{ \mathbf{W}^{(\gamma,\delta)} (s, t) : 0 \leq s, t \leq 1 \big\}$, the (restriction to $[0, 1]^2$ of the) weighted process $(s, t) \mapsto s^\gamma t^\delta \mathbf{W} (s, t)$, where $\mathbf{W}(\cdot, \cdot)$ denotes, as in (2.14), a (standard) bivariate Wiener process (or Brownian sheet) on $[0, \infty]^2$, and we set by convention $\mathbf{W}^{(\gamma,\delta)} (s, t) = 0$, whenever $s \wedge t = 0$. We drop the indexes to write $\mathbf{W}^{(0,0)} = \mathbf{W}$ when $\gamma = \delta = 0$. We set further

$$\begin{aligned} \mathbf{B}^{(\gamma,\delta)} &= \Big\{ \mathbf{B}^{(\gamma,\delta)} (s, t) : 0 \leq s, t \leq 1 \Big\}, && (3.3) \\ \mathbf{B}_*^{(\gamma,\delta)} &= \Big\{ \mathbf{B}_*^{(\gamma,\delta)} (s, t) : 0 \leq s, t \leq 1 \Big\}, && (3.4) \\ \mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)} &= \Big\{ \mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)} (s, t) : 0 \leq s, t \leq 1 \Big\}. && (3.5) \end{aligned}$$

to be respectively a *weighted bivariate Brownian bridge*, a *weighted bivariate tied-down Brownian bridge* and a *weighted asymmetric bivariate Brownian bridge*, also known as a *weighted Kiefer process* (see, for instance, [12]). Namely, in agreement with the notation (2.15)–(2.16), we set, for $0 < s, t \leq 1$,

$$\begin{aligned} \mathbf{B}^{(\gamma,\delta)} (s, t) &= s^\gamma t^\delta \ \mathbf{W} (s, t) - s^{\gamma+1} t^{\delta+1} \mathbf{W} (1, 1) \\ &= s^\gamma t^\delta \{ \mathbf{W} (s, t) - st \mathbf{W} (1, 1) \} = s^\gamma t^\delta \mathbf{B} (s, t) , && (3.6) \\ \mathbf{B}_*^{(\gamma,\delta)} (s, t) &= s^\gamma t^\delta \mathbf{W} (s, t) - s^{\gamma+1} t^\delta \mathbf{W} (1, t) - s^\gamma t^{\delta+1} \mathbf{W} (s, 1) + s^{\gamma+1} t^{\delta+1} \mathbf{W} (1, 1) \\ &= s^\gamma t^\delta \{ \mathbf{W} (s, t) - s \mathbf{W} (1, t) - t \mathbf{W} (s, 1) + st \mathbf{W} (1, 1) \} = s^\gamma t^\delta \mathbf{B}_* (s, t) , && (3.7) \\ \mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)} (s, t) &= s^\gamma t^\delta \mathbf{W} (s, t) - s^{\gamma+1} t^\delta \mathbf{W} (1, t) \\ &= s^\gamma t^\delta [\mathbf{W} (s, t) - s \mathbf{W} (1, t)] , && (3.8) \end{aligned}$$

and we define $\mathbf{B}^{(\gamma,\delta)}(s,t) = \mathbf{B}_*^{(\gamma,\delta)}(s,t) = \mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}(s,t) = 0$ whenever $s \wedge t = 0$. In agreement with (2.15)–(2.16), we set $\mathbf{B}^{(0,0)} = \mathbf{B}$, $\mathbf{B}_*^{(0,0)} = \mathbf{B}_*$, $\mathbf{B}_{\mathcal{A}}^{(0,0)} = \mathbf{B}_{\mathcal{A}}$. For $\gamma > -1/2$ and $\delta > -1/2$, we will denote further by

$$\widetilde{\mathbf{W}}^{(\gamma,\delta)} = \left\{ \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t) : 0 \le s,t \le 1 \right\}, \tag{3.9}$$

the process

$$(s,t) \mapsto \int_{[s,1]\times[t,1]} u^\gamma v^\delta \mathbf{W}(du,dv). \tag{3.10}$$

It is readily checked that, for $\gamma, \delta > -1/2$, the distributional identity between processes

$$\widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t) \overset{\text{law}}{=} \mathbf{W}\left(1 - s^{2\gamma+1}, 1 - t^{2\delta+1}\right) / \left\{(2\gamma+1)(2\delta+1)\right\} \tag{3.11}$$

holds globally on $[0,1]^2$.

3.2 Main Distributional Identities

The aim of this subsection is to prove the following Theorems 3.1 and 3.2. In the first theorem, we establish distributional identities involving the conditioned processes $\mathbf{B}^{(\gamma,\delta)}$, $\mathbf{B}_*^{(\gamma,\delta)}$ and $\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}$, for $\gamma > -1/2$ and $\delta > -1/2$ (the role played by this assumption will be clarified in § 3.3 below). In the second theorem, distributional identities are established for three different realizations of the *path-variance* of the bivariate process $\mathbf{W}^{(\gamma,\delta)}$. As a corollary, for the case $\gamma = \delta = 0$ we obtain a generalization of some known results holding in the univariate case. For our needs, the following notation will turn out to be convenient. Given two centered, real valued Gaussian processes $\mathbf{Z}_1 = \left\{\mathbf{Z}_1(\mathbf{t}) : \mathbf{t} \in [0,1]^d\right\}$ and $\mathbf{Z}_2 = \left\{\mathbf{Z}_2(\mathbf{t}) : \mathbf{t} \in [0,1]^d\right\}$, defined on $[0,1]^d$, we write (with "Quad" for "Quadratic")

$$\mathbf{Z}_1 \overset{\text{Quad}}{\sim} \mathbf{Z}_2, \tag{3.12}$$

whenever the identity in law (3.13) below holds:

$$\int_{[0,1]^d} \mathbf{Z}_1^2(\mathbf{t})\, d\mathbf{t} \overset{\text{law}}{=} \int_{[0,1]^d} \mathbf{Z}_2^2(\mathbf{t})\, d\mathbf{t}. \tag{3.13}$$

To introduce our forthcoming theorems, we recall the following non-trivial example of distributional identity of the form (3.13) for $d = 1$ (see, e.g., [17] and [16]). In view of the notation (3.1)–(3.2), we fix an arbitrary $\gamma \in (-1, -\frac{1}{2})$, and define

$$\varrho(\gamma) := -1 - \frac{\gamma+1}{2\gamma+1}, \tag{3.14}$$

so that, under these assumptions, it holds that

$$W_\gamma \overset{\text{Quad}}{\sim} \left\{\frac{1}{2\gamma+1}\right\} B_{\varrho(\gamma)}. \tag{3.15}$$

We refer to § 1.7 of [16] for a proof and discussion of relation (3.15) based on Karhunen-Loève expansions.

Under the notation above, we may state the main results of the section.

Theorem 3.1 *For each $\gamma > -1/2$ and $\delta > -1/2$ the following relations hold.*

$$\mathbf{B}^{(\gamma,\delta)} \overset{\text{Quad}}{\sim} \left\{ \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t) - \int_0^1 \int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(u,v)\, dudv : (s,t) \in [0,1]^2 \right\} \tag{3.16}$$

$$\overset{\text{Quad}}{\sim} \left\{ \frac{(1-s)^{-\frac{\gamma}{2\gamma+1}} (1-t)^{-\frac{\delta}{2\delta+1}}}{(2\gamma+1)^{\frac{3}{2}} (2\delta+1)^{\frac{3}{2}}} \left[\mathbf{W}(s,t) - \frac{1}{(2\gamma+1)(2\delta+1)} \right. \right. \tag{3.17}$$

$$\left. \left. \times \int_0^1 \int_0^1 (1-u)^{-\frac{2\gamma}{2\gamma+1}} (1-v)^{-\frac{2\delta}{2\delta+1}} \mathbf{W}(u,v) \right] : (s,t) \in [0,1]^2 \right\};$$

$$\mathbf{B}_*^{(\gamma,\delta)} \overset{\text{Quad}}{\sim} \left\{ \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t) - \int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,v)\, dv - \int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(u,t)\, du \right. \tag{3.18}$$

$$\left. + \int_0^1 \int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(u,v)\, dudv : (s,t) \in [0,1]^2 \right\}$$

$$\overset{\text{Quad}}{\sim} \left\{ \frac{(1-s)^{-\frac{\gamma}{2\gamma+1}} (1-t)^{-\frac{\delta}{2\delta+1}}}{(2\gamma+1)^{\frac{3}{2}} (2\delta+1)^{\frac{3}{2}}} \left[\mathbf{W}(s,t) - \frac{1}{2\gamma+1} \int_0^1 (1-u)^{-\frac{2\gamma}{2\gamma+1}} \mathbf{W}(u,t)\, du \right. \right.$$

$$- \frac{1}{2\delta+1} \int_0^1 (1-v)^{-\frac{2\delta}{2\delta+1}} \mathbf{W}(s,v)\, dv + \frac{1}{(2\gamma+1)(2\delta+1)} \tag{3.19}$$

$$\left. \left. \times \int_0^1 \int_0^1 (1-u)^{-\frac{2\gamma}{2\gamma+1}} (1-v)^{-\frac{2\delta}{2\delta+1}} \mathbf{W}(u,v)\, dudv \right] : (s,t) \in [0,1]^2 \right\};$$

$$\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)} \overset{\text{Quad}}{\sim} \left\{ \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t) - \int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(u,t)\, du : (s,t) \in [0,1]^2 \right\} \tag{3.20}$$

$$\overset{\text{Quad}}{\sim} \left\{ \frac{(1-s)^{-\frac{\gamma}{2\gamma+1}} (1-t)^{-\frac{\delta}{2\delta+1}}}{(2\gamma+1)^{\frac{3}{2}} (2\delta+1)^{\frac{3}{2}}} \left[\mathbf{W}(s,t) \right. \right.$$

$$\left. \left. - \frac{1}{2\gamma+1} \int_0^1 (1-u)^{-\frac{2\delta}{2\delta+1}} \mathbf{W}(u,t)\, du \right] : (s,t) \in [0,1]^2 \right\}. \tag{3.21}$$

Theorem 3.2 *For each $\gamma > -1/2$ and $\delta > -1/2$ the following relations hold.*

$$\left\{ \mathbf{W}^{(\gamma,\delta)}(s,t) - \int_0^1 \int_0^1 \mathbf{W}^{(\gamma,\delta)}(u,v)\, dudv : (s,t) \in [0,1]^2 \right\} \tag{3.22}$$

$$\overset{\text{Quad}}{\sim} \left\{ \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t) - \frac{\left(1-s^{\gamma+1}\right)\left(1-t^{\delta+1}\right)}{(\gamma+1)(\delta+1)} \mathbf{W}(1,1) : (s,t) \in [0,1]^2 \right\};$$

$$\left\{\mathbf{W}^{(\gamma,\delta)}(s,t)-\int_0^1\mathbf{W}^{(\gamma,\delta)}(s,v)\,dv-\int_0^1\mathbf{W}^{(\gamma,\delta)}(u,t)\,du\right.$$

$$\left.+\int_0^1\!\!\int_0^1\mathbf{W}^{(\gamma,\delta)}(u,v)\,dudv:(s,t)\in[0,1]^2\right\} \tag{3.23}$$

$$\overset{\text{Quad}}{\sim}\left\{\widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)-\frac{1-s^{\gamma+1}}{\gamma+1}\int_t^1 v^\delta\mathbf{W}(1,dv)-\frac{1-t^{\delta+1}}{\delta+1}\int_s^1 u^\gamma\mathbf{W}(du,1)\right.$$

$$\left.+\frac{\left(1-s^{\gamma+1}\right)\left(1-t^{\delta+1}\right)}{(\gamma+1)(\delta+1)}\mathbf{W}(1,1):(s,t)\in[0,1]^2\right\};$$

$$\left\{\mathbf{W}^{(\gamma,\delta)}(s,t)-\int_0^1\mathbf{W}^{(\gamma,\delta)}(u,t)\,du:(s,t)\in[0,1]^2\right\} \tag{3.24}$$

$$\overset{\text{Quad}}{\sim}\left\{\widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)-\frac{1-s^{\gamma+1}}{\gamma+1}\int_t^1 v^\delta\mathbf{W}(1,dv):(s,t)\in[0,1]^2\right\}.$$

By specializing Theorems 3.1 and 3.2 to the case where $\gamma=\delta=0$, we obtain the following corollary.

Corollary 3.1 *Under the above assumptions and notation,*

$$\mathbf{B}\overset{\text{Quad}}{\sim}\left\{\mathbf{W}(s,t)-\int_0^1\!\!\int_0^1\mathbf{W}(u,v)\,dudv:(s,t)\in[0,1]^2\right\}; \tag{3.25}$$

$$\mathbf{B}_*\overset{\text{Quad}}{\sim}\left\{\mathbf{W}(s,t)-\int_0^1\mathbf{W}(s,v)\,dv-\int_0^1\mathbf{W}(u,t)\,du\right.$$

$$\left.+\int_0^1\!\!\int_0^1\mathbf{W}(u,v)\,dudv:(s,t)\in[0,1]^2\right\}, \tag{3.26}$$

$$\mathbf{B}_{\mathcal{A}}\overset{\text{Quad}}{\sim}\left\{\mathbf{W}(s,t)-\int_0^1\mathbf{W}(u,t)\,du:(s,t)\in[0,1]^2\right\}. \tag{3.27}$$

Remark 3.1 (a) *Conditionally on the event that $\mathbf{W}(1,\lambda)=\mathbf{W}(\lambda,1)=0$ for all $\lambda\in[0,1]$, the process $\widetilde{\mathbf{W}}^{(\gamma,\delta)}$ has the same distribution as the (unconditioned) process*

$$(s,t)\mapsto\widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)-\frac{1-s^{\gamma+1}}{\gamma+1}\int_t^1 v^\delta\mathbf{W}(1,dv)$$

$$-\frac{1-t^{\delta+1}}{\delta+1}\int_s^1 u^\gamma\mathbf{W}(du,1)+\frac{\left(1-s^{\gamma+1}\right)\left(1-t^{\delta+1}\right)}{(\gamma+1)(\delta+1)}\mathbf{W}(1,1). \tag{3.28}$$

(b) *Conditionally on the event that $\mathbf{W}(1,1)=0$, the process $\widetilde{\mathbf{W}}^{(\gamma,\delta)}$ has the same distribution as the (unconditioned) process*

$$(s,t)\mapsto\widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)-\frac{\left(1-s^{\gamma+1}\right)\left(1-t^{\delta+1}\right)}{(\gamma+1)(\delta+1)}\mathbf{W}(1,1). \tag{3.29}$$

(c) *Conditionally on the event that $\mathbf{W}(1,\lambda)=0$ for every $\lambda\in[0,1]$, the process $\widetilde{\mathbf{W}}^{(\gamma,\delta)}$ has the same distribution as the (unconditioned) process*

$$(s,t)\mapsto\widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)-\frac{1-s^{\gamma+1}}{\gamma+1}\int_{[t,1]}v^\delta\mathbf{W}(1,dv). \tag{3.30}$$

The proof of Theorems 3.1 and 3.2 requires some refinement of the techniques discussed in [17] to the case of a general Gaussian measure. This will be detailed in the next section.

3.3 Generalized Fubini-Wiener Techniques

3.3.1 A General Fubini Theorem

Let $(A, \mathcal{A}, \mu)$ and $(B, \mathcal{B}, \nu)$ be two measurable spaces, with μ and ν denoting positive and σ-finite measures. Consider two *isonormal Gaussian processes* (or *Gaussian measures*)

$$\left\{\mathbf{G}_{\mu}(h) : h \in L^2(A, \mathcal{A}, \mu)\right\} \quad \text{and} \quad \left\{\mathbf{G}_{\nu}(p) : p \in L^2(B, \mathcal{B}, \nu)\right\}. \tag{3.31}$$

Namely, $\mathbf{G}_{\mu}$ and $\mathbf{G}_{\nu}$ are two centered Gaussian processes, indexed respectively by functions in $L^2(A, \mathcal{A}, \mu)$ and $L^2(B, \mathcal{B}, \nu)$, respectively, and fulfilling

$$\mathbb{E}\left(\mathbf{G}_{\mu}(h)\,\mathbf{G}_{\mu}(k)\right) = \int_A h(a)\,k(a)\,\mu(da) \quad \forall h, k \in L^2(A, \mathcal{A}, \mu), \tag{3.32}$$

$$\mathbb{E}\left(\mathbf{G}_{\nu}(p)\,\mathbf{G}_{\nu}(q)\right) = \int_B p(b)\,q(b)\,\nu(db) \quad \forall p, q \in L^2(B, \mathcal{B}, \nu). \tag{3.33}$$

The key of the subsequent discussion is stated in the following theorem.

Theorem 3.3 (Fubini Theorem for Gaussian Measures) *Under the assumptions above, for every* $\phi \in L^2(A \times B, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$,

$$\int_A \left\{ \int_B \phi(a, b)\,\mathbf{G}_{\nu}(db) \right\}^2 \mu(da) \overset{\text{law}}{=} \int_B \left\{ \int_A \phi(a, b)\,\mathbf{G}_{\mu}(da) \right\}^2 \nu(db). \tag{3.34}$$

Proof. Without loss of generality, we can and do assume in our proof that the processes $\mathbf{G}_{\mu}$ and $\mathbf{G}_{\nu}$ are independent. For every $\phi \in L^2(A \times B, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$, the stochastic integral

$$\int_A \left\{ \int_B \phi(a, b)\,\mathbf{G}_{\nu}(db) \right\} \mathbf{G}_{\mu}(da)$$

is well defined as the L^2 limit of linear combinations of random variables of the form

$$\int_A \phi_1(a)\,\mathbf{G}_{\mu}(da) \int_B \phi_2(b)\,\mathbf{G}_{\nu}(db),$$

with $\phi_1 \in L^2(A, \mathcal{A}, \mu)$ and $\phi_2 \in L^2(B, \mathcal{B}, \nu)$. This, in turn, implies the almost sure relation

$$\int_A \left\{ \int_B \mathbf{G}_{\nu}(db)\,\phi(a, b) \right\} \mathbf{G}_{\mu}(da) = \int_B \left\{ \int_A \phi(a, b)\,\mathbf{G}_{\mu}(da) \right\} \mathbf{G}_{\nu}(db). \tag{3.35}$$

From the equality in (3.35) we infer, in view of (3.32)-(3.33), that, for every $u \in \mathbb{R}$,

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_A \left\{ \int_B \phi(a, b)\,\mathbf{G}_{\nu}(db) \right\}^2 \mu(da)\right)\right] \\
= {} & \mathbb{E}\left[\exp\left(iu \int_A \left\{ \int_B \phi(a, b)\,\mathbf{G}_{\nu}(db) \right\} \mathbf{G}_{\mu}(da)\right)\right] \\
= {} & \mathbb{E}\left[\exp\left(iu \int_B \left\{ \int_A \phi(a, b)\,\mathbf{G}_{\mu}(da) \right\} \mathbf{G}_{\nu}(db)\right)\right] \\
= {} & \mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_B \left\{ \int_A \phi(a, b)\,\mathbf{G}_{\mu}(da) \right\}^2 \nu(db)\right)\right],
\end{aligned} \tag{3.36}$$

from where (3.34) is straightforward.□

3.3.2 The Special Case of the Brownian Sheet

Let $\mu(da) = \nu(db)$ be the Lebesgue measure on $A = [0,1]^n$ for some $n \geq 1$, so that

$$(A, \mathcal{A}, \mu) = (B, \mathcal{B}, \nu) = \left([0,1]^n, \mathcal{B}\left([0,1]^n\right), dt_1 \ldots dt_n\right). \tag{3.37}$$

Then $\mathbf{G}_\mu \stackrel{\text{law}}{=} \mathbf{G}_\nu$ is the Gaussian measure generated by an n-variate Wiener process (or Brownian sheet) $\mathbf{W}(t_1, \ldots, t_n)$. Recalling (2.14), by an n-variate Wiener process is meant a centered Gaussian process $\{\mathbf{W}(t_1, \ldots, t_n) : t_1, \ldots, t_n \geq 0\}$ with continuous sample paths and covariance function fulfilling

$$\mathbb{E}\left(\mathbf{W}(t_1', \ldots, t_n')\mathbf{W}(t_1'', \ldots, t_n'')\right) = \prod_{j=1}^{n} \left(t_j' \wedge t_j''\right). \tag{3.38}$$

Consider now a function $\phi(s_1, \ldots, s_n, t_1, \ldots, t_n) \in L^2\big([0,1]^{2n}, ds_1 \ldots ds_n dt_1 \ldots dt_n\big)$, as in Theorem 3.3. Introduce the random variables

$$Z_1(s_1, \ldots, s_n) = \int_{[0,1]^n} \phi(s_1, \ldots, s_n, t_1, \ldots, t_n) \mathbf{W}(dt_1, \ldots, dt_n) \tag{3.39}$$

$$Z_2(t_1, \ldots, t_n) = \int_{[0,1]^n} \phi(s_1, \ldots, s_n, t_1, \ldots, t_n) \mathbf{W}(ds_1, \ldots, ds_n). \tag{3.40}$$

In this special setup, the conclusion (3.34) of Theorem 3.3 may be rewritten into

$$\int_{[0,1]^n} Z_1^2(s_1, \ldots, s_n)\, ds_1 \ldots ds_n \stackrel{\text{law}}{=} \int_{[0,1]^n} Z_2^2(t_1, \ldots, t_n)\, dt_1 \ldots dt_n. \tag{3.41}$$

Formula (3.41), for a suitable function ϕ (for example, continuous), can be directly obtained by using the theory of Karhunen-Loève [KL] expansions. We refer to [1] and to the discussion in the forthcoming section for details and limit ourselves to the following fact concerning [KL] expansions. For $k = 1, 2$, there exist sequences $\lambda_{1,k} \geq \lambda_{2,k} \geq \ldots \geq 0$, $k = 1, 2$, of positive constants, such that, if $\omega_1, \omega_2, \ldots$ denotes independent standard normal $N(0,1)$ random variables, then

$$\int_{[0,1]^n} Z_k^2(u_1, \ldots, u_n)\, du_1 \ldots du_n \stackrel{\text{law}}{=} \sum_{j \geq 1} \lambda_{j,k} \omega_j^2. \tag{3.42}$$

For $k = 1, 2$, the coefficients $\lambda_{1,k} \geq \lambda_{2,k} \geq \ldots \geq 0$, are the eigenvalues λ of the Hilbert-Schmidt operator, from $L^2\big([0,1]^n, ds_1 ... ds_n\big)$ onto itself, associated to the covariance function of Z_k, with eigenfunctions f such that

$$\lambda f(t_1, \ldots, t_n) = \int_{[0,1]^n} f(s_1, \ldots, s_n)\mathbb{E}\left(Z_k(s_1, \ldots, s_n) Z_k(t_1, \ldots, t_n)\right) ds_1 \ldots ds_n. \tag{3.43}$$

In the present case, it can be checked (by using for instance the results stated in [28], pp. 246-248) that the eigenvalues of the two operators are identical for $k = 1, 2$, thus providing another proof of Theorem 3.3.

3.3.3 Integration by Parts Formulæ

Theorem 3.3 turns out to have a series of applications of independent interest. Here, we present two examples yielding *integration by parts* formulae following from this result. The first one was previously given on p. 22 of [53], whereas the second one is new and will be used later on, and in particular at the end of §5.

Example 3.1 Select x and y in such a way that $0 \leq x < y < \infty$ and choose two positive continuous functions $f(\cdot)$ and $g(\cdot)$, with f nonincreasing, and g nondecreasing. Let us now choose $A = B$ in Theorem 3.3 as the closed bounded interval $A = B = [x, y]$. Denoting, in general, by δ_z the Dirac measure at $z \in \mathbb{R}$, we define the measures μ and ν in §3.3 by

$$\mu(da) = -df(a) + \delta_y(da) f(y) \quad \text{and} \quad \nu(db) = \delta_x(db) g(x) + dg(b). \tag{3.44}$$

Then, we set

$$\begin{aligned} \mathbf{G}_\mu(h) &= h(y) W_1(f(y)) - \int_x^y h(a) dW_1(f(a)), \quad h \in L^2(A, \mathcal{A}, \mu) & (3.45) \\ \mathbf{G}_\nu(k) &= k(x) W_2(g(x)) + \int_x^y k(b) dW_2(g(b)), \quad k \in L^2(B, \mathcal{B}, \nu), & (3.46) \end{aligned}$$

where W_1 and W_2 denote two independent standard Wiener processes on $[0, \infty)$, and the stochastic integrals with respect to the processes $\{W_1(f(a)) : a \in [x, y]\}$ and $\{W_2(g(b)) : b \in [x, y]\}$ are defined by a time reversal. Clearly, the isonormal processes $\mathbf{G}_\mu$ and $\mathbf{G}_\nu$ in (3.45)–(3.46) are independent. Moreover,

$$\begin{aligned} \int_A \left\{ \int_B 1_{\{b \leq a\}} \mathbf{G}_\nu(db) \right\}^2 \mu(da) &= \int_x^y \{ -df(a) + \delta_y(da) f(y) \} W_2^2(g(a)) & (3.47) \\ &= -\int_x^y W_2^2(g(a)) df(a) + f(y) W_2^2(g(y)), \end{aligned}$$

whereas

$$\begin{aligned} \int_B \left\{ \int_A 1_{\{a \geq b\}} \mathbf{G}_\mu(da) \right\}^2 \nu(db) &= \int_x^y \{ \delta_x(db) g(x) + dg(b) \} W_1^2(f(b)) & (3.48) \\ &= g(x) W_1^2(f(x)) + \int_x^y W_1^2(f(b)) dg(b). \end{aligned}$$

We so obtain the distributional equality (see, e.g., [53])

$$-\int_x^y W^2(g(a)) df(a) + f(y) W^2(g(y) \overset{\text{law}}{=} g(x) W^2(f(x)) + \int_x^y W^2(f(b)) dg(b). \tag{3.49}$$

where here, W denotes the restriction of a standard Wiener process to $[0, 1]$.

Example 3.2 Choose $A = B = [x, y] \times [w, z]$ in Theorem 3.3, with x, y, w, z such that $0 \leq x, w < y, z < +\infty$. Select positive and continuous functions f_1, f_2, g_1, g_2 such that f_1 and f_2 are nonincreasing and g_1 and g_2 are nondecreasing. Set further, in this theorem,

$$\begin{aligned} \mu(da_1, da_2) &= \Big\{ -df_1(a_1) + \delta_y(da_1) f_1(y) \Big\} \Big\{ -df_2(a_2) + \delta_z(da_2) f_2(z) \Big\}, & (3.50) \\ \nu(db_1, db_2) &= \Big\{ \delta_x(db_1) g_1(x) + dg_1(b_1) \Big\} \Big\{ \delta_w(db_2) g_2(w) + dg_2(b_2) \Big\}, & (3.51) \end{aligned}$$

and define $\{\mathbf{W}_i(s, t) : (s, t) \in \mathbb{R}_+^2\}$ $i = 1, 2$ to be a pair of independent standard bivariate Wiener processes (or Brownian sheets) on $\mathbb{R}_+^2$, as in (2.14). Then, we define

$$\begin{aligned} \mathbf{G}_\mu(h) &= \int_x^y \int_w^z h(a_1, a_2) d\mathbf{W}_1(f_1(a_1), f_2(a_2)) & (3.52) \\ &\quad + \int_w^z h(y, a_2) d_{a_2} \mathbf{W}_1(f_1(y), f_2(a_2)) \\ &\quad + \int_x^y h(a_1, z) d_{a_1} \mathbf{W}_1(f_1(a_1), f_2(z)) \\ &\quad + h(y, z) \mathbf{W}_1(f_1(y), f_2(z)), \quad h \in L^2(A, \mathcal{A}, \mu) \end{aligned}$$

where the stochastic integration is again performed by means of a time reversal, and d_u means integration with respect to the variable u. Set further

$$\begin{aligned}\mathbf{G}_\nu(k) &= \int_x^y\int_w^z k(b_1,b_2)\,d\mathbf{W}_2(g_1(b_1),g_2(b_2)) \\ &\quad+\int_w^z k(x,b_2)\,d_{b_2}\mathbf{W}_2(g_1(x),g_2(b_2)) \\ &\quad+\int_x^y k(b_1,w)\,d_{b_1}\mathbf{W}_2(g_1(b_1),g_2(w)) \\ &\quad+k(x,w)\,\mathbf{W}_2(g_1(x),g_2(w)),\quad k\in L^2(B,\mathcal{B},\nu),\end{aligned}\tag{3.53}$$

where the stochastic integration is taken in the usual sense. Then, by setting

$$\phi(a_1,a_2,b_1,b_2)=1_{\{a_1>b_1\}}1_{\{a_2>b_2\}},$$

we obtain, via an easy application of (3.34), that

$$\begin{aligned}&\int_x^y\int_w^z \mu(da_1,da_2)\left(\int_x^y\int_w^z \mathbf{G}_\nu(db_1,db_2)\,1_{\{a_1>b_1\}}1_{\{a_2>b_2\}}\right)^2 \\ =&\int_x^y\int_w^z df_1(a_1)\,df_2(a_2)\,\mathbf{W}_2^2(g_1(a_1),g_2(a_2)) \\ &-f_1(y)\int_w^z df_2(a_2)\,\mathbf{W}_2^2(g_1(y),g_2(a_2)) \\ &-f_2(z)\int_x^y df_1(a_1)\,\mathbf{W}_2^2(g_1(a_1),g_2(z))+f_1(y)\,f_2(z)\,\mathbf{W}_2^2(g_1(y),g_2(z)).\end{aligned}\tag{3.54}$$

Likewise, we get

$$\begin{aligned}&\int_x^y\int_w^z \nu(db_1,db_2)\left(\int_x^y\int_w^z \mathbf{G}_\mu(da_1,da_2)\,1_{\{a_1>b_1\}}1_{\{a_2>b_2\}}\right)^2 \\ =&\int_x^y\int_w^z dg_1(b_1)\,dg_2(b_2)\,\mathbf{W}_1^2(f_1(b_1),f_2(b_2)) \\ &+g_1(x)\int_w^z dg_2(b_2)\,\mathbf{W}_1^2(f_1(x),f_2(b_2)) \\ &+g_2(w)\int_x^y dg_1(b_1)\,\mathbf{W}_1^2(f_1(b_1),f_2(w))+g_1(x)\,g_2(w)\,\mathbf{W}_1^2(f_1(x),f_2(w)).\end{aligned}\tag{3.55}$$

This, in turn, shows that for any standard bivariate Wiener process $\mathbf{W}(\cdot,\cdot)$ on $\mathbb{R}_+^2$,

$$\begin{aligned}&\int_x^y\int_w^z df_1(a_1)\,df_2(a_2)\,\mathbf{W}^2(g_1(a_1),g_2(a_2))-f_1(y)\int_w^z df_2(a_2)\,\mathbf{W}^2(g_1(y),g_2(a_2)) \\ &-f_2(z)\int_x^y df_1(a_1)\,\mathbf{W}^2(g_1(a_1),g_2(z))+f_1(y)\,f_2(z)\,\mathbf{W}^2(g_1(y),g_2(z)) \\ \overset{law}{=}&\int_x^y\int_w^z dg_1(b_1)\,dg_2(b_2)\,\mathbf{W}^2(f_1(b_1),f_2(b_2))+g_1(x)\int_w^z dg_2(b_2)\,\mathbf{W}^2(f_1(x),f_2(b_2)) \\ &+g_2(w)\int_x^y dg_1(b_1)\,\mathbf{W}^2(f_1(b_1),f_2(w))+g_1(x)\,g_2(w)\,\mathbf{W}^2(f_1(x),f_2(w)).\end{aligned}\tag{3.56}$$

In the special case where $g_1(x)=g_2(w)=f_1(y)=f_2(z)=0$, (3.56) may be rewritten into

$$\int_x^y\int_w^z \mathbf{W}^2(g_1(a_1),g_2(a_2))\,df_1(a_1)\,df_2(a_2)\overset{\text{law}}{=}\int_x^y\int_w^z \mathbf{W}^2(f_1(b_1),f_2(b_2))\,dg_1(b_1)\,dg_2(b_2).\tag{3.57}$$

A special case of (3.57) is obtained by taking $x = w = 0$, $y = z = 1$, $g_1(a) = g_2(a) = a^2$ and $f_1(b) = f_2(b) = \log(1/b)$. We so obtain the distributional identity

$$\left\{\frac{1}{2\sqrt{st}}\mathbf{W}\left(s^2,t^2\right):(s,t)\in[0,1]^2\right\}\overset{\text{Quad}}{\sim}\left\{\sqrt{st}\mathbf{W}\left(\log\frac{1}{s},\log\frac{1}{t}\right):(s,t)\in[0,1]^2\right\},$$

where the notation is the same as in the previous paragraph.

In the next subsection, we apply Theorem 3.3 to prove Theorems 3.1 and 3.2.

3.4 Proofs of Theorems 3.1 and 3.2, via a Projection Principle

We keep the notation and assumptions of §3.3. Consider a measurable space $(A,\mathcal{A},\mu)$, where μ is positive and σ-finite. For every closed subspace $H\subset L^2(A,\mathcal{A},\mu)$, we define $\pi[h(\cdot),H](a_1)$ to be the canonical projection operator, mapping $h\in L^2(A,\mathcal{A},\mu)$ into $\pi[h(\cdot),H](\cdot)\in H$. For every $h_2(a_1,a_2)\in L^2\left(A^2,\mathcal{A}^2,\mu\otimes\mu\right)$, we write

$$\begin{aligned}\pi_1[h_2,H](a_1,a_2) &= \pi[h_2(\cdot,a_2),H](a_1) &(3.58)\\ \pi_2[h_2,H](a_1,a_2) &= \pi[h_2(a_1,\cdot),H](a_2). &(3.59)\end{aligned}$$

Then, with the notation of §3.3, we may apply Theorem 3.3, in the case $(A,\mathcal{A},\mu)=(B,\mathcal{B},\nu)$, to obtain that for every $h_2\in L^2\left(A^2,\mathcal{A}^2,\mu\otimes\mu\right)$

$$\begin{aligned}&\int_A\left\{\int_A\pi_2[h_2,H](a_1,a_2)\,\mathbf{G}_\mu(da_2)\right\}^2\mu(da_1)\\ &\overset{\text{law}}{=}\int_A\left\{\int_A\pi_2[h_2,H](a_1,a_2)\,\mathbf{G}_\mu(da_1)\right\}^2\mu(da_2)\\ &\overset{\text{a.s}}{=}\int_A\left\{\pi\left(\int_A\mathbf{G}_\mu(da_1)\,h_2(a_1,\cdot)\right)(a_2)\right\}^2\mu(da_2),\end{aligned}\tag{3.60}$$

where $\pi\left(\int_A\mathbf{G}_\mu(da_1)\,h_2(a_1,\cdot)\right)(a_2)$ stands for the operator $\pi[\cdot,H]$ applied to the function

$$a_2\mapsto\int_A\mathbf{G}_\mu(da_1)\,h_2(a_1,a_2).\tag{3.61}$$

We note that formula (3.60) follows from the fact that, with probability 1, for every $k\in H^\perp$, where $H^\perp$ denotes the orthogonal of H in $L^2(A,\mathcal{A},\mu)$,

$$\begin{aligned}&\int_A\left\{\int_A\pi_2[h_2,H](a_1,a)\,\mathbf{G}_\mu(da_1)\right\}k(a)\,\mu(da) &(3.62)\\ =\;&\int_A\left\{\int_A\pi_2[h_2,H](a_1,a)\,k(a)\,\mu(da)\right\}\mathbf{G}_\mu(da_1)\\ =\;&\int_A\mathbf{G}_\mu(da_1)\int_A\mu(da)\,k(a)\,\pi[h_2(a_1,\cdot),H](a)=0.\end{aligned}$$

Since a_1 and a_2 play a symmetric role in (3.60) we have just proved the following proposition.

Proposition 3.1 (Projection principle) *Under the above notation and assumptions, we have*

$$\begin{aligned}&\int_A\left\{\int_A\pi_2[h_2,H](a_1,a_2)\,\mathbf{G}_\mu(da_2)\right\}^2\mu(da_1)\\ &\overset{\text{law}}{=}\int_A\left\{\pi\left(\int_A\mathbf{G}_\mu(da_1)\,h_2(a_1,\cdot)\right)(a_2)\right\}^2\mu(da_2)\end{aligned}\tag{3.63}$$

and

$$\int_A \left\{ \int_A \mathbf{G}_\mu (da_1)\, \pi_1 [h_2, H] (a_1, a_2) \right\}^2 \mu (da_2) \tag{3.64}$$
$$\stackrel{\text{law}}{=} \int_A \left\{ \pi \left(\int_A \mathbf{G}_\mu (da_2)\, h_2 (\cdot, a_2) \right) (a_1) \right\}^2 \mu (da_1) .$$

Now take a real valued kernel $K(\cdot,\cdot;\cdot,\cdot)$ on $[0,1]^4$, satisfying

$$\int_{[0,1]^4} K(s,t;a,b)^2 \, dsdtdadb < \infty. \tag{3.65}$$

Given a bivariate Wiener process (or Brownian sheet) $\mathbf{W}$ on $[0,1]^2$, and with K as above, we define two *Volterra sheets* V_K and $\widetilde{V}_K$ by setting

$$V_K(s,t) = \int_0^1 \int_0^1 K(s,t;a,b)\, \mathbf{W}(da,db), \quad (s,t) \in [0,1]^2, \tag{3.66}$$
$$\widetilde{V}_K(s,t) = \int_0^1 \int_0^1 K(a,b;s,t)\, \mathbf{W}(da,db), \quad (s,t) \in [0,1]^2. \tag{3.67}$$

Remark 3.2 *We note for further use that, whenever $\gamma > -1/2$ and $\delta > -1/2$, then the kernel*

$$K(s,t;a,b) = s^\gamma t^\delta 1_{\{a \le s\}} 1_{\{b \le t\}},$$

satisfies (3.65). Moreover, in this case, with the notation of §3.1, $V_K = \mathbf{W}^{(\gamma,\delta)}$ and $\widetilde{V}_K = \widetilde{\mathbf{W}}^{(\gamma,\delta)}$.

The following technical Corollary of Proposition 3.1 will turn out to imply Theorems 3.1 and 3.2.

Proposition 3.2 *Under the above notation and assumptions, let K satisfy (3.65). Then, for every closed linear subspace H of $L^2\big([0,1]^2, \mathcal{B}\left([0,1]^2\right), dsdt\big)$, and for every $u \in \mathbb{R}$,*

$$\mathbb{E}\left[\exp\Big(-\frac{u^2}{2} \int_0^1 \int_0^1 V_K^2(s,t)\, dsdt \Big) \Big| \mathbf{W}(h) = 0, \quad h \in H^\perp \right] \tag{3.68}$$
$$= \mathbb{E}\left[\exp\Big(-\frac{u^2}{2} \int_0^1 \int_0^1 \Big(\pi\big[\widetilde{V}_K, H\big](s,t) \Big)^2 \Big) dsdt \right],$$

where $\pi\big[\widetilde{V}_K, H\big](s,t)$ denotes the orthogonal projection on H of the random function

$$(s,t) \mapsto \widetilde{V}_K(s,t).$$

Proof of Theorem 3.1. We apply Proposition 3.2 to the special case where

$$K(s,t;a,b) = s^\gamma t^\delta 1_{\{a \le s\}} 1_{\{b \le t\}},$$

and $\gamma, \delta > -1/2$. In this case, via Remark 3.2, the Volterra sheet V_K coincides with $\mathbf{W}^{(\gamma,\delta)}$, and likewise, $\widetilde{V}_K$ with $\widetilde{\mathbf{W}}^{(\gamma,\delta)}$. Now consider the three Hilbert subspaces H_1, H_2 and H_3, of $L^2\big([0,1]^2, \mathcal{B}\big([0,1]^2\big), dsdt\big)$, defined by

$$H_1 = \left\{h : \int_0^1\int_0^1 h(x,y)\,dxdy = 0\right\}, \tag{3.69}$$

$$H_2 = \left\{h : \int_0^1\int_0^\lambda h(x,y)\,dxdy = \int_0^\lambda\int_0^1 h(x,y)\,dxdy = 0, \quad \forall\lambda\in[0,1]\right\}, \tag{3.70}$$

$$H_3 = \left\{h : \int_0^1\int_0^\lambda h(x,y)\,dxdy = 0, \quad \forall\lambda\in[0,1]\right\}. \tag{3.71}$$

Note for further use that $H_2 \subset H_3 \subset H_1$. It is easily seen that, for every real $u\in\mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_0^1\int_0^1 \mathbf{W}^{(\gamma,\delta)}(s,t)^2\,dsdt\right)\Big|\mathbf{W}(h)=0, \quad h\in H_1^\perp\right]\\ =\ &\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_0^1\int_0^1 \mathbf{B}^{(\gamma,\delta)}(s,t)^2\,dsdt\right)\right],\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_0^1\int_0^1 \mathbf{W}^{(\gamma,\delta)}(s,t)^2\,dsdt\right)\Big|\mathbf{W}(h)=0, \quad h\in H_2^\perp\right]\\ =\ &\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_0^1\int_0^1 \mathbf{B}_*^{(\gamma,\delta)}(s,t)^2\,dsdt\right)\right],\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_0^1\int_0^1 \mathbf{W}^{(\gamma,\delta)}(s,t)^2\,dsdt\right)\Big|\mathbf{W}(h)=0, \quad h\in H_3^\perp\right]\\ =\ &\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2} \mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}(s,t)^2\,dsdt\right)\right].\end{aligned}$$

Since for every $k\in L^2\big([0,1]^2,\mathcal{B}\big([0,1]^2\big),dsdt\big)$,

$$\begin{aligned}\pi[k,H_1](s,t) &= k(s,t) - \int_0^1\int_0^1 k(u,v)\,dudv,\\ \pi[k,H_2](s,t) &= k(s,t) - \int_0^1 k(s,v)\,dv\\ &\quad - \int_0^1 k(u,t)\,du + \int_0^1\int_0^1 k(u,v)\,dudv,\\ \pi[k,H_3](s,t) &= k(s,t) - \int_0^1 k(u,t)\,du,\end{aligned}$$

so that the conclusions (3.16)-(3.21) of Theorem 3.1 readily follow from Proposition 3.2, when combined with 3.11), and the change of variables $x = 1 - s^{2\gamma+1}$, $y = 1 - t^{2\delta+1}$ $\square$.

Proof of Theorem 3.2 The result follows from an application of Proposition 3.2 to the kernel

$$K(s,t;a,b) = a^\gamma b^\delta 1_{\{a\geq s\}} 1_{\{b\geq t\}},$$

with $\gamma > -1/2$ and $\delta > -1/2$. In view of Remark 3.2, V_K and $\widetilde{V}_K$ coincide in this case with $\widetilde{\mathbf{W}}^{(\gamma,\delta)}$ and $\mathbf{W}^{(\gamma,\delta)}$, respectively. In particular, as already pointed out, one has obviously the equalities

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2}\int_0^1\int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)^2\,dsdt\Big)\Big|\mathbf{W}(h)=0,\quad h\in H_1^\perp\Big]\\
=\;&\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2}\int_0^1\int_0^1\Big\{\widetilde{\mathbf{W}}^{(\gamma,\delta)}(a,b)-\frac{\left(1-a^{\gamma+1}\right)\left(1-b^{\delta+1}\right)}{(\gamma+1)(\delta+1)}\mathbf{W}(1,1)\Big\}^2 dadb\Big)\Big],\\
&\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2}\int_0^1\int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)^2\,dsdt\Big)\Big|\mathbf{W}(h)=0,\quad h\in H_2^\perp\Big]\\
=\;&\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2}\int_0^1\int_0^1\Big\{\widetilde{\mathbf{W}}^{(\gamma,\delta)}(a,b)-\frac{1-a^{\gamma+1}}{\gamma+1}\int_b^1 t^\delta\mathbf{W}(1,dt)\\
&-\frac{1-b^{\delta+1}}{\delta+1}\int_a^1 s^\gamma\mathbf{W}(ds,1)+\frac{\left(1-a^{\gamma+1}\right)\left(1-b^{\delta+1}\right)}{(\gamma+1)(\delta+1)}\mathbf{W}(1,1)\Big\}^2 dadb\Big)\Big],\\
&\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2}\int_0^1\int_0^1 \widetilde{\mathbf{W}}^{(\gamma,\delta)}(s,t)^2\,dsdt\Big)\Big|\mathbf{W}(h)=0,\quad h\in H_3^\perp\Big]\\
=\;&\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2}\int_0^1\int_0^1\Big\{\widetilde{\mathbf{W}}^{(\gamma,\delta)}(a,b)-\frac{1-a^{\gamma+1}}{\gamma+1}\int_b^1 t^\delta\mathbf{W}(1,dt)\Big\}^2 dadb\Big)\Big],
\end{aligned}$$

and the conclusion is straightforward.□

Remark 3.3 *(A counterexample) It is natural to ask whether different representations of the same Gaussian process may lead to different distributional equalities of the form* (3.34)*. The following example gives a positive answer to this question. Let W denote a standard Wiener process on $[0,1]$. Now, observe that the following distributional equality holds between processes on $[0,1]$:*

$$W(t)\overset{\text{law}}{=}W^{(1)}(t):=W(t)-\int_0^t\frac{W(1)-W(a)}{1-a}da\overset{\text{a.s.}}{=}\int_0^1\left[1_{\{s\le t\}}-\int_0^{s\wedge t}\frac{da}{1-a}\right]dW(s). \tag{3.72}$$

Next, we infer from (3.34) *that*

$$\int_0^1\left\{W(s)-\int_0^1 W(u)du\right\}^2 ds\overset{\text{law}}{=}\int_0^1 B^2(s)ds$$

where $\{B(t):0\le t\le 1\}$ denotes a standard (univariate) Brownian bridge. Now, since via (3.72),

$$\int_0^1 W^{(1)}(s)ds\overset{\text{a.s.}}{=}\int_0^1(1-2s)\,dW(s),$$

we may also write

$$\begin{aligned}
&\int_0^1\left\{W(t)-\int_0^1 W(u)du\right\}^2 dt\overset{\text{law}}{=}\int_0^1\left\{W^{(1)}(t)-\int_0^1 W^{(1)}(s)ds\right\}^2 dt\\
&\overset{\text{a.s.}}{=}\int_0^1\left\{\int_0^1\left[1_{\{s\le t\}}-\int_0^{s\wedge t}\frac{da}{1-a}-1+2s\right]dW(s)\right\}^2 dt\\
&\overset{\text{law}}{=}\int_0^1\left\{\int_0^1\left[1_{\{s\le t\}}-\int_0^{s\wedge t}\frac{da}{1-a}-1+2s\right]dW(t)\right\}^2 ds\\
&=\int_0^1 ds\left[W^{(1)}(s)-2\left(W(s)-sW(1)\right)\right]^2.
\end{aligned}$$

To conclude, we observe that the process

$$C(s):=W^{(1)}(s)-2\left(W(s)-sW(1)\right),\quad s\in[0,1], \tag{3.73}$$

is obviously not a Brownian bridge (it fulfills $C(0)=0$, but $C(1)=W^{(1)}(1)\neq 0$, a.s.).

4 Karhunen-Loève Expansions and Laplace Transforms

4.1 Statement of the Main Results

The aim of this section is, for each $\gamma > -1$ and $\delta > -1$, to evaluate explicit expressions for the Laplace transform of quadratic functionals of the weighted processes $\mathbf{W}^{(\gamma,\delta)}$, $\mathbf{B}^{(\gamma,\delta)}$, $\mathbf{B}_*^{(\gamma,\delta)}$ and $\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}$, as defined in §3.1. In particular, we will show that, our results extend the results obtained in the univariate case by [36] when $\gamma = \delta = 0$ and by [16] for $\gamma > -1$ and $\delta > -1$. We will now state our main theorems, whereas the remainder of the section is devoted to their formal proof, mainly through the classical technique of Karhunen-Loève [KL] expansions (see, e.g., Ch.1 in [32]).

To achieve our goal, for any real $\nu > -1$ define $J_\nu(\cdot)$ to be the Bessel function of first order and index ν (see, e.g., [35], or [16], for any unexplained definition) and denote by $\{z_{\nu,k} : k \geq 1\}$ the ordered sequence of positive zeros of J_ν. For any $\gamma > -1$, and by setting $\nu = 1/(2(\gamma+1))$, we define

$$\begin{aligned} \lambda_{k,\gamma} &= \left(\frac{2\nu}{z_{\nu-1,k}}\right)^2, \quad e_{k,\gamma}(t) = t^{\frac{1}{2\nu}-\frac{1}{2}}\left(\frac{J_\nu\left(z_{\nu-1,k}t^{\frac{1}{2\nu}}\right)}{\sqrt{\nu}J_\nu\left(z_{\nu-1,k}\right)}\right), \quad t \in (0,1], \quad k \geq 1 \\ \zeta_{k,\gamma} &= \left(\frac{2\nu}{z_{\nu,k}}\right)^2, \quad h_{k,\gamma}(t) = t^{\frac{1}{2\nu}-\frac{1}{2}}\left(\frac{J_\nu\left(z_{\nu,k}t^{\frac{1}{2\nu}}\right)}{\sqrt{\nu}J_{\nu-1}\left(z_{\nu,k}\right)}\right), \quad t \in (0,1], \quad k \geq 1. \end{aligned} \tag{4.1}$$

Then, we have the following

Theorem 4.1 *For $\gamma, \delta > -1$, let $\mathbf{W}^{(\gamma,\delta)}$, $\mathbf{B}_*^{(\gamma,\delta)}$ and $\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}$ be defined as above. Then,*

(i) the KL expansions of $\mathbf{W}^{(\gamma,\delta)}$, $\mathbf{B}_*^{(\gamma,\delta)}$ and $\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}$ are respectively given by

$$\mathbf{W}^{(\gamma,\delta)}(s,t) = \sum_{j,k=1}^{\infty} \omega_{j,k}\sqrt{\lambda_{j,\gamma}\lambda_{k,\delta}}e_{j,\gamma}(s)\,e_{k,\delta}(t), \quad 0 < s,t \leq 1,$$

$$\mathbf{B}_*^{(\gamma,\delta)}(s,t) = \sum_{j,k=1}^{\infty} \theta_{j,k}\sqrt{\zeta_{j,\gamma}\zeta_{k,\delta}}h_{j,\gamma}(s)\,h_{k,\delta}(t), \quad 0 < s,t \leq 1$$

and

$$\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}(s,t) = \sum_{j,k=1}^{\infty} \xi_{j,k}\sqrt{\zeta_{j,\gamma}\lambda_{k,\delta}}h_{j,\gamma}(s)\,e_{k,\delta}(t), \quad 0 < s,t \leq 1$$

where $\{\omega_{j,k}\}$, $\{\theta_{j,k}\}$ and $\{\xi_{j,k}\}$ are three doubly indexed sequences of independent standard Gaussian random variables and the $\lambda_{\cdot,\cdot}$ and $\zeta_{\cdot,\cdot}$ are defined according to (4.1).

(ii) For any $u \in \mathbb{C}$ with $\mathrm{Re}(u) \geq 0$

$$\begin{aligned} \mathbb{E}\left[\exp\left(-u\int_{[0,1]^2}\left(\mathbf{W}^{(\gamma,\delta)}(s,t)\right)^2 dsdt\right)\right] &= \prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(\frac{1}{1+2u\lambda_{j,\gamma}\lambda_{k,\delta}}\right)^{\frac{1}{2}} \\ \mathbb{E}\left[\exp\left(-u\int_{[0,1]^2}\left(\mathbf{B}_*^{(\gamma,\delta)}(s,t)\right)^2 dsdt\right)\right] &= \prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(\frac{1}{1+2u\zeta_{j,\gamma}\zeta_{k,\delta}}\right)^{\frac{1}{2}} \\ \mathbb{E}\left[\exp\left(-u\int_{[0,1]^2}\left(\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}(s,t)\right)^2 dsdt\right)\right] &= \prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(\frac{1}{1+2u\zeta_{j,\gamma}\lambda_{k,\delta}}\right)^{\frac{1}{2}} \end{aligned}$$

Remark 4.1 *Note that, point (i) of Theorem 4.1 implies that, for every i, j, l, m,*

$$\begin{aligned} &\int_0^1\int_0^1 e_{j,\gamma}(s)\,e_{k,\delta}(t)\,e_{l,\gamma}(s)\,e_{m,\delta}(t)\,dtds \\ = &\int_0^1\int_0^1 h_{j,\gamma}(s)\,h_{k,\delta}(t)\,h_{l,\gamma}(s)\,h_{m,\delta}(t)\,dtds \\ = &\int_0^1\int_0^1 h_{j,\gamma}(s)\,e_{k,\delta}(t)\,h_{l,\gamma}(s)\,e_{m,\delta}(t)\,dtds = \delta_{j,l}\times\delta_{k,m}, \end{aligned} \tag{4.2}$$

where δ is the Kronecker symbol.

When applied to the case $\gamma = \delta = 0$, the above results give some interesting extensions of classic computations due to P. Lévy (see [36], but also [27]). To this end, introduce, the following remarkable class of functions, defined for every $a \in \mathbb{R}$,

1. $C(a) = \prod_{j=1}^{\infty} \cosh\left(\frac{a}{j\pi}\right)$;
2. $C_{\text{odd}}(a) = \prod_{j=0}^{\infty} \cosh\left[\frac{a}{(2j+1)\pi}\right]$;
3. $C_{\text{even}}(a) = \prod_{j=1}^{\infty} \cosh\left[\frac{a}{2j\pi}\right] = C\left(\frac{a}{2}\right)$;
4. $S(a) = \prod_{j=1}^{\infty}\left[\pi j \sinh\left(\frac{a}{\pi j}\right)/a\right]$;
5. $S_{\text{even}}(a) = \prod_{j=1}^{\infty}\left[\pi 2j \sinh\left(\frac{a}{\pi 2j}\right)/a\right] = S(a/2)$;
6. $S_{\text{odd}}(a) = \prod_{j=1}^{\infty}\left[\pi(2j-1)\sinh\left(\frac{a}{\pi(2j-1)}\right)/a\right] = C(a/2)$, where the last equality comes from the relations
$$\begin{aligned} S(a) &= S_{\text{odd}}(a)\,S_{\text{even}}(a) \\ \sinh\left(\frac{a}{\pi j}\right) &= 2\cosh\left(\frac{a}{2\pi j}\right)\sinh\left(\frac{a}{2\pi j}\right); \end{aligned}$$
7. $\mathcal{T}(a) = \sum_{j=0}^{\infty}\left\{\tanh\left(\frac{2a}{(2j+1)\pi}\right)\left[(2j+1)\pi\right]^{-1}\right\}$.

Remark 4.2 *Observe that*

$$\frac{d}{da}\log\left[C_{odd}(a)\right] = \frac{d}{da}\sum_{j=0}^{\infty}\log\left[\cosh\left(\frac{2a}{(2j+1)\pi}\right)\right] = 2\mathcal{T}(a) \tag{4.3}$$

and therefore

$$\frac{C'_{odd}(a)}{C_{odd}(a)} = 2\mathcal{T}(a). \tag{4.4}$$

The following result provides an exhaustive characterization of the case $\gamma = \delta = 0$. Note that, to simplify the notation, we will write $e_{j,0} = e_j$ and $h_{k,0} = h_k$ for every $j, k \geq 1$ where the $e_{\cdot,\cdot}$ and $h_{\cdot,\cdot}$ are defined in (4.1).

Theorem 4.2 *Let* $\mathbf{W}$ *be a standard Brownian sheet, then, for every* $a>0$*,*

(i) $\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\right]=\{C_{odd}(2a)\}^{-\frac{1}{2}}$*;*

(ii) for every real x*,*

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\mid \mathbf{W}(1,1)=x\right]\\ =\ &\left(C_{odd}(2a)\,\frac{4\mathcal{T}(a)}{a}\right)^{-\frac{1}{2}}\exp\left[-\frac{x^2}{2}\left(\frac{a}{4\mathcal{T}(a)}-1\right)\right];\end{aligned}$$

(iii) for every continuous, deterministic function $y(s,t)$ *on* $[0,1]^2$

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\mid \mathbf{W}(1,\cdot)=y(1,\cdot),\mathbf{W}(\cdot,1)=y(\cdot,1)\right]\\ =\ &\{S(a)\}^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{k,j\geq 1}\langle y^*,h_kh_j\rangle^2_{L^2([0,1]^2)}\frac{\left(akj\pi^2\right)^2}{a^2+\left(kj\pi^2\right)^2}\right]\end{aligned}$$

where $y^*(s,t)=sy(1,t)+ty(s,1)-sty(1,1)$ *by definition;*

(iv) for every deterministic and square integrable function ϕ *on* $[0,1]$*,*

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\mid \mathbf{W}(\cdot,1)=\phi(\cdot)\right]\\ =\ &\{S_{odd}(2a)\}^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{j\geq 0}\langle\phi,e_j\rangle^2_{L^2([0,1])}\left[\frac{(2j+1)\pi a}{2}\coth\left(\frac{2a}{(2j+1)\pi}\right)-1\right]\right].\end{aligned}$$

Remark 4.3 *Observe that point (ii) of Theorem 4.2 gives the analogue, in the case of a Brownian sheet, of the well known formula due to P. Lévy (see [36])*

$$\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_0^1 W^2(t)\,dt\right)\mid W(1)=x\right]=\left(\frac{a}{\sinh a}\right)^{\frac{1}{2}}\exp\left[-\frac{x^2}{2}(a\coth a-1)\right],\tag{4.5}$$

where $x\in\mathbb{R}$*, and* $\{W(t):t\geq 0\}$ *is a standard, real valued Brownian motion.*

The following corollary is obtained by setting $x=y(\cdot,\cdot)=\phi(\cdot)=0$, in Theorem 4.2.

Proposition 4.1 *For any* $u>0$*,*

(i) $\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2}\mathbf{B}(s,t)^2\,dsdt\right)\right]=\left(C_{odd}(2u)\,\frac{4\mathcal{T}(u)}{u}\right)^{-\frac{1}{2}}$*;*

(ii) $\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2}\mathbf{B}_*(s,t)^2\,dsdt\right)\right]=\{S(u)\}^{-\frac{1}{2}}$*;*

(iii) $\mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{[0,1]^2}\mathbf{B}_\natural(s,t)^2\,dsdt\right)\right]=\{S_{odd}(2u)\}^{-\frac{1}{2}}$*.*

We now use the same notation as in Paragraph 2.2. In particular, one immediate consequence of Corollaries 2.2 and 2.3, as well as Theorem 4.1 and Proposition 4.1 is a precise description of the limiting behavior of some of the statistics introduced in Sections 1 and 2.

Proposition 4.2 *Let the notation of Section 2 prevail. Then, for every $\gamma, \delta > -\frac{1}{2}$ and for every $u > 0$,*

$$\begin{aligned} \lim_{n\to+\infty} \mathbb{E}\left[\exp\left(-u\Omega^2_{n;C;\gamma,\delta}\right)\right] &= \lim_{n\to+\infty} \mathbb{E}\left[\exp\left(-u\Omega^2_{n;T;\gamma,\delta}\right)\right] \\ &= \prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(\frac{1}{1+2u\zeta_{j,\gamma}\zeta_{k,\delta}}\right)^{\frac{1}{2}}. \end{aligned} \tag{4.6}$$

In particular,

$$\begin{aligned} \lim_{n\to+\infty} \mathbb{E}\left[\exp\left(-u\Omega^2_{n;C}\right)\right] &= \lim_{n\to+\infty} \mathbb{E}\left[\exp\left(-u\Omega^2_{n;T}\right)\right] \\ &= \left\{S\left(\sqrt{2u}\right)\right\}^{-\frac{1}{2}}. \end{aligned} \tag{4.7}$$

4.2 KL Expansions and Proof of Theorem 4.1

The reader is referred to [1, Chapter III] or [32, Chapter 1] for any definition or proof concerning KL expansions.

To prove Theorem 4.1, we shall use once again, for every $\gamma > -1$, the two processes

$$W_\gamma(t) = t^\gamma W(t), \quad t \in (0,1], \quad W_\gamma(0) = 0, \tag{4.8}$$
$$B_\gamma(t) = t^\gamma B(t), \quad t \in (0,1], \quad B_\gamma(0) = 0, \tag{4.9}$$

as defined in (3.1) and (3.2). We use the following result, due to Deheuvels and Martynov (see [16, Theorems 1.3 and 1.4]).

Proposition 4.3 *For $\gamma > -1$, let the processes W_γ and B_γ be defined as above. Then, the KL expansions of W_γ and B_γ are respectively given by*

$$W_\gamma(t) = \sum_{k=1}^{\infty} \omega_k \sqrt{\lambda_{k,\gamma}}\, e_{k,\gamma}(t), \quad t \in [0,1] \tag{4.10}$$

and

$$B_\gamma(t) = \sum_{k=1}^{\infty} \theta_k \sqrt{\zeta_{k,\gamma}}\, h_{k,\gamma}(t), \quad t \in [0,1] \tag{4.11}$$

where $\{\omega_k\}$ and $\{\theta_k\}$ are two sequences of independent standard Gaussian random variables and, by setting $\nu = 1/(2(\gamma+1))$, the sequences $\{\lambda_{k,\gamma}\}$, $\{e_{k,\gamma}\}$, $\{\gamma_{k,\gamma}\}$ and $\{h_{k,\gamma}\}$ are given in (4.1).

Remark 4.4 *Note that the following relation holds for every $\gamma > -1$ and every $j, k \geq 1$,*

$$\int_0^1 e_{j,\gamma}(t)\, e_{k,\gamma}(t)\, dt = \int_0^1 h_{j,\gamma}(t)\, h_{k,\gamma}(t)\, dt = \delta_{j,k}$$

where δ is the Kronecker symbol.

In particular, the proof of Theorem 4.1 can be easily deduced from Proposition 4.3, by using an elementary result about KL expansions of bivariate processes. To this end, let $X_a = \{X_a(t) : t \in [0,1]\}$, $a = 1, 2$, be two centered Gaussian processes, and suppose moreover that, for $a = 1, 2$, the covariance function

$$(s,t) \mapsto R_a(s,t) = \mathbb{E}\left[X_a(s) X_a(t)\right], \quad (s,t) \in [0,1]^2, \tag{4.12}$$

is continuous on $[0,1]^2$. The KL expansions of X_1 and X_2 will be denoted by

$$X_a(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_{i,a}}\xi_{i,a} e_{i,a}(t), \quad t \in [0,1], \tag{4.13}$$

where, for $a = 1, 2$, $\{\xi_{i,a}\}$ is a sequence of independent standard Gaussian random variables, and $\{\lambda_{i,a}\}$ and $\{e_{i,a}\}$ are respectively the eigenvalues and eigenfunctions associated to the operator from $L^2([0,1])$ to itself $f \mapsto \int f(x) R_a(x, \cdot)\, dx$.

Lemma 4.1 *Let the process $Y = \left\{Y(s,t) : (s,t) \in [0,1]^2\right\}$ be Gaussian, centered and such that*

$$\mathbb{E}\left[Y(s,t) Y(u,v)\right] = R_1(s,u) \times R_2(t,v) \tag{4.14}$$

for every $(s,t,u,v) \in [0,1]^4$, where R_1 and R_2 are defined in (4.12). Then, the KL expansion of Y is given by

$$Y(s,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \xi_{i,j} \sqrt{\lambda_{i,1}\lambda_{j,2}} e_{i,1}(s) e_{j,2}(t), \quad (s,t) \in [0,1]^2, \tag{4.15}$$

where $\{\xi_{i,j}\}$ is a doubly indexed family of independent, standard Gaussian random variables, and the sequences $\{\lambda_{i,a}\}$ and $\{e_{i,a}\}$, $a = 1, 2$, are defined in (4.13).

End of the Proof of Theorem 4.1 Note that, for $\gamma, \delta > -1$, the covariance functions of $\mathbf{W}^{(\gamma,\delta)}$, $\mathbf{B}_*^{(\gamma,\delta)}$ and $\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}$ are respectively given by

$$\begin{aligned} \mathbb{E}\left[\mathbf{W}^{(\gamma,\delta)}(s,t)\, \mathbf{W}^{\gamma,\delta}(u,v)\right] &= (su)^\gamma (s \wedge u) (tv)^\delta (t \wedge v) \\ &= \mathbb{E}\left[(su)^\gamma W(s) W(u)\right] \mathbb{E}\left[(tv)^\delta W(t) W(v)\right], \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left[\mathbf{B}_*^{(\gamma,\delta)}(s,t)\, \mathbf{B}_*^{\gamma,\delta}(u,v)\right] &= (su)^\gamma (s \wedge u - su) (tv)^\delta (t \wedge v - tv) \\ &= \mathbb{E}\left[(su)^\gamma B(s) B(u)\right] \mathbb{E}\left[(tv)^\delta B(t) B(v)\right] \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left[\mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}(s,t)\, \mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}(u,v)\right] &= (su)^\gamma (s \wedge u - su) (tv)^\delta (t \wedge v) \\ &= \mathbb{E}\left[(su)^\gamma B(s) B(u)\right] \mathbb{E}\left[(tv)^\delta W(t) W(v)\right] \end{aligned}$$

where, as usual, $\{W(t) : t \geq 0\}$ and $\{B(t) : 0 \leq t \leq 1\}$ are respectively a standard Brownian motion and a standard Brownian bridge from 0 to 0. The first part of Theorem 4.1 can now be immediately deduced from Proposition 4.3 and Lemma 4.1 in the special cases: (a) $R_1(s,t) = R_2(s,t) = (s \wedge t)$, (b) $R_1(s,t) = R_2(s,t) = [(s \wedge t) - st]$, and (c) $R_1(s,t) = [(s \wedge t) - st]$ and $R_2(s,t) = (s \wedge t)$. Point (ii) of Theorem 4.1 derives from standard calculations.

Remark 4.5 *(a) Clearly, the functions s^{γ} and t^{δ} play an immaterial role in the previous discussion, and we might as well have replaced them by general functions $\phi(s)$ and $\psi(t)$ subject to the condition that*

$$\int_0^1 t\phi(t)^2\,dt < +\infty \quad \text{and} \quad \int_0^1 t\psi(t)^2\,dt < +\infty \tag{4.16}$$

or

$$\int_0^1 t(1-t)\,\phi(t)^2\,dt < +\infty \quad \text{and} \quad \int_0^1 t(1-t)\,\psi(t)^2\,dt < +\infty. \tag{4.17}$$

More precisely, whenever the functions ϕ and ψ verify (4.16), one can show that, for every $u \in \mathbb{C}$ with $\mathrm{Re}(u) \geq 0$,

$$\mathbb{E}\left[\exp\left(-u\int_{[0,1]^2}(\phi(s)\,\psi(t)\,\mathbf{W}(s,t))^2\,dsdt\right)\right] = \prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(\frac{1}{1+2u\lambda_{j,\phi}\lambda_{k,\psi}}\right)^{\frac{1}{2}} \tag{4.18}$$

where $\{\lambda_{j,\phi} : j \geq 1\}$ (resp. $\{\lambda_{k,\psi} : k \geq 1\}$) denotes the ordered sequence of eigenvalues associated to the Hilbert-Schmidt operator on $L^2([0,1], tdt)$

$$f \mapsto \int_0^1 K(s,\cdot)\,f(s)\,ds, \tag{4.19}$$

with kernel $K(s,t) = \phi(s)\,\phi(t)\,(s \wedge t)$ (resp. $K(s,t) = \psi(s)\,\psi(t)\,(s \wedge t)$). Likewise, whenever ϕ and ψ verify (4.17), for every $u > 0$,

$$\mathbb{E}\left[\exp\left(-u\int_{[0,1]^2}(\phi(s)\,\psi(t)\,\mathbf{B}_*(s,t))^2\,dsdt\right)\right] = \prod_{j,k}\left(\frac{1}{1+2u\gamma_{j,\phi}\gamma_{k,\psi}}\right)^{\frac{1}{2}} \tag{4.20}$$

where $\{\gamma_{j,\phi} : j \geq 1\}$ (resp. $\{\gamma_{k,\psi} : k \geq 1\}$) denotes the ordered sequence of eigenvalues associated to the Hilbert-Schmidt operator on $L^2([0,1], t(1-t)\,dt)$, with kernel $K(s,t) = \phi(s)\,\phi(t)\,(s \wedge t - st)$ (resp. $K(s,t) = \psi(s)\,\psi(t)\,(s \wedge t - st)$. Similar conclusions can be obtained for weighted asymmetric bivariate Brownian bridge.

(b) Let ϕ and ψ be such that (4.16) is verified. Then, from (4.18) we infer the relation

$$\begin{aligned}\int_{[0,1]^2}(\phi(s)\,\psi(t)\,\mathbf{W}(s,t))^2\,dsdt &\overset{law}{=} \sum_j \lambda_{j,\phi}\int_0^1 dt\,(\psi(t)\,W_j(t))^2 \\ &\overset{law}{=} \sum_j \lambda_{j,\psi}\int_0^1 dt\,(\phi(t)\,W_j(t))^2\end{aligned} \tag{4.21}$$

where $\{W_j(t) : t \geq 0\}$, $j = 1, 2, \ldots$, is a sequence of independent standard Wiener processes and we keep the notation of the previous remark. Similarly, if (4.17) is satisfied, (4.20) gives

$$\begin{aligned}\int_{[0,1]^2}(\phi(s)\,\psi(t)\,\mathbf{B}_*(s,t))^2\,dsdt &\overset{law}{=} \sum_j \gamma_{j,\phi}\int_0^1 dt\,(\psi(t)\,B_j(t))^2 \\ &\overset{law}{=} \sum_j \gamma_{j,\psi}\int_0^1 dt\,(\phi(t)\,B_j(t))^2\end{aligned} \tag{4.22}$$

where $\{B_j(t) : 0 \leq t \leq 1\}$, $j = 1, 2, \ldots$, is a sequence of independent standard Brownian bridges from 0 to 0. Moreover, since the above formulae hold for $\phi = \psi$ satisfying either (4.16) or (4.17), a Laplace *transform argument yields immediately*

$$\left\{\int_{[0,1]^2}(\phi(s)\,\mathbf{W}(s,t))^2\,ds : t\in[0,1]\right\} \overset{law}{=} \left\{\sum_j \lambda_{j,\phi}(W_j(t))^2 : t\in[0,1]\right\} \tag{4.23}$$

$$\left\{\int_{[0,1]^2}(\phi(s)\,\mathbf{B}_*(s,t))^2\,ds : t\in[0,1]\right\} \overset{law}{=} \left\{\sum_j \gamma_{j,\phi}(B_j(t))^2 : t\in[0,1]\right\} \tag{4.24}$$

where the first identity holds for any $\phi\in L^2([0,1],tdt)$, and the second for any $\phi\in L^2([0,1],t(1-t)\,dt)$.

4.3 Proof of Theorem 4.2

In what follows, we note $\{\lambda_k : k\geq 1\}$ and $\{e_k : k\geq 1\}$ respectively the ordered sequence of eigenvalues and the corresponding sequence of eigenfunctions associated to the Hilbert-Schmidt operator (4.19) with kernel $K(s,t)=s\wedge t$. Observe that, in the notation of (4.1), for every $k\geq 1$, $\lambda_k=\lambda_{k,0}$ and $e_k=e_{k,0}$, and in particular (see e.g. [1, p. 77])

$$\begin{aligned}\lambda_k &= \lambda_{k,0}=\left(\frac{2}{(2k-1)\pi}\right)^2, \quad k=1,2,...\\ e_k(t) &= e_{k,0}(t)=\sqrt{2}\sin\left[\left(k-\frac{1}{2}\right)\pi t\right], \quad t\in(0,1], \quad k=1,2,...\end{aligned}$$

[Proof of Theorem 4.2-(i)] We apply for instance (4.21) in the case $\phi=\psi=1$ to obtain that for every real a

$$\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\right]=\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\sum_{k\geq 1}\lambda_k\int_0^1 W_k(t)^2\,dt\right)\right]$$

where $\{W_k\}$ is a sequence of independent, standard Wiener processes on $[0,1]$. Now, it is well known (see for instance [16]) that for any standard Wiener processes $\{W(t):t\geq 0\}$

$$\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_0^1 W(t)^2\,dt\right)\right]=\{\cosh(a)\}^{-\frac{1}{2}}$$

and therefore

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\right] &= \prod_{k\geq 1}\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\lambda_k\int_0^1 W_k(t)^2\,dt\right)\right]\\ &= \prod_{k\geq 1}\cosh\left(\frac{2a}{(2k-1)\pi}\right)^{-\frac{1}{2}}\\ &= C_{\mathrm{odd}}(2a)^{-\frac{1}{2}}.\end{aligned}$$

[Proof of Theorem 4.2-(ii)] According to Theorem 4.1-(i), there exists a sequence of i.i.d. standard Gaussian random variables $\{\omega_{i,j}\}$ such that

$$\mathbf{W}(s,t)=\sum_{i\geq 1}\sum_{j\geq 1}\sqrt{\lambda_i\lambda_j}\,\omega_{i,j}e_i(s)\,e_j(t).$$

Now define, for $i=1,2,...,$

$$W_i(t)=\sum_{j\geq 1}\sqrt{\lambda_j}\,\omega_{i,j}e_j(t), \quad t\in[0,1], \tag{4.25}$$

and observe that the sequence $\{W_i : i \geq 1\}$ is composed of independent, standard Brownian motions on $[0,1]$. In particular, from the above representation of $\mathbf{W}$ we obtain

$$\begin{aligned}\int_{[0,1]^2} \mathbf{W}(s,t)^2\, dsdt &= \sum_{i\geq 1}\sum_{j\geq 1} \lambda_i \lambda_j \omega_{i,j}^2 \\ &= \sum_{i\geq 1} \lambda_i \int_0^1 W_i(t)^2\, dt\end{aligned}$$

and also

$$\begin{aligned}\mathbf{W}(1,1) &= \sum_{i\geq 1}\sum_{j\geq 1} \sqrt{\lambda_i\lambda_j}\omega_{i,j} e_i(1)\, e_j(1) \\ &= \sum_{i\geq 1} \sqrt{\lambda_i} e_i(1)\, W_i(1) \\ &= \sum_{i\geq 1} \sqrt{2\lambda_i}\,(-1)^{i+1}\, W_i(1)\,.\end{aligned}$$

Now, the above formulae yield, for any $a, b \in \Re$

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2} \mathbf{W}(s,t)^2\, dsdt\right)\exp\left(b\mathbf{W}(1,1)\right)\right] \\ &= \prod_{i\geq 1} \mathbb{E}\left[\exp\left(-\frac{a^2}{2}\lambda_i \int_0^1 W_i(t)^2\, dt\right)\exp\left(b\sqrt{2\lambda_i}\,(-1)^{i+1}\, W_i(1)\right)\right] \\ &= \prod_{i\geq 1}\left(\frac{a\sqrt{\lambda_i}}{\sinh\left(a\sqrt{\lambda_i}\right)}\right)^{\frac{1}{2}} \mathbb{E}\left\{\exp\left[-\left(W_i(1)\right)^2 \frac{\mu_i}{2}\right]\exp\left(b\sqrt{2\lambda_i}\,(-1)^{i+1}\, W_i(1)\right)\right\}\end{aligned}$$

where $\mu_i = \left(a\sqrt{\lambda_i}\coth\left(a\sqrt{\lambda_i}\right) - 1\right)$, due to (4.5). Since, for every i,

$$\mathbb{E}\left\{\exp\left[-\left(W_i(1)\right)^2 \frac{\mu_i}{2}\right]\exp\left(b\sqrt{2\lambda_i}\,(-1)^{i+1}\, W_i(1)\right)\right\} = \frac{1}{\sqrt{1+\mu_i}}\exp\left(\frac{b^2\lambda_i}{1+\mu_i}\right)$$

one deduces from the definition of the μ_i

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2} \mathbf{W}(s,t)^2\, dsdt\right)\exp\left(b\mathbf{W}(1,1)\right)\right] \\ &= \prod_{i\geq 1}\left(\frac{1}{\cosh\left(a\sqrt{\lambda_i}\right)}\right)^{\frac{1}{2}} \exp\left[\frac{b^2}{2}\left(\sum_{i\geq 1}\frac{2\lambda_i}{1+\mu_i}\right)\right] \\ &= \prod_{i\geq 1}\left(\frac{1}{\cosh\left(a\sqrt{\lambda_i}\right)}\right)^{\frac{1}{2}} \times \\ &\quad \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx \exp\left(-\frac{x^2}{2} + bx\right)\frac{\exp\left[-\frac{x^2}{2}\left(\left(\sum_{i\geq 1}\frac{2\lambda_i}{1+\mu_i}\right)^{-1} - 1\right)\right]}{\left(\sum_{i\geq 1}\frac{2\lambda_i}{1+\mu_i}\right)^{\frac{1}{2}}} dx\end{aligned}$$

thus yielding

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\mid\mathbf{W}(1,1)=x\right]\\
&= \prod_{i\geq 1}\left(\frac{1}{\cosh\left(a\sqrt{\lambda_i}\right)}\right)^{\frac{1}{2}}\times\frac{\exp\left[-\frac{x^2}{2}\left(\left(\sum_{j\geq 1}\frac{2\lambda_j}{1+\mu_j}\right)^{-1}-1\right)\right]}{\left(\sum_{j\geq 1}\frac{2\lambda_j}{1+\mu_j}\right)^{\frac{1}{2}}}.
\end{aligned}$$

By substituting for every i the correct values of λ_i and μ_i, we obtain the desired conclusion.

We will write $\{\gamma_k : k\geq 1\}$ and $\{h_k : k\geq 1\}$ respectively for the ordered sequence of eigenvalues and the sequence of eigenfunctions associated to the operator (4.19) with kernel $K(s,t)=s\wedge t-st$. In particular, with the notation of (4.1), for every $k\geq 1$, $\gamma_k=\zeta_{k,0}$ and $h_k=h_{k,0}$, and therefore (see e.g. [46, pp. 213-214])

$$\begin{aligned}
\gamma_k &= \zeta_{k,0}=\frac{1}{k^2\pi^2},\quad k=1,2,...\\
h_k(t) &= h_{k,0}(t)=\sqrt{2}\sin(k\pi t),\quad t\in(0,1],\quad k=1,2,...
\end{aligned}$$

[Proof of Theorem 4.2-(iii)] Given $y(s,t)$, continuous on $[0,1]^2$, the law of $\mathbf{W}$, conditioned to equal y on $\partial[0,1]^2 := \{(s,t) : s\vee t=1\}$ coincides with the law of the process

$$\begin{aligned}
&\mathbf{B}_*(s,t)+y^*(s,t)\\
&= \sum_{i,j\geq 1}\left[\sqrt{\gamma_i\gamma_j}\theta_{i,j}+\langle y^*,h_ih_j\rangle_{L^2([0,1]^2)}\right]h_i(s)\,h_j(t),\quad (s,t)\in[0,1]^2
\end{aligned}$$

where $\{\theta_{i,j}\}$ is a doubly indexed sequence of independent standard Gaussian random variables. This implies in particular that, for any real a,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\mid\mathbf{W}(1,\cdot)=y(1,\cdot),\mathbf{W}(\cdot,1)=y(\cdot,1)\right]\\
&= \mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\left(\mathbf{B}_*(s,t)+y^*(s,t)\right)^2 dsdt\right)\right]\\
&= \prod_{k,j\geq 1}\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\left[\sqrt{\gamma_k\gamma_j}\theta_{k,j}+\langle y^*,h_kh_j\rangle\right]^2\right)\right].
\end{aligned}$$

Now fix k and j, and observe that standard calculations yield

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\left[\sqrt{\gamma_k\gamma_j}\theta_{k,j}+\langle y^*,h_kh_j\rangle\right]^2\right)\right]\\
&= \exp\left[-\frac{1}{2}\langle y^*,h_kh_j\rangle^2\frac{\left(akj\pi^2\right)^2}{a^2+\left(kj\pi^2\right)^2}\right]\times\left(\frac{1}{1+a^2\gamma_k\gamma_j}\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, according to Theorem 4.1-(ii) and (4.22)

$$\begin{aligned}
\prod_{k,j\geq 1}\left(\frac{1}{1+a^2\gamma_k\gamma_j}\right)^{\frac{1}{2}} &= \mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{B}_*(s,t)^2\,dsdt\right)\right]\\
&= \prod_{j\geq 1}\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\gamma_j\int_0^1 B_j(t)^2\,dt\right)\right]
\end{aligned}$$

where $\{B_j\}$ is a sequence of independent standard Brownian bridges on $[0,1]$, from 0 to 0, and (4.5) yields

$$\begin{aligned}\prod_{j\geq 1}\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\gamma_j\int_0^1 B_j(t)^2\,dt\right)\right] &= \prod_{j\geq 1}\left(\frac{a}{j\pi\sinh\left(\frac{a}{j\pi}\right)}\right)^{\frac{1}{2}}\\ &= S(a)^{-\frac{1}{2}},\end{aligned}$$

which yields immediately the desired conclusion.

[Proof of Theorem 4.2-(iv)] To prove this point, we shall use the sequence of independent Brownian motions $\{W_i : i \geq 1\}$ introduced in (4.25). We start by observing that

$$\sigma\{\mathbf{W}(s,1) : s\in[0,1]\} = \sigma\{W_i(1) : i\geq 1\}$$

as easily proved by the relations

$$\begin{aligned}\mathbf{W}(s,1) &= \sum_{i\geq 1}\sqrt{\lambda_i}e_i(s)\,W_i(1), \quad s\in[0,1] \qquad (4.26)\\ W_i(1) &= \frac{1}{\sqrt{\lambda_i}}\int_0^1 \mathbf{W}(s,1)\,e_i(s)\,ds\\ &= \frac{1}{\sqrt{\lambda_i}}\int_0^1 \mathbf{W}(du,1)\left(\int_u^1 dse_i(s)\right), \quad i=1,2,...\end{aligned}$$

We can therefore use (4.5) directly, to obtain that for any real a

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-\frac{a^2}{2}\int_{[0,1]^2}\mathbf{W}(s,t)^2\,dsdt\right)\mid\sigma\{\mathbf{W}(s,1):s\in[0,1]\}\right]\\ &= \mathbb{E}\left[\exp\left(-\frac{a^2}{2}\sum_{i\geq 1}\lambda_i\int_{[0,1]^2}[W_i(t)]^2\,dt\right)\mid W_i(1), \quad i=1,2...\right]\\ &= \prod_{j\geq 1}\left(\frac{a\sqrt{\lambda_j}}{\sinh\left(a\sqrt{\lambda_j}\right)}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\sum_{k\geq 1}[W_k(1)]^2\left[a\sqrt{\lambda_k}\coth\left(a\sqrt{\lambda_k}\right)-1\right]\right\}\\ &= \prod_{j\geq 1}\left(\frac{2a}{(2j-1)\,\pi\sinh\left(2a\left[(2j-1)\,\pi\right]^{-1}\right)}\right)^{\frac{1}{2}}\times\\ &\quad\times\exp\left\{-\frac{1}{2}\sum_{j\geq 1}[W_j(1)]^2\left[\frac{2a}{(2j-1)\,\pi}\coth\left(\frac{2a}{(2j-1)\,\pi}\right)-1\right]\right\},\end{aligned}$$

and the conclusion follows by using (4.26).

4.4 An Application: the Laws of some Double Stochastic Integrals

In this section, we use Theorem 4.2 and Proposition 4.1 to calculate the laws of double stochastic integrals involving two independent, bivariate and centered Gaussian processes. As discussed below, our results extend and generalize some classic computations due to Julia and Nualart ([31], but see also [39]).

4.4.1 The law of a double stochastic integral involving two independent Brownian sheets

Consider two independent, standard Brownian sheets

$$\left\{\mathbf{W}_1(s,t) : (s,t) \in [0,1]^2\right\} \quad \text{and} \quad \left\{\mathbf{W}_2(s,t) : (s,t) \in [0,1]^2\right\}.$$

In [31, Theorem 1], it is proved that the characteristic function of the random variable

$$X = \int_{[0,1]^2} \mathbf{W}_2(s,t)\, \mathbf{W}_1(ds,dt)$$

is given by the formula

$$\mathbb{E}\left[\exp\left(\mathrm{i}\lambda X\right)\right] = \left\{C_{\mathrm{odd}}(2\lambda)\right\}^{-\frac{1}{2}}.$$

Such a result can be obtained by means of the identities in law discussed in the previous section. As a matter of fact, a standard conditioning argument and Theorem 4.2-(i), yield

$$\mathbb{E}\left[\exp\left(\mathrm{i}\lambda X\right)\right] = \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\int_{[0,1]^2} \mathbf{W}_1(s,t)^2\, dsdt\right)\right] = \left\{C_{\mathrm{odd}}(2\lambda)\right\}^{-\frac{1}{2}}. \qquad (4.27)$$

4.4.2 The law of a double stochastic integral involving a Brownian sheet and an independent bivariate Brownian bridge

Consider a Brownian sheet $\left\{\mathbf{W}(s,t) : (s,t) \in [0,1]^2\right\}$, as well as an independent bivariate Brownian bridge $\left\{\mathbf{B}(s,t) : (s,t) \in [0,1]^2\right\}$, as defined above. We shall study the laws of the two random variables

$$\begin{aligned} T &= \int_{[0,1]^2} \mathbf{B}(s,t)\, \mathbf{W}(ds,dt) \\ U &= \int_{[0,1]^2} \mathbf{W}(s,t)\, \mathbf{B}(ds,dt) \\ &= \int_{[0,1]^2} \left(\mathbf{W}(s,t) - \overline{\mathbf{W}}\right) \mathbf{B}(ds,dt) \end{aligned}$$

where $\overline{\mathbf{W}} = \int_{[0,1]^2} \mathbf{W}(u,v)\, dudv$. Relation (3.25) yields that for every real λ

$$\begin{aligned} \mathbb{E}\left[\exp\left(\mathrm{i}\lambda T\right)\right] &= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\int_{[0,1]^2} \mathbf{B}(s,t)^2\, dsdt\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\int_{[0,1]^2} \left(\mathbf{W}(s,t) - \overline{\mathbf{W}}\right)^2 dsdt\right)\right] \\ &= \mathbb{E}\left[\exp\left(\mathrm{i}\lambda U\right)\right], \end{aligned}$$

thus implying that T and U are equal in distribution. Moreover, thanks to Proposition 4.1-(i), for every real λ,

$$\mathbb{E}\left[\exp\left(\mathrm{i}\lambda T\right)\right] = \mathbb{E}\left[\exp\left(\mathrm{i}\lambda U\right)\right] = \left(C_{\mathrm{odd}}(2\lambda)\, \frac{4\mathcal{T}(\lambda)}{\lambda}\right)^{-\frac{1}{2}}. \qquad (4.28)$$

4.4.3 The law of a double stochastic integral involving a Brownian sheet and an independent tied-down bivariate Brownian bridge

Define $\left\{\mathbf{W}(s,t) : (s,t) \in [0,1]^2\right\}$ and $\left\{\mathbf{B}_*(s,t) : (s,t) \in [0,1]^2\right\}$ to be two independent processes, and namely a standard Brownian sheet and a tied-down bivariate Brownian bridge, as defined above. We want to show that the two random variables

$$
\begin{aligned}
Y &= \int_{[0,1]^2} \mathbf{B}_* (s,t)\, \mathbf{W}(ds, dt) \\
Z &= \int_{[0,1]^2} \mathbf{W}(s,t)\, \mathbf{B}_* (ds, dt) \\
&= \int_{[0,1]^2} \left(\mathbf{W}(s,t) - \overline{\overline{\mathbf{W}}}(s,t) \right) \mathbf{B}_* (ds, dt),
\end{aligned}
$$

where

$$
\overline{\overline{\mathbf{W}}}(s,t) = \int_0^1 \mathbf{W}(a,t)\, da + \int_0^1 \mathbf{W}(s,b)\, db - \int_{[0,1]^2} \mathbf{W}(a,b)\, dadb,
$$

have the same law, and moreover we will compute their common characteristic function. Indeed, from (3.26) we deduce that for every real λ

$$
\begin{aligned}
\mathbb{E}[\exp(\mathrm{i}\lambda Y)] &= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2} \int_{[0,1]^2} \mathbf{B}_* (s,t)^2\, dsdt \right) \right] \\
&= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2} \int^2_{[0,1]^2} \left(\mathbf{W}(s,t) - \overline{\overline{\mathbf{W}}}(s,t) \right)^2 dsdt \right) \right] \\
&= \mathbb{E}[\exp(\mathrm{i}\lambda Z)]
\end{aligned}
$$

and finally, from Proposition 4.1,

$$
\mathbb{E}[\exp(\mathrm{i}\lambda Y)] = \mathbb{E}[\exp(\mathrm{i}\lambda Z)] = \{S(\lambda)\}^{-\frac{1}{2}}. \tag{4.29}
$$

4.4.4 The law of a double stochastic integral involving a Brownian sheet and an independent asymmetric bivariate Brownian bridge

Let $\left\{ \mathbf{W}(s,t) : (s,t) \in [0,1]^2 \right\}$ and $\left\{ \mathbf{B}_\natural (s,t) : (s,t) \in [0,1]^2 \right\}$ denote a standard Brownian sheet and an independent asymmetric bivariate Brownian bridge, as defined above. We define

$$
\begin{aligned}
Q &= \int_{[0,1]^2} \mathbf{B}_\natural (s,t)\, \mathbf{W}(ds, dt) \\
J &= \int_{[0,1]^2} \mathbf{W}(s,t)\, \mathbf{B}_\natural (ds, dt) \\
&= \int_{[0,1]^2} \left(\mathbf{W}(s,t) - \widehat{\mathbf{W}}(t) \right) \mathbf{B}_{\mathcal{A}} (ds, dt),
\end{aligned}
$$

where $\widehat{\mathbf{W}}(t) = \int_0^1 \mathbf{W}(u,t)\, du$. From the identity in law (3.27), we obtain that for every real λ

$$
\begin{aligned}
\mathbb{E}[\exp(\mathrm{i}\lambda Q)] &= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2} \int_{[0,1]^2} \mathbf{B}_{\mathcal{A}} (s,t)^2\, dsdt \right) \right] \\
&= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2} \int_{[0,1]^2} \left(\mathbf{W}(s,t) - \widehat{\mathbf{W}}(t) \right)^2 dsdt \right) \right] \\
&= \mathbb{E}[\exp(\mathrm{i}\lambda J)],
\end{aligned}
$$

and therefore that Q and J have the same law. Eventually, Proposition 4.1-(iii) yields

$$
\mathbb{E}[\exp(\mathrm{i}\lambda Q)] = \mathbb{E}[\exp(\mathrm{i}\lambda J)] = \{S_{\mathrm{odd}}(2\lambda)\}^{-\frac{1}{2}}. \tag{4.30}
$$

5 Weak convergence

5.1 Preliminaries

To simplify the notation, throughout this section we will write, for any $-1 < \gamma, \delta < +\infty$,

$$\Xi(\gamma,\delta) = \int_{[0,1]^2} \mathbf{W}^{(\gamma,\delta)}(s,t)^2\, dsdt \quad ; \quad \Xi_*(\gamma,\delta) = \int_{[0,1]^2} \mathbf{B}_*^{(\gamma,\delta)}(s,t)^2\, dsdt \tag{5.1}$$

$$\Xi_1(\gamma,\delta) = \int_{[0,1]^2} \mathbf{B}^{(\gamma,\delta)}(s,t)^2\, dsdt \quad ; \quad \Xi_{\mathcal{A}}(\gamma,\delta) = \int_{[0,1]^2} \mathbf{B}_{\mathcal{A}}^{(\gamma,\delta)}(s,t)^2\, dsdt. \tag{5.2}$$

Standard arguments (see e.g. [30, Lemma 1, page 44]) yield that, a.s.-$\mathbb{P}$,

$$\lim_{\gamma,\delta\downarrow -1} \Xi(\gamma,\delta) = \lim_{\gamma,\delta\downarrow -1} \Xi_*(\gamma,\delta) = \lim_{\gamma,\delta\downarrow -1} \Xi_1(\gamma,\delta) = \lim_{\gamma,\delta\downarrow -1} \Xi_{\mathcal{A}}(\gamma,\delta) = +\infty,$$

as well as

$$\lim_{\gamma,\delta\to +\infty} \Xi(\gamma,\delta) = \lim_{\gamma,\delta\to +\infty} \Xi_*(\gamma,\delta) = \lim_{\gamma,\delta\to +\infty} \Xi_1(\gamma,\delta) = \lim_{\gamma,\delta\to +\infty} \Xi_{\mathcal{A}}(\gamma,\delta) = 0.$$

The aim of this section is to study the speed at which the above quantities respectively diverge to infinity and converge to zero. In particular, we will prove the following

Theorem 5.1 *Let $N(0,1)$ denote a standard Gaussian random variable independent of $\mathbf{W}$. Then, as $\gamma, \delta \downarrow -1$,*

(i) $\left\{ \frac{4(\gamma+1)(\delta+1)\Xi(\gamma,\delta)-1}{4\sqrt{(\delta+1)(\gamma+1)}} \;\; ; \;\; \mathbf{W} \right\} \overset{law}{\to} \{N(0,1) \;\; ; \;\; \mathbf{W}\};$

(ii) $\left\{ \frac{4(\gamma+1)(\delta+1)\Xi_1(\gamma,\delta)-1}{4\sqrt{(\delta+1)(\gamma+1)}} \;\; ; \;\; \mathbf{W} \right\} \overset{law}{\to} \{N(0,1) \;\; ; \;\; \mathbf{W}\};$

(iii) $\left\{ \frac{4(\gamma+1)(\delta+1)\Xi_*(\gamma,\delta)-1}{4\sqrt{(\delta+1)(\gamma+1)}} \;\; ; \;\; \mathbf{W} \right\} \overset{law}{\to} \{N(0,1) \;\; ; \;\; \mathbf{W}\};$

(iv) $\left\{ \frac{4(\gamma+1)(\delta+1)\Xi_{\mathcal{A}}(\gamma,\delta)-1}{4\sqrt{(\delta+1)(\gamma+1)}} \;\; ; \;\; \mathbf{W} \right\} \overset{law}{\to} \{N(0,1) \;\; ; \;\; \mathbf{W}\}.$

A partial description of the asymptotic behavior of the above quadratic functionals, when $\gamma, \delta \to +\infty$, is the following non-central limit theorem.

Theorem 5.2 *Let the above notation prevail, and let $\widetilde{\mathbf{W}}$ denote a standard Brownian sheet independent of $\mathbf{B}_*$. Then, as $\gamma, \delta \to +\infty$*

$$\left\{ 4(\gamma+1)^2(\delta+1)^2\, \Xi_*(\gamma,\delta) \;\; ; \;\; \mathbf{B}_* \right\} \overset{law}{\to} \left\{ \frac{1}{4}\int_{[0,1]^2} \frac{dsdt}{st} \widetilde{\mathbf{W}}(s,t)^2 \;\; ; \;\; \mathbf{B}_* \right\} \tag{5.3}$$

The proofs of Theorem 5.1 and Theorem 5.2 are obtained in the next two paragraphs.

5.2 Asymptotic Study as $\gamma, \delta \downarrow -1$, and Proof of Theorem 5.1

In this paragraph, we use the apparatus of the Malliavin calculus, and we refer the reader to [39] for any unexplained notion or definition concerning this topic. We also use some standard convention: by noting $\{W_t : t \in [0,1]\}$ a standard Brownian motion and $\left\{\mathbf{W}(s,t) : (s,t) \in [0,1]^2\right\}$ a standard Brownian sheet, for any $h_1, h_2 \in L^2([0,1], ds)$ and $g \in L^2\left([0,1]^2, dsdt\right)$, we write

$$\begin{aligned} W(h_1) &:= \int_0^1 h_1(s)\, dW_s \\ \mathbf{W}(h_1, h_2) &:= \int_{[0,1]^2} h_1(s)\, h_2(t)\, \mathbf{W}(ds, dt) \\ \mathbf{W}(g) &:= \int_{[0,1]^2} g(s,t)\, \mathbf{W}(ds, dt) \end{aligned}$$

In particular, our main tool to prove Theorem 5.1 is the following consequence of Theorem 1 in [40] (but see also [42]).

Lemma 5.1 *Let ϕ_i, $i = 1,2$, be an operator from $L^2([0,1], dt)$ to itself, and let $\{\mu_\varepsilon^i : \varepsilon > 0\}$, $i = 1,2$, be a collection of finite measures on $([0,1], \mathcal{B}([0,1]))$. Define moreover, for $\varepsilon > 0$,*

$$Y_\varepsilon^i = \int_{[0,1]} \mu_\varepsilon^i(dt) \left[W\left(\phi_i \mathbf{1}_{[0,t]}\right)\right]^2, \quad i = 1,2 \tag{5.4}$$

$$Z_\varepsilon = \int_{[0,1]^2} \mu_\varepsilon^1(ds)\, \mu_\varepsilon^2(dt) \left[\mathbf{W}\left(\phi_1 \mathbf{1}_{[0,s]}, \phi_2 \mathbf{1}_{[0,t]}\right)\right]^2, \tag{5.5}$$

and suppose that, for $i = 1,2$,

$$\frac{Y_\varepsilon^i - \mathbb{E}\left(Y_\varepsilon^i\right)}{Var\left(Y_\varepsilon^i\right)^{\frac{1}{2}}} \xrightarrow{law} N(0,1), \tag{5.6}$$

as ε goes to 0, where $N(0,1)$ is a standard Gaussian random variable. Then,

$$\frac{Z_\varepsilon - \mathbb{E}(Z_\varepsilon)}{Var(Z_\varepsilon)^{\frac{1}{2}}} \xrightarrow{law} N(0,1). \tag{5.7}$$

Moreover, if there exist positive numbers $\{b_{i,\varepsilon} : \varepsilon > 0\}$, $i = 1,2$, such that

$$Var\left(Y_\varepsilon^i\right)^{\frac{1}{2}} \sim b_{i,\varepsilon} \tag{5.8}$$

as $\varepsilon \to 0$, then

$$Var(Z_\varepsilon)^{\frac{1}{2}} \sim b_{1,\varepsilon} \times b_{2,\varepsilon}. \tag{5.9}$$

Proof. A standard application of Stroock's formula (see [48]) gives the Wiener chaos expansion of Y_ε^i, for every $\varepsilon > 0$ and $i = 1,2$, namely

$$Y_\varepsilon^i = \int_{[0,1]} \mu_\varepsilon^i(dt) \int_{[0,1]} dx \left(\phi_i \mathbf{1}_{[0,t]}(x)\right)^2 + I_2^W\left[\int_{[0,1]} \mu_\varepsilon^i(dt) \left(\phi_i \mathbf{1}_{[0,t]}\right)^{\otimes_0}\right]$$

where I_2^W indicates a double Wiener stochastic integral with respect to W, and, for every $f \in L^2([0,1], dt)$, $(f)^{\otimes_0}$stands for the element of $L^2\left([0,1]^2, dsdt\right)$ given by $f(s)\, f(t)$. Moreover, the isometric properties of multiple stochastic integrals imply that

$$\mathrm{Var}\left(Y_\varepsilon^i\right) = 2 \int_{[0,1]^2} dxdy \left[\int_{[0,1]} \mu_\varepsilon^i(dt) \left(\phi_i \mathbf{1}_{[0,t]}\right)^{\otimes_0}(x,y)\right]^2$$

and therefore, since (5.6) holds, we deduce immediately from Theorem 1 in [40] that the quantity

$$\frac{\int_{[0,1]^2} dydy' \left[\int_{[0,1]} dx \int_{[0,1]} \mu_\varepsilon^i(dt) \left(\phi_i \mathbf{1}_{[0,t]}\right)^{\otimes_0}(x,y) \int_{[0.1]} \mu_\varepsilon^i(du) \left(\phi_i \mathbf{1}_{[0,u]}\right)^{\otimes_0}(x,y')\right]^2}{\int_{[0,1]^2} dxdy \left[\int_{[0,1]} \mu_\varepsilon^i(dt) \left(\phi_i \mathbf{1}_{[0,t]}\right)^{\otimes_0}(x,y)\right]^2} \tag{5.10}$$

converges to zero, as ε goes to zero. On the other hand, the chaotic decomposition of Z_ε, $\varepsilon > 0$, is given – thanks again to Stroock's formula – by

$$Z_\varepsilon = \int_{[0,1]^2} \mu_\varepsilon^1(ds)\, \mu_\varepsilon^2(dt) \int_{[0,1]^2} dxdy \left(\phi_1 \mathbf{1}_{[0,s]}(x)\, \phi_2 \mathbf{1}_{[0,t]}(y)\right)^2 + I_2^{\mathbf{W}}\left[\Psi_\varepsilon(\cdot,\cdot)\right]$$

where $I_2^{\mathbf{W}}$ stands for a double stochastic integral with respect to $\mathbf{W}$, and, for $\varepsilon > 0$, the functions Ψ_ε are symmetric on $[0,1]^2 \times [0,1]^2$ and given by

$$\begin{aligned} [(x,y),(a,b)] \quad &\mapsto \quad \Psi_\varepsilon(x,y;a,b) \\ &= \quad \int_{[0,1]} \mu_\varepsilon^1(ds) \left[\left(\phi_1 \mathbf{1}_{[0,s]}\right)^{\otimes_0}(x,a)\right] \times \\ &\qquad \int_{[0,1]} \mu_\varepsilon^2(dt) \left[\left(\phi_2 \mathbf{1}_{[0,t]}\right)^{\otimes_0}(y,b)\right] \end{aligned}$$

where the notation is as before. Observe also that

$$\begin{aligned} \mathrm{Var}(Z_\varepsilon) \quad = \quad & 2\int_{[0,1]^2} dxda \left[\int_{[0,1]} \mu_\varepsilon^1(ds) \left(\phi_1 \mathbf{1}_{[0,s]}\right)^{\otimes_0}(x,a)\right]^2 \times \\ & \int_{[0,1]^2} dydb \left[\int_{[0,1]} \mu_\varepsilon^2(dt) \left(\phi_2 \mathbf{1}_{[0,t]}\right)^{\otimes_0}(y,b)\right]^2. \end{aligned} \tag{5.11}$$

Thanks again to Theorem 1 in [40], to show (5.7) one shall only verify that

$$\lim_{\varepsilon \to 0} \frac{\left\|\Psi_\varepsilon^{\otimes 1}\right\|^2_{L^2\left([0,1]^4\right)}}{\left\|\Psi_\varepsilon\right\|^4_{L^2\left([0,1]^4\right)}} = 0, \tag{5.12}$$

where

$$\Psi_\varepsilon^{\otimes 1}(x,y;a,b) = \int_{[0,1]^2} dudz \Psi_\varepsilon(x,y;u,z)\, \Psi_\varepsilon(a,b;u,z)$$

as usual. Standard computations imply that

$$\left\|\Psi_\varepsilon^{\otimes 1}\right\|^2_{L^2\left([0,1]^4\right)} = K_{1,\varepsilon} \times K_{2,\varepsilon}$$

where, for $i = 1,2$,

$$K_{i,\varepsilon} = \int_{[0,1]^2} dydy' \left[\int_{[0,1]} dx \int_{[0,1]} \mu_\varepsilon^i(dt) \left(\phi_i \mathbf{1}_{[0,t]}\right)^{\otimes_0}(x,y) \int_{[0.1]} \mu_\varepsilon^i(du) \left(\phi_i \mathbf{1}_{[0,u]}\right)^{\otimes_0}(x,y')\right]$$

and $\left\|\Psi_\varepsilon\right\|^4_{L^2\left([0,1]^4\right)} = \mathrm{Var}(Z_\varepsilon)^2$, and therefore (5.12) is proved, since the sequence in (5.10) converges to zero. The last assertion in the statement follows immediately from (5.11). □

Proof of Theorem 5.1 We note $\{W_t : t \geq 0\}$ a standard Brownian motion, and $\{B_t : t \in [0,1]\}$ a standard Brownian bridge of length 1, from 0 to 0. We start by observing that Proposition 3.2 in [41] implies the following asymptotic relations

$$\frac{2(\gamma+1)\int_0^1 t^{2\gamma}W_t^2dt-1}{\sqrt{\gamma+1}} \underset{\gamma\to-1}{\overset{law}{\to}} N(0,1) \tag{5.13}$$
$$\frac{2(\gamma+1)\int_0^1 t^{2\gamma}B_t^2dt-1}{2\sqrt{\gamma+1}} \underset{\gamma\to-1}{\overset{law}{\to}} N(0,1).$$

Formula (5.13) yields immediately point (i) of Theorem 5.1, thanks to Lemma 5.1 in the special case $\phi_1 = \phi_2 = id.$, and

$$\begin{aligned}\mu^1_{\gamma+1}(ds) &= 2(\gamma+1)s^{2\gamma}ds \\ \mu^2_{\delta+1}(dt) &= 2(\delta+1)t^{2\delta}dt.\end{aligned} \tag{5.14}$$

Point (ii) derives again from (5.13) and Lemma 5.1 in the case

$$\phi_1 f(x) = \phi_2 f(x) = f(x) - \int_0^1 f(a)\,da,$$

and $\mu^1_{\gamma+1}$ and $\mu^2_{\delta+1}$ defined as in (5.14). Point (iii) is a consequence of the identity in law

$$\left\{\mathbf{B}(s,t):(s,t)\in[0,1]^2\right\} \overset{law}{=} \left\{\mathbf{W}(s,t)-st\mathbf{W}(1,1):(s,t)\in[0,1]^2\right\}.$$

Eventually, point (iv) of Theorem 5.1 comes from (5.13) and Lemma 5.1, with

$$\phi_1 f(x) = f(x) - \int_0^1 f(a)\,da \quad ; \quad \phi_2 = id.,$$

and $\mu^1_{\gamma+1}$ and $\mu^2_{\delta+1}$ as in (5.14). The asymptotic independence is an immediate consequence of Theorem 1 in [42]. This concludes the proof of Theorem 12. □

5.3 Asymptotic Study as $\gamma, \delta \uparrow +\infty$

We start by proving a useful analogue of Lemma 2.1 in [41].

Proposition 5.1 *Let* $\mathbf{W} = \left\{\mathbf{W}(s,t):(s,t)\in\mathbb{R}^2_+\right\}$ *be a standard Brownian sheet, and let* $\mathbf{B}_*$ *tied down bivariate Brownian bridge defined above. We write* $\widetilde{\mathbf{W}} = \left\{\widetilde{\mathbf{W}}(s,t):(s,t)\in\mathbb{R}^2_+\right\}$ *to indicate a standard Brownian sheet independent of* $\mathbf{W}$ *and* $\mathbf{B}_*$*. Then, as* γ *and* δ *tend to infinity,*

(i) the family

$$\left\{\sqrt{\gamma\delta}\left[\mathbf{W}\left(e^{-\frac{u}{\gamma}},e^{-\frac{v}{\delta}}\right)-\mathbf{W}\left(e^{-\frac{u}{\gamma}},1\right)-\mathbf{W}\left(1,e^{-\frac{v}{\delta}}\right)+\mathbf{W}(1,1)\right]:(u,v)\in\mathbb{R}^2_+ \;;\; \mathbf{W}\right\}$$

converges in distribution to

$$\left\{\widetilde{\mathbf{W}}(u,v):(u,v)\in\mathbb{R}^2_+ \;;\; \mathbf{W}\right\};$$

(ii) the family

$$\left\{\sqrt{\gamma\delta}\left[\mathbf{W}\left(e^{-\frac{u}{\gamma}},1-e^{-\frac{v}{\delta}}\right)-\mathbf{W}\left(1,1-e^{-\frac{v}{\delta}}\right)\right]:(u,v)\in\mathbb{R}^2_+ \;;\; \mathbf{W}\right\}$$

converges in distribution to

$$\left\{\widetilde{\mathbf{W}}(u,v):(u,v)\in\mathbb{R}^2_+ \;;\; \mathbf{W}\right\};$$

(iii) the family

$$\left\{\sqrt{\gamma\delta}\mathbf{B}_*\left(e^{-\frac{u}{\gamma}},e^{-\frac{v}{\delta}}\right):(s,t)\in\mathbb{R}_+^2\ \ ;\ \ \mathbf{B}_*\right\}$$

converges in distribution to

$$\left\{\widetilde{\mathbf{W}}(u,v):(u,v)\in\mathbb{R}_+^2\ \ ;\ \ \mathbf{B}_*\right\}.$$

Proof. Parts (i) and (ii) of the statement are an easy consequence of the fact that if $\widehat{\mathbf{W}}$ denotes a standard Brownian sheet, then as $\gamma,\delta\to+\infty$ the family

$$\left\{\sqrt{\gamma\delta}\widehat{\mathbf{W}}\left(1-e^{-\frac{u}{\gamma}},1-e^{-\frac{v}{\delta}}\right):(u,v)\in\mathbb{R}_+^2\ \ ;\ \ \widehat{\mathbf{W}}(s,t):(s,t)\in[0,1]^2\right\}$$

converges in distribution to

$$\left\{\widetilde{\mathbf{W}}(u,v):(u,v)\in\mathbb{R}_+^2\ \ ;\ \ \widehat{\mathbf{W}}\right\}$$

where $\widetilde{\mathbf{W}}$ is another standard Brownian sheet independent of $\widehat{\mathbf{W}}$. Now write

$$\mathbf{W}\left(e^{-\frac{u}{\gamma}},e^{-\frac{v}{\delta}}\right)-\mathbf{W}\left(e^{-\frac{u}{\gamma}},1\right)-\mathbf{W}\left(1,e^{-\frac{v}{\delta}}\right)+\mathbf{W}(1,1)=\widehat{\mathbf{W}}\left(1-e^{-\frac{u}{\gamma}},1-e^{\frac{v}{\delta}}\right)$$

where

$$\widehat{\mathbf{W}}(s,t)=\mathbf{W}(1-s,1-t)-\mathbf{W}(1-s,1)-\mathbf{W}(1,1-t)+\mathbf{W}(1,1),$$

to obtain point (i), whereas to prove point (ii) it is sufficient to write

$$\mathbf{W}\left(e^{-\frac{u}{\gamma}},1-e^{-\frac{v}{\delta}}\right)-\mathbf{W}\left(1,1-e^{-\frac{v}{\delta}}\right)=\widehat{\mathbf{W}}\left(1-e^{-\frac{u}{\gamma}},1-e^{\frac{v}{\delta}}\right)$$

for

$$\widehat{\mathbf{W}}(s,t)=\mathbf{W}(1-s,t)-\mathbf{W}(1,t).$$

To deal with point (iii), just use the identity in law (for the processes as a whole)

$$\mathbf{B}_*(s,t)\overset{law}{=}\mathbf{W}(s,t)-t\mathbf{W}(s,1)-s\mathbf{W}(1,t)+st\mathbf{W}(1,1)$$

as well as the fact that the process

$$\begin{aligned}
&\sqrt{\gamma\delta}\Big[\mathbf{W}\left(e^{-\frac{u}{\gamma}},e^{-\frac{v}{\delta}}\right)-\mathbf{W}\left(e^{-\frac{u}{\gamma}},1\right)-\mathbf{W}\left(1,e^{-\frac{v}{\delta}}\right)+\mathbf{W}(1,1)-\\
&\mathbf{W}\left(e^{-\frac{u}{\gamma}},e^{-\frac{v}{\delta}}\right)-e^{-\frac{v}{\delta}}\mathbf{W}\left(e^{-\frac{u}{\gamma}},1\right)-e^{-\frac{u}{\gamma}}\mathbf{W}\left(1,e^{-\frac{v}{\delta}}\right)+e^{-\frac{u}{\gamma}-\frac{v}{\delta}}\mathbf{W}(1,1)\Big]\\
=\ &\sqrt{\gamma\delta}\mathbf{W}(1,1)\left(1-e^{-\frac{u}{\gamma}}\right)\left(1-e^{-\frac{v}{\delta}}\right)+\sqrt{\gamma\delta}\left[\mathbf{W}\left(1,e^{-\frac{v}{\delta}}\right)-\mathbf{W}(1,1)\right]\left(1-e^{-\frac{u}{\gamma}}\right)\\
&+\sqrt{\gamma\delta}\left[\mathbf{W}\left(e^{-\frac{u}{\gamma}},1\right)-\mathbf{W}(1,1)\right]\left(1-e^{-\frac{v}{\delta}}\right)
\end{aligned}$$

trivially converges in distribution to the zero process. □

Proof of Theorem 5.2 Standard changes of variables yield

$$\begin{aligned}
&4(\gamma+1)^2(\delta+1)^2\,\Xi_*(\gamma,\delta)\\
=\ &\frac{(\gamma+1)^2(\delta+1)^2}{\gamma\delta}\int_{\mathbb{R}_+^2}dxdy\exp\left[-x\frac{2\gamma+1}{2\gamma}-y\frac{2\delta+1}{2\delta}\right]\mathbf{B}_*\left(e^{-\frac{x}{2\gamma}},e^{-\frac{y}{2\delta}}\right)^2
\end{aligned}$$

so that we can apply directly Proposition 5.1 to the left side to obtain

$$\left\{4(\gamma+1)^2(\delta+1)^2\,\Xi_*(\gamma,\delta)\ \ ;\ \ \mathbf{B}_*\right\}\overset{law}{\to}\left\{\frac{1}{4}\int_{\mathbb{R}_+^2}dxdye^{-(x+y)}\widetilde{\mathbf{W}}(x,y)^2\ \ ;\ \ \mathbf{B}_*\right\}.$$

The conclusion follows from an application of the Fubini type techniques developed in Section 3, yielding

$$\int_{\mathbb{R}_+^2} dxdye^{-(x+y)}\widetilde{\mathbf{W}}(x,y)^2 = \int_{[0,1]^2} dsdt\widetilde{\mathbf{W}}\left(\log\frac{1}{s},\log\frac{1}{t}\right)^2 \stackrel{law}{=} \int_{[0,1]^2} \frac{dsdt}{st}\widetilde{\mathbf{W}}(s,t)^2 . \qquad (5.15)$$

□

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