A Stochastic Asset Model for Fair Values in Pensions and Insurance

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Abstract

This paper is a first attempt to describe a new stochastic investment model. The model produces simulations for interest rates, inflation, equity income and capital gains, under a realistic probability law. The model also produces state price deflators, which are useful for pricing and for fair value calculations.

We have prepared an implementation of the model, in Excel / VBA, which the authors are pleased to release free of charge to interested researchers. The software requires a PC with Excel 97 or later.

1 Introduction – Requirements for a Fair Value Model

1.1 The need for stochastic models

Recent moves towards fair value accounting have provided a focus for the development of better modelling techniques, especially in the insurance and pensions industries. The fair value is readily calculated for insurance and pension products that have fixed cash flows: it suffices to discount cash flows at market risk-free rates. In the case of cash flows whose variability is independent of market risk, most orthodox financial theory dictates that the estimated mean cash flow should be discounted at risk-free rates.

The challenge of calculating fair values is greatest for cash flows carrying significant market risk, particularly those with embedded options, profit sharing or guarantees. Similar issues arise for enterprises whose profits are cyclical, for example, insurers affected by cycles in premium adequacy. In these cases, deterministic methods based on best estimate mean cash flows are, at best, a blunt valuation tool, because it is difficult to justify an appropriate risk-adjusted discount rate. Instead, it is necessary to consider at least two possible outcomes, based on good and bad business environments. Where multiple options are involved, some form of probability model is required.

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1.2 From distributions to prices

Having produced simulations of cash flows, it is by no means obvious where to go next. The task is to translate the distributions into fair values. One technical tool for this is the state price deflator. It is not our purpose here to discuss in detail the possible applications of the model. For a general overview, see Jarvis, Southall & Varnell. For applications in general insurance, see Christofides & Smith (2001), in life assurance see Bezooyen, Mehta & Smith (2001) or Hairs et al (2001). Chapman, Gordon & Speed (2001) gave a pensions example.

What then are the requirements for an asset model, in order to support fair value calculation? Plainly, it must have enough outputs to model the required cash flows; for example if cash flows relate to interest rates, equity markets or retail prices then these items must all be outputs from the model. Secondly, the model must support state price deflators. Finally, the model must be capable of calibration to market prices so that the resulting cash flow valuations are consistent with the market evaluation of similar cash flows. This also means that the model should be consistent with fair value approaches adopted by banks for the valuation of cash flows from financial instruments.

1.3 Monte Carlo versus backward induction and analytics

Many of the most difficult insurance and pension flows to model are path dependent. This means that the value of a maturing policy or pension payment will depend not only on the level of market prices on the maturity date, but also on the path taken to get there. This path dependent feature means that standard option pricing algorithms, for example based on backward induction, are computationally inefficient when applied to insurance and pension problems. Instead, we have found Monte Carlo simulation to be the most practical approach.

2 A Proposed Model

2.1 Notation

In this section we propose some formulas for a fair value model.

The simulation dependent state price deflator at time a is denoted by D_a. The price at time a of a zero coupon bond paying 1 at time b is denoted by P_{ab}. These parameters define the term structure of the model. Superscripts distinguish different numeraires (or asset categories) – in this model we consider cash, inflation and equity numeraires measured in a single currency, sterling. Our model works in discrete time, so that a and b must be integers.

We denote by B_a a 5-dimensional Brownian motion. The components of B_a are independent Brownian motions, with zero drift and unit variance, so that

$$B_{a+1} = B_a + N(0, I)$$

We define C_a to be an accumulation of historic values of B_a, so that C_a satisfies the recurrence relation:

$$C_{a+1} = C_a + B_a$$

We assume that deflators and term structures satisfy the following equations (for $b \geq a \geq 0$):

$$D_a(P_{ab} - P_{a:b+1}) = \frac{const}{(1+f)^b}$$
$$* \exp\left[\begin{array}{l} \sigma.(C_a - C_{\min\{a,b-\tau\}}) + \lambda.B_a + \min\{b-a,\tau\}\sigma.B_a - \tfrac{1}{2}(\lambda+\sigma\tau)^2 a \\ + \dfrac{\max\{0,\tau-b+a\}(\tau+1-b+a)}{12}\left[6\sigma.\lambda + (4\tau+2b-2a-1)\sigma^2\right] \end{array}\right]$$

Using the fact that P_{aa}=1, we can recover deflators and term structures from:

$$D_a = \sum_{u=a}^{\infty} D_a(P_{au} - P_{a:u+1})$$

$$P_{ab} = \frac{\sum_{u=b}^{\infty} D_a(P_{au} - P_{a:u+1})}{\sum_{u=a}^{\infty} D_a(P_{au} - P_{a:u+1})}$$

Here f>0 is the long forward rate, τ is a positive integer; λ and σ are 5-dimensional vectors. We offer no step-by-step derivation of this formula; we derived it by a combination of inspired guesswork and trial and error. By way of justification, we notice that the result has a number of attractive properties, including:

- Arbitrage-free because it satisfies the deflator relation: $D_a P_{ab} = \mathbf{E}_a(D_b)$.
- A flexible form that allows many combinations of risk premiums, volatilities and correlations and can fit a range of initial yield curve shapes.
- Interest rates are guaranteed positive, but rates of all terms can fall arbitrarily close to zero.
- The model is tractable analytically, and easy to compute (although the infinite sums look daunting, the summand is a geometric progression for $u \geq a+\tau$ so the formulas can be evaluated in finite form).
- Good news for learners - the model requires no hard maths, beyond manipulation of lognormal distributions. In particular we avoid the need for stochastic calculus (Ito's formula) and the partial differential equations that have become almost universal in the financial pricing literature.

We believe this is the first published multi-asset model to display this combination of characteristics.

2.2 Observable quantities

Using the deflators and term structures we can develop formulas for market observable quantities as below:

term u spot yield at time a	$\frac{1}{\sqrt[u]{P^{cash}_{a:a+u}}} - 1$
cum dividend equity price index at time a	$\frac{D^{equity}_{a}}{D^{cash}_{a}}$
equity income index at time a	$\frac{D^{equity}_{a}(1 - P^{equity}_{a:a+1})}{D^{cash}_{a}}$
equity dividend yield at time a (based on ex-dividend price)	$\frac{1}{P^{equity}_{a:a+1}} - 1$
term u real spot yield at time a	$\frac{1}{\sqrt[u]{P^{inflation}_{a:a+u}}} - 1$
return between a and a+1 on zero coupon bond of term u going to u-1:	$\frac{P^{cash}_{a+1:a+u}}{P^{cash}_{a:a+u}} - 1$
nominal return between a and a+1 on zero coupon index linked bond of term u becoming u-1	$\frac{D^{cash}_{a}}{D^{inflation}_{a}} \frac{D^{inflation}_{a+1}}{D^{cash}_{a+1}} \frac{P^{inflation}_{a+1:a+u}}{P^{inflation}_{a:a+u}} - 1$
equity capital return between a and a+1	$\frac{D^{cash}_{a}}{D^{equity}_{a} P^{equity}_{a:a+1}} \frac{D^{equity}_{a+1} P^{equity}_{a+1:a+2}}{D^{cash}_{a+1}} - 1$
equity income return between a and a+1	$\frac{D^{cash}_{a}}{D^{equity}_{a} P^{equity}_{a:a+1}} \frac{D^{equity}_{a+1}(1 - P^{equity}_{a+1:a+2})}{D^{cash}_{a+1}} - 1$
equity total return between a and a+1	$\frac{D^{cash}_{a}}{D^{equity}_{a} P^{equity}_{a:a+1}} \frac{D^{equity}_{a+1}}{D^{cash}_{a+1}} - 1$
term v nominal par yield at a	$\frac{1 - P^{cash}_{a:a+v}}{\sum_{r=1}^{v} P^{cash}_{a:a+r}}$
term v index-linked par yield at a	$\frac{1 - P^{inflation}_{a:a+v}}{\sum_{r=1}^{v} P^{inflation}_{a:a+r}}$

2.3 Calibrated parameters

We will see that reasonable parameters for this model for the UK are:

Cash		Inflation		Equity	
$\tau=20$		$\tau=20$		$\tau=20$	
$f=5.16\%$		$f=1.40\%$		$f=2.26\%$	
λ	σ	λ	σ	λ	σ
0.377147	-0.006418	0.246355	0.002959	0.313879	-0.009733
0	0.007629	0.031411	0.003570	0.106971	-0.009827
0	0	0.118461	-0.006846	-0.003257	0.000808
0	0	0	0.000285	0.000822	-0.000284
0	0	0	0	0.066525	-0.004170

A further part of the calibration process is to set the values of B_a for $-\tau \le a < 0$. These determine the initial shape of the yield curve. Given a starting yield curve, the back-solution for the C_a is elementary. However, the values will depend on the start date for the projections; we recommend that the model be re-calibrated for each use to fit exactly the market spot curve on the run date.

3 Calibration Issues

3.1 Observables and non-observables

As deflators are not observable, they are not amenable to standard statistical estimation procedures. There are two possible approaches to calibrating a model with deflators.

- Firstly, deflators could be added after the event, to a model whose observable statistical properties have already been established. In some cases (for example, see Chapman, Gordon & Speed, 2001) this is straightforward. But for most statistically motivated models, this route turns out to be very challenging. The resulting deflator formulas are seldom analytically tractable, and may require extensive and time-consuming numerical analysis at run time. See Guthrie and others (2001) for a description of deflators applied to the Wilkie model and the practical issues that arise.
- The second alternative is to develop a model algebraically with explicit deflators, including many unknown parameters. These parameters can then be tweaked so that the observable values behave with the desired volatilities, correlations and other statistical properties. That is the route we have followed here.

3.2 Linear approximations

Our model construction does not make it easy to extract the volatilities. Neither prices nor interest rates are exactly lognormal. Our approach, then, is to consider a lognormal approximation. We approximate log prices and yields by a linear expression in the Brownian innovation. Our coefficients are derived using a Taylor expansion. To evaluate these Taylor expansions, we need to define a starting position. For this we choose all yield curves to be flat and equal to the relevant long spot rate.

Taking first par yields, we have expressions of the form:

$$\frac{1-P_{a:a+v}}{\sum_{r=1}^{v} P_{a:a+r}} \approx \frac{P_{a-1:a}-P_{a-1:a+v}}{\sum_{r=1}^{v} P_{a-1:a+r}} \exp\left[\theta_1(v,\tau,f)\sigma.(B_a - B_{a-1})\right]$$

Similarly, for deflators we have expressions of the form:

$$D_a P_{a:a+v} \approx D_{a-1} P_{a-1:a+v} \exp\left[(\lambda + \theta_2(v,\tau,f)\sigma).(B_a - B_{a-1})\right]$$

where the sensitivity functions are given by:

$$\theta_1(v,\tau,f) = \left(1 - \frac{1}{(1+f)^v}\right)^{-1} \left(\frac{\min\{v,\tau\}}{(1+f)^\tau} - \left(1+\frac{1}{f}\right)\left(1 - \frac{1}{(1+f)^{\min\{v,\tau\}}}\right)\right)$$

$$\theta_2(v,\tau,f) = \min\{v,\tau\} + \frac{1}{f}\left(1 - \frac{1}{(1+f)^{\max\{\tau - v,0\}}}\right)$$

In the first part of the calibration we assume that the f and τ values are known and therefore the functions above can be evaluated. Using these functions, we can calculate approximately the variance-covariance matrix of increments in prices and yields.

We also require approximations for long term average values of yields. As we know that $D_a P_{ab}$ is a martingale in b, we can identify some constant terms in the deflator approximation:

$$\frac{P_{a:a+v}}{P_{a-1:a+v}} \approx \frac{D_{a-1}}{D_a} \exp\left[(\lambda + \theta_2(v,\tau,f)\sigma).(B_a - B_{a-1}) - \tfrac{1}{2}(\lambda + \theta_2(v,\tau,f)\sigma)^2\right]$$

By induction, we can compute the total return on a constant maturity bond index:

$$\prod_{a=1}^{b} \frac{P_{a:a+v}}{P_{a-1:a+v}} \approx \frac{D_0}{D_b} \exp\left[(\lambda + \theta_2(v,\tau,f)\sigma).(B_b - B_0) - \tfrac{b}{2}(\lambda + \theta_2(v,\tau,f)\sigma)^2\right]$$

Substituting in for the original deflator, and taking b large, we find the asymptotic forward rate.

3.3 Approximate Statistical Properties for our Chosen Model

Given our choices of parameters, we used our θ-approximations to estimate approximate volatilities and correlations, as given in the table below.

	Volatilities	Correlations				
Price inflation	**0.5%**	100%	-10%	0%	0%	10%
Real dividend growth	**7.0%**	**-10%**	100%	0%	-20%	-30%
Nominal par yield	**10.0%**	**0%**	**0%**	100%	10%	10%
Index-linked par yield	**10.0%**	**0%**	**-20%**	**10%**	100%	60%
MV total equity return	**20.0%**	**10%**	**-30%**	**10%**	**60%**	100%

Although we have presented these as outputs, these were in fact inputs; we back-solved for σ, λ parameters to fit these assumptions. In this section, we have indicated input assumptions in bold. The other figures are then imposed by the model structure.

We also calculated the risk premiums in excess of cash for three possible investments. The values (according to our approximations) are:

	Risk premium relative to cash (expressed as a force)
Bonds (Term 10)	**0.50%**
Index linked (Term 10)	-0.23%
Equity	**3.50%**

The bond and equity risk premiums were inputs. For index-linked bonds, where less historically credible data is available, our chosen risk premium maximises the efficiency of the model, using principles described in Bezooyen and others (2001). The negative risk premium should not be surprising – for many investors, the ability to lock real returns in represents a reduction in risk relative to cash.

In a fair value context, for reasons familiar to option pricing specialists, only the first of these is relevant. Fair values are unaffected by risk premium or expected return estimates. A change in the expected return does, of course, change prospective statistical distributions, but there is an offsetting change in the deflators so that cash flow present values are invariant. However, risk premiums are vital for individual portfolio selection, where risk is to be balanced against expected return. Furthermore, the idea that risk premiums are available on risky assets is fundamental to the rationale behind many savings products. Practitioners tend to reject as unrealistic any model failing to conjure up their chosen risk premiums.

We can identify (geometric) mean forward rates on various yield curves. These are tabulated below.

Geometric Mean	Sterling	Retail Prices	Equity
0-1 year forward	**4.50%**	**1.75%**	**2.00%**
5-6 year forward	4.83%	1.61%	
10-11 year forward	**5.02%**	1.50%	
15-16 year forward	5.12%	1.43%	
20-21 year forward	5.16%	1.40%	

The short yield of 4.5% and the risk premium of 0.5% for bonds of term 10 in excess of cash were input calibration items discussed earlier. The 5.02% highlighted above follows because 1.0502 = 1.0450 * exp(0.0050)

Comparing deflator ratios, we can deduce the following long run (geometric) rates of price increase:

Index	Long run geometric growth
Retail prices	**2.72%**
Equity capital	6.10%

The retail price assumption was an input; we can reconcile this to mean yields and mean inflation, for

1.0272 * 1.0150 = 1.045 * exp(-0.0023)

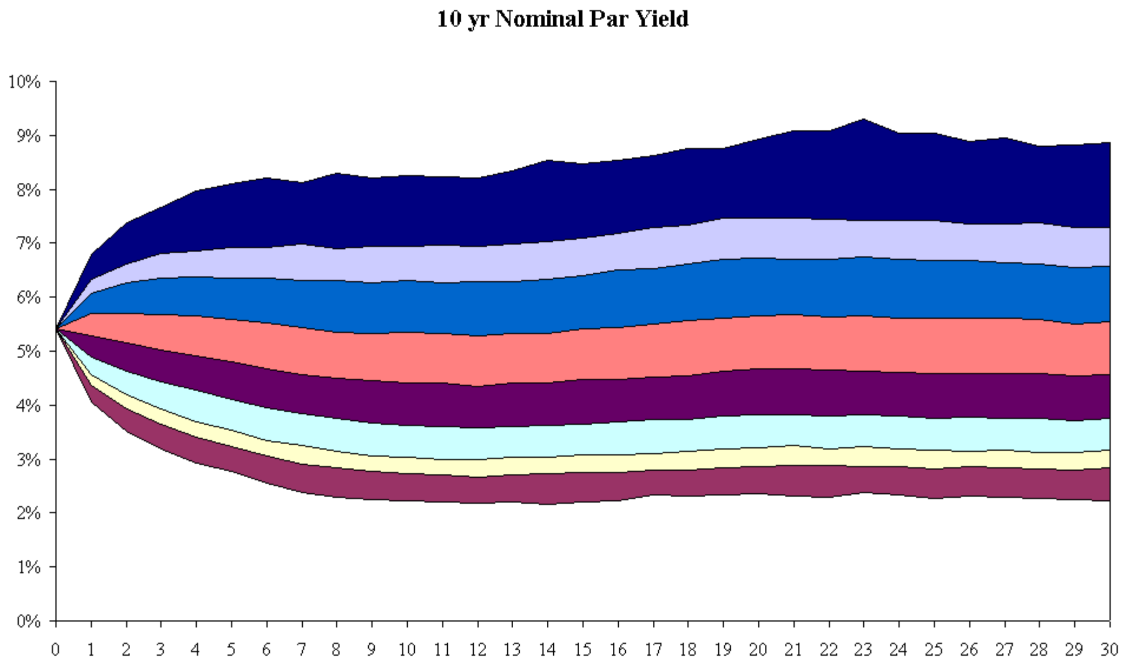
In the same way, the equity capital return relates to yields and risk premiums because:

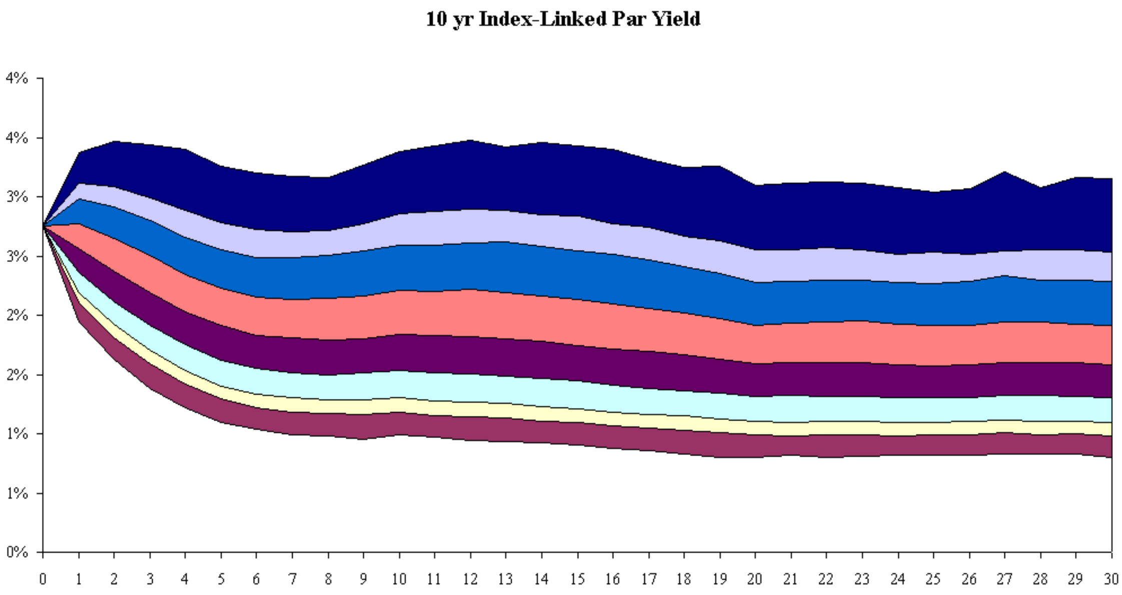
1.0610 * 1.0200 = 1.0450 * exp(0.0350)

4 Sample Model Output

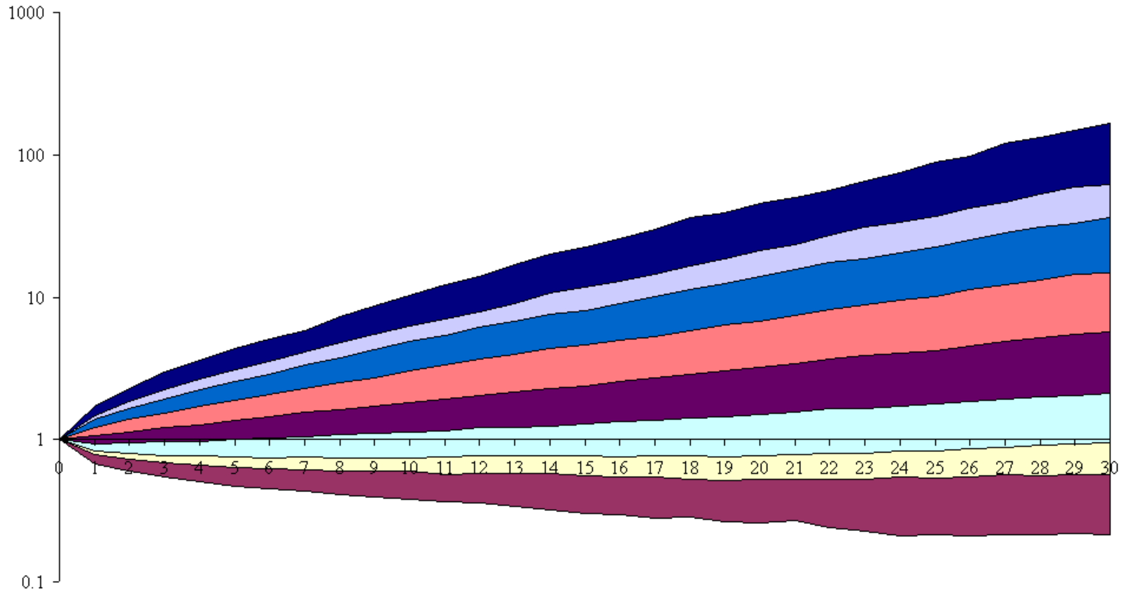
4.1 Sample distributions

In this section we have plotted projections of selected quantities of interest. We have used yield curves as at June 2001, and have shown the 1st, 5th, 10th, 25th, 50th, 75th, 90th, 95th and 99th percentiles based on 5000 simulations. Users can run the free software provided to generate many more outputs than the selections shown here.

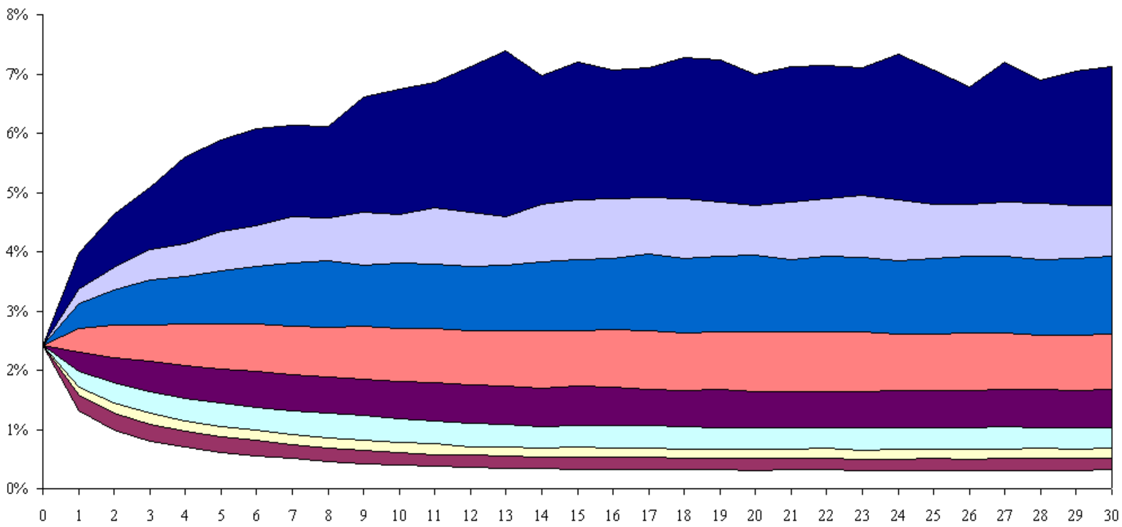


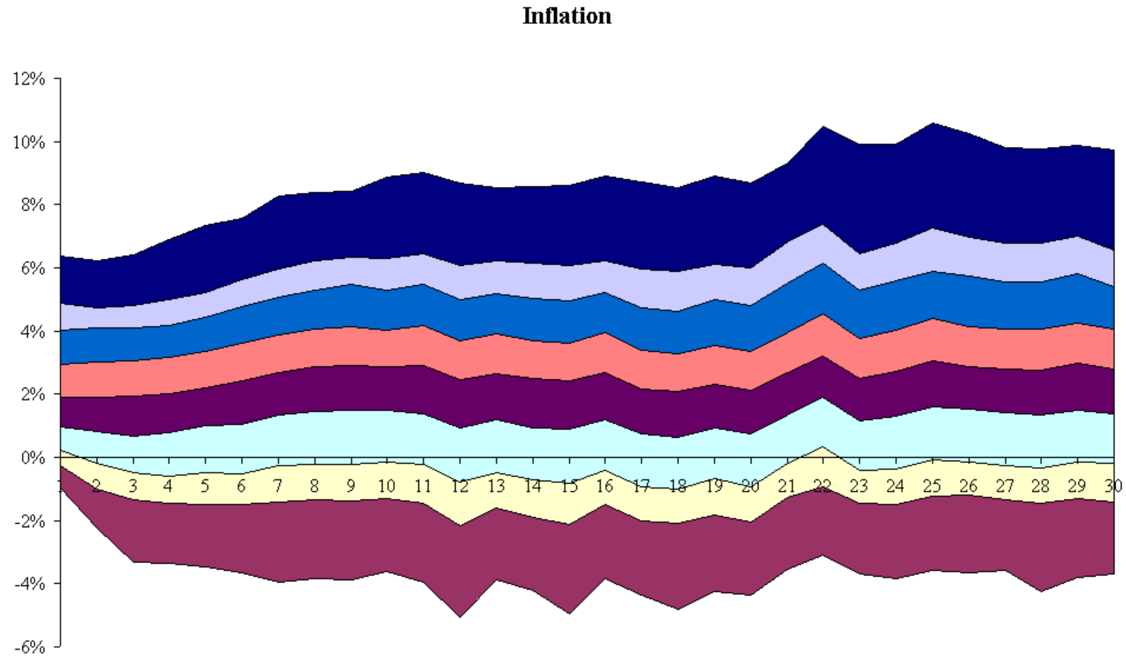


Equity Price Cum Dividend



Equity Dividend Yield





4.2 *Output analysis*

The actual volatilities and correlations of the simulated observable quantities will be different to those input. Quite apart from sampling error, systematic differences are due to higher order terms in our linear approximations. In many cases, these terms are small. However, in our model it appears that the second order terms are relatively significant in the inflation model. The volatility of inflation coming out of the model is therefore higher than was intended when the calibration inputs were chosen. It is possible that an iterative approach would solve this problem. This and other smaller issues serve to highlight the inherent difficulty in calibrating a model indirectly, that is, when the principal parameters cannot be observed in historic data.

5 Conclusions and Further Work

5.1 *Possible model enhancements*

We have published this model because it is useful for teaching purposes and it can give reasonable results to simple problems. It provides a transparent methodology for market-consistent valuation of cash flows.

This model is only the first step on a long journey. A number of proprietary models are available which improve materially on what we have made public. In this section we describe some of the ways in which our base model could be improved, to make it more acceptable for practical use.

In real business, cash flows occur frequently. Even when financial reporting is annual, the run date is often not a whole number of years from the first reporting date. It is useful to have a model, which can produce simulations at arbitrary time intervals. Our annual model does not easily generalise to continuous time. To create a continuous time model, the formulas would need to be rebuilt from scratch.

A second aspect relates to empirical distributions. We have used normal distributions because they are easy to handle. The empirical evidence is broadly unsupportive of our model, and equally unsupportive of several other popular models also based on normal distributions, or on the corresponding diffusion processes in continuous time.

Our model here has considered three asset classes – equities, bonds and index-linked bonds. There are many other asset classes, which investors might choose, including real estate, corporate bonds, mortgages or overseas assets. In some cases, extending our model to other assets is straightforward. The same matrix equations arise, but in a larger number of dimensions. In other cases, for example corporate bonds, inclusion raises a large number of essentially new modelling issues.

Simulation methods are useful for identifying values of many sorts of cash flows. The technique is especially useful for *path dependent* cash flows. A cash flow is *path dependent* if it is a function not only of market conditions on the cash flow date, but also of conditions at previous dates. Simulations are not so easy to use in the case of cash flows with embedded American options. In this case, it is necessary to find a suitable rule for the date at which these options might be exercised – dependent on market conditions. In low-dimensional applications other tools, such as lattice or finite difference methods, might be more appropriate.

There are a number of calibration subtleties associated with the tax system. In a model with deflators, it is axiomatic that the market value of an investment is the present value of its future cash flows. However, it is not always clear which cash flows are being valued – for example, five different investors in the same equity market might all receive different net dividend streams as a result of the investors' different tax positions. To calibrate a model, the user must choose some dominant tax rate implicit in market prices – and this choice will affect the relative attractiveness of different investments for investors with less common tax treatments.

5.2 Other issues in valuing cash flows

We should be aware that the valuation of corporate cash flows could be more complex than the simple application of deflators. The issue is that the planned business cash flows may not be those valued by shareholders – for example because of plan optimism, taxation, capital raising or agency costs. In aggregate, these and other flows are commonly called "frictional costs". Therefore, in a context of corporate valuations it is important that the cash flows valued are those that benefit shareholders; the business model should not omit any of the less obvious frictional costs.

This issue has become important in the context of property/casualty insurance. Many insurers argue that liability cash flows should be discounted at a rate lower than the risk free rate, on account of the variability of these cash flows, even when the cash flows concerned are uncorrelated with outer market movements. On the face of it, such a suggestion is inconsistent with capital market theory; it is arithmetically impossible for a willing buyer and a willing seller both to receive a positive premium for diversifiable risk. On the other hand, an increase in the valuation of liabilities could perhaps be justified on the basis of the inclusion of frictional costs as a liability. It is not yet clear whether accounting standards or fiscal authorities would permit the recognition of future frictional costs as a liability.

5.3 Conclusions

The calculation of fair values in pensions and insurance is a pressing problem, both from a theoretical and implementation perspective. Application of traditional option pricing formulas (such as Black-Scholes) is often not appropriate, because the cash flows to be valued do not have the form of traded options. The deflator approach combines the pricing insights of Black, Scholes and others with the tradition of realistic stochastic projections, which many actuaries already employ for financial management.

Some concerns have been expressed that deflators are available only in a small number of proprietary models, and are therefore out of the reach of many companies, consultants and auditing firms. This was never strictly true - the deflator methodologies have been in the public domain for at least twenty years - but it is true that some effort is required to make them work in practice. It is the purpose of this paper to make the methods practically accessible to a much wider audience. We look forward to seeing greater use of financial techniques in the management of insurance and pension business.

6 References

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