Portfolio selection by dynamic stochastic programming compared to stochastic optimal control

Bosch, Manuela (Universidad de Barcelona, Spain)
Devolder, Pierre (Université Catholique de Louvain, Belgium)
Domínguez, Inmaculada (Universidad de Extremadura, Spain)

Abstract

The aim of this paper is to compare optimal investment policy in a defined contribution pension plan when using two different techniques: dynamic stochastic programming or stochastic optimal control.

The analysis includes assumptions related to the following aspects:
a) keeping constant or changing strategies in each period of the time horizon,
b) using different utility functions of the accumulated fund
c) considering single or periodical contributions.

The optimal investment policy, for periods before and after retirement, is obtained applying both techniques; numerical results are presented and compared together with the main conclusions.

Keywords: asset allocation, optimal control, stochastic programming, asset liability management

1.- Introduction

This paper is an extension of Devolder et al. (2003) where the objective was to show how stochastic optimal control can be applied to obtain optimum investment policy in a pension fund made up of defined contribution plan. The analysis was made for periods before and after retirement. The yield of the risk-free assets was constant and the price index of the risk-bearing asset was defined by a geometric Brownian motion process with given drift and volatility.

A different approach is to consider the information available about the yields on the risky assets as a set of scenarios $s \in S, S = \{1,2,\ldots,m\}$ with an associated probability p_s. The scenarios are arranged into a tree structure for the different periods. In this case, the appropriate methodology for selecting the optimal strategy is dynamic stochastic programming.

The contribution of the present paper is to focus on comparing these two techniques - stochastic optimal control and dynamic stochastic programming - applied to the selection of the portfolio of the defined contribution plan and confirming numerically that the optimum selection is the same when the scenarios are obtained after the discretization of the distribution applied in the stochastic optimal control method.

The results represent a wide range of cases: decisions taken before and after retirement, periodic or single one-off contributions, the use of two different utility functions, and investment strategies that are constant along the time horizon in some cases and change in others.

2.- Model Assumptions

Before starting to apply the different methods in the model, some assumptions will be described:

2.1.- Two types of situation: Before and After Retirement

The investment policy of the pension fund affects both the participants and the beneficiary of the defined contribution plan. The implication of a given decision will differ between the two situations, so that in this sense we shall distinguish between a decision maker during his or her active life (i.e., pre-retirement) and a decision maker who has retired (post-retirement).

In both cases we shall analyze the optimal investment selection when $t \in (0, N)$ where N is the time horizon of simulations.

2.2.- Two types of time horizon : single period or multiple periods

The decisions taken in each of two previous situations can take into account either just the information of the current period or an expectation of the risky yields in future periods. In this sense we distinguish:

- Single period model: $N = 1$
- Multiperiod model: $N > 1$

2.3.- Two types of contributions: periodic or single contributions

The investment policy will depend on whether a single contribution is made or there are periodic contributions:

- Single contribution at $t = 0$: $P(0)$
- Periodic contributions during the active period: $P(t)$.

2.4.- Two types of assets: riskless and risky asset

The investment alternatives studied in this work are reduced to just two categories: a riskless asset and a risky asset:

- Monetary amount allocated to the risk-free asset at the beginning of the period *t*: $X_1(t)$
- Monetary amount allocated to the risky asset at the beginning of the period *t*: $X_2(t)$

The amount corresponding to the risky asset is the proportion $u(t)$ of the accumulated fund at the beginning of the each period $F^+(t)$, and $(1-u(t))$ the proportion corresponding to the riskless asset.

2.5.- Two types of strategies: constants or varying strategies along the time horizon

In this multiperiod context, the investment strategies are obtained for all the periods analyzed. The optimal policy obtained by the two optimization tools will then be given by:

- Constant strategies along the time horizon: $u(1) = u(2) = \cdots = u(t)$.
- Varying strategies along the time horizon: $u(1) \neq u(2) \neq \cdots \neq u(t)$

3.- The utility function of the accumulated fund

The aim of the manager is to maximize the accumulated fund at the end of the time horizon being analyzed. The fund is made up of the contributions and the yields obtained at the end of the period. We express the utility function of the fund at the end of the period t as $U(F(t))$ which can take different expressions as will be seen in the following subsections.

3.1.- HARA (Hyperbolic Absolute Risk Aversion) utility functions

From the Arrow-Pratt definition about the *absolute risk aversion measure* $A(F(t)) = -\dfrac{U''(F(t))}{U'(F(t))}, U'(F(t)) \neq 0$, the *relative risk aversion measure* $R(F(t)) = -\dfrac{U''(F(t))}{U'(F(t))} \cdot F(t)$ and *the index of risk tolerance* $T(F(t)) = -\dfrac{1}{A(F(t))}$, one defines the HARA[1] (hyperbolic absolute risk aversion) utility functions. This class of utility functions has the feature that the risk tolerance is linear or the absolute risk aversion is a hyperbolic function.

$$T(F(t)) = a + b \cdot F(t) \rightarrow A(F(t)) = \frac{1}{a + bF(t)}$$

That means the $U(F(t))$ satisfies the differential equation :

$$\big(a + b \cdot F(t)\big) \cdot U''(F(t)) + U'(F(t)) = 0 \qquad (1)$$

which has the general solution:

$$U(F(t)) = \frac{1}{b-1} \cdot (a + b \cdot F(t))^{1-\frac{1}{b}}, \; a + b \cdot F(t) > 0 \qquad (2)$$

This class includes power-law utility functions and the negative exponential utility functions as limiting cases. Both types will be used in this paper to study how the optimal portfolio is affected by this choice.

3.2.- Particular Case: The negative exponential utility function

When $A = \frac{1}{\alpha}, B = 0$ in (1) the solution of the differential equation is the negative exponential utility function:

$$U(F(t)) = \frac{1}{\alpha} \cdot \left(1 - e^{-\alpha \cdot F(t)}\right) \quad with\ \alpha > 0 \qquad (3)$$

This function has constant absolute risk aversion:

$$A(F(t)) = \alpha$$

and is often called CARA utility function.

3.3.- The power-law utility function

When $A = 0$ in (1) the solution of the differential equation is the power-law utility function:

$$U(F(t)) = \frac{1}{(1-\gamma)} \cdot F(t)^{(1-\gamma)} \quad with \quad \gamma > 0 \qquad (4)$$

[1] Also known as LRT or Lineal risk tolerance.

The power-law utility has decreasing absolute risk aversion, i.e., there is less aversion for bearing the same amount of risk when the accumulated fund is large. However, it has a constant relative aversion:

$$A(F(t)) = \frac{\gamma}{F(t)}$$

$$R(F(t)) = \gamma$$

This utility function is often called CRRA or constant relative risk aversion

3.4.- The objective function

The problem in the two situations (pre- and post-retirement) will be to optimize the expected utility of the final wealth: the total fund obtained at the retirement age and the surplus after payment of pensions for some periods.

$$\max_{u(t)} E(U(F(N)) \quad \forall \quad t \in (0, N) \tag{5}$$

4.- Optimal portfolio: stochastic optimal control and dynamic stochastic programming.

4.1- Optimal portfolio using stochastic programming

In this section, it is necessary to introduce the scenario tree for the uncertain parameter, the budget constrains and the non-anticipativity constrains, the decision variables and finally the optimization model.

4.1.1.- Scenario Tree

In dynamic stochastic programming, the uncertainty is represented by a number of different realizations. Each complete realization of all the uncertain parameters is a scenario along the multiperiod horizon.

In the dynamic stochastic programming model, the information available about the single uncertain parameter, the risky active yield, is a set of scenarios

$r_2^s(t)$ $s \in S, S = \{1,2,\ldots,m\}$ with an associated probability p_s and $\sum_{s=1}^{m} p_s = 1$. The yield of the free-risk asset for each scenario $r_1^s(t)$ is supposed to be constant r_1 .

The scenarios are arranged into a tree structure along the succession of periods so that tree has a depth equal to the length of the planning horizon N.

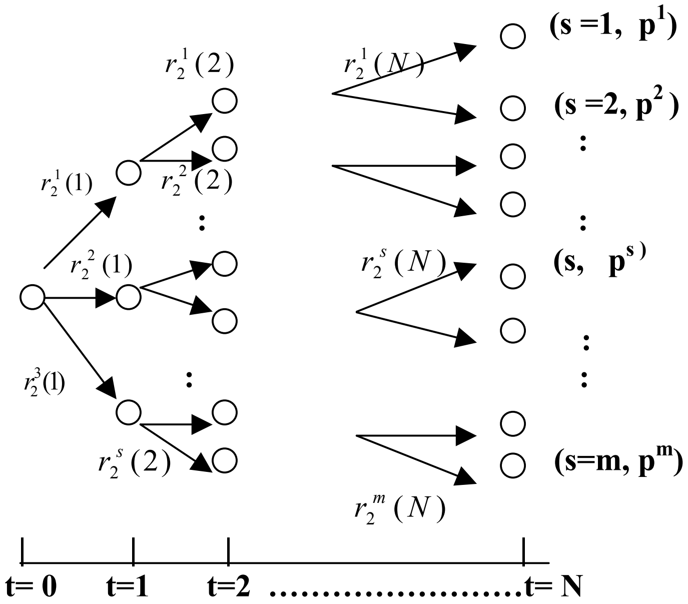

Figure 1

Each path represents one scenario in each period (r_1 , $r_2^s(t)$) and each node represents the moment when decisions are made taking into account the scenario obtained in the period before $X^s(t)$.

Scenario-dependent variables are represented after this moment with a super index s as the accumulated fund at the beginning $F^{s+}(t)$, or at the end, $F^s(t)$, of each period .

When the values of some uncertain parameters are observed in a multiperiod model, new information that has became available is introduced into the optimization model. The value of the decision variable is therefore not unique for each period. Instead, there is one value for each period under each scenario.

4.1.2.- Budget constrains and non-anticipativity constrains

In order to make decisions along the planning horizon, it is necessary to first introduce, the budget at the beginning of each period t under scenario s :

$$F^{s+}(t) = F^{s}(t-1) + P(t) = X_1^s(t) + X_2^s(t) \quad \forall\, t = \{0,1,2,\ldots,N\} \tag{6}$$

If there is only a single contribution rather than periodic contributions, then:

$$P(0) = P,\ P(1) = P(2) = \ldots = P(N) = 0$$

If the decision maker has retired and is receiving his or her benefits, we consider:

$$F^{s+}(t) = F^{s}(t-1) - B(t) = X_1^s(t) + X_2^s(t) \quad \forall\, t = \{0,1,2,\ldots,N\} \tag{7}$$

At the end of each period the accumulated fund is:

$$F^{s}(t) = X_1^s(t)\cdot(1+r_1) + X_2^s \cdot (1+r_2^s(t)) \qquad \forall\, t = \{0,1,2,\ldots,N\} \tag{8}$$

In some points where the decision maker has to take a decision, the information under different scenarios has a common past. Scenarios with common information history up to a particular period must yield the same decisions up to that period, a condition known as *non-anticipativity*[2]. In the Figure 1, for instance, $X_2^1(1) = X_2^2(1)$. A way to generalize this rule is that if two different scenarios s and s' are indistinguishable at time t, then

$$X_2^s(t) = X_2^{s'}(t) \tag{9}$$

4.1.3.- Decision variables

The amount invested in risky assets and riskless assets are a proportion of the Fund at the beginning of each period.

$$\begin{aligned} X_1^s(t) &= (1-u^s(t))\cdot F^{s+}(t) \\ X_2^s(t) &= u^s(t)\cdot F^{s+}(t) \end{aligned} \tag{10}$$

In this sense, we consider as decision variables of the model the proportion invested in risk free assets and risky assets for the period t under the scenario s .

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[2] Rockafeller, R.T. and Wets, R. J.-B. (1991)

4.1.4.- The optimization model

If the objective function (5) is substituted by the equation (3) and the constrains (6) and (7) are applied conjointly with constrains of the non-negativity of the variables, the optimization model with exponential utility function is:

$$
\begin{cases}
Max \quad E(U(F(N))) = Max \quad E\left(\frac{1}{\alpha}\cdot\left(1-e^{-\alpha\cdot F^s(N)}\right)\right) = Max \sum_{s=1}^{m}\left(\frac{1}{\alpha}\cdot\left(1-e^{-\alpha\cdot F^s(N)}\right)\right)\cdot p^s \\
subject \quad to: \\
\qquad F^{s+}(t) = F^s(t-1) + P(t) = X_1^s(t) + X_2^s(t) \quad \forall\ t = \{0,1,2\ldots,N\} \qquad (11) \\
\qquad X_2^s(t) = X_2^{s'}(t) \\
\qquad X_1^s(t) \geq 0 \\
\qquad X_2^s(t) \geq 0
\end{cases}
$$

If , on the other hand, equation (4) is introduced into the objective function, the new power-law utility function model is:

$$
\begin{cases}
Max \quad E(U(F(N))) = Max \quad E\left(\frac{(F(N))^{1-\gamma}}{1-\gamma}\right) = Max \sum_{s=1}^{m}\left(\frac{(F^s(N))^{1-\gamma}}{1-\gamma}\right)\cdot p^s \\
subject \quad to: \\
\qquad F^{s+}(t) = F^s(t-1) + P(t) = X_1^s(t) + X_2^s(t) \quad \forall\ t = \{0,1,2\ldots,N\} \qquad (12) \\
\qquad X_2^s(t) = X_2^{s'}(t) \\
\qquad X_1^s(t) \geq 0 \\
\qquad X_2^s(t) \geq 0
\end{cases}
$$

The solution may be obtained by making use of a software package that permits us to work with many scenarios and different periods. This kind of model is known as a large optimization problem. The decision variables $u^s(t)$ and $(1\text{-}u^s(t))$ are in concordance with the corresponding $u(t)$ and $(1\text{-}u(t))$ presented in the next subsection when the scenarios used come from discretization of the distribution applied in the stochastic optimal control.

4.2- Optimal Portfolio using Stochastic Optimal Control

The model of stochastic optimal control developed in Devolder et al.(2003) was based on the following assumptions :

1° continuous choice between two kinds of asset :

- riskless asset: X_1 solution of the following deterministic equation:

$$dX_1(t) = r\ X_1(t)\ dt$$

- risky asset : X_2 solution of the following stochastic differential equation :

$$dX_2(t) = \mu\ X_2(t)\ dt + \sigma\ X_2(t)\ dw(t)$$

where w is a standard Brownian motion.

- The proportion of the fund invested in the risky asset at time t is denoted u(t); 1-u(t) is the proportion in the riskless asset.

2° evolution equation :

The fund is solution of an evolution equation :

$$dF(t) = \big(F(t)\ (u(t)\ \mu + (1-u(t)r) + M\big)\, dt + F(t)\ u(t)\ \sigma\ dw(t) \qquad (13)$$

where M is a continuous rate of external income to the fund. The following particular cases can be considered :

-2.1 period before retirement with single contribution: F(0) = P (initial single contribution) M=0

-2.2 period before retirement with periodical contributions: F(0)=0 M= P (constant continuous premium)

-2.3 period after retirement : F(0) = C (initial amount accumulated at retirement age) M= -B (continuous rate of benefit)

3° optimization problem :

The problem is to find at each moment the best investment policy u(t) in order to optimize the expected utility of the final wealth at the end of the considered time horizon N:

$$\max E\, U(F(N))$$

4° solution:

General solutions have been obtained for power law utility and exponential utility.

- 4.1 Power-law utility function :

If we take as utility function the formula (4) the optimal asset allocation is given by: :

$$u(t) = \frac{F + M.a\,(N-t)}{F} . \frac{\mu - r}{\gamma.\sigma^2} \tag{14}$$

where a (N-t) is a continuous annuity computed at the riskfree rate :

$$a(N-t) = \frac{1 - \exp(-r(N-t)}{r}$$

In the particular case before retirement with single contribution this optimal policy is constant (Merton ratio):

$$u(t) = \frac{\mu - r}{\gamma \cdot \sigma^2} \tag{15}$$

- 4.2 Exponential utility :

If we take as utility function the formula (3) the optimal asset allocation is given by:

$$u(t) = \frac{\exp(r\ (t-N))}{F} . \frac{\mu - r}{\alpha.\sigma^2} \tag{16}$$

which is explicitly independent of the amount M.

In order to compare this model with the model of dynamic programming of section 4.1 we can at this stage already take the following conclusions in this optimal control framework:

- If the **power-law utility** is used :

- **Conclusion a)** in the case before retirement with a single contribution the optimal policy is constant and given by (15).

As numerical illustration and comparison with the following section let us consider the special case (we use discrete value of the parameters because of the comparison with the dynamic programming methodology):

- riskfree rate of 4% : $r = \ln(1.04) = 0.03922$
- mean return of the risky asset of 6% : $\mu = \ln(1.06) = 0.05827$
- volatility of the return of the risky asset of 0.15: taking into account the distribution of the return in the Brownian model (log normal distribution) the parameter of volatility is then given by :

$$\sigma = \sqrt{\ln\ (1 + \left(\frac{0.15^2}{1.06^2} \right)} = 0{,}1408$$

- risk aversion : $(1-\gamma) = -1$

Putting these different values of the parameters into formula (15) we obtain an optimal investment in risky asset of 48%.

- **<u>Conclusion b)</u>** in the case before retirement with periodical contributions (M positive equal to the continuous premium) the formula (14) shows that the optimal allocation will be bigger than in the case with single contribution . This optimal allocation will be a decreasing function of time starting generally at t=0 at 100% and converging to the Merton ratio (15) for t=N.

- **<u>Conclusion c)</u>** in the case after retirement (M negative equal to the continuous benefit) the formula (14) shows that the optimal allocation will be less than in the case before retirement with single contribution.

This optimal allocation will be an increasing function of time converging to the Merton ratio (15) for t=N.

- If **the exponential utility** is used :

- **<u>Conclusion d)</u>** the same formal expression (16) has been obtained and can be used before or after retirement and with single or periodical contributions The main conclusion with exponential utility is the fact that when the fund increases less (more) rapidly that the riskfree rate the proportion in risky asset must increase (decrease) to compensate this bad (good) performance.

This general statement can be adapted in different situations:

<u>Conclusion e)</u> In the case before retirement with periodical contributions the fund normally increases more rapidly than the riskfree rate because of the presence of contributions; so it can be expected that the proportion in risky asset must decrease over time.

<u>Conclusion f)</u> In the case after retirement the fund normally increases less rapidly than the riskfree rate because of the presence of benefits to pay; so it can be expected that the proportion in risky asset must increase over time.

Conclusion g) In the case before retirement with single contribution the proportion in risky asset from one year to the other will decrease for a good economic scenario and will increase for a bad scenario.

5.- Practical case

In this section, we shall deal with the selection of a portfolio formed by two assets: one risk-free and the other risky, using the dynamic stochastic programming technique and looking at the eventual coherence of the results with the main conclusions presented just before in section 4.2 when using an optimal control framework.

5.1. Initial data

-Two decision makers: One pre-retirement and the other post-retirement.

- Temporal horizon $N = 3$ periods.

- Contributions: $P(0) = P(1) = P(2) = P(3) = 5.292,2€$ made at the beginning of each period or $P(0) = 15.273,77€$ in the case of only a single contribution.

- The benefit is perceived in the form of a pension at the end of each of the periods $B(1) = B(2) = B(3) = 3000€$, with the accumulated fund at retirement age being $F(0) = 15.273,77€$

- The exponential and power-law utility functions will be used in the search for the optimal investment portfolio for both the pre- and the post-retirement cases.

- The risk aversion coefficients taken for the exponential utility function are:

$$\alpha = \{0,00004;\ 0,00008;\ 0,0001;\ 0,004;\ 0,009\}$$

- The risk aversion coefficients taken for the power-law utility function are:

$$(1-\gamma)=\{-2;\ -1{,}5;\ \ -1;\ \ -0{,}5;\ \ -0{,}25;\ \ 0{,}25\}$$

- In the application of dynamic stochastic programming, one starts from 3 scenarios for each of the three periods – a pessimistic, a neutral, and an optimistic scenario – with mean value 6% and volatility 0,15.

N=1

	$r_1^s(1)$	$r_2^s(1)$	p^s
Optimistic scenario:	4%	24,37 %	0,33
Neutral scenario:	4%	6 %	0,33
Pessimistic scenario:	4%	-12,37 %	0,33

N=3

The combination of the three scenarios for the remaining periods generates a total of 27 scenarios, structured in the form of a tree, each with a $p^s = 0{,}03704$, as shown in the following scenario tree:

	$r_2^s(1)$	$r_2^s(2)$	$r_2^s(3)$
Scenario 1	24,37%	24,37%	24,37%
Scenario 2	24,37%	24,37%	6,00%
Scenario 3	24,37%	24,37%	-12,37%
Scenario 4	24,37%	6,00%	24,37%
Scenario 5	24,37%	6,00%	6,00%
Scenario 6	24,37%	6,00%	-12,37%
Scenario 7	24,37%	-12,37%	24,37%
Scenario 8	24,37%	-12,37%	6,00%
Scenario 9	24,37%	-12,37%	-12,37%
Scenario 10	6,00%	24,37%	24,37%
Scenario 11	6,00%	24,37%	6,00%
Scenario 12	6,00%	24,37%	-12,37%
Scenario 13	6,00%	6,00%	24,37%
Scenario 14	6,00%	6,00%	6,00%
Scenario 15	6,00%	6,00%	-12,37%
Scenario 16	6,00%	-12,37%	24,37%
Scenario 17	6,00%	-12,37%	6,00%
Scenario 18	6,00%	-12,37%	-12,37%
Scenario 19	-12,37%	24,37%	24,37%
Scenario 20	-12,37%	24,37%	6,00%
Scenario 21	-12,37%	24,37%	-12,37%
Scenario 22	-12,37%	6,00%	24,37%
Scenario 23	-12,37%	6,00%	6,00%
Scenario 24	-12,37%	6,00%	-12,37%
Scenario 25	-12,37%	-12,37%	24,37%
Scenario 26	-12,37%	-12,37%	6,00%
Scenario 27	-12,37%	-12,37%	-12,37%

Figure 2

$$G_1^1 = \{1,2,3,4,5,6,7,8,\cdots,\ 27\}$$

We are working with a scenario tree of 27 scenarios, with the same amount invested at the beginning of t=2 for the first nine scenarios since they have the information up to that moment in common. The same is the case for the following group of nine scenarios and for the last nine. Hence, for t= 2 the following groups are established:

$$G_2^1 = \{1,2,3,4,5,6,7,8,9\} \quad G_2^2 = \{10,11,12,13,14,15,16,17,18\} \quad G_2^3 = \{19,20,21,22,23,24,25,26,27\}$$

and for t= 3:

$$G_3^1 = \{1,2,3\} \qquad G_3^2 = \{4,5,6\} \qquad G_3^3 = \{7,8,9\}$$
$$G_3^4 = \{10,11,12\} \qquad G_3^5 = \{13,14,15\} \qquad G_3^6 = \{16,17,18\}$$
$$G_3^7 = \{19,20,21\} \qquad G_3^8 = \{22,23,24\} \qquad G_3^9 = \{25,26,27\}$$

The *non-anticipativity constraints* are:

- t=1

$$X_2^1(1) = X_2^2(1) = X_2^3(t) = X_2^4(t) = X_2^5(t) = X_2^6(t) = X_2^7(t) = X_2^8(t) = X_2^9(t) =$$
$$= X_2^{10}(1) = X_2^{11}(1) = X_2^{12}(t) = X_2^{13}(t) = X_2^{14}(t) = X_2^{15}(t) = X_2^{16}(t) = X_2^{17}(t) = X_2^{18}(t) =$$
$$= X_2^{19}(1) = X_2^{20}(1) = X_2^{21}(t) = X_2^{22}(t) = X_2^{23}(t) = X_2^{24}(t) = X_2^{25}(t) = X_2^{26}(t) = X_2^{27}(t) = u^{G_1^1}(1) \cdot C(1)$$

- t=2

$$X_2^1(2) = X_2^2(2) = X_2^3(t) = X_2^4(t) = X_2^5(t) = X_2^6(t) = X_2^7(t) = X_2^8(t) = X_2^9(t) = u^{G_2^1}(2) \cdot \left(F^{G_2^1}(1) + C(2)\right)$$
$$X_2^{10}(2) = X_2^{11}(2) = X_2^{12}(2) = X_2^{13}(2) = X_2^{14}(2) = X_2^{15}(2) = X_2^{16}(2) = X_2^{17}(2) = X_2^{18}(2) = u^{G_2^2}(2) \cdot \left(F^{G_2^2}(1) + C(2)\right)$$
$$X_2^{19}(2) = X_2^{20}(2) = X_2^{21}(2) = X_2^{22}(2) = X_2^{23}(2) = X_2^{24}(2) = X_2^{25}(2) = X_2^{26}(2) = X_2^{27}(2) = u_2^{G_2^3}(2) \cdot \left(F^{G_2^3}(1) + C(2)\right)$$

- t=3

$$X_2^1(3) = X_2^2(3) = X_2^3(2) = u^{G_3^1}(3) \cdot \left(F^{G_3^1}(2) + C(3)\right)$$
$$X_2^4(3) = X_2^5(3) = X_2^6(3) = u^{G_3^2}(3) \cdot \left(F^{G_3^2}(2) + C(3)\right)$$
$$X_2^7(3) = X_2^8(3) = X_2^9(3) = u^{G_3^3}(3) \cdot \left(F^{G_3^3}(2) + C(3)\right)$$
$$X_2^{10}(3) = X_2^{11}(3) = X_2^{12}(3) = u^{G_3^4}(3) \cdot \left(F^{G_3^4}(2) + C(3)\right)$$
$$X_2^{13}(3) = X_2^{14}(3) = X_2^{15}(3) = u^{G_3^5}(3) \cdot \left(F^{G_3^5}(2) + C(3)\right)$$
$$X_2^{16}(3) = X_2^{17}(3) = X_2^{18}(3) = u^{G_3^6}(3) \cdot \left(F^{G_3^6}(2) + C(3)\right)$$
$$X_2^{19}(3) = X_2^{20}(3) = X_2^{21}(3) = u^{G_3^7}(3) \cdot \left(F^{G_3^7}(2) + C(3)\right)$$
$$X_2^{22}(3) = X_2^{23}(3) = X_2^{24}(3) = u^{G_3^8}(3) \cdot \left(F^{G_3^8}(2) + C(3)\right)$$
$$X_2^{25}(3) = X_2^{26}(3) = X_2^{27}(3) = u^{G_3^9}(3) \cdot \left(F_2^{G_3^9}(2) + C(3)\right)$$

The Figure 3 summarizes <u>the calculations</u> that were made using the dynamic stochastic programming.

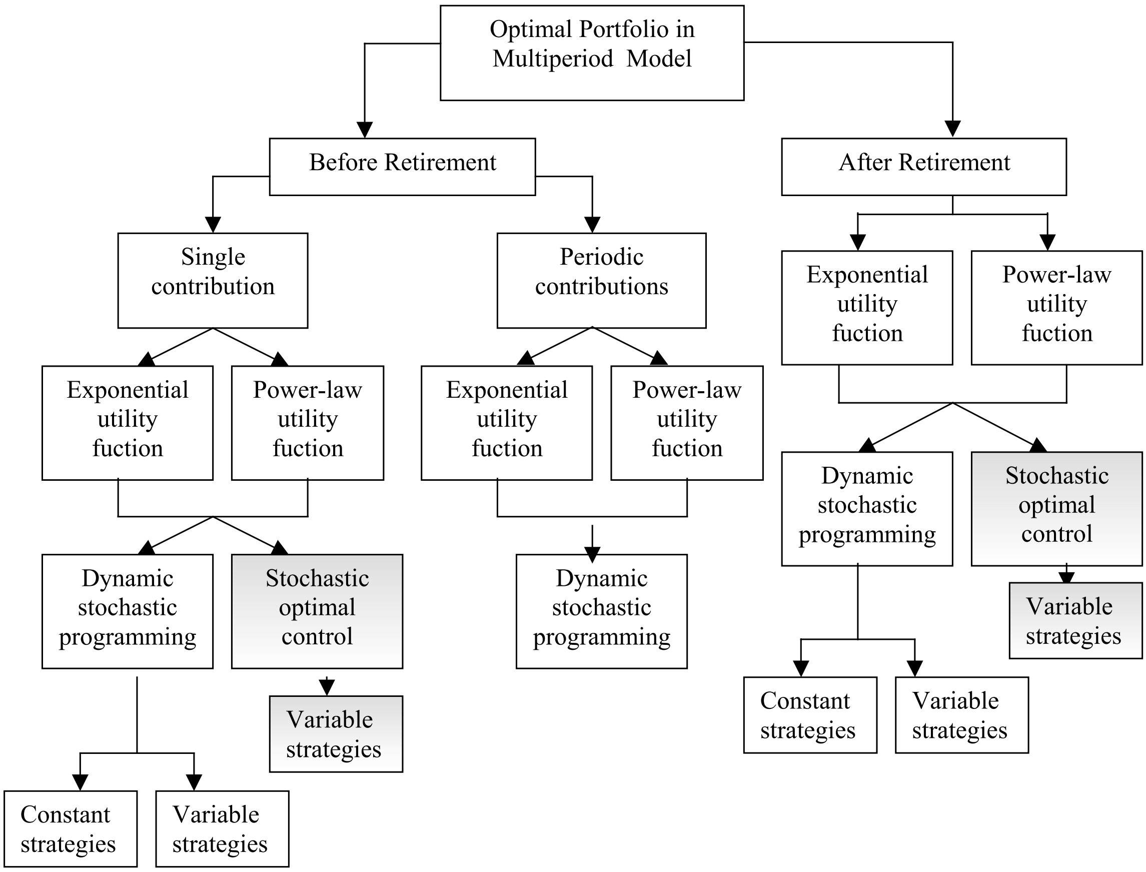

Figure 3

A wide range of cases were analysed with the aim of achieving conclusions with respect to the application of the two methods in three major blocks:

(i) Differences in the investment policy of a pre-retirement individual with respect to a post-retirement individual, taking into account the two types of utility functions in the assumption of strategies that are variable along the temporal horizon and a single contribution.

(ii) Differences between starting from constant or variable investment strategies in dynamic stochastic programming, and implicit variable strategies in stochastic optimal control.

(iii) Differences between making a single contribution or periodic contributions during the pre-retirement period.

5.2. Optimal portfolio pre- and post-retirement for the two utility functions

Though a sensitivity analysis of the aversion coefficients of the CARA utility function or exponential function α and of the CRRA utility or power-law function $(1-\gamma)$, in this subsection we shall start from certain particular values, for instance, $\alpha = 0{,}0001$ and $(1-\gamma) = -1$.

The said aversion coefficients are substituted into the optimization models developed in (12) and (13), incorporating the non-anticipativity constraints specified in the data subsection.

The analysis is performed for single contributions and when the investment strategies are allowed to vary along the planning horizon.

The results are summarized in Table 1:

Table 1: Percentage invested in risky assets with variable strategies.

T = 3	Before retirement[*]		After retirement
	Stochastic optimal control	Dynamic stochastic programming	Dynamic stochastic programming
Exponential utility function $\alpha = 0{,}0001$		$u^{G_1^1}(1) = 54{,}05\%$ $u^{G_2^1}(2) = 48{,}87\%$ $u^{G_2^2}(2) = 53{,}49\%$ $u^{G_2^3}(2) = 59{,}09\%$ $u^{G_3^1}(3) = 44{,}60\%$ $u^{G_3^2}(3) = 48{,}58\%$ $u^{G_3^3}(3) = 52{,}95\%$ $u^{G_3^4}(3) = 48{,}42\%$ $u^{G_3^5}(3) = 52{,}95\%$ $u^{G_3^6}(3) = 58{,}41\%$ $u^{G_3^7}(3) = 52{,}95\%$ $u^{G_3^8}(3) = 58{,}41\%$ $u^{G_3^9}(3) = 65{,}13\%$	$u^{G_1^1}(1) = 53{,}84\%$ $u^{G_2^1}(2) = 58{,}96\%$ $u^{G_2^2}(2) = 65{,}79\%$ $u^{G_2^3}(2) = 73{,}68\%$ $u^{G_3^1}(3) = 64{,}27\%$ $u^{G_3^2}(3) = 72{,}51\%$ $u^{G_3^3}(3) = 83{,}16\%$ $u^{G_3^4}(3) = 72{,}47\%$ $u^{G_3^5}(3) = 83{,}11\%$ $u^{G_3^6}(3) = 97{,}41\%$ $u^{G_3^7}(3) = 83{,}20\%$ $u^{G_3^8}(3) = 97{,}37\%$ $u^{G_3^9}(3) = 100\%$
Power-law utility function $(1-\gamma) = -1$	$u(t) = 48\%$	$u^{G_1^1}(1) = 46{,}64\%$ $u^{G_2^1}(2) = 46{,}64\%$ $u^{G_2^2}(2) = 46{,}67\%$ $u^{G_2^3}(2) = 46{,}61\%$ $u^{G_3^1}(3) = 46{,}85\%$ $u^{G_3^2}(3) = 46{,}74\%$ $u^{G_3^3}(3) = 46{,}63\%$ $u^{G_3^4}(3) = 46{,}47\%$ $u^{G_3^5}(3) = 46{,}64\%$ $u^{G_3^6}(3) = 46{,}55\%$ $u^{G_3^7}(3) = 46{,}64\%$ $u^{G_3^8}(3) = 46{,}55\%$ $u^{G_3^9}(3) = 46{,}47\%$	$u^{G_1^1}(1) = 29{,}36\%$ $u^{G_2^1}(2) = 36{,}89\%$ $u^{G_2^2}(2) = 36{,}27\%$ $u^{G_2^3}(2) = 35{,}57\%$ $u^{G_3^1}(3) = 46{,}65\%$ $u^{G_3^2}(3) = 46{,}64\%$ $u^{G_3^3}(3) = 46{,}64\%$ $u^{G_3^4}(3) = 46{,}64\%$ $u^{G_3^5}(3) = 46{,}64\%$ $u^{G_3^6}(3) = 46{,}64\%$ $u^{G_3^7}(3) = 46{,}64\%$ $u^{G_3^8}(3) = 46{,}64\%$ $u^{G_3^9}(3) = 46{,}64\%$

(*) Single contribution

5.3. Constant and temporally variable strategies

In the method of stochastic control theory, one starts from a percentage allocated to risky assets that varies over time. In dynamic stochastic programming, however, one fixes beforehand whether the strategies will remain constant or be variable. In Table 1, the strategies were variable over time.

The common information of certain scenarios in the past is irrelevant for the determination of the constant percentages, so that the non-anticipativity constraints are not applied.

Table 2 gives the above scheme for the case of dynamic stochastic programming with constant coefficients.

Table 2: Percentage invested in risky assets with constant strategies.

T = 3	Before retirement(*)	After retirement
	Dynamic stochastic programming	**Dynamic stochastic programming**
Exponential utility function $\alpha = 0{,}0001$	$u(1) = u(2) = u(3) = 53\%$	$u(1) = u(2) = u(3) = 63\%$
Power-law utility function $(1-\gamma) = -1$	$u(1) = u(2) = u(3) = 47\%$	$u(1) = u(2) = u(3) = 35\%$

(*) Single contribution

5.4. Single contribution versus periodic contributions

In this subsection, the objective consists of analysing how the investment policy varies when periodic contributions are made instead of a single contribution. The results are summarized in Table 3.

Table 3: Percentage invested in risky assets with variable strategies.

T = 3	Before retirement(*) Dynamic stochastic programming
Exponential utility function $\alpha = 0{,}0001$	$u^{G_1^1}(1) = 100\%$ $u^{G_2^1}(2) = 72{,}30\%$ $u^{G_2^2}(2) = 78{,}75\%$ $u^{G_2^3}(2) = 86{,}46\%$ $u^{G_3^1}(3) = 46{,}05\%$ $u^{G_3^2}(3) = 50{,}12\%$ $u^{G_3^3}(3) = 54{,}49\%$ $u^{G_3^4}(3) = 48{,}58\%$ $u^{G_3^5}(3) = 53{,}14\%$ $u^{G_3^6}(3) = 58{,}14\%$ $u^{G_3^7}(3) = 51{,}41\%$ $u^{G_3^8}(3) = 56{,}54\%$ $u^{G_3^9}(3) = 62{,}82\%$
Power-law utility function $(1-\gamma) = -1$	$u^{G_1^1}(1) = 100\%$ $u^{G_2^1}(2) = 66{,}62\%$ $u^{G_2^2}(2) = 68{,}41\%$ $u^{G_2^3}(2) = 70{,}54\%$ $u^{G_2^1}(3) = 46{,}64\%$ $u^{G_2^2}(3) = 46{,}64\%$ $u^{G_2^3}(3) = 46{,}64\%$ $u^{G_2^4}(3) = 46{,}64\%$ $u^{G_2^5}(3) = 46{,}64\%$ $u^{G_2^6}(3) = 46{,}64\%$ $u^{G_2^7}(3) = 46{,}64\%$ $u^{G_2^8}(3) = 46{,}64\%$ $u^{G_2^9}(3) = 46{,}64\%$

(*) Periodic contributions

Table 4 gives the optimal percentages if the strategies were constant.

Table 4: Percentage invested in risky assets with constant strategies.

T = 3	Before Retirement (*)
	Dynamic stochastic programming
Exponential utility function $\alpha = 0{,}0001$	$u(1) = u(2) = u(3) = 69\%$
Power-law utility function $(1-\gamma) = -1$	$u(1) = u(2) = u(3) = 60\%$

(*) Periodic contributions

6. Comparative results using the two techniques and the main conclusions

The principal conclusions drawn from the numerical example refer both to the calculations performed using the two optimization techniques and to the additional results using exclusively dynamic stochastic programming since, once programmed, this method has greater flexibility in increasing the range of cases that it is feasible to analyse. These conclusions are:.

Conclusion h) **For the pre-retirement period with single contribution**, assuming that the strategies may vary along the temporal horizon, if one works with the power-law function the percentage invested in risky assets remains constant independently of the scenario, while for the exponential function this percentage decreases as retirement age is approached. Also, in this latter case, for pessimistic scenarios there is a tendency to increase the assets that have a greater profitability at the same time as a greater level of risk.

This conclusion obtained by using the dynamic stochastic programming is exactly the same as conclusion a) and conclusion g) obtained in section 4.2 in stochastic optimal control.

<u>Conclusion i)</u> The post-retirement individual, as his or her initial fund diminishes as a consequence of the pension payments, is converted into a riskier investor for an exponential utility. If we consider the scenario level, for pessimistic scenarios the investment in risky assets is greater than for optimistic scenarios. With the power law utility, the case is precisely the opposite : the pre-retirement investor is riskier.

This conclusion obtained by using the dynamic stochastic programming is exactly the same as conclusion a) and conclusion g) obtained in section 4.2 in stochastic optimal control.

<u>Conclusion j)</u> In the case of periodical contributions before retirement, the investor has a riskier attitude than in the case of a single contribution.

This conclusion obtained by using the dynamic stochastic programming is exactly the same as conclusion b) and conclusion e) obtained in section 4.2 in stochastic optimal control.

<u>Conclusion k)</u> For strategies that are constant over time, the post-retirement investor is riskier than the pre-retirement investor for the exponential utility function. For the power-law utility function, the result is the contrary: the pre-retirement investor is riskier.

This conclusion was reached by applying the dynamic stochastic programming model. The methodological philosophy of stochastic optimal control theory makes that method inapplicable to this case.

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