PRICING GUARANTEED LIFE INSURANCE PARTICIPATING POLICIES WITH PERIODICAL PREMIUMS AND SURRENDER OPTION

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Abstract

In this paper we analyse a life insurance endowment policy, paid by periodical premiums, in which the benefit is annually adjusted according to the performance of a special investment portfolio (*reference portfolio*, henceforth) and a minimum return is guaranteed to the policyholder. In particular, we consider both the case in which the periodical premium is constant and the case, very common in Italy, in which also the premium is adjusted according to the performance of the reference portfolio. Moreover, the policy under scrutiny is characterized by the presence of a *surrender option*, i.e., of an American-style put option that enables the policyholder to give up the contract and to receive the *surrender value*. The aim of the paper is to give sufficient conditions under which there exists a (unique) *fair* premium, i.e., a (constant or initial) periodical premium that makes the contract fair in the sense of no-arbitrage. This premium is implicitly defined by an equation (or, alternatively, can be viewed as a fixed point of a suitable function) based on a recursive binomial tree à la Cox, Ross and Rubinstein (1979). An iterative algorithm is then implemented in order to compute it.

Keywords: life insurance with profits, surrender option, periodical premiums, minimum guarantee, fair pricing, multinomial tree.

*Research on "Modelli Matematici Innovativi per lo Studio dei Rischi Finanziari e Assicurativi" supported by Regione Autonoma Friuli-Venezia Giulia.

1 Introduction

Participating life insurance policies (or policies *with profits*) have been very popular in Europe since the seventies. These policies are characterized by the fact that the insurer's profits are shared with the policyholders. There are several ways in which this profit-sharing is realized. Usually "dividends" are credited to the mathematical reserves of the policies at the end of each year, and this implies the "purchase" of additional insurance. The benefits are then "adjusted" in consequence of the "adjustment" of the mathematical reserves. Sometimes, in the case of periodical premium contracts, also the policyholders are called for contributing to the increase of the benefits by means of an increase in the periodical premiums. That is because the (prospective) mathematical reserve, as it is well known, represents the value of future benefits net of the value of future premiums. Therefore the only increase of the mathematical reserve does not imply a proportional increase in the benefits when there are still premiums to be paid to the insurer. This mechanism allows the policies to "follow" the market returns on investments by keeping up-to-date both benefits and premiums. The popularity of participating policies, especially before the large diffusion of equity-linked contracts [1], was probably due to the fact that the policyholders relied on the (rather conservative) financial management of life insurance companies. On the other hand, equity-linked products without (with) minimum guarantees were considered too risky by the policyholders (by the insurers, respectively). In particular, the insurers were reluctant to allow such guarantees because they did not know how to hedge and to price them. A fundamental push in this direction was due to the option pricing theory initiated by the results of Black and Scholes (1973) and Merton (1973), and, later, by the connection of this theory with guaranteed equity-linked products recognized by Brennan and Schwartz (1976) and Boyle and Schwartz (1977). Coming back to participating policies, they are usually coupled with a minimum interest rate guaranteed. However, this rate used to be far lower than the market rates, so that the risk associated to the issue of the minimum guarantee seemed to be quite negligible and was not seriously considered a threat to the solvency of a life insurance company. Only in recent years,

[1] See, e.g., Bacinello and Persson (2002) and the references therein.

after the drop in the market interest rates observed in many industrial countries, this threat has become impending [2]. Then the problem of accurately assessing all the parameters characterizing the guarantees and the participation mechanism has attracted the interest both of researchers and of practitioners. There is now a great attention towards the options embedded in a participating life insurance contract, in particular on the "bonus option", implied by the participation mechanism, and on the "surrender option", i.e., the policyholder's right to early terminate the contract and to receive the *surrender value*. The literature concerning the bonus option, as well as some important issues connected with participating policies, is rather abundant. We recall, for instance, Wilkie (1987), Briys and de Varenne (1994, 1997), Norberg (1999, 2001), Grosen and Jørgensen (2000, 2002), Bacinello (2001a), Consiglio, Cocco and Zenios (2001a, 2001b), Hansen and Miltersen (2002), and Miltersen and Persson (2002). As far as the surrender option is concerned, the first treatments analysing it of which we are aware are due to Albizzati and Geman (1994) and Grosen and Jørgensen (1997) (that, however, do not apply to participating contracts), to Grosen and Jørgensen (2000), Jensen, Jørgensen and Grosen (2001) and Bacinello (2001b) (with reference to these specific contracts), and to Steffensen (2001) (in a more general framework). In particular, Bacinello (2001a) analyses a life insurance product introduced in Italy at the end of the seventies, the so-called *rivalutabile*, for which a special portfolio of investments, covering at least the mathematical reserves of all the policies with profits issued by the same insurance company, is constituted and kept apart from the other assets of the company. At the end of each year the rate of return on this portfolio (*reference portfolio*, henceforth) in the preceding year is assigned to the policyholder, provided that it does not fall below the *technical* rate. Bacinello (2001a) considers both the case in which the policy is paid by a single premium at issuance, and the case in which it is paid by a sequence of periodical premiums annually adjusted according to the performance of the reference portfolio, and obtains a very simple closed-form relation that characterizes "fair" contracts in the Black-Merton-Scholes framework. However, this analysis does not take into account the presence of the surrender option. This gap is partially filled

[2] See, e.g., Grosen and Jørgensen (2000).

by Bacinello (2001b), that employs a recursive binomial formula patterned after the Cox, Ross and Rubinstein (1979) model for pricing the single premium contract. The aim of the present paper is to complete the analysis by pricing also periodical premium contracts. The structure of these contracts is rather complex, and the fair premium is expressed as a fixed point of a function defined by a backward recursive procedure. We prove the existence and uniqueness of such a premium, and implement an iterative algorithm for computing it. This premium is also split into various components. Moreover, we consider both the case in which it is annually adjusted according to the performance of the reference portfolio and the case in which it remains constant. The paper is organized as follows. In Section 2 we describe the structure of the contract and define all the liabilities that the insurer has to face. Section 3 is devoted to the introduction of our valuation framework. The core of the paper is constituted by Section 4, where we price the contract and all its components. Finally, in Section 5 we present some numerical results for the fair premium of the contract and of all its components.

2 The structure of the contract

Consider a life insurance endowment policy issued at time 0 and maturing T years after (at time T). As it is well known, under this contract the insurer is obliged to pay a specified amount of money (*benefit* or *sum insured*) to the beneficiary if the insured dies within the term of the contract or survives the maturity date. More in detail, we assume that, in case of death during the t-th year of contract, the benefit is paid at the end of the year, i.e., at time t ($t = 1, ..., T$); otherwise it is paid at maturity T. We denote by x the age of the insured at time 0, by C_1 the "initial" sum insured, payable in case of death during the first year of contract, and by C_t the benefit payable at time t ($t = 2, ..., T$). While C_1 is given, for $t>1$ C_t is contingent on the performance of the reference portfolio. We assume that the contract is paid by a sequence of periodical premiums, due at the beginning of each year, if the insured is alive. We denote by P_0 the "initial" (net) premium, due at the inception of the contract. It is standard, at least for Italian insurance companies, to compute P_0 in the following way, as like as everything remained unchanged in the future:

$$P_0 = C_1 P^{(i)}_{x:T\rceil} = C_1 \frac{A^{(i)}_{x:T\rceil}}{\ddot{a}^{(i)}_{x:T\rceil}} = C_1 \frac{\sum_{t=1}^{T-1}(1+i)^{-t}\ {}_{t-1/1}q_x \ + \ (1+i)^{-T}\ {}_{T-1}p_x}{\sum_{t=0}^{T-1}(1+i)^{-t}\ {}_t p_x}. \tag{1}$$

Here i represents the (annual compounded) *technical* interest rate, ${}_{t-1/1}q_x$ the probability that the insured dies within the t-th year of contract (i.e., between times $t-1$ and t), and ${}_tp_x$ the probability that the insured is still alive at time t. As usual, these probabilities depend on the age of the insured (and on his(her) sex as well), and are extracted from a mortality table that constitutes, together with i, the so-called *first-order technical bases.* Observe that the premium P_0 defined by relation (1) makes the expected value at time 0 of the benefit C_1 of a standard life insurance endowment policy, discounted from the random time of payment to time 0 with the technical rate i ($=C_1 A^{(i)}_{x:T\rceil}$), equal to the expected value at time 0 of the stream of constant periodical premiums P_0, discounted with i as well ($=P_0\, \ddot{a}^{(i)}_{x:T\rceil}$). Then, on the ground of the first-order technical bases, P_0 makes the contract "fair" at inception. Moreover, if P_0 is the premium actually paid by the policyholder, the technical rate i can be interpreted as the rate of return credited to the policy since the beginning. However, as we have previously said, not everything remains unchanged in the future. The benefit is annually adjusted, and it is often stated that also the periodical premium is contingent on the performance of the reference portfolio. Anyway, as we will see in a moment, all adjustments are made in such a way that the contract remains fair, always on the ground of the first-order technical bases, with regard to the residual policy period. To see how these adjustments are realized, we first denote by $P_t, t = 1, 2, ..., T-1$, the (net) premium paid at time t, if the insured is alive, and by V_t^- (V_t^+) the mathematical reserve of the policy at time t ($t = 1, 2, ..., T-1$) just before the payment of the premium P_t and before (after respectively) the annual adjustment. Given C_t and P_{t-1}, V_t^- is computed as

$$\begin{aligned} V_t^- &= C_t\, A^{(i)}_{x+t:T-t\rceil} - P_{t-1}\, \ddot{a}^{(i)}_{x+t:T-t\rceil} \\ &= C_t \left[\sum_{h=1}^{T-t} (1+i)^{-h}\ {}_{h-1/1}q_{x+t} \ + \ (1+i)^{-(T-t)}\ {}_{T-t}p_{x+t} \right] \\ &\quad - P_{t-1} \sum_{h=0}^{T-t-1} (1+i)^{-h}\ {}_h p_{x+t}, \qquad t = 1, 2, ..., T-1, \end{aligned} \tag{2}$$

where ${}_{h-1/1}q_{x+t}$ represents the probability that the insured dies within the $(t+h)$-th year of contract (i.e., between times $t+h-1$ and $t+h$) conditioned on the event that he(she) is alive at time t, and ${}_{h}p_{x+t}$ is the probability that the insured is still alive at time $t+h$ conditioned on the same event. This reserve is immediately adjusted at a rate δ_t, so that

$$V_t^{+} = V_t^{-}(1+\delta_t), \qquad t = 1, 2, ..., T-1. \tag{3}$$

The rate δ_t is defined as follows:

$$\delta_t = \max\left\{\frac{\eta g_t - i}{1+i}, 0\right\}, \qquad t = 1, 2, ..., T-1, \tag{4}$$

where g_t denotes the rate of return on the reference portfolio during the t-th year of contract, and η, between 0 and 1, identifies a participation coefficient. As we have previously observed, if the initial premium P_0 is expressed by relation (1), a return at the technical rate i has already been credited to the policy. Then, taking into account the adjustment of the mathematical reserve (and disregarding the surrender option), we can argue that the total return granted to the policyholder during the t-th year of contract (except the year at the end of which the benefit is paid) is given by

$$(1+i)(1+\delta_t) - 1 = \max\left\{i, \eta g_t\right\},$$

so that, in this case, i can also be interpreted as a minimum interest rate guaranteed. Moreover, denoting by α_t and β_t the adjustment rates of the benefit and of the premium respectively, i.e.,

$$C_{t+1} = C_t(1+\alpha_t), \qquad t = 1, 2, ..., T-1 \tag{5}$$

and

$$P_t = P_{t-1}(1+\beta_t), \qquad t = 1, 2, ..., T-1, \tag{6}$$

the previously mentioned fairness on the ground of the first-order technical bases and with regard to the residual policy period is expressed by the following relation:

$$V_t^{+} = C_{t+1}\, A^{(i)}_{x+t:\,\overline{T-t}|} - P_t\, \ddot{a}^{(i)}_{x+t:\,\overline{T-t}|}, \qquad t = 1, 2, ..., T-1. \tag{7}$$

Exploiting expressions (2), (3), (5), (6), one can immediately see that relation (7) implies a constraint among the adjustment rates α_t, β_t and δ_t. In particular, α_t turns out to be a suitable mean of the remaining two rates:

$$\alpha_t = w_t\,\delta_t + (1 - w_t)\beta_t, \qquad t = 1, 2, ..., T - 1, \tag{8}$$

with

$$w_t = \frac{V_t^-}{C_t A^{(i)}_{x+t:\,T-t\rceil}}\,.$$

In what follows we will consider two extreme cases, that are actually the most common in Italy.

(a) *Identical adjustment rates* We assume $\alpha_t = \beta_t = \delta_t$ for any t, so that the mathematical reserve, the benefit and the periodical premium are adjusted in the same measure. This situation has also been considered by Bacinello (2001a) that, however, does not take into account the presence of a surrender option. To sum up, in this case we have:

$$C_{t+1} = C_t(1 + \delta_t), \qquad t = 1, 2, ..., T - 1, \tag{9}$$

$$P_t = P_{t-1}(1 + \delta_t), \qquad t = 1, 2, ..., T - 1, \tag{10}$$

with C_1 given and δ_t expressed by relation (4). Our goal is then, in this case, to determine an initial premium P_0 (not necessarily equal to the one defined by expression (1)) that makes the contract *fair*, at inception, on the ground of the market assumptions presented in the next section, and includes also a compensation for the surrender option. Since we will split this premium into various components, it is also useful to express both the benefit and the periodical premium by means of the following iterative relations:

$$C_t = C_1 \prod_{k=1}^{t-1}(1 + \delta_k), \qquad t = 2, 3, ..., T, \tag{11}$$

$$P_t = P_0 \prod_{k=1}^{t}(1 + \delta_k), \qquad t = 1, 2, ..., T - 1. \tag{12}$$

(b) *Constant periodical premiums* We assume $\beta_t = 0$ for any t, so that the premium P_t is constant; hence, when we will treat this case apart, we will denote it by P. Exploiting relations (2), (5) and (8) we have, in this case:

$$C_{t+1} = C_t(1 + w_t\,\delta_t) = C_t(1+\delta_t) - \; C_t\,\delta_t(1-w_t) = C_t(1+\delta_t) - \; \frac{P\delta_t}{P^{(i)}_{x+t:\,\overline{T-t}|}}\,,$$
$$t = 1, 2, ..., T-1, \qquad (13)$$

where

$$P^{(i)}_{x+t:\,\overline{T-t}|} = \frac{A^{(i)}_{x+t:\,\overline{T-t}|}}{\ddot{a}^{(i)}_{x+t:\,\overline{T-t}|}}\,.$$

Note that the adjustment rate of the benefit depends on the pair $(x+t, T-t)$. To simplify the aggregation of policies belonging to a same portfolio, in recent years the "exact" relation (13) has been approximated by the following [3]:

$$C_{t+1} = C_t(1+\delta_t) - \; C_1\,\delta_t\left(1 - \frac{t}{T}\right), \qquad t = 1, 2, ..., T-1, \qquad (14)$$

in which the adjustment rate depends on the duration t and the maturity T, but not on the age of the insured x. More precisely, relation (14) is obtained from (13) by replacing P with the premium defined in expression (1) ($=C_1 P^{(i)}_{x:\,\overline{T}|}$) and by approximating $P^{(i)}_{y:\,\overline{z}|}$ with $1/z$. It can easily be proved, by induction, that relations (13) and (14) imply, respectively:

$$C_t = C_1 \prod_{k=1}^{t-1}(1+\delta_k) \; - \; P\sum_{k=1}^{t-1}\left[\frac{\delta_k}{P^{(i)}_{x+k:\,\overline{T-k}|}} \prod_{h=k+1}^{t-1}(1+\delta_h)\right], \quad t = 2, 3, ..., T, \qquad (15)$$

$$C_t = C_1\left\{\prod_{k=1}^{t-1}(1+\delta_k) \; - \; \sum_{k=1}^{t-1}\left[\delta_k\left(1-\frac{k}{T}\right)\prod_{h=k+1}^{t-1}(1+\delta_h)\right]\right\}, \quad t = 2, 3, ..., T, \qquad (16)$$

with the convention

$$\prod_{h=t}^{t-1}(1+\delta_h) = 1.$$

[3] See Pacati (2000).

From now on, in the case of constant premiums we will assume that the approximated relations (14) and (16) hold. Also in this case our goal is to define a constant premium P which makes the contract fair, at inception, on the ground of our market assumptions, and includes a compensation for the surrender option.

Coming now to the surrender conditions, we assume that surrender takes place (if the contract is still in force) at the beginning of the year, just after the announcement of the benefit for the coming year and before the payment of the periodical premium. Let the surrender value at time t, denoted by R_t, be defined as follows:

$$R_t = \begin{cases} 0 & t = 1, 2 \\ C_{t+1}(1+\rho)^{-(T-t)}\, t/T & t = 3, 4, ..., T-1 \end{cases}, \tag{17}$$

where ρ represents an annual compounded discount rate, usually greater than i. Relation (17) is consistent with Italian practice, according to which nothing is paid back to the policyholder until the insurer has collected at least three periodical premiums.

3 The valuation framework

The contract described in the previous section is a typical example of contingent-claim since it is affected by both the mortality and the financial risk. While the mortality risk determines the expiration time of the contract, the financial risk not only affects the amounts of the benefit and, if not constant, of the periodical premiums, but also the surrender decision. We assume, in fact, that financial and insurance markets are perfectly competitive, frictionless [4], and free of arbitrage opportunities. Moreover, all the agents are supposed to be rational and non-satiated, and to share the same information. Therefore, in this framework, the surrender decision can only be the consequence of a rational choice, taken after comparison, at any time, between the total value of the policy (including the option of surrendering

[4] In particular there are no taxes, no transaction costs such as, e.g., expenses and relative loadings of the insurance premiums, and short-sale is allowed.

it in the future) and the surrender value. As it is standard in actuarial practice, we assume that mortality does not affect (and is not affected by) the financial risk, and that the mortality probabilities introduced in the previous section are extracted from a *risk-neutral* mortality measure, i.e., that all insurance prices are computed as expected values with respect to this specific measure. If, in particular, the insurance company is able to extremely diversify its portfolio in such a way that mortality fluctuations are completely eliminated, then the above probabilities coincide with the "true" ones. Otherwise, if mortality fluctuations do occur, then the "true" probabilities are "adjusted" in such a way that the premium, expressed as an expected value, is implicitly charged by a *safety loading* which represents a compensation for accepting mortality risk. In this case the adjusted probabilities derive from a *change of measure*, as usually occurs in the Financial Economics environment; that is why we have called them "risk-neutral". Coming now to the financial set-up, we assume that the rate of return on risk-free assets is deterministic and constant, and denote by r the annual compounded riskless rate. The financial risk which affects the policy under scrutiny is then generated by a stochastic evolution of the rates of return on the reference portfolio. In this connection, we assume that it is a well-diversified portfolio, split into *units*, and that any kind of yield is immediately reinvested and shared among all its units. Therefore the reinvested yields increase only the unit-price of the portfolio but not the total number of units, that changes when new investments or withdrawals are made. These assumptions imply that the rates of return on the reference portfolio are completely determined by the evolution of its unit price. Denoting by G_τ this unit-price at time τ (≥ 0), we have then:

$$g_t = \frac{G_t}{G_{t-1}} - 1, \quad t = 1, ..., T-1. \tag{18}$$

For describing the stochastic evolution of G_τ, we choose the discrete model by Cox, Ross and Rubinstein (1979), universally acknowledged for its important properties. In particular it may be seen either as an "exact" model under which "exact" values for both European and American-style contingent-claims can be computed, or as an approximation of the Black and Scholes (1973) and Merton (1973) model to which it asymptotically converges. More in detail, we divide each policy year into N sub-periods of equal length, let $\Delta{=}1/N$, fix a volatility parameter $\sigma{>}\sqrt{\Delta}\ln(1{+}r)$, set

u=exp$(\sigma\sqrt{\Delta})$ and d=1/u. Then we assume that G_τ can be observed at the discrete times τ=t+$h\Delta$, t=0, 1, ...; h=0, 1, ..., N−1 and that, conditionally on all relevant information available at time τ, $G_{\tau+\Delta}$ can take only two possible values: uG_τ ("up" value) and dG_τ ("down" value). As it is well known, in this discrete setting absence of arbitrage is equivalent to the existence of a risk-neutral probability measure under which all financial prices, discounted by means of the risk-free rate, are martingales. Under this risk-neutral measure, the probability of the event $\{G_{\tau+\Delta} = uG_\tau\}$ conditioned on all information available at time τ (that is, in particular, on the knowledge of the value taken by G_τ), is given by

$$q = \frac{(1+r)^\Delta - d}{u-d}, \tag{19}$$

while

$$1 - q = \frac{u - (1+r)^\Delta}{u-d}$$

represents the risk-neutral (conditioned) probability of $\{G_{\tau+\Delta} = dG_\tau\}$. We observe that, in order to prevent arbitrage opportunities, we have fixed σ in such a way that $d<(1+r)^\Delta<u$, which implies a strictly positive value for both q and $1-q$. The above assumptions imply that g_t, t=1, 2, ..., T−1, are i.i.d. and take one of the following N+1 possible values:

$$\gamma_j = u^{N-j}d^j - 1, \quad j = 0, 1, ..., N \tag{20}$$

with (risk-neutral) probability

$$Q_j = \binom{N}{j} q^{N-j}(1-q)^j, \quad j = 0, 1, ..., N. \tag{21}$$

Moreover, also the adjustment rates of the mathematical reserve, δ_t, t=1, 2, ..., T−1, are i.i.d., and can take n+1 possible values, given by

$$\mu_j = \frac{\eta\gamma_j - i}{1+i}, \quad j = 0, 1, ..., n-1 \tag{22}$$

with probability Q_j, and 0 with probability $1-\sum_{j=0}^{n-1} Q_j$. Here

$$n = \left\lfloor \frac{N}{2} + 1 - \frac{ln(1+i/\eta)}{2ln(u)} \right\rfloor, \tag{23}$$

with $\lfloor y \rfloor$ the integer part of a real number y, represents the minimum number of "downs" such that a call option on the rate of return on the reference portfolio in a given year with exercise price i/η does not expire in the money.

4 Fair pricing of the contract and its components

In this section we will determine an initial periodical premium P_0 (not necessarily coinciding with that expressed by relation (1)) that makes the contract "fair" on the ground of our assumptions and, in particular, in the sense of no-arbitrage. More precisely, it is defined as the (initial) premium P_0 that makes the market value at time 0 of the insurer's liabilities equal to the market value at time 0 of the stream of periodical premiums. We will consider both the case in which this premium is annually adjusted at the rate δ_t (see relations (10) and (4)) and the case in which it remains constant. In both cases we will split it into various components, that represent the compensation for

a) the *basic contract* (i.e., without profits and without surrender),

b) the *bonus option*,

c) the *surrender option*.

Although these embedded options are not traded separately from the other elements of the contract, we believe that such decomposition can be extremely useful to an insurance company since it allows it to understand the incidence of the various components on the premium and, if necessary, to identify possible changes in the design of the policy. More in detail, we will directly price the basic contract, the non-surrendable participating contract (i.e., with profits but without surrender), and the whole contract. In particular, the basic contract and the non-surrendable participating contract are of European style, while the whole contract is of American style. The premium for the bonus option is then given by the difference between the premium for the non-surrendable participating contract and the premium for the basic contract. Similarly, the premium for the surrender option is given by the difference between the premium for the whole contract and the premium for the non-surrendable participating contract. As we will see in a moment, the premium for each European-style component is expressed by a closed formula. It must be pointed out that the premium for the basic contract is the same both in the case of constant periodical premiums and in the case of adjustable premiums, and remains constant as time goes by. In the latter case, however, the premiums for the non-surrendable participating contract and for the whole contract are both adjusted at the rate δ_t. Therefore, in this case, our decomposition applies only to the first premium, and the incidence of the various components on the total premium changes stochastically with time.

4.1 Fair pricing of the basic contract

The basic contract is a standard endowment policy with constant benefit C_1 and constant premium P^{basic} (to be determined). The insurer's liability is represented by the *deterministic* benefit C_1, payable at the random time of death (end of the year) or, at the latest, at maturity. Then its market value at time 0 is given by

$$C_1 A^{(r)}_{x:\overline{T}|} = C_1 \left[\sum_{t=1}^{T-1} (1+r)^{-t} \ {}_{t-1/1}q_x \ + \ (1+r)^{-T} \ {}_{T-1}p_x \right].$$

Observe that this is the expected value, with respect to the risk-neutral mortality measure introduced in the previous sections, of the benefit discounted from the random time of payment to time 0 with the risk-free rate r. Similarly, the stream of constant periodical premiums P^{basic}, payable at the beginning of each year of contract, if the insured is alive, has market value at time 0 given by

$$P^{basic}\, \ddot{a}^{(r)}_{x:\overline{T}|} = P^{basic} \sum_{t=0}^{T-1} (1+r)^{-t} \ {}_t p_x.$$

Finally, the premium P^{basic} which equals the two above values is given by

$$P^{basic} = C_1 \frac{A^{(r)}_{x:\overline{T}|}}{\ddot{a}^{(r)}_{x:\overline{T}|}} = C_1 P^{(r)}_{x:\overline{T}|} \, . \tag{24}$$

4.2 Fair pricing of the non-surrendable participating contract

The insurer's liability is now represented by the *stochastic* benefit C_t payable at the random time of death of the insured (t=$1, 2, ..., T$) or, at the latest, at maturity (t=T). The assumptions described in Section 3, in particular the risk-neutrality of all the probabilities introduced so far and the stochastic independence between mortality and the financial elements, imply that its time 0 value can be computed in two separate stages: in the first stage the market value at time 0 of the benefit C_t, supposed to be due with certainty at time t, is computed for all t=$1, 2, ..., T$; in the second stage these values are "averaged" with the probabilities of payment at each possible date. We denote by $\pi(C_t)$, t=$1, 2, ..., T$ the values computed in the first stage. While

$$\pi(C_1) = C_1(1 + r)^{-1}, \tag{25}$$

for t>1

$$\pi(C_t) = E^Q[(1 + r)^{-t}C_t], \tag{26}$$

where E^Q denotes expectation taken with respect to the (financial) risk-neutral measure introduced in Section 3. The fair value of the insurer's liability is then given by

$$U^P = \sum_{t=1}^{T-1} \pi(C_t)\, {}_{t-1/1}q_x \, + \, \pi(C_T)\, {}_{T-1}p_x. \tag{27}$$

As far as the time 0 value of the stream of periodical premiums is concerned, it is also computed in the same way. However, we have now to distinguish between the case in which the premiums are adjusted and the case in which they are constant.

(a) *Identical adjustment rates* In this case C_t, for $t>1$, is defined by relation (11), so that, exploiting the stochastic independence of δ_k, $k=1,2,...,T-1$, we can first rewrite (26) as

$$\pi(C_t) = C_1(1+r)^{-t}\prod_{k=1}^{t-1} E^Q[1+\delta_k].$$

Then, taking into account that δ_k, $k=1,2,...,T-1$, are also identically distributed, we have

$$\pi(C_t) = C_1(1+r)^{-t}(1+\mu)^{t-1} = \frac{C_1}{1+\mu}\left(\frac{1+r}{1+\mu}\right)^{-t}, \qquad t = 2,3,...,T, \quad (28)$$

where

$$\mu = E^Q[\delta_k] = \sum_{j=0}^{n-1}\mu_j Q_j, \qquad k = 1,2,...,T-1,$$

with Q_j, μ_j and n defined in relations (19) to (23). Observe that

$$\frac{\mu(1+i)}{\eta(1+r)} = \frac{1+i}{\eta}\, E^Q[(1+r)^{-1}\delta_k] = E^Q\left[(1+r)^{-1}\, max\left\{g_k - \frac{i}{\eta}\,,\,0\right\}\right]$$

represents the market price, at the beginning of each year of contract, of a European call option on the rate of return on the reference portfolio with maturity the end of the year and exercise price i/η. Finally, we can rewrite (27) as

$$U^P = \frac{C_1}{1+\mu}\left[\sum_{t=1}^{T-1}(1+\lambda)^{-t}\ {}_{t-1/1}q_x\ +\ (1+\lambda)^{-T}\ {}_{T-1}p_x\right] = \frac{C_1}{1+\mu}\,A^{(\lambda)}_{x:T\rceil}, \quad (29)$$

where

$$\lambda = \frac{r-\mu}{1+\mu}\,.$$

The periodical premiums have exactly the same structure as the benefit, because they are adjusted in the same measure (see relation (12)). Denoting by P_0^{part} the initial premium (to be determined), with market value

$$\pi(P_0^{part}) = P_0^{part}, \quad (30)$$

and by

$$P_t^{part} = P_0^{part}\prod_{k=1}^{t}(1+\delta_k), \qquad t = 1,2,...,T-1, \quad (31)$$

we have then

$$\pi(P_t^{part}) = E^Q[(1+r)^{-t}P_t^{part}] = P_0^{part}(1+r)^{-t}(1+\mu)^t = P_0^{part}(1+\lambda)^{-t},$$
$$t = 1, 2, ..., T-1. \tag{32}$$

Therefore the fair value at time 0 of the stream of periodical premiums P_t^{part}, given by

$$\sum_{t=0}^{T-1} \pi(P_t^{part})\,{}_tp_x = P_0^{part}\sum_{t=0}^{T-1}(1+\lambda)^{-t}\,{}_tp_x = P_0^{part}\,\ddot{a}^{(\lambda)}_{x:\overline{T|}},$$

equals the fair value of the insurer's liability U^P if and only if

$$P_0^{part} = \frac{C_1 A^{(\lambda)}_{x:\overline{T|}}}{(1+\mu)\,\ddot{a}^{(\lambda)}_{x:\overline{T|}}} = \frac{C_1}{1+\mu}\,P^{(\lambda)}_{x:\overline{T|}}. \tag{33}$$

(b) *Constant periodical premiums* In this case C_t, for $t>1$, is defined by relation (16). Once again we exploit, first of all, the fact that δ_k, $k=1,2,...,T-1$, are i.i.d. and rewrite (26) as

$$\pi(C_t) = C_1(1+r)^{-t}\left[(1+\mu)^{t-1} - \sum_{k=1}^{t-1}\mu(1+\mu)^{t-k-1}\left(1-\frac{k}{T}\right)\right].$$

Then, after some simple algebraic manipulations of the expression between square brackets, we obtain

$$\pi(C_t) = C_1\left(1-\frac{1}{\mu T}\right)(1+r)^{-t} + \frac{C_1}{\mu T}(1+\lambda)^{-t} - \frac{C_1}{T}\,t\,(1+r)^{-t},$$
$$t = 2, 3, ..., T. \tag{34}$$

Finally, we rewrite (27) as

$$U^P = C_1\left(1-\frac{1}{\mu T}\right)A^{(r)}_{x:\overline{T|}} + \frac{C_1}{\mu T}A^{(\lambda)}_{x:\overline{T|}} - \frac{C_1}{T}(IA)^{(r)}_{x:\overline{T|}}, \tag{35}$$

where

$$(IA)^{(r)}_{x:\overline{T|}} = \sum_{t=1}^{T-1} t(1+r)^{-t}\,{}_{t-1/1}q_x + T(1+r)^{-T}\,{}_{T-1}p_x.$$

We denote by P^{part} the (constant) periodical premium, payable at the beginning of each year of contract, if the insured is alive. As in the case of the basic contract, the time 0 value of the stream of constant periodical premiums P^{part} is given by $P^{part}\,\ddot{a}^{(r)}_{x:\,T\rceil}$, so that the fair premium for the non-surrendable participating contract is now

$$P^{part} = P^{basic}\left(1 - \frac{1}{\mu T}\right) + \frac{C_1\left[A^{(\lambda)}_{x:\,T\rceil} - \mu(IA)^{(r)}_{x:\,T\rceil}\right]}{\mu T \ddot{a}^{(r)}_{x:\,T\rceil}}, \tag{36}$$

where P^{basic} is the fair premium for the basic contract.

4.3 Fair pricing of the whole contract

The insurer's liabilities are now represented by the *stochastic* benefit C_t payable at the random time of death of the insured (t=$1, 2, ..., T$) or, at the latest, at maturity (t=T), if the policyholder does not surrender the contract, and by the *stochastic* surrender value R_t (defined by relation (17)) payable at the random time of surrender (t=$1, 2, ..., T-1$), otherwise. On the other hand, the policyholder's liabilities are given by the periodical premiums P_t (constant or adjusted according to relation (10)), payable at the beginning of each year of contract until either maturity, or death of the insured, or surrender, whichever comes first. We do not need to distinguish between the case of constant periodical premiums and the case of adjustable premiums. As already said, our goal is to determine an initial premium P_0 that makes the contract "fair", at inception, in the sense of no-arbitrage. To this end we completely forget relation (1) (and the consequent interpretation of the technical rate i as a minimum interest rate guaranteed). Only in the numerical section we will compare our premium with that computed according to relation (1). Moreover, in the case of constant premiums, we assume that the benefit is adjusted according to the approximated relation (14). Under our assumptions, the stochastic evolution of the benefit $\{C_t, t$=$1, 2, ...,T\}$ can be represented by means of an (n+1)-nomial tree. In the root of this tree we represent the initial benefit C_1 (given); then each node of the tree has n+1 branches that connect it to n+1 following nodes. In the nodes at time t we represent the possible values of C_{t+1}. The possible trajectories that the stochastic process of the benefit can follow from time 0 to time t ($t = 1, 2, ..., T-1$) are $(n+1)^t$, but not all these trajectories lead to different nodes. The tree is, in fact, *recombining*, and the different nodes (or *states of nature*) at time t are only $\binom{n+t}{n}$.

In the same tree we can also represent the surrender values R_t defined by relation (17) and, for any given initial premium P_0, the periodical premiums P_t [5], the value of the whole contract given by the difference between the value of the insurer's liabilities and that of the policyholder's liabilities, and a *continuation* value that we are going to define immediately. The last two values can be computed by means of a backward recursive procedure operating from time $T{-}1$ to time 0. To see how, we denote, first of all, by $\{F_t,\, t=1,...,T{-}1\}$ and $\{W_t,\, t=0,1,...,T{-}1\}$ the stochastic processes with components the values of the whole contract, and the continuation values respectively, at the beginning of the $(t{+}1)$-th year of contract (time t). Then we observe that in each node at time $T{-}1$ (if the insured is alive) the continuation value is given by

$$W_{T-1} = (1+r)^{-1} C_T - P_{T-1} \tag{37}$$

since the benefit C_T is due with certainty at time T. The value of the whole contract is therefore the maximum between the continuation and the surrender value, since the (rational and non-satiated) policyholder chooses between continuation and surrender in order to maximize his(her) profit:

$$F_{T-1} = max\{W_{T-1},\, R_{T-1}\}. \tag{38}$$

Assume now to be, at time $t{<}T{-}1$, in a given node K. For ease of notation we have not indexed so far either the benefit, or the surrender value, or the periodical premium, or the value of the whole contract, or the continuation value, in a given node. Now, in order to catch the link between values at time t and values at time $t{+}1$, we denote by C^K_{t+1}, R^K_t, P^K_t, F^K_t, W^K_t all of them in the node K, and by $F^{K(j)}_{t+1}$, $W^{K(j)}_{t+1}$ $j{=}0,1,...,n$, the value of the whole contract and the continuation value at time $t{+}1$ in each node following K. More in detail, $F^{K(j)}_{t+1}$ ($W^{K(j)}_{t+1}$ respectively) , $j{=}0,1,...,n{-}1$, represent the value when $\delta_{t+1}{=}\mu_j$ (with risk-neutral probability Q_j), while $F^{K(n)}_{t+1}$ ($W^{K(n)}_{t+1}$) represents the value corresponding to $\delta_{t+1}{=}0$ (with probability $1{-}\sum_{j=0}^{n-1} Q_j$). We observe that, in the node K, to continue the contract means to pay immediately the premium P^K_t and to receive, at time $t{+}1$,

[5] In the case of constant premiums, they are deterministic and therefore the same in each node of the tree.

the benefit C^K_{t+1}, if the insured dies within 1 year, or to be entitled to a contract whose total random value (including the option of surrendering it in the future) equals F_{t+1}, if the insured survives. The continuation value at time t (in the node K) is then given by the difference between the risk-neutral expectation of the latter payoff, discounted for 1 year with the risk-free rate, and the premium:

$$\begin{aligned} W^K_t &= (1+r)^{-1}\left\{ q_{x+t}C^K_{t+1} + p_{x+t}\left[\sum_{j=0}^{n-1} F^{K(j)}_{t+1}Q_j\ + \right.\right. \\ &+ \left.\left. F^{K(n)}_{t+1}\left(1-\sum_{j=0}^{n-1}Q_j\right)\right]\right\} - P^K_t, \quad t=0,1,...,T-2. \end{aligned} \tag{39}$$

Here q_{x+t} denotes the probability that the insured, alive at time t, dies within 1 year, and $p_{x+t} = 1 - q_{x+t}$. Finally, we have:

$$F^K_t = max\{W^K_t, R^K_t\}, \quad t = 1,2,...,T-2. \tag{40}$$

Recall that P_t, F_t and W_t are computed, in each node of the tree, under the assumption that P_0 is given, hence they are functions of P_0. In particular, we let

$$W_0 = f(P_0). \tag{41}$$

Recall, moreover, that we have defined a contract "fair" when the market value at time 0 of the insurer's liabilities equals that of the policyholder's liabilities. Then we can state that the whole contract is fairly priced if and only if

$$f(P_0) = 0. \tag{42}$$

Our problem is now to establish if there exists a (unique) initial premium P_0 that makes the contract fair. To this end, we first observe that the function f is continuous in the interval $[\,0,+\infty[\,$, that $f(0) > 0$ (since $C_1 > 0$), and

$$\lim_{P_0\to+\infty} f(P_0) = -\infty.$$

The existence of such a premium is then guaranteed. Its uniqueness is instead an immediate consequence of the following result:

Proposition 1 *The function f defined by relation (41) (together with relations (37) to (40)) is strictly monotonic with respect to P_0.*

Proof 1 *Observe, first of all, that C_t and R_t are independent of P_0 both in the case of constant periodical premiums and in the case of adjustable premiums (see relations (11), (16) and (17)) while, in both cases, the periodical premium P_t is strictly increasing with P_0. Then W_{T-1} is strictly decreasing with P_0 (see relation (37)). Now assume that, in each node at time t+1 ($t = 0, 1, ..., T-2$), W_{t+1} is strictly decreasing with respect to P_0. Therefore, from relations (38) and (40), we argue that F_{t+1} is weakly decreasing in each node at time t+1. Given this, from relation (39) we have that W_t is strictly decreasing in each node at time t. Finally, by backward induction, we conclude that $W_0 = f(P_0)$ is strictly decreasing with P_0.*

In the following section we will implement an iterative algorithm in order to find the zero of the function f. To conclude this section, we observe that the fair premium P_0 can also be viewed as the (unique) fixed point of the function

$$g(P_0) = (1+r)^{-1}\left\{q_x C_1 + p_x\left[\sum_{j=0}^{n-1} F_1^{(j)} Q_j + F_1^{(n)}\left(1 - \sum_{j=0}^{n-1} Q_j\right)\right]\right\},$$

where we have denoted by $F_1^{(j)}$, j=0,1,...,n, the value of the whole contract in each node at time 1.

5 Numerical results

In this section we present some numerical results for the initial fair premium of the contract and of all its components, both in the case in which this premium is annually adjusted at the rate δ_t and in the case in which it remains constant. To obtain these results we have extracted the mortality probabilities from the Italian Statistics for Females Mortality in 1991, fixed C_1=1, T=5, N=250, and considered different values for the remaining parameters. We observe that our choice for N implies a daily change in the unit price of the reference portfolio since there are about 250 trading days in a year. Moreover, this choice guarantees a very good approximation to the Black and Scholes (1973) and Merton (1973) model. In fact, when the unit price of the reference portfolio follows a geometric Brownian motion with volatility parameter σ, the market value, at the beginning of each year of contract, of a European call option written on the rate of return on the reference portfolio with maturity the end of the year and exercise price i/η is given by

$$\phi(a) - \frac{1+i/\eta}{1+r}\,\phi(b),$$

where

$$a = \frac{\ln(1+r) - \ln(1+i/\eta)}{\sigma} + \frac{\sigma}{2}, \qquad b = a - \sigma,$$

and ϕ denotes the cumulative distribution function of a standard normal variate. In a very large amount of numerical experiments carried out with different sets of parameters we have found that the difference between this Black and Scholes (1973) price and the one obtained in our model (with N=250) is less than 1 basis point (bp). However, this high number of steps in each year requires a large amount of CPU time; that is why we have not fixed a high value for T. In order to get some numerical feeling and to catch some comparative statics properties of the model, we have fixed a basic set of values for the parameters x, r, i, η, σ, ρ, and after we have moved each parameter one at a time. For comparison, we have also computed the premium defined by relation (1). As already discussed in Section 2, when this is the (initial) premium paid by the policyholder, the technical rate i can be interpreted as a minimum interest rate guaranteed. To avoid confusion, here we denote by P_0^{whole} (P^{whole}) the initial premium of the whole contract in the case of adjustable premiums (of constant premiums respectively), and by P_0^{comp} the premium defined by relation (1). Moreover, we denote by B_0 (B) the initial premium for the bonus option, given by $P_0^{part} - P^{basic}$ ($P^{part} - P^{basic}$ respectively), and by S_0 (S) the initial premium for the surrender option, given by $P_0^{whole} - P_0^{part}$ ($P^{whole} - P^{part}$ respectively). The basic set of parameters is as follows:

$$x = 50,\ r = 0.05,\ i = 0.03,\ \eta = 0.5,\ \sigma = 0.15,\ \rho = 0.035.$$

With these parameters we have obtained the following results:

$$P^{basic} = 0.1734,\ B_0 = 0.0102,\ P_0^{part} = 0.1836,\ S_0 = 0.0010,\ P_0^{whole} = 1.1846,$$
$$B = 0.0100,\ P^{part} = 0.1834,\ S = 0.0002,\ P^{whole} = 0.1836,\ P_0^{comp} = 0.1839.$$

Note that, in this case, the premium for the surrender option is very low (almost negligible if the premiums are constant), while the premium for the bonus option is about 5.88% of the basic premium (5.77% respectively). Moreover, the premium defined by relation (1) is too low in the case of adjustable periodical premiums ($P_0^{comp} < P_0^{whole}$) and too high in the case of constant premiums ($P_0^{comp} > P^{whole}$). Our results are reported in Tables 1 to 6. More in detail, in Table 1 we present the results obtained when x varies between 40 and 60 and in Table 2 those obtained when r varies between 3% and 10% with step 0.5%. In Table 3 i varies between 0 and 5% with step 0.5%; in Table 4 η varies between 5% and 100% with step 5%; in Table 5 σ varies between 5% and 50% with step 5%. Finally, in Table 6 we move the surrender parameter ρ from 0 to 5% with step 0.5%.

TABLE 1

The whole premium and all its components versus the age of the insured x

		ADJUSTABLE PREMIUMS				CONSTANT PREMIUMS				
x	P^{basic}	B_0	P_0^{part}	S_0	P_0^{whole}	B	P^{part}	S	P^{whole}	P_0^{comp}
40	0.1728	0.0102	0.1830	0.0008	0.1838	0.0100	0.1828	0.0001	0.1829	0.1833
41	0.1728	0.0102	0.1830	0.0008	0.1838	0.0100	0.1828	0.0002	0.1830	0.1833
42	0.1728	0.0102	0.1830	0.0009	0.1839	0.0100	0.1828	0.0002	0.1830	0.1834
43	0.1729	0.0102	0.1831	0.0009	0.1840	0.0100	0.1829	0.0002	0.1831	0.1834
44	0.1730	0.0102	0.1832	0.0009	0.1841	0.0100	0.1830	0.0002	0.1832	0.1835
45	0.1730	0.0102	0.1832	0.0009	0.1841	0.0100	0.1830	0.0002	0.1832	0.1835
46	0.1731	0.0102	0.1833	0.0009	0.1842	0.0100	0.1831	0.0002	0.1833	0.1836
47	0.1732	0.0102	0.1834	0.0009	0.1843	0.0100	0.1832	0.0002	0.1834	0.1837
48	0.1732	0.0102	0.1834	0.0009	0.1843	0.0100	0.1832	0.0002	0.1834	0.1837
49	0.1733	0.0102	0.1835	0.0010	0.1845	0.0100	0.1833	0.0002	0.1835	0.1838
50	0.1734	0.0102	0.1836	0.0010	0.1846	0.0100	0.1834	0.0002	0.1836	0.1839
51	0.1735	0.0102	0.1837	0.0010	0.1847	0.0100	0.1835	0.0002	0.1837	0.1840
52	0.1736	0.0102	0.1838	0.0010	0.1848	0.0100	0.1836	0.0002	0.1838	0.1841
53	0.1737	0.0102	0.1839	0.0011	0.1850	0.0100	0.1837	0.0002	0.1839	0.1842
54	0.1739	0.0101	0.1840	0.0011	0.1851	0.0100	0.1839	0.0002	0.1841	0.1844
55	0.1740	0.0101	0.1841	0.0011	0.1852	0.0100	0.1840	0.0002	0.1842	0.1845
56	0.1742	0.0101	0.1843	0.0012	0.1855	0.0100	0.1842	0.0002	0.1844	0.1847
57	0.1744	0.0101	0.1845	0.0012	0.1857	0.0099	0.1843	0.0003	0.1846	0.1848
58	0.1746	0.0101	0.1847	0.0013	0.1860	0.0099	0.1845	0.0003	0.1848	0.1850
59	0.1748	0.0101	0.1849	0.0013	0.1862	0.0099	0.1847	0.0003	0.1850	0.1853
60	0.1750	0.0101	0.1851	0.0014	0.1865	0.0099	0.1849	0.0003	0.1852	0.1855

From the results reported in Table 1 we notice that the age of the insured has a very small influence on the premiums, at least in the range of values here considered. As expected, the basic premium (P^{basic}) and the premium computed by Italian insurance companies (P_0^{comp}) are increasing with x, while the premiums for the bonus option (B_0, B) are decreasing. However, the increasing trend of the basic premium "beats" the decreasing trend of the premium for the bonus option, so that the premiums for the non-surrendable participating contract (P_0^{part}=P^{basic}+B_0, P^{part}=P^{basic}+B) increase with x. The premiums for the surrender option (S_0, S) are also increasing with x, and so are the premiums for the whole contract (P_0^{whole}=P^{part}+S_0, P^{whole}= P^{part}+S). The incidence of the premium for the bonus option on the total premium decreases from 5.55% to 5.42% in the case of adjustable premiums, and from 5.47% to 5.35% in the case of constant premiums, while the surrender option is very cheap in the first case (its incidence on P_0^{whole} increases from 0.44% to 0.75%) and almost valueless in the second case (from 0.05% to 0.16% of P^{whole}). Finally, the premium P_0^{whole} is always greater than P_0^{comp}, while P^{whole} is always lower than (and indeed very close to) P_0^{comp}.

TABLE 2

The whole premium and all its components versus the risk-free rate r

$\boxed{P_0^{comp} = 0.1839}$

		ADJUSTABLE PREMIUMS				CONSTANT PREMIUMS			
r	P^{basic}	B_0	P_0^{part}	S_0	P_0^{whole}	B	P^{part}	S	P^{whole}
0.030	0.1839	0.0088	0.1927	0.0000	0.1927	0.0088	0.1927	0.0000	0.1927
0.035	0.1812	0.0092	0.1904	0.0000	0.1904	0.0091	0.1903	0.0000	0.1903
0.040	0.1786	0.0095	0.1881	0.0000	0.1881	0.0094	0.1880	0.0000	0.1880
0.045	0.1760	0.0098	0.1858	0.0004	0.1862	0.0097	0.1857	0.0001	0.1858
0.050	0.1734	0.0102	0.1836	0.0010	0.1846	0.0100	0.1834	0.0002	0.1836
0.055	0.1709	0.0105	0.1814	0.0015	0.1829	0.0103	0.1812	0.0003	0.1815
0.060	0.1684	0.0109	0.1793	0.0020	0.1813	0.0106	0.1790	0.0005	0.1795
0.065	0.1660	0.0112	0.1772	0.0025	0.1797	0.0109	0.1769	0.0007	0.1776
0.070	0.1636	0.0115	0.1751	0.0030	0.1781	0.0112	0.1748	0.0010	0.1758
0.075	0.1612	0.0119	0.1731	0.0036	0.1767	0.0115	0.1727	0.0012	0.1739
0.080	0.1589	0.0123	0.1712	0.0040	0.1752	0.0118	0.1707	0.0016	0.1723
0.085	0.1566	0.0126	0.1692	0.0046	0.1738	0.0121	0.1687	0.0019	0.1706
0.090	0.1544	0.0130	0.1674	0.0050	0.1724	0.0124	0.1668	0.0022	0.1690
0.095	0.1522	0.0133	0.1655	0.0055	0.1710	0.0127	0.1649	0.0026	0.1675
0.100	0.1500	0.0137	0.1637	0.0060	0.1697	0.0130	0.1630	0.0030	0.1660

As expected, all the results reported in Table 2 are very sensitive with respect to the market rate r. The basic premium is obviously decreasing with r, and so are the premiums for the non-surrendable participating contract and for the whole contract, in spite of the increasing trend of the premiums for the bonus option and for the surrender option. The surrender option is valueless when $r \leq 4\%$ and goes up to 3.54% of the total premium in the case of adjustable premiums, to 1.81% in the case of constant premiums. The premium for the bonus option, instead, increases from 4.57% to 8.07% in the first case, from 4.57% to 7.83% in the second one. Finally, there is a value of r between 5% and 5.5% such that P_0^{whole}=P_0^{comp}, and between 4.5% and 5% such that P^{whole}=P_0^{comp}.

TABLE 3

The whole premium and all its components versus the technical rate i

$\boxed{P^{basic} = 0.1734}$

i	ADJUSTABLE PREMIUMS				CONSTANT PREMIUMS				P_0^{comp}
	B_0	P_0^{part}	S_0	P_0^{whole}	B	P^{part}	S	P^{whole}	
0.000	0.0161	0.1895	0.0025	0.1920	0.0161	0.1895	0.0003	0.1898	0.2010
0.005	0.0150	0.1884	0.0022	0.1906	0.0149	0.1883	0.0003	0.1886	0.1980
0.010	0.0139	0.1873	0.0019	0.1892	0.0138	0.1872	0.0003	0.1875	0.1951
0.015	0.0129	0.1863	0.0017	0.1880	0.0128	0.1862	0.0002	0.1864	0.1922
0.020	0.0119	0.1853	0.0014	0.1867	0.0118	0.1852	0.0002	0.1854	0.1894
0.025	0.0110	0.1844	0.0012	0.1856	0.0109	0.1843	0.0002	0.1845	0.1866
0.030	0.0102	0.1836	0.0010	0.1846	0.0100	0.1834	0.0002	0.1836	0.1839
0.035	0.0094	0.1828	0.0008	0.1836	0.0092	0.1826	0.0001	0.1827	0.1812
0.040	0.0086	0.1820	0.0006	0.1826	0.0084	0.1818	0.0001	0.1819	0.1786
0.045	0.0079	0.1813	0.0005	0.1818	0.0077	0.1811	0.0001	0.1812	0.1760
0.050	0.0072	0.1806	0.0003	0.1809	0.0071	0.1805	0.0000	0.1805	0.1734

From Table 3 we observe that the technical rate i has a discrete influence on the premium for the bonus option, as expected, and it also affects the premium for the surrender option, at least in the case of adjustable premiums. All the values reported in the table are decreasing, and there is a level of i that makes the total premium equal to P_0^{comp} (between 2.5% and 3% in the case of adjustable premiums, between 3% and 3.5% in the case of constant premiums). The incidence of the bonus option on the total premium decreases from 8.39% to 3.98% in the first case, from 8.48% to 3.93% in the second one, while that of the surrender option decreases from 1.30% to 0.17% (from 0.16% to 0 respectively).

TABLE 4

The whole premium and all its components versus the participation coefficient η

$\boxed{P^{basic} = 0.1734,\ P_0^{comp} = 0.1839}$

η	ADJUSTABLE PREMIUMS B_0	P_0^{part}	S_0	P_0^{whole}	CONSTANT PREMIUMS B	P^{part}	S	P^{whole}
0.05	0.0000	0.1734	0.0000	0.1734	0.0000	0.1734	0.0000	0.1734
0.10	0.0002	0.1736	0.0000	0.1736	0.0002	0.1736	0.0000	0.1736
0.15	0.0009	0.1743	0.0000	0.1743	0.0009	0.1743	0.0000	0.1743
0.20	0.0019	0.1753	0.0000	0.1753	0.0018	0.1752	0.0000	0.1752
0.25	0.0031	0.1765	0.0000	0.1765	0.0030	0.1764	0.0000	0.1764
0.30	0.0044	0.1778	0.0000	0.1778	0.0043	0.1777	0.0000	0.1777
0.35	0.0058	0.1792	0.0000	0.1792	0.0056	0.1790	0.0000	0.1790
0.40	0.0072	0.1806	0.0003	0.1809	0.0071	0.1805	0.0000	0.1805
0.45	0.0087	0.1821	0.0007	0.1828	0.0085	0.1819	0.0001	0.1820
0.50	0.0102	0.1836	0.0010	0.1846	0.0100	0.1834	0.0002	0.1836
0.55	0.0117	0.1851	0.0013	0.1864	0.0115	0.1849	0.0002	0.1851
0.60	0.0132	0.1866	0.0017	0.1883	0.0130	0.1864	0.0003	0.1867
0.65	0.0147	0.1881	0.0021	0.1902	0.0146	0.1880	0.0004	0.1884
0.70	0.0162	0.1896	0.0025	0.1921	0.0162	0.1896	0.0004	0.1900
0.75	0.0177	0.1911	0.0029	0.1940	0.0178	0.1912	0.0006	0.1918
0.80	0.0193	0.1927	0.0033	0.1960	0.0194	0.1928	0.0006	0.1934
0.85	0.0208	0.1942	0.0038	0.1980	0.0210	0.1944	0.0008	0.1952
0.90	0.0224	0.1958	0.0042	0.2000	0.0227	0.1961	0.0008	0.1969
0.95	0.0239	0.1973	0.0046	0.2019	0.0244	0.1978	0.0009	0.1987
1.00	0.0255	0.1989	0.0050	0.2039	0.0261	0.1995	0.0010	0.2005

As far as the participation coefficient η is concerned, we notice, from Table 4, a very strong influence on the premium for the bonus option, as expected, and a discrete influence also on the premium for the surrender option in the case of adjustable premiums. All the premiums reported in the table are increasing, even those for the surrender option, and this is a bit surprising. In particular, such option is valueless when $\eta \leq 35\%$ and goes up to 2.45% of the total premium in the case of adjustable premiums (valueless when $\eta \leq 40\%$ and until the 0.50% of the total premium in the case of constant premiums respectively). The bonus option is valueless, in both cases, when η=5%, and goes up to 12.51% of the total premium in the first case, to 13.02% in the second one. Finally, there is a value also for η that makes the total premium equal to P_0^{comp} (between 45% and 50% in the case of adjustable premiums, between 50% and 55% in the case of constant premiums).

TABLE 5

The whole premium and all its components versus the volatility coefficient σ

$$\boxed{P^{basic} = 0.1734,\ P_0^{comp} = 0.1839}$$

	ADJUSTABLE PREMIUMS				CONSTANT PREMIUMS			
σ	B_0	P_0^{part}	S_0	P_0^{whole}	B	P^{part}	S	P^{whole}
0.05	0.0029	0.1763	0.0000	0.1763	0.0028	0.1762	0.0000	0.1762
0.10	0.0065	0.1799	0.0001	0.1800	0.0063	0.1797	0.0000	0.1797
0.15	0.0102	0.1836	0.0010	0.1846	0.0100	0.1834	0.0002	0.1836
0.20	0.0139	0.1873	0.0019	0.1892	0.0137	0.1871	0.0004	0.1875
0.25	0.0175	0.1909	0.0029	0.1938	0.0175	0.1909	0.0007	0.1916
0.30	0.0212	0.1946	0.0038	0.1984	0.0214	0.1948	0.0009	0.1957
0.35	0.0248	0.1982	0.0049	0.2031	0.0253	0.1987	0.0013	0.2000
0.40	0.0284	0.2018	0.0060	0.2078	0.0293	0.2027	0.0016	0.2043
0.45	0.0320	0.2054	0.0071	0.2125	0.0333	0.2067	0.0020	0.2087
0.50	0.0356	0.2090	0.0082	0.2172	0.0374	0.2108	0.0024	0.2132

All the comments concerning the behaviour of the premiums with respect to the participation coefficient η are still valid when referred to the volatility parameter σ (see Table 5). In particular, in the case of adjustable premiums, the surrender option is valueless when σ=5% and reaches the 3.78% of the total premium when σ=50% (valueless when $\sigma \leq 10\%$ and up to 1.13% of the total premium in the case of constant premiums respectively), and the bonus option increases from 1.64% to 16.39% of the total premium in the first case (from 1.59% to 17.54% in the second one). Finally, a value of σ between 10% and 15% makes P_0^{whole}=P_0^{comp}, while P^{whole}=P_0^{comp} when σ is between 15% and 20%.

TABLE 6

The whole premium and all its components versus the surrender parameter ρ

$$\boxed{\begin{gathered} P^{basic} = 0.1734,\ P_0^{comp} = 0.1839 \\ B_0 = 0.0102,\ P_0^{part} = 0.1836 \\ B = 0.0100,\ P^{part} = 0.1834 \end{gathered}}$$

	ADJUST. PR.		CONST. PR.	
ρ	S_0	P_0^{whole}	S	P^{whole}
0.000	0.0096	0.1932	0.0060	0.1894
0.005	0.0077	0.1913	0.0045	0.1879
0.010	0.0058	0.1894	0.0033	0.1867
0.015	0.0046	0.1882	0.0023	0.1857
0.020	0.0036	0.1872	0.0016	0.1850
0.025	0.0028	0.1864	0.0009	0.1843
0.030	0.0018	0.1854	0.0005	0.1839
0.035	0.0010	0.1846	0.0002	0.1836
0.040	0.0001	0.1837	0.0000	0.1834
$\geq$ 0.045	0.0000	0.1836	0.0000	0.1834

As expected, the discount rate ρ used for computing the surrender values has a negative influence on the premium for the surrender option and thus on the total premium (see Table 6). However, the surrender option is not very expensive even when ρ=0. This is certainly due to the fact that surrender is only theoretically admitted when the duration of the policy is less than 3 years (see relation (17)), and here we are considering a contract that matures after only 5 years! In particular S_0 equals the 4.97% of the total premium P_0^{whole} when ρ=0 and becomes null when $\rho \geq 4.5\%$, while S equals the 3.17% of P^{whole} when ρ=0 and becomes null when $\rho \geq 4\%$. Finally, ρ=3% is such that the total premium equals P_0^{comp} in the case of constant premiums; in the case of adjustable premiums this happens when ρ is between 3.5% and 4%.

References

Albizzati, M.-O. and H. Geman (1994): "Interest Rate Risk Management and Valuation of the Surrender Option in Life Insurance Policies", *The Journal of Risk and Insurance* 61: 616-637.

Bacinello, A.R. (2001a): "Fair Pricing of Life Insurance Participating Policies with a Minimum Interest Rate Guaranteed", *Astin Bulletin* 31: 275-297.

Bacinello, A.R. (2001b): "Fair Valuation of the Surrender Option Embedded in a Guaranteed Life Insurance Participating Policy", Quaderni del Dipartimento di Matematica Applicata alle Scienze Economiche Statistiche e Attuariali "Bruno de Finetti", Università degli Studi di Trieste, n. 8/2001.

Bacinello, A.R. and S.-A. Persson (2002): "Design and Pricing of Equity-Linked Life Insurance under Stochastic Interest Rates", *The Journal of Risk Finance* 3(2): 6-21.

Black, F. and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy* 81: 637-654.

Boyle, P.P. and E.S. Schwartz (1977): "Equilibrium Prices of Guarantees under Equity-Linked Contracts", *The Journal of Risk and Insurance* 44: 639-660.

Brennan, M.J. and E.S. Schwartz (1976): “The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee”, *Journal of Financial Economics* 3: 195-213.

Briys, E. and F. de Varenne (1994): “Life Insurance in a Contingent Claim Framework: Pricing and Regulatory Implications”, *The Geneva Papers on Risk and Insurance Theory* 19: 53-72.

Briys, E. and F. de Varenne (1997): “On the Risk of Life Insurance Liabilities: Debunking Some Common Pitfalls”, *The Journal of Risk and Insurance* 64: 673-694.

Consiglio, A., Cocco, F. and S.A. Zenios (2001a): “The Value of Integrative Risk Management for Insurance Products with Guarantees”, *The Journal of Risk Finance* 3(2): 6-16.

Consiglio, A., Cocco, F. and S.A. Zenios (2001b): “Asset and Liability Modeling for Participating Policies with Guarantees”, Working Paper n. 00-41-C, Financial Institutions Center, The Wharton School, University of Pennsylvania.

Cox, J.C., Ross, S.A. and M. Rubinstein (1979): “Option Pricing: A Simplified Approach”, *Journal of Financial Economics* 7: 229-263.

Grosen, A. and P.L. Jørgensen (1997): “Valuation of Early Exercisable Interest Rate Guarantees”, *The Journal of Risk and Insurance* 64: 481-503.

Grosen, A. and P.L. Jørgensen (2000): “Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies”, *Insurance: Mathematics and Economics* 26: 37-57.

Grosen, A. and P.L. Jørgensen (2002): “Life Insurance Liabilities at Market Values: An Analysis of Insolvency Risk, Bonus Policy, and Regulatory Intervention Rules in a Barrier Option Framework”, Forthcoming in *The Journal of Risk and Insurance.*

Hansen, M. and K.R. Miltersen (2002): “Minimum Rate of Return Guarantees: The Danish Case”, Forthcoming in *Scandinavian Actuarial Journal.*

Jensen, B., Jørgensen, P.L. and A. Grosen (2001): "A Finite Difference Approach to the Valuation of Path Dependent Life Insurance Liabilities", *The Geneva Papers on Risk and Insurance Theory* 26: 57-84.

Merton, R.C. (1973): "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science* 4: 141-183.

Miltersen, K.R. and S.-A. Persson (2002): "Guaranteed Investment Contracts: Distributed and Undistributed Excess Returns", Forthcoming in *Scandinavian Actuarial Journal*.

Norberg, R. (1999): "A Theory of Bonus in Life Insurance", *Finance and Stochastics* 3: 373-390.

Norberg, R. (2001): "On Bonus and Bonus Prognoses in Life Insurance", *Scandinavian Actuarial Journal* 2001: 126-147.

Pacati, C. (2000): "Valutazione di Portafogli di Polizze Vita con Rivalutazione agli Ennesimi", Working Paper n. 38, Gruppo di Ricerca su "Modelli per la Finanza Matematica".

Steffensen, M. (2001): "Intervention Options in Life and Pension Insurance", Working Paper n. 177, Laboratory of Actuarial Mathematics, University of Copenhagen.

Wilkie, A.D. (1987): "An Option Pricing Approach to Bonus Policy", *Journal of the Institute of Actuaries* 114: 21-77.