Pension-Fund Mathematics

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1 Introduction

In this article we will review the role of mathematics as a tool in the running of pension funds. We will consider separately defined benefit (DB) and defined contribution (DC) pension funds.

2 Defined benefit pension funds

2.1 Deterministic methods

We will concentrate here on final salary schemes, and in order to illustrate the various uses of mathematics we will consider some simple cases. In this section we will describe the traditional use of mathematics first, before moving on to more recent developments using stochastic modelling. We will use a model pension scheme with a simple structure:

- Salaries are increased each year in line with a cost-of-living index $CLI(t)$.
- At the start of each year $(t, t+1)$ one new member joins the scheme at age 25, with a salary of €$10,000 \times CLI(t)/CLI(0)$.
- All members stay with the scheme until age 65 and mortality before age 65 is assumed to be zero.

- At age 65 the member retires and receives a pension equal to 1/60 of final salary for each year as a member of the scheme, payable annually in advance for life. This pension is secured by the purchase of an annuity from a life insurer, so that the pension fund has no further responsibility for making payments to the member. Final salary is defined as the salary rate at age 65 including the cost-of-living salary increase at age 65.[1]

- Salaries are increased at the start of each year and include a combination of age-related and cost-of-living increases.

Let $S(t,x)$ represent the salary at time t for a member aged x at that time. The scheme structure described above indicates that

$$S(t+1,x+1) = S(t,x) \times \frac{w(x+1)}{w(x)} \times \frac{CLI(t+1)}{CLI(t)}$$

where the function $w(x)$ is called the wage profile and determines age-related increases. Initially we assume that $S(0,x) = 10000w(x)/w(25)$ for $x = 25, \dots, 65$. It follows that we also have the identity

$$S(t,x) = S(0,x)\frac{CLI(t)}{CLI(0)}.$$

The traditional role of the actuary is twofold: to determine the actuarial liability at time t (which is then compared with the value of the assets) and to recommend a contribution rate.[2] There are numerous approaches (*funding methods*) to answering these two questions. We will describe two of the most popular.

One approach (Called the Entry Age Method in the UK) treats the contract like a life insurance policy and poses the question: *what contribution rate (as a percentage of salary) should a new member pay throughout his career in order to have the right amount of cash available at age 65 to buy the promised pension?* Once this question has been answered the overall fund liability can be calculated.

The second approach (the Projected Unit Method) calculates the actuarial liability first. The accrued liability values only service accrued to date and makes no allowance for benefits arising from continued service in the future. The valuation

[1] This definition of final salary is slightly generous, but it simplifies some of our agruments below.

[2] Unlike a traditional life insurance policy it is not essential to get the contribution rate exactly right at the outset. For a DB fund, if it turns out that the contribution rate has been too low in the past then the fund sponsors can pay more later. This somewhat comfortable position probably stifled progress in pension mathematics for decades because there was essentially no penalty for poor advice.

also takes into account projected future salary increases (age-related and cost-of-living). In this case the actuarial liability at time t is

$$
\begin{aligned}
AL(t) &= \sum_{x=25}^{64} \frac{x-25}{60} \left(S(t,x) \frac{w(65)}{w(x)} (1+e)^{65-x} \right) \; v^{65-x} \; \ddot{a}_{65} \\
&\quad + \frac{40}{60} S(t,65) \; \ddot{a}_{65}.
\end{aligned}
$$

Recall that pensions are bought out at age 65. The component of $AL(t)$ in the second line represents the liability for the member who has just attained age 65 but for whom a pension has not yet been purchased.

In this equation e represents the assumed rate of growth of $CLI(t)$, $v = 1/(1+i)$ where i is the valuation rate of interest and $\ddot{a}_{65}$ is the assumed price for a unit of pension from age 65. It is straightforward, but messy, to incorporate a variety of decrements before age 65 such as mortality, ill-health retirement, resignation from the company, along with associated other benefits such as death benefits, ill-health pensions, early-leavers benefits etc.

Since salaries at each constant age x increase each year in line with $CLI(t)$ we can note that

$$AL(t+1) = AL(t)CLI(t+1)/CLI(t).$$

The normal contribution rate, $NC(t)$, payable at time t is the rate which will ensure the following:

- if we start at t with assets equal to liabilities;
- if the sponsor pays in $NC(t)$ times the total salary roll at t, $TSR(t) = \sum_{x=25}^{64} S(t,x)$;
- if the pension-fund trustees immediately secure the purchase of a pension (for a price $B(t)$) for the member who has just attained age 65;
- if experience is as anticipated in the valuation basis (investment return, salary increase at time $t+1$ and no deaths);

  then

- the assets will still be equal to the liabilities at time $t+1$.

Thus,

$$\big(AL(t) + NC(t)TSR(t) - B(t)\big)(1+i) = \hat{AL}(t+1) = AL(t)(1+e).$$

This means that

$$NC(t) = (B(t) - AL(t)(1 - v_v))/TSR(t)$$

where $v_v = (1+e)/(1+i)$ is the assumed real discount factor.

The remaining element of a funding valuation is the recommendation of a contribution rate (bearing in mind that $NC(t)$ is only appropriate if assets equal liabilities). This leads us to the concept of *amortisation of surplus or deficit*. The simplest approach is that adopted in the United Kingdom amongst other places. Let $F(t)$ represent the fund size at time t, so that the deficit on the funding basis is $AL(t) - F(t)$. The recommended contribution rate is

$$\begin{aligned} RCR(t) &= NC(t) + \frac{AL(t) - F(t)}{TSR(t)\ddot{a}_{\overline{m|}\,,i_v}} \\ \text{where } \ddot{a}_{\overline{m|}\,,i_v} &= \sum_{k=0}^{m-1} v_v^k = \frac{1 - v_v^m}{1 - v_v}. \end{aligned}$$

Here the constant m is the *amortisation period*. This could be set according to how rapidly the fund sponsor wants to get rid of surplus or deficit. Often, though, it is set equal to the expected future working lifetime of the active membership, in line with certain accounting guidelines.

A related approach to amortisation is used in North America. The adjustment is divided into m components each relating to the amortisation of the surplus/deficit arising in each of the last m years. Additionally there may be a corridor around the actuarial liability within which surplus or deficit is not amortised. Both of these differences (compared with the UK approach) lead to greater volatility in contribution rates.

Within this deterministic framework the actuary has control over the assumptions. Traditionally, for example, there have been relatively few external controls over how the valuation basis should be set. As a consequence actuaries have tended to err on the side of caution: that is, low i, high e, low mortality in retirement. This results in prudent overestimates of both the value of the liabilities and the required contribution rate. As mentioned before, traditionally this has not caused a problem as over-payment in one year can be balanced by reduced payments in later years when surplus arises as a result of the use of a cautious basis. A problem with this approach is that there is no rationale behind the level of caution in the valuation basis: that is, no attempt has ever been made to link the level of caution in the individual assumptions to the level of risk.

2.2 Stochastic methods

Since the late 1980's we have seen the development of a number of possible new approaches to pension fund management using stochastic models. In the literature to date, these have focused on relatively simple models, with a view to understanding the stochastic nature of pension fund dynamics and their interaction with the funding methods used. The aim of this fundamental work is to give us an understanding of some of the factors that are likely to be important in the more complex world that we really work in.

Early work on this was conducted by Dufresne in a series of papers (Dufresne, 1988, 1989, 1990). Assumptions made in this early work were later relaxed (see, for example, Haberman, 1993) and it was found that the original conclusions remained broadly intact. Dufresne took the series of actuarial liabilities $AL(0), AL(1), \ldots$ as given: the valuation method and basis were assumed to be given. In particular, all elements of the basis were best estimates of the relevant quantities.

We focus on the dynamics of the fund size and the contribution rate

$$F(t+1) = (1 + i(t+1))[F(t) + RCR(t)TSR(t) - B(t)]$$

where $i(t+1)$ was the achieved return on the fund from t to $t+1$, and

$$RCR(t) = NC(t) + \frac{(AL(t) - F(t))}{TSR(t)\ddot{a}_{\overline{m}|,i_v}}.$$

Simple models for $i(t)$ allow us to derive analytical (or semi-analytical) formulae for the unconditional mean and variance of both the $F(t)$ and $RCR(t)$. The key feature of these investigations was the assessment of how these values depend upon the amortisation period m. It was found that if m is too large (typically greater than 10 years) then the amortisation strategy was *inefficient*: that is, a lower value for m would reduce the variance of both the fund size and the contribution rate. Below a certain threshold for m, however, there would be a tradeoff between continued reductions in $Var[F(t)]$ and an increasing $Var[RCR(t)]$.

A number of further advances were made by Cairns & Parker (1997) and Huang (2000). In the earlier works the only control variable was the amortisation period. Cairns & Parker extended this to include the valuation rate of interest, while Huang extended this further to include the asset strategy as a control variable. They found that having a valuation rate of interest different from $E[i(t)]$ enriched the analysis somewhat since a cautious basis then results in the systematic generation of surplus. This meant that the decision process now had to take into account the mean contribution rate as well as the variances. Cairns & Parker also conducted a comprehensive sensitivity analysis with respect to various model parameters and took a close look at conditional in addition to unconditional means and variances with finite time horizons.

2.3 Dynamic stochastic control

Up to this point the decision-making process was still relatively subjective. There was no formal objective which would result in the emergence of one specific strategy out of the range of efficient strategies. This then led to the introduction of stochastic control theory as a means of assisting in the decision making process. This approach has been taken using both continuous time modelling (Boulier *et al.*, 1995, and Cairns, 2000) and discrete time (Haberman & Sung, 1994). The former approach yields some stronger results and that is what we will describe here.

Once the objective function has been specified, dynamic stochastic control (at least in theory) identifies the dynamic control strategy which is optimal at all times in the future and in all possible future states of the world. This is in contrast to the previous approach where a limited range of controls might be considered and which might only result in the identification of a strategy which is optimal for one state of the world only.

In stochastic control there is no automatic requirement to conduct actuarial valuations or set contribution rates with respect to a normal contribution rate augmented by rigid amortisation guidelines. Instead we start with a clean sheet and formulate an objective function which takes into account the interests of the various stakeholders in the pension fund. Thus, let $F(t)$ the fund size at time t and $c(t)$ be the corresponding contribution rate. Cairns (2000) proposed that the dynamics of $F(t)$ be governed by the stochastic differential equation

$$dF(t) \quad = \quad F(t)d\delta(t) + c(t)dt - (Bdt + \sigma_b dZ_b(t)). \tag{1}$$

This assumes that the benefit outgo has a constant mean with fluctuations around this to account for demographic and other uncertainty in the benefit payments. The first element in (1) gives us the instantaneous investment gain on the fund from t to $t+dt$. The $d\delta(t)$ term represents the instantaneous gain per unit invested and contains the usual mixture of drift (dt) and Brownian motion $dZ(t)$ terms. The second element represents the contributions paid in by the fund sponsor. The third term in brackets represents the benefit outgo. Boulier *et al.* (1995) and Cairns (2000) assumed that both the contribution rate, $c(t)$, and the (possibly dynamic) asset-allocation strategy, $p(t)$, were both control variables which could be used to optimise the fund's objective function. The objective function was similar to that introduced by Merton (1969, 1971) (see also Merton, 1990) and can be described as the discounted expected loss function:

$$\Lambda(t,f)(c,p) = E\left[\int_t^\infty e^{-\beta s} L(s, c(s), F(s))ds \;\middle|\; F(t) = f\right].$$

From the current fund size $F(t) = f$ the expected discounted loss depends on the choice of control strategies c and p. These strategies can depend upon the time of application, s, in the future as well as the "state of the world" at that time (that is, $F(s)$).

The function $L(t, c, f)$ is a loss function which measures how unhappy (in a collective sense) the stakeholders are about what is happening at time s given that

$F(s) = f$ and that the contribution rate will be $c(s) = c$. The discount function $e^{-\beta s}$ determines the relative weight attached to outcomes at various points in the future: for example, a large value of β will place more emphasis on the short term.

The aim of the exercise is then to determine what strategies c and p minimise $\Lambda(t, f)(c, p)$. Thus let

$$V(t, f) = \inf_{c,p} \Lambda(t, f)(c, p).$$

For problems of this type it is well known that $V(t, f)$ can be solved using the Hamilton-Jacobi-Bellman equation (HJB) (see, for example, Fleming & Rishel, 1975, Korn, 1997, or Björk, 1998) and Cairns (2000) used this to determine the optimal contribution and asset strategies. He then analysed examples where the loss function is quadratic in c and f as well as power and exponential loss in c only. These led to some interesting conclusions about the optimal c and p. Some of these were intuitively sensible but others were less so and this could be connected to aspects of the original loss function. Cairns concluded that alternative loss functions needed to be developed to address these problems and this is the subject of ongoing work.

3 Defined contribution pensions

DC pensions operate in quite a different way from DB pensions. In the latter the company sponsoring the fund usually takes on the majority of the risk (especially investment risk). In a DC pension fund the individual members take on all of the risk. In a typical occupational DC pension fund the contribution rate payable by both the member and the employer is a fixed percentage of salary. This is invested in a variety of managed funds with some control over the choice of funds in the hands of the member. The result is that there is considerable uncertainty over the amount of pension that might be achieved at the time of retirement. Again this is in contrast to a DB pension which delivers a well-defined level of pension.

Individual DC pensions (that is, personal pensions) offer additional flexibility over occupational schemes through variation of the contribution rate. This means that policyholder might choose to pay more if their pension fund investments have not been performing very well.

Typically there has never been any formal requirement for actuarial or other advice to help DC fund members choose how to invest or what level of contributions they should pay. An exception to this (for example, in the UK, 2003) is that at the point of sale of a personal pension policy, the insurer may be obliged to provide the potential policyholder with *deterministic projections* to help decide the right contribution rate. Similarly existing DC pension fund members (personal pension policyholders and occupational fund members) may be provided with, again, *deterministic* projections. This lamentable situation has left DC fund members largely ignorant about the risk that they are exposed to.

So the present situation is that they just have to accept the risks that face them. However, the growing use of stochastic methods in DB pension funds has spawned similar work in DC pensions. For DC pensions the aim of stochastic modelling is to

- inform existing members of the potential risks that they face if they continue with their present strategy;
- inform potential new DC fund members or new personal pension policy-holders of the risks that they face to allow them to choose between the DC pension and some alternatives;
- allow existing members to manage the risks that they face by choosing an investment and contribution strategy which is consistent with their appetite for risk and with the current status of their personal DC-pension account;
- allow members to adopt a strategy which will, with high probability, permit them to retire by a certain age with a comfortable level of pension.

We will focus here on occupational DC pension funds, where the contribution rate is a fixed percentage of salary. This type of scheme has been analysed extensively by Blake, Cairns & Dowd (2001), Haberman & Vigna (2002) and Cairns, Blake & Dowd (2000, 2003).

Blake, Cairns & Dowd (2001) (BCD) look at the problem in discrete time from an empirical point of view. The fund dynamics are

$$F(t+1) = \big(1 + i(t+1)\big)\left(F(t) + C(t)\right) \tag{2}$$

where the annual contribution $C(t) = k \times S(t)$ is a fixed proportion, k, of the member's current salary, $S(t)$, and $i(t+1)$ is the investment return from t to $t+1$. The salary process itself is stochastic and correlated with investment returns. At the time, T, of retirement (age 65, say) $F(T)$ is used to purchase a pension at the prevailing market rate $a_{65}(T)$. The market rate will depend primarily on long-term interest rates at T but could also incorporate changes in mortality expectations. This pension is then compared with a benchmark DB pension of 2/3rds of final salary to give the pension ratio

$$PR(T) = \frac{PEN(T)}{\frac{2}{3}S(T)} = \frac{3F(T)}{2a_{65}(T)S(T)}.$$

BCD investigate a variety of investment strategies which are commonly used in practice and offered by pension funds and pension providers. They consider a variety of different models for investment returns on 6 asset classes in order to assess the extent of model risk. They conclude that if the models are all calibrated to the same historical data set then differences between models are relative small when compared with the differences which arise as a result of adopting different investment strategies.

The analysis is empirical in nature due to the relative complexity of the individual asset models and they use the model to generate (by simulation) the distribution of the pension ratio at retirement for each investment strategy. This then allows users to compare strategies using a variety of different measures of risk (although BCD concentrate on Value at Risk measures). They conclude that some of the more sophisticated strategies (such as the *Lifestyle* strategy[3] popular with insurers) do not deliver superior performance over the long period of the contract. They conclude instead that a static strategy with a high equity content is likely to be best for all but the most risk-averse policyholders.

Haberman & Vigna (2002) also consider a discrete-time model as in equation (2) but restrict themselves to a fixed contribution rate in monetary terms in a framework which is consistent with constant salary over the working lifetime of the policyholder. They use a simpler model for investment returns and this allows them to conduct a more rigorous analysis using stochastic control with quadratic or mean-shortfall loss functions.

Cairns, Blake & Dowd (2000, 2003) (CBD) formulate the DC pension problem in continuous time. Their aim is to tackle the same problem as Blake, Cairns & Dowd (2001) with the requirement to optimise the expected value of a terminal utility function. This requires a simpler model (multivariate geometric Brownian motion) than before for asset values and salary growth but also includes a full, arbitrage-free model for the term structure of interest rates which allows us to calculate accurately the price of the annuity at time T. Terminal utility is assumed to be of the power-utility form

$$\frac{1}{\gamma}PR(T)^{\gamma}$$

for $\gamma < 1$ and $\gamma \neq 0$.

CBD apply the same HJB technology as used by Cairns (2000) in tackling the DB pension problem to determine the optimal dynamic asset-allocation strategy for policyholder with a given level of risk aversion. A crucial feature of the DC pension policy is that the policyholder needs to take into account future contributions. The result of this feature is that the optimal strategy should vary significantly over

[3] The Lifestyle Strategy is a deterministic but time varying investment strategy covering the period from inception to the date of retirement. At young ages the policyholder's DC account is invested in a high-return, high-risk fund consisting mainly of equities. Over the final 5, 10 or 20 years these assets are gradually transferred into a bond fund which matches the annuity purchase price.

time, starting with a high proportion in high-return-high-risk assets, gradually reducing to a mixed portfolio consistent with their degree of risk aversion. They find that the optimal strategy depends significantly on the current fund size. This is then compared with a variety of static and deterministic-but-dynamic asset-allocation strategies, and CBD find that the optimal stochastic strategy delivers a significantly higher expected terminal utility.

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