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Yaari dual theory without the completeness axiom*

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Abstract

This note shows how Yaari's dual theory of choice under risk naturally extends to the case of incomplete preferences. This also provides an axiomatic characterization of a large and widely studied class of stochastic orders used to rank the riskiness of random variables or the dispersion of income distributions (including, e.g., second order stochastic dominance, dispersion, location independent riskiness).

Keywords and Phrases: Yaari's dual theory, incomplete preferences, stochastic orders.

1 Introduction

One of the most successful nonexpected utility models is the dual theory of choice under risk due to Yaari (1987). Monetary lotteries are represented by bounded random variables on a probability space $(\Omega, \mathcal{A}, P)$. Given a complete preference $\succsim$ among the lotteries, the "dual expected utility theorem" gives necessary and sufficient conditions under which there exists a continuous nondecreasing function $f : [0,1] \to [0,1]$ such that for all lotteries X and Y

$$X \succsim Y \Leftrightarrow \int_\Omega X d(f \circ P) \geq \int_\Omega Y d(f \circ P). \tag{1}$$

The function f is often interpreted as an adjustement of the underlying objective probability due to the subjective risk perception of the decision maker. For this reason, it is called probability distortion or perception function. Then, Eq. (1) reads as follow: lottery X is preferred to lottery

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Y if and only if the Choquet expected value of X with respect to the distorted probability $f \circ P$ is greater than the Choquet expected value of Y with respect to the distorted probability $f \circ P$.

This theory, while eliminating some of expected utility's drawbacks (and introducing new ones), shares with expected utility the completeness assumption: the decision maker must be able to rank any pair X, Y of lotteries.

Many contributions, for example, Aumann (1962), Kannai (1963), Fishburn (1971), Bewley (1986), Shapley and Baucells (1998), Mandler (1999), Mitra and Ok (2000), and Dubra, Maccheroni, and Ok (2001), have pointed out that the completeness assumption is difficult to be justified both from a normative and a descriptive perspective; incompleteness naturally arising from indecisiveness or lack of information of a single decision maker, from discordance among many decision makers, etc. The aforementioned contributions have suggested ways to deal without completeness in the expected utility setting. In the present note we face incompleteness in Yaari's dual setting.

Specifically, we consider an incomplete preference $\succsim$ among lotteries and we obtain necessary and sufficient conditions under which there exists a family $\mathcal{F}$ of probability distortions such that for all lotteries X and Y

$$X \succsim Y \Leftrightarrow \int_{\Omega} X d\left(f \circ P\right) \geq \int_{\Omega} Y d\left(f \circ P\right) \quad \forall f \in \mathcal{F}. \tag{2}$$

The non-singleton nature of $\mathcal{F}$ captures the indecisiveness of the decision maker. In fact, she is not able (or not willing) to compare X and Y whenever there exist two perceptions f and g such that $\int_{\Omega} X d\left(f \circ P\right) > \int_{\Omega} Y d\left(f \circ P\right)$ and $\int_{\Omega} X d\left(g \circ P\right) < \int_{\Omega} Y d\left(g \circ P\right)$. In other words, incompleteness may be seen as a consequence of the multiplicity of perceptions of the decision maker (represented by the elements of $\mathcal{F}$), preference resulting when the different perceptions agree, indecision arising when they do not. This interpretation is supported by the fact that preference $\succsim'$ is "less incomplete" than preference $\succsim$, if and only if it is based on "fewer perceptions" (for a formal statement see Proposition 2).

When many decision makers are considered, Eq. (2) represents the unanimous (incomplete) preference of all the Yaari decision makers whose individual (complete) preferences are described by elements of $\mathcal{F}$; notice that, again, incomparability springs from disagreement, and the fewer the decision makers, the finer the relation. As observed by Chateauneuf, Cohen, and Meilijson (1997), several orders used to rank the riskiness of random variables or the dispersion of income distributions have this form. For example, Rothshild and Stiglitz (1970)'s second order stochastic dominance is obtained when $\mathcal{F}$ is the set of all convex probability distortions; Bickel and Lehman (1976)'s dispersion is obtained when $\mathcal{F}$ is the set of all probability distortions that are majorized by the identity; Jewitt (1989)'s location independent riskiness is obtained when $\mathcal{F}$ is the set of all probability distortions that are star-shaped at 1. Our main result may thus be seen as an axiomatic characterization of stochastic orders of this kind.

2 Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A random variable is simply a measurable function $X : \Omega \to \mathbb{R}$. We assume that P is nonatomic.[1]

Two random variables X and Y are *comonotonic* if

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$$

for all $\omega_1, \omega_2 \in \Omega$.

For each random variable X the *decumulative distribution function* of X is defined by

$$G_X(t) = P\{X > t\}$$

for all $t \in \mathbb{R}$. It is always nonincreasing and right-continuous.

A random variable X *first-order stochastically dominates* a random variable Y if $G_X(t) \geq G_Y(t)$ for all $t \in \mathbb{R}$; we write $X \geq_{FSD} Y$ (or $G_X \geq G_Y$). A random variable X *second-order stochastically dominates* a random variable Y if $\int_{-\infty}^{x} (G_X(t) - 1)\, dt \geq \int_{-\infty}^{x} (G_Y(t) - 1)\, dt$ for all $x \in \mathbb{R}$; we write $X \geq_{SSD} Y$.

A sequence X_n of random variables is said to *converge in distribution* to a random variable X if $G_{X_n}(t)$ converges to $G_X(t)$, for all $t \in \mathbb{R}$ at which G_X is continuous; we write $X_n \Rightarrow X$ (or $G_{X_n} \Rightarrow G_X$).

Let $C([0,1])$ be the set of all continuous functions on the unit interval $[0,1]$, endowed with the supnorm. An nondecreasing function f in $C([0,1])$ such that $f(0) = 0$ and $f(1) = 1$, is called *probability distortion* or *perception*.

We denote by $\mathcal{L}_\infty$ the set of all almost surely bounded random variables. If $X \in \mathcal{L}_\infty$, and $f : [0,1] \to [0,1]$ is a probability distortion, the *Choquet expected value* of X with respect to the distorted probability measure $f \circ P$ is defined by

$$\int_\Omega X d(f \circ P) = \int_{-\infty}^{0} [f(G_X(t)) - 1]\, dt + \int_0^\infty f(G_X(t))\, dt.$$

If $f(p) = p$ for all $p \in [0,1]$, the above equation is the classical expectation formula. Finally, random variables X_j, $j \in J$, are *uniformly bounded* if there exists $c \in \mathbb{R}$ such that $\|X_j\|_\infty \leq c$ for all $j \in J$.[2]

3 Axioms

Random variables can be interpreted as monetary lotteries that a decision maker is comparing. Let $\succsim$ be a binary relation representing the decision maker's preferences on a convex set $\mathcal{X}$ of

[1] This is equivalent to assume, as in Yaari (1987), that all the probability distributions on the real line can be generated from random variables on Ω.

[2] We recall that $\|X\|_\infty = \inf\{d \in \mathbb{R} : |X(\omega)| \leq d \text{ almost surely}\}$, with the convention, $\|X\|_\infty = \infty$ if $\{d \in \mathbb{R} : |X(\omega)| \leq d \text{ almost surely}\} = \emptyset$.

random variables; $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$, respectively. We will make use of the following axioms. We will not discuss them extensively, since they are widely used and well studied in the literature.

A0. *Completeness:* $X \succsim Y$ or $Y \succsim X$, for all $X, Y \in \mathcal{X}$.

This is exactly what we will *not* assume.

A1. *Nontrivial Preorder:* $\succsim$ is reflexive, transitive, and not symmetric, that is:

(a) $X \sim X$ for all $X \in \mathcal{X}$;

(b) if $X, Y, Z \in \mathcal{X}$, $X \succsim Y$ and $Y \succsim Z$, then $X \succsim Z$;

(c) $X \succ Y$ for some $X, Y \in \mathcal{X}$.

This is a natural way to speak of incomplete preferences.

A2. *Stochastic Dominance:* if $X, Y \in \mathcal{X}$ and $X \geq_{FSD} Y$, then $X \succsim Y$.

In words, if, for each amount t of money, the probability that lottery X yields more than t is greater than the probability that lottery Y yields more than t, then X is preferred to Y. This implies that identically distributed random variables are indifferent to the decision maker.

A3. *Continuity:* if $X_n, Y_n, X, Y \in \mathcal{X}$ are uniformly bounded, $X_n \succsim Y_n$ for all $n \in \mathbb{N}$, $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$, then $X \succsim Y$.

This is clearly a technical assumption, however it is common and clean.

A4. *Comonotonic Independence:* if $X, Y, Z \in \mathcal{X}$ are pairwise comonotonic, $\alpha \in [0, 1]$ and $X \succsim Y$, then $\alpha X + (1 - \alpha) Z \succsim \alpha Y + (1 - \alpha) Z$.

This is the core of rank-dependent expected utility. As proved by Chateauneuf, Kast and Lapied (1994), if $\mathcal{A}$ contains all the singletons, two bounded random variables X and Y are comonotonic if and only if their covariance is nonnegative for any probability measure on $(\Omega, \mathcal{A})$. Therefore "...When two random variables are comonotonic, then it can be said that neither of them is a *hedge* against the other...Suppose, for example, that X and Y are random variables such that $X \succsim Y$. Would this preference be retained when both X and Y are mixed, half and half, with some third random variable, say Z?...If the agent whose preferences are being discussed is risk averse, and Z is a hedge against Y but not against X, then this agent might well have reason ...[not to retain]... the direction of preference: i.e., the assertions $X \succsim Y$ and ...[not $\frac{1}{2}X + \frac{1}{2}Z \succsim \frac{1}{2}Y + \frac{1}{2}Z$]... will both be true. Similarly, if the agent for whom $X \succsim Y$ is risk seeking, and Z is a hedge against X but not against Y, then, once again, there will be reason for the agent ...[not to retain]... the direction of preference as above. Thus, the demand that $X \succsim Y$ should imply $\alpha X + (1 - \alpha) Z \succsim \alpha Y + (1 - \alpha) Z$ seems to be justified only in the case where Z is neither a hedge against X nor a hedge against Y. This is precisely what ...[comonotonic independence]... says..." Yaari (1987).[3]

[3]Random variables were renamed for consistency with our notation.

4 Results

We can now state the main contribution of the note (Subsection 4.1) and some ancillary results (Subsections 4.2 and 4.3).

4.1 Representation theorem

Next theorem characterizes Yaari incomplete preferences.

Theorem 1 *A binary relation* $\succsim$ *on* $\mathcal{L}_\infty$ *satisfies axioms A1-A4 if and only if there exists a closed and convex set* $\mathcal{F}$ *of probability distortions such that, for all* X *and* Y *in* $\mathcal{L}_\infty$*,*

$$X \succsim Y \Leftrightarrow \int_\Omega X d\left(f \circ P\right) \geq \int_\Omega Y d\left(f \circ P\right) \quad \forall f \in \mathcal{F}.$$

$\mathcal{F}$ *is unique.*
Moreover, $\succsim$ *satisfies also axiom A0 if and only if* $\mathcal{F}$ *is a singleton.*

That is, dropping the completeness assumption, a Yaari decision maker prefers lottery X over lottery Y if and only if X yields an expected payoff greater than Y with respect to all her risk perceptions; while she cannot choose between X and Y if her perceptions disagree on which of the two lotteries yields a greater expected payoff. As anticipated in the Introduction, the result also provides an axiomatic foundation to the use of dual integral stochastic orders in risk evaluation and inequality measures.

4.2 Extensions and completions of an incomplete preference

In this subsection we consider relations $\succsim$, $\succsim'$, $\succsim''$ on $\mathcal{L}_\infty$ satisfying A1-A4 and hence represented by closed and convex sets of probability distortions respectively denoted by $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$.

Given two relations on $\mathcal{L}_\infty$, we say that the relation $\succsim'$ *extends* $\succsim$ if $X \succsim Y$ implies $X \succsim' Y$.

Proposition 2 *Let* $\succsim$ *and* $\succsim'$ *be two relations on* $\mathcal{L}_\infty$ *both satisfying A1-A4 and represented by* $\mathcal{F}$ *and* $\mathcal{F}'$*. The relation* $\succsim'$ *extends* $\succsim$ *if and only if* $\mathcal{F}' \subseteq \mathcal{F}$*.*

In words, some comparative statics is possible: $\succsim'$ is "less incomplete" than $\succsim$ if, and only if, it builds on "fewer" perceptions. The concept of extension can be strengthened into the one of completion: a relation $\succsim''$ *completes* $\succsim$ if it is complete and $X \succ Y$ (resp. $X \sim Y$) implies $X \succ'' Y$ (resp. $X \sim'' Y$). As a consequence of Theorem 1 and Proposition 2, $\succsim''$ must be a complete Yaari ordering represented by a function $g \in \mathcal{F}$ such that

$$X \succ Y \Rightarrow \int_\Omega X d\left(g \circ P\right) > \int_\Omega Y d\left(g \circ P\right), \text{ and} \tag{3}$$

$$X \sim Y \Rightarrow \int_\Omega X d\left(g \circ P\right) = \int_\Omega Y d\left(g \circ P\right). \tag{4}$$

Denote by $\mathcal{F}^\bullet$ the set of all probability deformations such that Eq. (3) and Eq. (4) hold, as just observed $\mathcal{F}^\bullet \subseteq \mathcal{F}$, and $\mathcal{F}^\bullet$ can be identified with the set of all completions of $\succsim$. The next Proposition shows that, not only $\mathcal{F}^\bullet$ is nonempty, but it is dense in $\mathcal{F}$.

Proposition 3 *Let $\succsim$ be a relation on $\mathcal{L}_\infty$ satisfying A1-A4 and represented by $\mathcal{F}$. The set $\mathcal{F}^\bullet$ is a dense convex subset of $\mathcal{F}$. In particular*

$$X \succsim Y \Leftrightarrow \int_\Omega X d\left(g \circ P\right) \geq \int_\Omega Y d\left(g \circ P\right) \ \ \forall g \in \mathcal{F}^\bullet.$$

Hence, an incomplete Yaari ordering $\succsim$ is the intersection of its completions, and thus it may be seen as a first decisional step towards a complete one.

4.3 Risk aversion

As in the complete case, a decision maker satysfying A1-A4 displays constant absolute risk aversion and constant relative risk aversion, in fact, for all X and Y in $\mathcal{L}_\infty$,

$$X \succsim Y \Leftrightarrow aX + b \geq aY + b,$$

provided $a > 0$ and $b \in \mathbb{R}$. Notice that this has nothing to do with risk neutrality which means that $\mathcal{F}$ is a singleton consisting of the identity function. As pointed out by various contributions, the many equivalent ways in which risk aversion can be defined in an expected utility setting lead to different notions of risk aversion in rank-dependent settings. For example, $\succsim$ is said to be *strongly risk averse* if $X \geq_{SSD} Y$ implies $X \succsim Y$, while $\succsim$ is said to be *weakly risk averse* if $\int_\Omega X dP \succsim X$ for all $X \in \mathcal{L}_\infty$.

Proposition 4 *If a binary relation $\succsim$ on $\mathcal{L}_\infty$ satisfies A1-A4 and is represented by $\mathcal{F}$, then*

- *$\succsim$ is strongly risk averse if and only if all the elements of $\mathcal{F}$ are convex.*
- *$\succsim$ is weakly risk averse if and only if all the elements of $\mathcal{F}$ are majorized by the identity.*

Similar considerations hold for monotone risk aversion and left monotone risk aversion (see Chateauneuf, Cohen, and Meilijson, 1997).

5 Conclusions

Following a hint of Schmeidler (1989) "...the completeness of the preferences seems to me the most restrictive and most imposing assumption of the theory. One can view the weakening of the completeness assumption as a main contribution of all other axioms...From this point of view, the independence axiom seems the most powerful axiom for extending partial preferences...However after additional retrospection this implication may be too powerful to be acceptable...Qualifying the comparisons and the application of independence to comonotonic acts rules out the possibility of contradiction....", in this note we showed what happens in the case of choice under risk. Another interesting issue would be investigating the same problem under uncertainty. This is the subject of future research.

6 Proofs

If X is a random variable we call *pseudoinverse* of G_X the function defined, for all $p \in (0,1)$, by

$$G_X^{-1}(p) = \min\{t \in \mathbb{R} : G_X(t) \leq p\}.$$

Next proposition collects some known properties of pseudoinverses (see e.g. Letta, 1993, and Denneberg, 1994).

Proposition 5 *Let X, X_n, Y be random variables.*

1. *G_X^{-1} is well defined, nonincreasing and right-continuous.*
2. *For all $a, b \in \mathbb{R}$ and $p \in (0,1)$, $G_X(a) > p \geq G_X(b)$ if and only if $a < G_X^{-1}(p) \leq b$.*
3. *G_X^{-1} as a random variable on $((0,1), \mathcal{B}, \beta)$ has decumulative distribution function G_X.*[4]
4. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function and assume $g(X)$ is summable, then*
$$\int_\Omega g(X)\,dP = -\int_{\mathbb{R}} g dG_X = \int_0^1 g\left(G_X^{-1}(u)\right) du.$$
5. *$X_n \Rightarrow X$ if and only if $G_{X_n}^{-1}(p)$ converges to $G_X^{-1}(p)$ for all $p \in (0,1)$ at which G_X^{-1} is continuous.*
6. *$G_X \geq G_Y$ if and only if $G_X^{-1} \geq G_Y^{-1}$.*
7. *If $a \geq 0$ and $b \in \mathbb{R}$, then $G_{aX+b}^{-1} = aG_X^{-1} + b$.*
8. *If X and Y are comonotonic, $G_{X+Y}^{-1} = G_X^{-1} + G_Y^{-1}$.*

Let $\mathcal{V}$ be the set of all random variables taking values in $[0,1]$.

Lemma 6 *A binary relation $\succsim$ on $\mathcal{V}$ satisfies axioms A1-A4 if and only if there exists a closed and convex set $\mathcal{F}$ of probability distortions such that, for all X and Y in $\mathcal{V}$,*

$$X \succsim Y \Leftrightarrow \int_\Omega X d(f \circ P) \geq \int_\Omega Y d(f \circ P) \quad \forall f \in \mathcal{F}.$$

$\mathcal{F}$ is unique.
Moreover, $\succsim$ satisfies also axiom A0 if and only if $\mathcal{F}$ is a singleton.

Proof. Let Γ be the set of all nonincreasing, right-continuous functions $G : [0,1] \to [0,1]$ such that $G(1) = 0$. By the nonatomicity of P, it is the set of all restrictions to $[0,1]$ of the decumulative distribution functions of all random variables in $\mathcal{V}$; this justifies the convention $G(0^-) = 1$ for all $G \in \Gamma$. For all $X \in \mathcal{V}$, we extend G_X^{-1} to the whole $[0,1]$ by setting $G_X^{-1}(0) = \min\{t \in \mathbb{R} : G_X(t) \leq 0\}$ and $G_X^{-1}(1) = 0$. Given $G = (G_X)_{|[0,1]} \in \Gamma$, we intend $G^{-1} = G_X^{-1}$. For all $G \in \Gamma$, the following properties hold:

[4] $\mathcal{B}$ is the Borel σ-field and β the Borel measure on $(0,1)$.

- $G^{-1}(p) = \min\{t \in [0,1] : G(t) \leq p\} = \min\{t \in [0,1] : G(t) \leq p \leq G(t^-)\}$,
- $G^{-1} \in \Gamma$,
- $\left(G^{-1}\right)^{-1} = G$.

The vector space generated by Γ is $RBV_1([0,1])$,[5] which is (isomorphic to) the topological dual of $C([0,1])$, the duality being

$$\langle f, F \rangle = -\int_{[0,1]} f dF.$$

If $G, H \in \Gamma$ and $\alpha \in [0,1]$, the *harmonic convex combination* of G and H is

$$\alpha G \boxplus (1-\alpha) H := \left(\alpha G^{-1} + (1-\alpha) H^{-1}\right)^{-1} \in \Gamma.$$

If $X, Y \in \mathcal{V}$ and $(G_X)_{|[0,1]} = (G_Y)_{|[0,1]}$, then $X \sim Y$ (by A2). Therefore, for all $G, H \in \Gamma$, it is only a little abuse to write $G \succsim H$ if there exists $X, Y \in \mathcal{V}$ such that $(G_X)_{|[0,1]} = G$, $(G_Y)_{|[0,1]} = H$, and $X \succsim Y$. The binary relation $\succsim$ on Γ has the following properties, transliterations of A1-A4.

(i) $\succsim$ is reflexive, transitive and not symmetric.

(ii) $G \geq H$ implies $G \succsim H$.

(iii) If $G_n, G, H_n, H \in \Gamma$, $G_n \succsim H_n$, $G_n \Rightarrow G$ and $H_n \Rightarrow H$, then $G \succsim H$.

(iv) If $F, G, H \in \Gamma$, $\alpha \in [0,1]$ and $F \succsim G$, then $\alpha F \boxplus (1-\alpha) H \succsim \alpha G \boxplus (1-\alpha) H$.

For the relation between A4 and (iv), see Yaari (1987) Proposition 3.

Define $\succsim^*$ on Γ as follows:

$$G \succsim^* H \Leftrightarrow G^{-1} \succsim H^{-1}.$$

It is easily seen that $\succsim^*$ satisfies (i)-(iii) and the classical independence axiom:

(v) If $F, G, H \in \Gamma$, $\alpha \in [0,1]$ and $F \succsim^* G$, then $\alpha F + (1-\alpha) H \succsim^* \alpha G + (1-\alpha) H$.

In fact, $F \succsim^* G$ implies $F^{-1} \succsim G^{-1}$, so

$$\alpha F^{-1} \boxplus (1-\alpha) H^{-1} \succsim \alpha G^{-1} \boxplus (1-\alpha) H^{-1}$$

that is

$$(\alpha F + (1-\alpha) H)^{-1} \succsim (\alpha G + (1-\alpha) H)^{-1}$$

and

$$\alpha F + (1-\alpha) H \succsim^* \alpha G + (1-\alpha) H.$$

[5] The set of all functions $F : [0^-, 1] \to \mathbb{R}$ such that F is of bounded variation, right continuous, and $F(1) = 0$.

Therefore the premises (reflexivity, transitivity, continuity, and independence) of the Expected Multi-Utility Theorem of Dubra, Maccheroni, and Ok (2001) are satisfied and there exists a unique closed and convex cone $\mathcal{U}$ in $C([0,1])$ containing all the constant functions such that

$$G \succsim^* H \Leftrightarrow -\int_{[0,1]} udG \geq -\int_{[0,1]} udH \;\; \forall u \in \mathcal{U}.$$

Since $\succsim^*$ satisfies (ii), $\mathcal{U}$ consists of nondecreasing functions. By contradiction, assume $u(x) > u(y)$ for some $x, y \in [0,1]$ with $x < y$, then $-\int_{[0,1]} ud1_{[0,x)} = u(x) > u(y) = -\int_{[0,1]} ud1_{[0,y)}$ and it cannot be $1_{[0,y)} \succsim^* 1_{[0,x)}$; on the other hand, $1_{[0,y)} \geq 1_{[0,x)}$ and (ii) imply $1_{[0,y)} \succsim^* 1_{[0,x)}$, a contradiction. If $\mathcal{U}$ consisted of constant functions, we had $G \sim^* H$ for all $G, H \in \Gamma$, which contradicts (i). Hence, if we denote by $\mathbb{R}1_{[0,1]}$ the set of all the constant function on $[0,1]$, we have

$$G \succsim^* H \Leftrightarrow -\int_{[0,1]} udG \geq -\int_{[0,1]} udH \;\; \forall u \in \mathcal{U} - \mathbb{R}1_{[0,1]}.$$

Let $\mathcal{F} = \{f \in \mathcal{U} : f(0) = 0 \text{ and } f(1) = 1\}$. By the observation above, $\mathcal{F}$ is nonempty and

$$G \succsim^* H \Leftrightarrow -\int_{[0,1]} fdG \geq -\int_{[0,1]} fdH \;\; \forall f \in \mathcal{F},$$

moreover $\mathcal{F}$ is a closed and convex set of probability distortions.

Hence,

$$\begin{aligned}
G \succsim H \quad &\Leftrightarrow \quad G^{-1} \succsim^* H^{-1} \\
&\Leftrightarrow \quad -\int_{[0,1]} fdG^{-1} \geq -\int_{[0,1]} fdH^{-1} \;\; \forall f \in \mathcal{F} \\
&\Leftrightarrow \quad \int_0^1 f(G(t))\,dt \geq \int_0^1 f(H(t))\,dt \;\; \forall f \in \mathcal{F}
\end{aligned}$$

Finally, remember that for all $X \in \mathcal{V}$, $\int_\Omega Xd(f \circ P) = \int_0^1 f(G_X(t))\,dt$.

Let $\mathcal{G}$ be another closed and convex set of probability distortions such that

$$G \succsim H \Leftrightarrow \int_0^1 g(G(t))\,dt \geq \int_0^1 g(H(t))\,dt \;\; \forall g \in \mathcal{G}.$$

Suppose there exists $g \in \mathcal{G} \backslash \mathcal{F}$. The cone $\mathcal{H}$ generated by $\mathcal{F}$ is closed, convex, it does not contains g and $h(0) = 0$ for all $h \in \mathcal{H}$. If $\alpha f, \alpha' f' \in \mathcal{H}$ (with $\alpha, \alpha' \in \mathbb{R}^+$ and $f, f' \in \mathcal{F}$) either $\alpha = \alpha' = 0$ and $\alpha f + \alpha' f' = 0 \in \mathcal{H}$, or $\alpha f + \alpha' f' = (\alpha + \alpha')\left(\frac{\alpha}{\alpha+\alpha'} f + \frac{\alpha'}{\alpha+\alpha'} f'\right) \in \mathcal{H}$, hence $\mathcal{H}$ is convex. If $\alpha_n f_n \in \mathcal{H}$ (with $\alpha_n \in \mathbb{R}^+$ and $f_n \in \mathcal{F}$) and $\alpha_n f_n \to h$, then $\alpha_n = \alpha_n f_n(1) \to h(1)$; therefore, either $h(1) = 0$ and $\alpha_n f_n \to 0 \in \mathcal{H}$, or $f_n = \frac{\alpha_n f_n}{\alpha_n} \to \frac{h}{h(1)} \in \mathcal{F}$ and $h \in \mathcal{H}$, hence $\mathcal{H}$ is closed. If $g \in \mathcal{H}$, then $g = \alpha f$ with $\alpha \in \mathbb{R}^+$ and $f \in \mathcal{F}$, in particular $1 = g(1) = \alpha f(1) = \alpha$, and $g \in \mathcal{F}$, a contradiction. The set $\{h + \alpha : h \in \mathcal{H} \text{ and } \alpha \in \mathbb{R}\}$ is a convex cone too, let $\mathcal{K}$ denote its closure. If $g \in \mathcal{K}$ there exist sequences $h_n \in \mathcal{H}$ and $\alpha_n \in \mathbb{R}$ s.t. $h_n + \alpha_n \to g$. Hence $h_n(0) + \alpha_n \to g(0)$, but $h_n(0) = g(0) = 0$, consequently $\alpha_n \to 0$, and $h_n = (h_n + \alpha_n) - \alpha_n \to g$.

So $g \in \overline{\mathcal{H}} = \mathcal{H}$, which is absurd. By the separating hyperplane theorem there exists a nonzero function $F \in RBV_1([0,1])$ such that

$$-\int_{[0,1]} g dF > 0 \geq -\int_{[0,1]} k dF \quad \text{ for all } k \in \mathcal{K}.$$

Since the constant functions belong to $\mathcal{K}$, then $\alpha F(0^-) = -\int_{[0,1]} \alpha dF \leq 0$ for all $\alpha \in \mathbb{R}$, and $F(0^-) = 0$. Therefore there exist $G_1, G_2 \in \Gamma$ and $\gamma > 0$ such that $F = \gamma(G_1 - G_2)$, so

$$-\gamma \int_{[0,1]} g d(G_1 - G_2) > 0 \geq -\gamma \int_{[0,1]} k d(G_1 - G_2) \quad \text{ for all } k \in \mathcal{K}.$$

For $i = 1, 2$, set $H_i = G_i^{-1} \in \Gamma$, since $\mathcal{F} \subseteq \mathcal{K}$, we have $-\int_{[0,1]} f d\left(H_1^{-1} - H_2^{-1}\right) \leq 0$ for all $f \in \mathcal{F}$, that is

$$\int_0^1 f(H_2(t))\, dt = -\int_{[0,1]} f dH_2^{-1} \geq -\int_{[0,1]} f dH_1^{-1} = \int_0^1 f(H_1(t))\, dt \;\; \forall f \in \mathcal{F}$$

and $H_2 \succsim H_1$. But $-\int_{[0,1]} g d\left(H_1^{-1} - H_2^{-1}\right) > 0$, that is $\int_0^1 g(H_1(t))\, dt > \int_0^1 g(H_2(t))\, dt$, which is absurd.

The converse is trivial.

Clearly, if $\mathcal{F}$ is a singleton $\succsim$ is complete; the converse, as in Yaari (1987), can be proved applying the standard expected utility result of Grandmont (1972) instead of the expected multi-utility one. ■

Proof of Theorem 1. Since all that matters in the choice problem we are facing are decumulative distribution functions, we can assume $\Omega = (0,1)$, $\mathcal{A}$ to be the Borel σ-field and P to be the Borel measure. If $X, Y \in \mathcal{L}_\infty$, the bounded random variables $Z = G_X^{-1}$ and $W = G_Y^{-1}$ belong to $\mathcal{L}_\infty$; moreover $G_Z = G_X$ and $G_W = G_Y$, hence, $X \succsim Y$ if and only if $G_X^{-1} \succsim G_Y^{-1}$.

The set $\mathcal{X}$ of the pseudoinverses of the elements of $\mathcal{L}_\infty$ is a convex subset of $\mathcal{L}_\infty$ consisting of pairwise comonotonic random variables. Clearly, the pseudoinverses, being nonincreasing are pairwise comonotonic; moreover if $\alpha \in [0,1]$ and $X, Y \in \mathcal{L}_\infty$, $\alpha G_X^{-1} + (1-\alpha) G_Y^{-1} = \alpha G_{G_X^{-1}}^{-1} + (1-\alpha) G_{G_Y^{-1}}^{-1} = G_{\alpha G_X^{-1}}^{-1} + G_{(1-\alpha)G_Y^{-1}}^{-1}$, but αG_X^{-1} and $(1-\alpha) G_Y^{-1}$ are comonotonic, so $G_{\alpha G_X^{-1}}^{-1} + G_{(1-\alpha)G_Y^{-1}}^{-1} = G_{\alpha G_X^{-1} + (1-\alpha)G_Y^{-1}}^{-1}$ with $\alpha G_X^{-1} + (1-\alpha) G_Y^{-1} \in \mathcal{L}_\infty$.

Take $U, V, W, Z \in \mathcal{X}$, the set $A = \{\alpha \in [0,1] : \alpha U + (1-\alpha) V \succsim \alpha W + (1-\alpha) Z\}$ is closed. In fact, let $\alpha_n \in A$ such that $\alpha_n \to \alpha$, then $\alpha_n U + (1-\alpha_n) V \succsim \alpha_n W + (1-\alpha_n) Z$ for all $n \in \mathbb{N}$, but $\alpha_n U + (1-\alpha_n) V$ converges pointwise to $\alpha U + (1-\alpha) V$, *a fortiori*, $\alpha_n U + (1-\alpha_n) V \Rightarrow \alpha U + (1-\alpha) V$. Analogously, $\alpha_n W + (1-\alpha_n) Z \Rightarrow \alpha W + (1-\alpha) Z$. Since all the involved random variables are uniformly bounded, by A3, $\alpha U + (1-\alpha) V \succsim \alpha W + (1-\alpha) Z$, and $\alpha \in A$. As a consequence, for any $G_X^{-1}, G_Y^{-1}, G_Z^{-1} \in \mathcal{X}$ and any $a \in (0,1]$, $aG_X^{-1} + (1-a)G_Z^{-1} \succsim aG_Y^{-1} + (1-a)G_Z^{-1}$ implies $G_X^{-1} \succsim G_Y^{-1}$ (see, e.g., Shapley and Baucells (1998) Lemma 1.2).

Let $a \in (0,1)$ and $b \in \mathbb{R}$,

$$X \succsim Y \quad \Leftrightarrow \quad G_X^{-1} \succsim G_Y^{-1}$$

$$
\begin{aligned}
&\Leftrightarrow\quad aG_X^{-1}+(1-a)\,G_{\frac{b}{1-a}}^{-1}\succsim aG_Y^{-1}+(1-a)\,G_{\frac{b}{1-a}}^{-1}\\
&\Leftrightarrow\quad aG_X^{-1}+b\succsim aG_Y^{-1}+b\\
&\Leftrightarrow\quad G_{aX+b}^{-1}\succsim G_{aY+b}^{-1}\\
&\Leftrightarrow\quad aX+b\succsim aY+b.
\end{aligned}
$$

By Lemma 6, there exists a unique closed and convex set $\mathcal{F}$ of probability distortions such that

$$X\succsim Y\Leftrightarrow\int_\Omega Xd\,(f\circ P)\geq\int_\Omega Yd\,(f\circ P)\quad\forall f\in\mathcal{F}$$

holds for all $X,Y\in\mathcal{V}\subseteq\mathcal{L}_\infty$. We have just proved that if $X,Y\in\mathcal{L}_\infty$, $a\in(0,1)$ and $b\in\mathbb{R}$, then

$$X\succsim Y\Leftrightarrow aX+b\succsim aY+b.$$

If $X=X'$ almost certainly, then $G_X=G_{X'}$ and $X\sim X'$. Therefore, for all X and Y in $\mathcal{L}_\infty$, $X\succsim Y$ if and only if there exist two bounded random variables $X'=X$ almost certainly and $Y'=Y$ almost certainly such that $X'\succsim Y'$. But there exist $a\in(0,1)$ and $b\in\mathbb{R}$ such that $aX'+b$ and $aY'+b$ belong to $\mathcal{V}$; so

$$
\begin{aligned}
X\succsim Y\quad&\Leftrightarrow\quad X'\succsim Y'\\
&\Leftrightarrow\quad aX'+b\succsim aY'+b\\
&\Leftrightarrow\quad\int_\Omega\left(aX'+b\right)d\,(f\circ P)\geq\int_\Omega\left(aY'+b\right)d\,(f\circ P)\quad\forall f\in\mathcal{F}\\
&\Leftrightarrow\quad\int_\Omega X'd\,(f\circ P)\geq\int_\Omega Y'd\,(f\circ P)\quad\forall f\in\mathcal{F}\\
&\Leftrightarrow\quad\int_\Omega Xd\,(f\circ P)\geq\int_\Omega Yd\,(f\circ P)\quad\forall f\in\mathcal{F}
\end{aligned}
$$

as wanted. The rest is trivial. ■

Proof of Proposition 2. Clearly, if $\mathcal{F}'\subseteq\mathcal{F}$, the relation $\succsim'$ extends $\succsim$. To prove the converse, apply the separation argument used in the uniqueness part of Lemma 6. ■

Proof of Proposition 3. Take a dense subset $\{f_n\}_{n\geq1}$ of $\mathcal{F}$, it is easy to prove that $g=\sum_{n\geq1}\frac{1}{2^n}f_n$ induces a completion of $\succsim$, hence $\mathcal{F}^\bullet$ is nonempty. Clearly $\mathcal{F}^\bullet$ is convex. The observation that $f\in\mathcal{F}$ and $g\in\mathcal{F}^\bullet$ imply $\alpha f+(1-\alpha)\,g\in\mathcal{F}^\bullet$ for all $\alpha\in(0,1)$ yields $\overline{\mathcal{F}^\bullet}=\mathcal{F}$. ■

Proof of Proposition 4. If $X,Y\in\mathcal{L}_\infty$, then $X\geq_{SSD}Y$ if and only if $\int_\Omega Xd\,(f\circ P)\geq\int_\Omega Yd\,(f\circ P)$ for all convex probability distortions (see Chateauneuf, 1991). Now $\succsim$ is strongly risk averse if and only if it is an extension of $\geq_{SSD}$, apply Proposition 2.

Clearly, if all the elements of $\mathcal{F}$ are majorized by the identity, then $\succsim$ is weakly risk averse. Conversely, assume that there exist $f\in\mathcal{F}$ and $t'\in I$ such that $f\,(t')>t'$, then $t'\in(0,1)$. Let

$$G_X\,(t)=\begin{cases}1 & t<t'\\ t' & t\in[t',1)\\ 0 & t\geq1\end{cases}.$$

$\int_\Omega XdP = \int_0^\infty G_X(t)\,dt = t' + t'(1-t')$, and $\int_\Omega Xd(f \circ P) = \int_0^\infty f(G_X(t))\,dt = t' + f(t')(1-t')$, hence

$$\int_\Omega \left(\int_\Omega XdP \right) d(f \circ P) = \int_\Omega XdP < \int_\Omega Xd(f \circ P)$$

and it cannot be $\int_\Omega XdP \succsim X$. So $\succsim$ is not weakly risk averse. ■

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