# STOCHASTIC DEFLATOR FOR AN ECONOMIC SCENARIO GENERATOR WITH FIVE FACTORS

## SUPPLEMENTARY MATERIAL


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## Summary




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## 1. APPENDIX 1: CONSTRUCT CORRELATED BROWNIAN MOTIONS

In Appendix 1, we provide details of the calculations to construct correlated Brownian motions in our model. We start with the Brownian motion parts of interest rate process and stock price process. Then we derive the Brownian motion parts of default intensity process and convenience yield process step by step. Readers are referred to Shreve (2004) Chapter 3.4 for discussions about quadratic variation.

Step 1

Let $dW_r = dW_0$, then $dW_S = \rho_{rS} dW_r + \sqrt{1-\rho_{rS}^2}\, dW_1$.

*pf.*

$$dW_r dW_S = dW_0\left(\rho_{rS} dW_0 + \sqrt{1-\rho_{rS}^2}\, dW_1\right) = \rho_{rS} dt$$

$$\begin{aligned} dW_S dW_S &= \left(\rho_{rS} dW_0 + \sqrt{1-\rho_{rS}^2}\, dW_1\right)\left(\rho_{rS} dW_0 + \sqrt{1-\rho_{rS}^2}\, dW_1\right) \\ &= \left(\rho_{rS}^2 + 1 - \rho_{rS}^2\right) dt = dt \end{aligned}$$

Step 2

$$dW_\chi = \rho_{r\chi} dW_r + \rho'_{S\chi} dW_1 + \rho'_{\chi\chi} dW_2,$$

where $\rho'_{S\chi} = \dfrac{\rho_{S\chi} - \rho_{rS}\rho_{r\chi}}{\sqrt{1-\rho_{rS}^2}}$, $\rho'_{\chi\chi} = \sqrt{\dfrac{1-\rho_{rS}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}{1-\rho_{rS}^2}}$.

*pf.*

$$dW_r dW_\chi = dW_0\left(\rho_{r\chi} dW_0 + \rho'_{S\chi} dW_1 + \rho'_{\chi\chi} dW_2\right) = \rho_{r\chi} dt$$

$$\begin{aligned} dW_S dW_\chi &= \left(\rho_{rS} dW_0 + \sqrt{1-\rho_{rS}^2}\, dW_1\right)\left(\rho_{r\chi} dW_0 + \rho'_{S\chi} dW_1 + \rho'_{\chi\chi} dW_2\right) \\ &= \left(\rho_{rS}\rho_{r\chi} + \rho'_{S\chi}\sqrt{1-\rho_{rS}^2}\right) dt = \rho_{S\chi} dt \end{aligned}$$

$\rho_{rS}\rho_{r\chi} + \rho'_{S\chi}\sqrt{1-\rho_{rS}^2} = \rho_{S\chi}$, then $\rho'_{S\chi} = \dfrac{\rho_{S\chi} - \rho_{rS}\rho_{r\chi}}{\sqrt{1-\rho_{rS}^2}}$.

$$\begin{aligned} dW_\chi dW_\chi &= \left(\rho_{r\chi} dW_0 + \rho'_{S\chi} dW_1 + \rho'_{\chi\chi} dW_2\right)\left(\rho_{r\chi} dW_0 + \rho'_{S\chi} dW_1 + \rho'_{\chi\chi} dW_2\right) \\ &= \left(\rho_{r\chi}^2 + \rho'^2_{S\chi} + \rho'^2_{\chi\chi}\right) dt = dt \end{aligned}$$

$\rho_{r\chi}^2 + \rho'^2_{S\chi} + \rho'^2_{\chi\chi} = 1$, then $\rho'^2_{\chi\chi} = 1 - \rho_{r\chi}^2 - \rho'^2_{S\chi} = 1 - \rho_{r\chi}^2 - \dfrac{\left(\rho_{S\chi} - \rho_{rS}\rho_{r\chi}\right)^2}{1-\rho_{rS}^2}$.

Then, $\rho'_{\chi\chi} = \sqrt{\dfrac{1-\rho_{rS}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}{1-\rho_{rS}^2}}$.

## Step 3

$$dW_\gamma = \rho_{r\gamma} dW_r + \rho''_{S\gamma} dW_1 + \rho''_{\chi\gamma} dW_2 + \rho''_{\gamma\gamma} dW_3,$$

where $\rho''_{S\gamma} = \dfrac{\rho_{S\gamma} - \rho_{rS}\rho_{r\gamma}}{\sqrt{1-\rho_{rS}^2}}$,

$$\rho''_{\chi\gamma} = \frac{\rho_{\chi\gamma} - \rho_{r\chi}\rho_{r\gamma} - \rho_{S\chi}\rho_{S\gamma} - \rho_{rS}^2\rho_{\chi\gamma} + \rho_{rS}\rho_{r\chi}\rho_{S\gamma} + \rho_{rS}\rho_{r\gamma}\rho_{S\chi}}{\sqrt{1+\rho_{rS}^4 - 2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{r\chi}^2 + \rho_{rS}^2\rho_{S\chi}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}},$$

$$\rho''_{\gamma\gamma} = \sqrt{1-\rho_{r\gamma}^2 - \rho''^2_{S\gamma} - \rho''^2_{\chi\gamma}}.$$

*pf*.

$$dW_r dW_\gamma = dW_0\left(\rho_{r\gamma} dW_0 + \rho''_{S\gamma} dW_1 + \rho''_{\chi\gamma} dW_2 + \rho''_{\gamma\gamma} dW_3\right) = \rho_{r\gamma} dt$$

$$dW_S dW_\gamma = \left(\rho_{rS} dW_0 + \sqrt{1-\rho_{rS}^2}\, dW_1\right)\left(\rho_{r\gamma} dW_0 + \rho''_{S\gamma} dW_1 + \rho''_{\chi\gamma} dW_2 + \rho''_{\gamma\gamma} dW_3\right)$$

$$= \left(\rho_{rS}\rho_{r\gamma} + \rho''_{S\gamma}\sqrt{1-\rho_{rS}^2}\right) dt = \rho_{S\gamma} dt$$

Then, $\rho''_{S\gamma} = \dfrac{\rho_{S\gamma} - \rho_{rS}\rho_{r\gamma}}{\sqrt{1-\rho_{rS}^2}}$.

$$dW_\chi dW_\gamma = \left(\rho_{r\chi} dW_0 + \rho'_{S\chi} dW_1 + \rho'_{\chi\chi} dW_2\right)\left(\rho_{r\gamma} dW_0 + \rho''_{S\gamma} dW_1 + \rho''_{\chi\gamma} dW_2 + \rho''_{\gamma\gamma} dW_3\right)$$

$$= \left(\rho_{r\chi}\rho_{r\gamma} + \rho'_{S\chi}\rho''_{S\gamma} + \rho'_{\chi\chi}\rho''_{\chi\gamma}\right) dt = \rho_{\chi\gamma} dt$$

$\rho_{r\chi}\rho_{r\gamma} + \rho'_{S\chi}\rho''_{S\gamma} + \rho'_{\chi\chi}\rho''_{\chi\gamma} = \rho_{\chi\gamma}$, then $\rho''_{\chi\gamma} = \dfrac{\rho_{\chi\gamma} - \rho_{r\chi}\rho_{r\gamma} - \rho'_{S\chi}\rho''_{S\gamma}}{\rho'_{\chi\chi}}$.

$$\rho''_{\chi\gamma} = \frac{\rho_{\chi\gamma} - \rho_{r\chi}\rho_{r\gamma} - \rho_{S\chi}\rho_{S\gamma} - \rho_{rS}^2\rho_{\chi\gamma} + \rho_{rS}\rho_{r\chi}\rho_{S\gamma} + \rho_{rS}\rho_{r\gamma}\rho_{S\chi}}{\sqrt{1+\rho_{rS}^4 - 2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{r\chi}^2 + \rho_{rS}^2\rho_{S\chi}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}}$$

$$dW_\gamma dW_\gamma = \left(\rho_{r\gamma} dW_0 + \rho''_{S\gamma} dW_1 + \rho''_{\chi\gamma} dW_2 + \rho''_{\gamma\gamma} dW_3\right)\left(\rho_{r\gamma} dW_0 + \rho''_{S\gamma} dW_1 + \rho''_{\chi\gamma} dW_2 + \rho''_{\gamma\gamma} dW_3\right)$$

$$= \left(\rho_{r\gamma}^2 + \rho''^2_{S\gamma} + \rho''^2_{\chi\gamma} + \rho''^2_{\gamma\gamma}\right) dt = dt$$

$\rho_{r\gamma}^2 + \rho''^2_{S\gamma} + \rho''^2_{\chi\gamma} + \rho''^2_{\gamma\gamma} = 1$, then $\rho''_{\gamma\gamma} = \sqrt{1-\rho_{r\gamma}^2 - \rho''^2_{S\gamma} - \rho''^2_{\chi\gamma}}$.

$$\rho''_{\gamma\gamma} = \sqrt{\frac{1-\rho_{rS}^2 - \rho_{r\gamma}^2 - \rho_{S\gamma}^2 + 2\rho_{rS}\rho_{r\gamma}\rho_{S\gamma}}{1-\rho_{rS}^2} - \frac{\left(\rho_{\chi\gamma} - \rho_{r\chi}\rho_{r\gamma} - \rho_{S\chi}\rho_{S\gamma} - \rho_{rS}^2\rho_{\chi\gamma} + \rho_{rS}\rho_{r\chi}\rho_{S\gamma} + \rho_{rS}\rho_{r\gamma}\rho_{S\chi}\right)^2}{1+\rho_{rS}^4 - 2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{r\chi}^2 + \rho_{rS}^2\rho_{S\chi}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}}$$

$$= \sqrt{\frac{D}{C}},\ \text{where } C = \left(1-\rho_{rS}^2\right)\left(1+\rho_{rS}^4 - 2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{r\chi}^2 + \rho_{rS}^2\rho_{S\chi}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}\right)$$

$$\begin{aligned}
&= 1-\rho_{rS}^6 + 2\rho_{rS}^5\rho_{r\chi}\rho_{S\chi} - \rho_{rS}^4\rho_{r\chi}^2 - \rho_{rS}^4\rho_{S\chi}^2 + 3\rho_{rS}^4 - 4\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} + 2\rho_{rS}^2\rho_{r\chi}^2 + 2\rho_{rS}^2\rho_{S\chi}^2 \\
&\quad - 3\rho_{rS}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi},
\end{aligned}$$

$$\begin{aligned}
D =\ & 1-\rho_{rS}^6 + \rho_{rS}^6\rho_{\chi\gamma}^2 - 2\rho_{rS}^5\rho_{r\chi}\rho_{S\gamma}\rho_{\chi\gamma} - 2\rho_{rS}^5\rho_{r\gamma}\rho_{S\chi}\rho_{\chi\gamma} + 2\rho_{rS}^5\rho_{r\chi}\rho_{S\chi} + 2\rho_{rS}^5\rho_{r\gamma}\rho_{S\gamma} \\
& - 2\rho_{rS}^4\rho_{r\chi}\rho_{r\gamma}\rho_{S\chi}\rho_{S\gamma} + 2\rho_{rS}^4\rho_{r\chi}\rho_{r\gamma}\rho_{\chi\gamma} + 2\rho_{rS}^4\rho_{S\chi}\rho_{S\gamma}\rho_{\chi\gamma} + \rho_{rS}^4\rho_{r\chi}^2\rho_{S\gamma}^2 + \rho_{rS}^4\rho_{r\gamma}^2\rho_{S\chi}^2 \\
& - \rho_{rS}^4\rho_{r\chi}^2 - \rho_{rS}^4\rho_{r\gamma}^2 - 3\rho_{rS}^4\rho_{\chi\gamma}^2 - \rho_{rS}^4\rho_{S\chi}^2 - \rho_{rS}^4\rho_{S\gamma}^2 + 3\rho_{rS}^4 + 4\rho_{rS}^3\rho_{r\chi}\rho_{S\gamma}\rho_{\chi\gamma} + 4\rho_{rS}^3\rho_{r\gamma}\rho_{S\chi}\rho_{\chi\gamma} \\
& - 4\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} - 4\rho_{rS}^3\rho_{r\gamma}\rho_{S\gamma} + 4\rho_{rS}^2\rho_{r\chi}\rho_{r\gamma}\rho_{S\gamma}\rho_{S\chi} - 4\rho_{rS}^2\rho_{r\chi}\rho_{r\gamma}\rho_{\chi\gamma} - 4\rho_{rS}^2\rho_{S\chi}\rho_{S\gamma}\rho_{\chi\gamma} \\
& - 2\rho_{rS}^2\rho_{r\chi}^2\rho_{S\gamma}^2 - 2\rho_{rS}^2\rho_{r\gamma}^2\rho_{S\chi}^2 + 2\rho_{rS}^2\rho_{r\chi}^2 + 2\rho_{rS}^2\rho_{r\gamma}^2 + 2\rho_{rS}^2\rho_{S\chi}^2 + 2\rho_{rS}^2\rho_{S\gamma}^2 + 3\rho_{rS}^2\rho_{\chi\gamma}^2 + \rho_{r\chi}^2\rho_{S\gamma}^2 \\
& + \rho_{r\gamma}^2\rho_{S\chi}^2 - 3\rho_{rS}^2 - \rho_{r\chi}^2 - \rho_{r\gamma}^2 - \rho_{S\chi}^2 - \rho_{S\gamma}^2 - \rho_{\chi\gamma}^2 - 2\rho_{rS}\rho_{r\chi}\rho_{S\gamma}\rho_{\chi\gamma} - 2\rho_{rS}\rho_{r\gamma}\rho_{S\chi}\rho_{\chi\gamma} \\
& - 2\rho_{r\chi}\rho_{r\gamma}\rho_{S\chi}\rho_{S\gamma} + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi} + 2\rho_{rS}\rho_{r\gamma}\rho_{S\gamma} + 2\rho_{r\chi}\rho_{r\gamma}\rho_{\chi\gamma} + 2\rho_{S\chi}\rho_{S\gamma}\rho_{\chi\gamma}
\end{aligned}$$

Following this recursive method, we could construct *n* correlated Brownian motions. However, the coefficients for each independent Brownian motions would become more and more tedious. In addition, if two variables are negative linearly correlated, then add a negative sign in front of one of the two variables would make them positive linearly correlated (such that the formulas here work).

Alternatively, if the correlation matrix of $n$ correlated Brownian motions $\mathbf{C}_{n\times n}$ is symmetric and positive definite, we could decompose $\mathbf{C}$ into $\mathbf{LL}^T$ by Cholesky decomposition, where $\mathbf{L}_{n\times n}$ is a lower triangular matrix and $\mathbf{L}^T$ is the transpose of $\mathbf{L}$. Then we could generate $n$ correlated Brownian motions $\mathbf{B}_{n\times 1}$ by $\mathbf{LW}$, here $\mathbf{W}_{n\times 1}$ is a vector of $n$ independent Brownian motions and $\mathbf{BB}^T = \mathbf{LWW}^T\mathbf{L}^T = \mathbf{C}$.

## 2. APPENDIX 2: DERIVE DRIFT TERM AND DIFFUSION TERM OF $P(t,T,r(t))$

In Appendix 2, we derive the drift term $\tilde{\mu}(t,r(t))$ and the diffusion term $\tilde{\sigma}(t,r(t))$ for $P(t,T,r(t))$. Let the process of zero coupon bond of no risk with maturity $T$ be $\frac{dP(t,T,r(t))}{P(t,T,r(t))} = \tilde{\mu}(t,r(t))dt + \tilde{\sigma}(t,r(t))dW_r(t)$. Since $P(t,T,r(t))$ is a function of $t$ and $r(t)$, by Itô formula, we have

$$dP(t,T,r(t)) = \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr(t) + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}dr(t)dr(t)$$

where $\frac{\partial P}{\partial t}$ is the first partial derivative of $P(t,T,r(t))$ with respect to $t$, $\frac{\partial P}{\partial r}$ and $\frac{\partial^2 P}{\partial r^2}$ are the first and second partial derivative of $P(t,T,r(t))$ with respect to $r(t)$. Thus, the dynamics of $P(t,T,r(t))$ could also be written as

$$dP(t,T,r(t)) = \left(P_t + \alpha(t,r(t))P_r + \frac{1}{2}P_{rr}\beta^2(t,r(t))\right)dt + P_r\beta(t,r(t))dW_r(t),$$

$P_t = \frac{\partial P}{\partial t}$, $P_r = \frac{\partial P}{\partial r}$, $P_{rr} = \frac{\partial^2 P}{\partial r^2}$. Comparing these two representations of $dP(t,T,r(t))$, we have $P(t,T,r(t))\tilde{\sigma}(t,r(t)) = P_r\beta(t,r(t))$. As a result, we could calculate $\tilde{\sigma}(t,r(t))$ by $\frac{P_r\beta(t,r(t))}{P(t,T,r(t))}$.

Next step is to derive the drift term $\tilde{\mu}(t,r(t))$ in the physical world. $\delta(t)$ equals $e^{-\int_0^t r(s)ds}$, and let $I(t)$ equal $\int_0^t r(s)ds$, and $f(x) = e^{-x}$. Then, $dI(t)$ is equal to $r(t)dt$, $f'(x) = -f(x)$, and $f''(x) = f(x)$. By Itô formula again,

$$d\delta(t) = df(I(t)) = f'(I(t))dI(t) + \frac{1}{2}f''(I(t))dI(t)dI(t) = -r(t)\delta(t)dt.$$

$d\left[\delta(t)P(t,T,r(t))\right] = \delta(t)dP(t,T,r(t)) + P(t,T,r(t))d\delta(t) + d\delta(t)dP(t,T,r(t))$ by Itô product rule. We introduce the process of market price of risk $\theta(t)$ here, and let $d\tilde{W}_r(t)=\theta(t)dt+dW_r(t)$. Let $\frac{d\mathbf{Q}'}{d\mathbf{P}}=Z(t)=\mathbf{exp}\left\{-\int_0^t\theta(s)dW_r-\frac{1}{2}\int_0^t\theta^2(s)ds\right\}$, $Z=Z(T)$, and assume $E\left(\int_0^T\theta^2(s)Z^2(s)ds\right)<\infty$. Then by Girsanov's Theorem, $E(Z)=1$, and $\tilde{W}_r(t)$, $0\le t\le T$, under probability measure $\mathbf{Q}'$ is a Brownian motion. Then

$$
\begin{aligned}
&d\left[\delta(t)P(t,T,r(t))\right]\\
&=\left[\tilde{\mu}(t,r(t))-r(t)\right]\delta(t)P(t,T,r(t))dt+\tilde{\sigma}(t,r(t))\delta(t)P(t,T,r(t))dW_r(t)\\
&=\tilde{\sigma}(t,r(t))\delta(t)P(t,T,r(t))d\tilde{W}_r(t)
\end{aligned}
\quad \text{(A2.1)}
$$

and $\theta(t)$ equal $\frac{\tilde{\mu}(t,r(t))-r(t)}{\tilde{\sigma}(t,r(t))}$, mathematically speaking.

We rewrite $\frac{dP(t,T,r(t))}{P(t,T,r(t))}=\left(\tilde{\mu}(t,r(t))-\tilde{\sigma}(t,r(t))\theta(t)\right)dt+\tilde{\sigma}(t,r(t))d\tilde{W}_r(t)$.

$P(t,T,r(t))$ is the zero coupon bond of no risk with maturity $T$, so the drift term of $\frac{dP(t,T,r(t))}{P(t,T,r(t))}$ under probability measure $\mathbf{Q}'$ is equal to $r(t)$.[1] Hence, we have $\tilde{\mu}(t,r(t))$ equals $r(t)+\tilde{\sigma}(t,r(t))\theta(t)$.

## 3. APPENDIX 3: DERIVE GENERAL FORM OF DEFLATOR $D(t)$

In Appendix 3, we provide details of the calculations to derive the general form of deflator $D(t)$ step by step.

$$dD(t)=\Omega(t)dt+\Phi(t)dW_r(t)+\Psi(t)dW_1(t)+\Gamma(t)dW_2(t)+\mathrm{I}(t)dW_3(t) \quad \text{(A3.1)}$$

Recall that we would like to have $E^{\mathbf{Q}}\left(\delta(t)X\right)=E\left[D(t)X\right]$ for a nonnegative random variable $X$. We let $D(t)B(t)$, $D(t)P(t,T,r(t))$, $D(t)S(t)$, $D(t)\chi(t)$, and $D(t)\gamma(t)$ be $\mathbf{P}$-martingales, then their drift terms are all equal to zero.

[1] We could verify this by the intuition of change of measure with $\delta(t)$ and also the solution of $\delta(t)P(t,T,r(t))$ being exponential functions. Alternatively, we could plug $\tilde{\mu}(t,r(t))$ back to $d\left[\delta(t)P(t,T,r(t))\right]$ to verify the statement.

A3.1 Short-term saving, $D(t)B(t)$

$$\begin{aligned} d\left[D(t)B(t)\right] &= B(t)dD(t)+D(t)dB(t)+dB(t)dD(t) \\ &= \left[\Omega(t)B(t)+D(t)B(t)r(t)\right]dt+\Phi(t)B(t)dW_r(t)+\Psi(t)B(t)dW_1(t)+\Gamma(t)B(t)dW_2(t) \\ &\quad +\mathrm{I}(t)B(t)dW_3(t) \end{aligned}$$

Let the drift term of $d\left[D(t)B(t)\right]$ equal zero, $\Omega(t)B(t)+D(t)B(t)r(t)=0$, then $\Omega(t)$ is equal to $-D(t)r(t)$.

A3.2 Zero coupon bond of no risk with maturity $T$, $D(t)P(t,T,r(t))$

$$\begin{aligned} d\left[D(t)P(t,T,r(t))\right] &= P(t,T,r(t))dD(t)+D(t)dP(t,T,r(t))+dP(t,T,r(t))dD(t) \\ &= \begin{bmatrix} \Omega(t)P(t,T,r(t))+\Phi(t)P(t,T,r(t))\tilde{\sigma}(t,r(t))+D(t)P(t,T,r(t))r(t) \\ +D(t)P(t,T,r(t))\tilde{\sigma}(t,r(t))\theta(t) \end{bmatrix}dt \\ &\quad +\left[\Phi(t)P(t,T,r(t))+D(t)P(t,T,r(t))\tilde{\sigma}(t,r(t))\right]dW_r(t) \\ &\quad +\Psi(t)P(t,T,r(t))dW_1(t)+\Gamma(t)P(t,T,r(t))dW_2(t)+\mathrm{I}(t)P(t,T,r(t))dW_3(t) \end{aligned}$$

Let the drift term of $d\left[D(t)P(t,T,r(t))\right]$ equal zero and plug $\Omega(t)$ into the drift term, we have $\Phi(t)=-D(t)\theta(t)$.

A3.3 Stock, $D(t)S(t)$

$$\begin{aligned} d\left[D(t)S(t)\right] &= S(t)dD(t)+D(t)dS(t)+dD(t)dS(t) \\ &= \left[\Omega(t)S(t)+\Phi(t)S(t)\sigma_S(t)\rho_{rS}+\Psi(t)S(t)\sigma_S\sqrt{1-\rho_{rS}^2}+D(t)S(t)\mu_S(t)\right]dt \\ &\quad +\left[\Phi(t)S(t)+D(t)S(t)\sigma_S(t)\rho_{rS}\right]dW_r(t)+\left[\Psi(t)S(t)+D(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}\right]dW_1(t) \\ &\quad +\Gamma(t)S(t)dW_2(t)+\mathrm{I}(t)S(t)dW_3(t) \end{aligned}$$

Let the drift term of $d\left[D(t)S(t)\right]$ equal zero and plug $\Omega(t)$, $\Phi(t)$ into the drift term, then $\Psi(t)$ is equal to $\dfrac{D(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}-\mu_S(t)\right]}{\sigma_S(t)\sqrt{1-\rho_{rS}^2}}$.

A3.4 Default density, $D(t)\chi(t)$

$$\begin{aligned} d\left[D(t)\chi(t)\right] &= \chi(t)dD(t)+D(t)d\chi(t)+d\chi(t)dD(t) \\ &= \left[\Omega(t)\chi(t)+\Phi(t)\sigma_\chi\rho_{r\chi}\sqrt{\chi(t)}+\Psi(t)\sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)}+\Gamma(t)\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}+D(t)e-D(t)f\chi(t)\right]dt \\ &\quad +\left[\Phi(t)\chi(t)+D(t)\sigma_\chi\rho_{r\chi}\sqrt{\chi(t)}\right]dW_r(t)+\left[\Psi(t)\chi(t)+D(t)\sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)}\right]dW_1(t) \\ &\quad +\left[\Gamma(t)\chi(t)+D(t)\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}\right]dW_2(t)+\mathrm{I}(t)\chi(t)dW_3(t) \end{aligned}$$

Let the drift term of $d\left[D(t)\chi(t)\right]$ equal zero and plug $\Omega(t)$, $\Phi(t)$, $\Psi(t)$ into the drift term, we have

$$\Gamma(t)=D(t)\left\{\frac{\theta(t)\rho_{r\chi}}{\rho'_{\chi\chi}}+\frac{r(t)\chi(t)-e+f\chi(t)}{\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}}+\frac{\rho'_{S\chi}\left[\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right]}{\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}}\right\}.$$

A3.5 Convenience yield, $D(t)\gamma(t)$

$$\begin{aligned}d\left[D(t)\gamma(t)\right]&=\gamma(t)dD(t)+D(t)d\gamma(t)+d\gamma(t)dD(t)\\&=\left[\Omega(t)\gamma(t)+\Phi(t)\eta\rho_{r\gamma}+\Psi(t)\eta\rho''_{S\gamma}+\Gamma(t)\eta\rho''_{\chi\gamma}+\mathrm{I}(t)\eta\rho''_{\gamma\gamma}\right]dt+\left[\Phi(t)\gamma(t)+D(t)\eta\rho_{r\gamma}\right]dW_r(t)\\&\quad+\left[\Psi(t)\gamma(t)+D(t)\eta\rho''_{S\gamma}\right]dW_1(t)+\left[\Gamma(t)\gamma(t)+D(t)\eta\rho''_{\chi\gamma}\right]dW_2(t)\\&\quad+\left(\mathrm{I}(t)\gamma(t)+D(t)\eta\rho''_{\gamma\gamma}\right)dW_3(t)\end{aligned}$$

Let the drift term of $d\left[D(t)\gamma(t)\right]$ equal zero and plug $\Omega(t)$, $\Phi(t)$, $\Psi(t)$, $\Gamma(t)$ into the drift term, then

$$\mathrm{I}(t)=D(t)\left\{\begin{aligned}&\frac{\rho_{r\gamma}\theta(t)}{\rho''_{\gamma\gamma}}+\frac{r(t)\gamma(t)}{\eta\rho''_{\gamma\gamma}}-\frac{\rho''_{\chi\gamma}\rho_{r\chi}\theta(t)}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}}+\frac{\rho''_{\chi\gamma}\left[e-r(t)\chi(t)-f\chi(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_\chi\sqrt{\chi(t)}}\\&+\frac{\left(\rho''_{S\gamma}\rho'_{\chi\chi}-\rho''_{\chi\gamma}\rho'_{S\chi}\right)\left[\mu_S(t)-r(t)-\rho_{rS}\theta(t)\sigma_S(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_S(t)\sqrt{\left(1-\rho_{rS}^2\right)}}\end{aligned}\right\}.$$

A3.6 Deflator, $D(t)$

We could derive the general form of deflator $D(t)$ now. Let $y(t)$ equal $\ln D(t)$, then the first partial derivative ($\frac{\partial y}{\partial D}$) and the second partial derivative ($\frac{\partial^2 y}{\partial D^2}$) of $y(t)$ with respect to $D(t)$ are $\frac{1}{D(t)}$ and $-\frac{1}{D^2(t)}$ respectively. By Itô formula, $dy(t)=\frac{\partial y}{\partial D}dD(t)+\frac{1}{2}\frac{\partial^2 y}{\partial D^2}dD(t)dD(t)$.

$$\begin{aligned}dy(t)&=\left[\frac{\Omega(t)}{D(t)}-\frac{\Phi^2(t)}{2D^2(t)}-\frac{\Psi^2(t)}{2D^2(t)}-\frac{\Gamma^2(t)}{2D^2(t)}-\frac{\mathrm{I}^2(t)}{2D^2(t)}\right]dt+\frac{\Phi(t)}{D(t)}dW_r(t)\\&\quad+\frac{\Psi(t)}{D(t)}dW_1(t)+\frac{\Gamma(t)}{D(t)}dW_2(t)+\frac{\mathrm{I}(t)}{D(t)}dW_3(t)\end{aligned}\qquad\text{(A3.2)}$$

Let
$$\begin{cases} K_{\Psi}(t) = \dfrac{\Psi(t)}{D(t)} = \dfrac{r(t)+\theta(t)\sigma_S(t)\rho_{rS}-\mu_S(t)}{\sigma_S(t)\sqrt{1-\rho_{rS}^2}} \\ K_{\Gamma}(t) = \dfrac{\Gamma(t)}{D(t)} = \dfrac{\theta(t)\rho_{r\chi}}{\rho'_{\chi\chi}} + \dfrac{r(t)\chi(t)-e+f\chi(t)}{\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}} + \dfrac{\rho'_{S\chi}\left[\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right]}{\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}} \\ K_{\mathrm{I}}(t) = \dfrac{\mathrm{I}(t)}{D(t)} = \dfrac{\rho_{r\gamma}\theta(t)}{\rho''_{\gamma\gamma}} + \dfrac{r(t)\gamma(t)}{\eta\rho''_{\gamma\gamma}} - \dfrac{\rho''_{\chi\gamma}\rho_{r\chi}\theta(t)}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}} + \dfrac{\rho''_{\chi\gamma}\left[e-r(t)\chi(t)-f\chi(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_\chi\sqrt{\chi(t)}} \\ \qquad + \dfrac{\left(\rho''_{S\gamma}\rho'_{\chi\chi}-\rho''_{\chi\gamma}\rho'_{S\chi}\right)\left[\mu_S(t)-r(t)-\rho_{rS}\theta(t)\sigma_S(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}} \end{cases}$$
, and

integrate both sides of $dy(t)$, plugging $\dfrac{\Omega(t)}{D(t)} = -r(t)$ and $\dfrac{\Phi(t)}{D(t)} = -\theta(t)$ into $y(t)$. We have $y(t)$ as follows.

$$\begin{aligned} y(t) &= y(0) + \int_0^t \left[-r(s) - \frac{1}{2}\theta^2(s) - \frac{1}{2}K_{\Psi}^2(s) - \frac{1}{2}K_{\Gamma}^2(s) - \frac{1}{2}K_{\mathrm{I}}^2(s)\right] ds \\ &\quad - \int_0^t \theta(s)\,dW_r(t) + \int_0^t K_{\Psi}(s)\,dW_1(s) + \int_0^t K_{\Gamma}(s)\,dW_2(s) + \int_0^t K_{\mathrm{I}}(s)\,dW_3(s) \end{aligned} \tag{A3.3}$$

Take exponential both sides of $y(t)$, we have the general form of deflator.

$$\begin{aligned} D(t) &= D(0)\,\mathbf{exp}\left\{-\int_0^t r(s)\,ds - \int_0^t \frac{1}{2}\left[\theta^2(s) + K_{\Psi}^2(s) + K_{\Gamma}^2(s) + K_{\mathrm{I}}^2(s)\right] ds\right\} \\ &\quad \times \mathbf{exp}\left[-\int_0^t \theta(s)\,dW_r(s) + \int_0^t K_{\Psi}(s)\,dW_1(s) + \int_0^t K_{\Gamma}(s)\,dW_2(s) + \int_0^t K_{\mathrm{I}}(s)\,dW_3(s)\right] \end{aligned} \tag{A3.4}$$

## 4. APPENDIX 4: REGULARITY CONDITIONS FOR DEFLATOR $D(t)$

In Appendix 4, we provide an analysis for regularity conditions regarding the drift coefficient $\mu_S(t)$ of stock prices, $e$ in the drift term of default density $\chi(t)$, and the diffusion coefficient $\eta$ of convenience yield $\gamma(t)$. In Appendix 3, we derive the general form of deflator $D(t)$ by letting $D(t)B(t)$, $D(t)P(t,T,r(t))$, $D(t)S(t)$, $D(t)\chi(t)$, and $D(t)\gamma(t)$ be $\mathbf{P}$ -martingales. However, recall that we'd like to have $E^{\mathbf{Q}}\left(\delta(t)X\right) = E\left[D(t)X\right]$ for a nonnegative random variable $X$, in which $\delta(t)$ is a discount process equalling $e^{-\int_0^t r(s)ds}$. In our model, $B(t)$, $P(t,T,r(t))$, and $S(t)$ are the nonnegative random variables $X$ in $E^{\mathbf{Q}}\left(\delta(t)X\right) = E\left[D(t)X\right]$. In addition, we also have to consider $\chi(t)$ and $\gamma(t)$ under probability measure $\mathbf{Q}$ when deriving the deflator $D(t)$. We discuss $\delta(t)B(t)$, $\delta(t)P(t,T,r(t))$, $\delta(t)S(t)$, $\delta(t)\chi(t)$, and $\delta(t)\gamma(t)$ under probability measure $\mathbf{Q}$ as follows.

A4.1 Short-term saving, $\delta(t)B(t)$

$$d\left[\delta(t)B(t)\right] = \delta(t)\,dB(t) + B(t)\,d\delta(t) + d\delta(t)\,dB(t) = 0$$

Thus, $\delta(t)B(t)$ is a $\mathbf{Q}$-martingale.

A4.2 Zero coupon bond of no risk with maturity $T$, $\delta(t)P(t,T,r(t))$

From (A2.1), $d\left[\delta(t)P(t,T,r(t))\right]=\tilde{\sigma}(t,r(t))\delta(t)P(t,T,r(t))d\tilde{W}_r(t)$.

Thus, $\delta(t)P(t,T,r(t))$ is a $\mathbf{Q}$-martingale.

A4.3 Stock, $\delta(t)S(t)$

$$\begin{aligned}
d\left[\delta(t)S(t)\right]&=\delta(t)dS(t)+S(t)d\delta(t)+d\delta(t)dS(t)\\
&=\delta(t)S(t)\left[\mu_S(t)-r(t)\right]dt+\delta(t)S(t)\sigma_S(t)\rho_{rS}dW_r(t)\\
&\quad+\delta(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t)\\
&=\delta(t)S(t)\left[\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right]dt+\delta(t)S(t)\sigma_S(t)\rho_{rS}d\tilde{W}_r(t)\\
&\quad+\delta(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t),\ \text{under }\mathbf{Q}
\end{aligned}$$

We require $\mu_S(t)=r(t)+\theta(t)\sigma_S(t)\rho_{rS}$ as regularity condition, such that $\delta(t)S(t)$ is a $\mathbf{Q}$-martingale. As a result, $K_\Psi(t)=0$.

Alternatively, let $\delta_S(t)$ be a process equal to $e^{-\int_0^t\left[\mu_S(s)-r(s)-\theta(s)\sigma_S(s)\rho_{rS}\right]ds}$. We could see that $\delta(t)\left[\delta_S(t)S(t)\right]$ is a $\mathbf{Q}$-martingale as follows.

Let $\delta_S(t)=e^{-\int_0^t\left[\mu_S(s)-r(s)-\theta(s)\sigma_S(s)\rho_{rS}\right]ds}$, $I_S(t)=\int_0^t\left[\mu_S(s)-r(s)-\theta(s)\sigma_S(s)\rho_{rS}\right]ds$, and $f(x)=e^{-x}$,
then $dI_S(t)=\left[\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right]dt$, $f'(x)=-f(x)$, and $f''(x)=f(x)$.

$$d\delta_S(t)=df(I_S(t))=f'(I_S(t))dI_S(t)+\frac{1}{2}f''(I_S(t))dI_S(t)dI_S(t)=-\left[\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right]\delta_S(t)dt$$

$$\begin{aligned}
d\left[\delta_S(t)S(t)\right]&=\delta_S(t)dS(t)+S(t)d\delta_S(t)+d\delta_S(t)dS(t)\\
&=\delta_S(t)S(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}\right]dt+\delta_S(t)S(t)\sigma_S(t)\rho_{rS}dW_r(t)+\delta_S(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t)
\end{aligned}$$

$$\begin{aligned}
d\left\{\delta(t)\left[\delta_S(t)S(t)\right]\right\}&=\delta(t)d\left[\delta_S(t)S(t)\right]+\delta_S(t)S(t)d\delta(t)+d\delta(t)d\left[\delta_S(t)S(t)\right]\\
&=\delta(t)\delta_S(t)S(t)\theta(t)\sigma_S(t)\rho_{rS}dt+\delta(t)\delta_S(t)S(t)\sigma_S(t)\rho_{rS}dW_r(t)+\delta(t)\delta_S(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t)\\
&=\delta(t)\delta_S(t)S(t)\sigma_S(t)\rho_{rS}d\tilde{W}_r(t)+\delta(t)\delta_S(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t),\ \text{under }\mathbf{Q}
\end{aligned}$$

Let $D(t)\left[\delta_S(t)S(t)\right]$ be a $\mathbf{P}$-martingale, then $\Psi(t)=0$ (so that $K_\Psi(t)=0$).

$$\begin{aligned}
d\left\{D(t)\left[\delta_S(t)S(t)\right]\right\}&=\delta_S(t)S(t)dD(t)+D(t)d\left[\delta_S(t)S(t)\right]+dD(t)d\left[\delta_S(t)S(t)\right]\\
&=\delta_S(t)S(t)\left\{D(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}\right]+\Omega(t)+\Phi(t)\sigma_S(t)\rho_{rS}+\Psi(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}\right\}dt\\
&\quad+\delta_S(t)S(t)\left[\Phi(t)+D(t)\sigma_S(t)\rho_{rS}\right]dW_r(t)+\delta_S(t)S(t)\left[\Psi(t)+D(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}\right]dW_1(t)\\
&\quad+\delta_S(t)S(t)\Gamma(t)dW_2(t)+\delta_S(t)S(t)\mathrm{I}(t)dW_3(t)
\end{aligned}$$

Plug $\Omega(t),\Phi(t)$ into $D(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}\right]+\Omega(t)+\Phi(t)\sigma_S(t)\rho_{rS}+\Psi(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}=0$, then $\Psi(t)=0$.

A4.4 Default density, $\delta(t)\chi(t)$

$$
\begin{aligned}
d\left[\delta(t)\chi(t)\right] &= \delta(t)d\chi(t)+\chi(t)d\delta(t)+d\delta(t)d\chi(t) \\
&= \delta(t)\left[e-r(t)\chi(t)-f\chi(t)\right]dt+\sigma_\chi\rho_{r\chi}\delta(t)\sqrt{\chi(t)}dW_r(t) \\
&\quad +\sigma_\chi\rho'_{S\chi}\delta(t)\sqrt{\chi(t)}dW_1(t)+\sigma_\chi\rho'_{\chi\chi}\delta(t)\sqrt{\chi(t)}dW_2(t) \\
&= \delta(t)\left[e-r(t)\chi(t)-f\chi(t)-\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)}\right]dt+\sigma_\chi\rho_{r\chi}\delta(t)\sqrt{\chi(t)}d\tilde{W}_r(t) \\
&\quad +\sigma_\chi\rho'_{S\chi}\delta(t)\sqrt{\chi(t)}dW_1(t)+\sigma_\chi\rho'_{\chi\chi}\delta(t)\sqrt{\chi(t)}dW_2(t), \text{ under } \mathbf{Q}
\end{aligned}
$$

We require $e=r(t)\chi(t)+f\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)}$ as regularity condition, such that $\delta(t)\chi(t)$ is a $\mathbf{Q}$-martingale. As a result, $K_\Gamma(t)=0$. In addition, we also have to consider if the Feller condition holds, i.e. $r(t)\chi(t)+f\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)}>\frac{1}{2}\sigma_\chi^2$.

A4.5 Convenience yield, $\delta(t)\gamma(t)$

$$
\begin{aligned}
d\left[\delta(t)\gamma(t)\right] &= \delta(t)d\gamma(t)+\gamma(t)d\delta(t)+d\delta(t)d\gamma(t) \\
&= -\delta(t)\gamma(t)r(t)dt+\eta\rho_{r\gamma}\delta(t)dW_r(t)+\eta\rho''_{S\gamma}\delta(t)dW_1(t) \\
&\quad +\eta\rho''_{\chi\gamma}\delta(t)dW_2(t)+\eta\rho''_{\gamma\gamma}\delta(t)dW_3(t) \\
&= \delta(t)\left[-\gamma(t)r(t)-\eta\rho_{r\gamma}\theta(t)\right]dt+\eta\rho_{r\gamma}\delta(t)d\tilde{W}_r(t)+\eta\rho''_{S\gamma}\delta(t)dW_1(t) \\
&\quad +\eta\rho''_{\chi\gamma}\delta(t)dW_2(t)+\eta\rho''_{\gamma\gamma}\delta(t)dW_3(t), \text{ under } \mathbf{Q}
\end{aligned}
$$

We require $\eta=-\dfrac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}$ as regularity condition, such that $\delta(t)\gamma(t)$ is a $\mathbf{Q}$-martingale. As a result, $K_I(t)=0$.

From (A3.4), we rewrite the general form of deflator as

$$D(t)=D(0)\mathbf{exp}\left[-\int_0^t r(s)ds-\frac{1}{2}\int_0^t\theta^2(s)ds-\int_0^t\theta(s)dW_r(s)\right].$$

## 5. APPENDIX 5: EUROPEAN PUT OPTION PRICING UNDER CIR INTEREST RATE IN KIM (2002)

In Kim (2002), the process of CIR interest rate $dr(t)$ under probability measure $\mathbf{Q}'$ is as follows.

$$dr(t)=\left[\kappa_{Kim}\theta_{Kim}-(\kappa_{Kim}+\delta_{Kim}\lambda_{Kim})r(t)\right]dt+\delta_{Kim}\sqrt{r(t)}d\tilde{W}_r(t) \qquad \text{(A5.1)}$$

Let $\lambda_{Kim}=1$, then $\delta_{Kim}=\sigma_r$, $\kappa_{Kim}=b_r-\sigma_r$, $\theta_{Kim}=a_r/(b_r-\sigma_r)$. Then, we could calculate the price of a European call option of stock S with strike $K$ and maturity $T$ at time zero, $Call_{Kim}(0,S(0),T,K)$.

$$\begin{aligned} Call_{Kim}\left(0,S(0),T,K\right) &= \left[S(0)\Phi(d_1) - K\exp\left(-\int_0^T r_t^* dt\right)\Phi(d_2)\right] \\ &+ \delta_{Kim}C_0\left[S(0)\phi(d_1) - K\exp\left(-\int_0^T r_t^* dt\right)\left(\phi(d_2) - \sigma_S\sqrt{T}\Phi(d_2)\right)\right] \quad (17) \\ &+ \delta_{Kim}C_1\left[d_2 S(0)\phi(d_1) - d_1 K\exp\left(-\int_0^T r_t^* dt\right)\phi(d_2)\right] + o(\delta_{Kim}), \end{aligned}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative density function and probability density function of the standard normal distribution respectively;

$$r_t^* = r_0 e^{-\kappa_{Kim}t} + \theta_{Kim}\left(1 - e^{-\kappa_{Kim}t}\right),\ \exp\left(-\int_0^T r_t^* dt\right) = \exp\left[-\frac{(r_0 - \theta_{Kim})}{\kappa_{Kim}}\left(1 - e^{-\kappa_{Kim}T}\right) - \theta_{Kim}T\right];$$

$$d_1 = \frac{1}{\sigma_S\sqrt{T}}\left[\ln\frac{S(0)}{K} + \frac{(r_0 - \theta_{Kim})}{\kappa_{Kim}}\left(1 - e^{-\kappa_{Kim}T}\right) + \theta_{Kim}T + \frac{\sigma_S^2}{2}T\right],\ d_2 = d_1 - \sigma_S\sqrt{T};$$

$$C_0 = \frac{1}{\kappa_{Kim}\sigma_S\sqrt{T}}\left[(r_0 - \theta_{Kim})\left(\frac{1 - e^{-\kappa_{Kim}T}}{\kappa_{Kim}} - Te^{-\kappa_{Kim}T}\right) + \theta_{Kim}T\left(1 - \frac{1 - e^{-\kappa_{Kim}T}}{\kappa_{Kim}}\right)\right];$$

$$C_1 = -\frac{\rho_{rS}}{\sigma_S T}C_{11},\ C_{11} = \frac{2\sqrt{\theta_{Kim}}\left[\left(1 + 2e^{\kappa_{Kim}T}\right)\sqrt{r_0} - 3e^{\frac{\kappa_{Kim}T}{2}}\sqrt{r_0 - \theta_{Kim}\left(1 - e^{-\kappa_{Kim}T}\right)}\right] + \psi_{Kim}\left[\theta_{Kim}\left(1 + 2e^{\kappa_{Kim}T}\right) - r_0\right]}{2e^{\kappa_{Kim}T}\kappa_{Kim}^2\sqrt{\theta_{Kim}}},$$

$$\psi_{Kim} = \ln\left[\frac{\theta_{Kim}\left(2e^{\kappa_{Kim}T} - 1\right) + r_0 + 2e^{\frac{\kappa_{Kim}T}{2}}\sqrt{\theta_{Kim}^2\left(e^{\kappa_{Kim}T} - 1\right) + \theta_{Kim}r_0}}{\left(\sqrt{r_0} + \sqrt{\theta_{Kim}}\right)^2}\right].$$

By Put-Call parity, $Call\left(0,S(0),T,K\right) + Ke^{-\int_0^T r_u du} = Put\left(0,S(0),T,K\right) + S(0)$. We could then calculate the price of the European put option at time zero $Put_{Kim}\left(0,S(0),T,K\right)$ as $Call_{Kim}\left(0,S(0),T,K\right) + KP\left(0,T,r(0)\right) - S(0)$.

## 6. Appendix 6: Implementation of simplified Second Milstein method

In Appendix 6, we present the implementation of simplified Second Milstein method in our example. $dX_t = a(t,X_t)dt + b(t,X_t)dW_t$ where $X_t$, $W_t$, $a(t,X_t)$, and $b(t,X_t)$ are as follows.

$$X_t = \begin{bmatrix} r(t) \\ \theta(t) \\ B(t) \\ P(t,T,r(t)) \\ S(t) \\ \chi(t) \\ \gamma(t) \\ D(t) \end{bmatrix}, W_t = \begin{bmatrix} W_r(t) \\ W_1(t) \\ W_2(t) \\ W_3(t) \\ W_\theta(t) \end{bmatrix} \quad \text{(A6.1)}$$

$$a(t,X_t)=\begin{bmatrix} a_r-b_r r(t)+\theta(t)\sigma_r\sqrt{r(t)} \\ a_\theta-b_\theta\theta(t) \\ B(t)r(t) \\ P(t,T,r(t))r(t)+P_r\sigma_r\sqrt{r(t)}\theta(t) \\ S(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}\right] \\ r(t)\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)} \\ 0 \\ -r(t)D(t) \end{bmatrix} \quad \text{(A6.2)}$$

$$b(t,X_t)=\begin{bmatrix} \sigma_r\sqrt{r(t)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\theta\sqrt{\theta(t)} \\ 0 & 0 & 0 & 0 & 0 \\ P_r\sigma_r\sqrt{r(t)} & 0 & 0 & 0 & 0 \\ S(t)\sigma_S(t)\rho_{rS} & S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2} & 0 & 0 & 0 \\ \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)} & 0 & 0 \\ -\dfrac{\gamma(t)r(t)}{\theta(t)} & -\dfrac{\rho''_{S\gamma}\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)} & -\dfrac{\rho''_{\chi\gamma}\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)} & -\dfrac{\rho''_{\gamma\gamma}\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)} & 0 \\ -\theta(t)D(t) & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(A6.3)}$$

For each $i=1,\ldots,d$ ,

$$Y_{n+1,i}=Y_{n,i}+a_i(n,Y_n)\Delta t+\sum_{k=1}^{m}b_{ik}(n,Y_n)\Delta W_{n,k}+\frac{1}{2}L^0a_i(n,Y_n)(\Delta t)^2$$
$$+\frac{1}{2}\sum_{k=1}^{m}\left[L^ka_i(n,Y_n)+L^0b_{ik}(n,Y_n)\right]\Delta W_{n,k}\Delta t+\frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}L^jb_{ik}(n,Y_n)\left(\Delta W_{n,j}\Delta W_{n,k}-V_{jk}\right).$$

(A6.4)

In order to calculate $Y_{n+1}$ at each time step $n+1$, we have to calculate operators $L^0$ and $L^k$ for each $a_i(n,X_n)$ and $b_{ik}(n,X_n)$. We present two illustrative examples below as follows.

A6.1 $a_1(t, X_t)$

$$a_1(t,X_t) = a_r - b_r r(t) + \theta(t)\sigma_r\sqrt{r(t)},\ \frac{\partial a_1(t,X_t)}{\partial x_1} = -b_r + \frac{1}{2}\theta(t)\sigma_r\frac{1}{\sqrt{r(t)}},\ \frac{\partial a_1(t,X_t)}{\partial x_2} = \sigma_r\sqrt{r(t)}$$

$$\frac{\partial^2 a_1(t,X_t)}{\partial x_1\partial x_1} = -\frac{1}{4}\theta(t)\sigma_r\frac{1}{\sqrt{r^3(t)}},\ \frac{\partial^2 a_1(t,X_t)}{\partial x_1\partial x_2} = \frac{\partial^2 a_1(t,X_t)}{\partial x_2\partial x_1} = \frac{1}{2}\sigma_r\frac{1}{\sqrt{r(t)}}$$

$$\begin{aligned}L^0 a_1(t,X_t) &= \frac{\partial a_1(t,X_t)}{\partial t} + \sum_{i=1}^{8} a_i(t,X_t)\frac{\partial a_1(t,X_t)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{8}\Sigma_{t,ij}\frac{\partial^2 a_1(t,X_t)}{\partial x_i\partial x_j}\\ &= a_1(t,X_t)\left[-b_r + \frac{1}{2}\theta(t)\sigma_r\frac{1}{\sqrt{r(t)}}\right] + a_2(t,X_t)\sigma_r\sqrt{r(t)}\\ &\quad -\frac{1}{8}\Sigma_{t,11}\sigma_r\theta(t)\frac{1}{\sqrt{r^3(t)}} + \frac{1}{2}\Sigma_{t,12}\sigma_r\frac{1}{\sqrt{r(t)}}\end{aligned}$$

$$\begin{aligned}L^k a_1(t,X_t) &= \sum_{i=1}^{d} b_{ik}(t,X_t)\frac{\partial a_1(t,X_t)}{\partial x_i}\\ &= b_{1k}(t,X_t)\left[-b_r + \frac{1}{2}\theta(t)\sigma_r\frac{1}{\sqrt{r(t)}}\right] + b_{2k}(t,X_t)\sigma_r\sqrt{r(t)}\end{aligned}$$

A6.2 $b_{11}(t, X_t)$

$$b_{11}(t,X_t) = \sigma_r\sqrt{r(t)},\ \frac{\partial b_{11}(t,X_t)}{\partial x_1} = \frac{1}{2}\sigma_r\frac{1}{\sqrt{r(t)}}$$

$$\frac{\partial^2 b_{11}(t,X_t)}{\partial x_1\partial x_1} = -\frac{1}{4}\sigma_r\frac{1}{\sqrt{r^3(t)}}$$

$$\begin{aligned}L^0 b_{11}(t,X_t) &= \frac{\partial b_{11}(t,X_t)}{\partial t} + \sum_{i=1}^{8} a_i(t,X_t)\frac{\partial b_{11}(t,X_t)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{8}\Sigma_{t,ij}\frac{\partial^2 b_{11}(t,X_t)}{\partial x_i\partial x_j}\\ &= \frac{1}{2}a_1(t,X_t)\sigma_r\frac{1}{\sqrt{r(t)}} - \frac{1}{8}\Sigma_{t,11}\sigma_r\frac{1}{\sqrt{r^3(t)}}\end{aligned}$$

$$\begin{aligned}L^k b_{11}(t,X_t) &= \sum_{i=1}^{d} b_{ik}(t,X_t)\frac{\partial b_{11}(t,X_t)}{\partial x_i}\\ &= \frac{1}{2}b_{1k}(t,X_t)\sigma_r\frac{1}{\sqrt{r(t)}}\end{aligned}$$

## 7. APPENDIX 7: CORPORATE COUPON BOND PRICING IN LONGSTAFF ET AL. (2005)

In Longstaff et al. (2005), the formula to calculate price of a corporate coupon bond is as follows.

$$CB(c,\omega,T)=c\int_0^T A_{CB}(t)\exp\left(B_{CB}(t)\chi_0\right)C_{CB}(t)P(0,t,r(0))e^{-\gamma_0 t}dt$$
$$+A_{CB}(T)\exp\left(B_{CB}(T)\chi_0\right)C_{CB}(T)P(0,T,r(0))e^{-\gamma_0 T}$$
$$+(1-\omega)\int_0^T \exp\left(B_{CB}(t)\chi_0\right)C_{CB}(t)P(0,t,r(0))\left[G_{CB}(t)+H_{CB}(t)\chi_0\right]e^{-\gamma_0 t}dt$$

$$A_{CB}(t)=\exp\left[\frac{e_\chi\left(f_\chi+\phi\right)}{\sigma_\chi^2}t\right]\left(\frac{1-\kappa}{1-\kappa e^{\phi t}}\right)^{\frac{2e_\chi}{\sigma_\chi^2}},\ B_{CB}(t)=\frac{f_\chi-\phi}{\sigma_\chi^2}+\frac{2\phi}{\sigma_\chi^2\left(1-\kappa e^{\phi t}\right)},$$

$$C_{CB}(t)=\exp\left(\frac{\eta^2t^3}{6}\right),\ G_{CB}(t)=\frac{e_\chi}{\phi}\left(e^{\phi t}-1\right)\exp\left[\frac{e_\chi\left(f_\chi+\phi\right)}{\sigma_\chi^2}t\right]\left(\frac{1-\kappa}{1-\kappa e^{\phi t}}\right)^{\frac{2e_\chi}{\sigma_\chi^2}+1},$$

$$H_{CB}(t)=\exp\left[\frac{e_\chi\left(f_\chi+\phi\right)+\phi\sigma_\chi^2}{\sigma_\chi^2}t\right]\left(\frac{1-\kappa}{1-\kappa e^{\phi t}}\right)^{\frac{2e_\chi}{\sigma_\chi^2}+2},\ \kappa=\left(f_\chi+\phi\right)/\left(f_\chi-\phi\right),$$

$$\phi=\sqrt{2\sigma_\chi^2+f_\chi^2}$$

In our numerical example, $e_\chi$ is equal to $r(0)\chi(0)+f\chi(0)+\sigma_\chi\rho_{r\chi}\theta(0)\sqrt{\chi(0)}$ and $f_\chi$ equals $f$, in which $f$ is equal to 0.1.

## 8. APPENDIX 8: EXAMPLE WITH CIR MODEL AND CORPORATE COUPON BOND IN SECTION 4.1.2

For the example with CIR model and corporate coupon bond in Section 4.2, the discount process $\delta(t)$ is equal to $e^{-\int_0^t[r(s)+\chi(s)+\gamma(s)]ds}$ and $d\delta(t)=-\left[r(t)+\chi(t)+\gamma(t)\right]\delta(t)dt$. To accommodate the three risk factors (interest rate, default intensity, and convenience yield) with deflator, we let $dB(t)=B(t)\left[r(t)+\chi(t)+\gamma(t)\right]dt$. We repeat the calculation in Appendix 3.1, then we have $\Omega(t)$ equalling $-D(t)\left[r(t)+\chi(t)+\gamma(t)\right]$. Similarly, we have $\Phi(t)$ equalling $-D(t)\theta(t)$ and $\Psi(t)$, $\Gamma(t)$, $\mathrm{I}(t)$ equalling zero. We show the detailed calculations as follows.

A8.1 Short-term saving, $D(t)B(t)$

$$d\left[D(t)B(t)\right]=B(t)dD(t)+D(t)dB(t)+dB(t)dD(t)$$
$$=\left\{\Omega(t)B(t)+D(t)B(t)\left[r(t)+\chi(t)+\gamma(t)\right]\right\}dt+\Phi(t)B(t)dW_r(t)$$
$$+\Psi(t)B(t)dW_1(t)+\Gamma(t)B(t)dW_2(t)+\mathrm{I}(t)B(t)dW_3(t)$$

Let the drift term of $d\left[D(t)B(t)\right]$ equal zero, $\Omega(t)B(t)+D(t)B(t)\left[r(t)+\chi(t)+\gamma(t)\right]=0$, then $\Omega(t)$ is equal to $-D(t)\left[r(t)+\chi(t)+\gamma(t)\right]$.

Also, $d\left[\delta(t)B(t)\right]=\delta(t)dB(t)+B(t)d\delta(t)+d\delta(t)dB(t)=0$.

Thus, $\delta(t)B(t)$ is a $\mathbf{Q}$-martingale.

A8.2 Zero coupon bond of no risk with maturity $T$, $D(t)P(t,T,r(t))$

Similar to Appendix 2 (A2.1), $d\left[\delta(t)P(t,T,r(t))\right]=\tilde{\sigma}(t,r(t))\delta(t)P(t,T,r(t))d\tilde{W}_r(t)$ with $\theta(t)$ equalling $\frac{\tilde{\mu}(t,r(t))-\left[r(t)+\chi(t)+\gamma(t)\right]}{\tilde{\sigma}(t,r(t))}$ here. $\delta(t)P(t,T,r(t))$ is a $\mathbf{Q}$-martingale.

We rewrite $\frac{dP(t,T,r(t))}{P(t,T,r(t))}=\left(\tilde{\mu}(t,r(t))-\tilde{\sigma}(t,r(t))\theta(t)\right)dt+\tilde{\sigma}(t,r(t))d\tilde{W}_r(t)$.

$P(t,T,r(t))$ is the zero coupon bond of no risk with maturity $T$, so the drift term of $\frac{dP(t,T,r(t))}{P(t,T,r(t))}$ under probability measure $\mathbf{Q}'$ is equal to $r(t)+\chi(t)+\gamma(t)$. Hence, we have $\tilde{\mu}(t,r(t))$ equal $r(t)+\chi(t)+\gamma(t)+\tilde{\sigma}(t,r(t))\theta(t)$.

$$\frac{dP(t,T,r(t))}{P(t,T,r(t))}=\left[r(t)+\chi(t)+\gamma(t)+\tilde{\sigma}(t,r(t))\theta(t)\right]dt+\tilde{\sigma}(t,r(t))dW_r(t),\ \tilde{\sigma}(t,r(t))=\frac{P_r\beta(t,r(t))}{P(t,r(t))} \quad \text{(A8.1)}$$

$$dP(t,T,r(t))=P(t,T,r(t))\left[r(t)+\chi(t)+\gamma(t)+\tilde{\sigma}(t,r(t))\theta(t)\right]dt+P(t,T,r(t))\tilde{\sigma}(t,r(t))dW_r(t)$$

$$\begin{aligned}d\left[D(t)P(t,T,r(t))\right]&=P(t,T,r(t))dD(t)+D(t)dP(t,T,r(t))+dP(t,T,r(t))dD(t)\\&=\left\{\begin{aligned}&\Omega(t)P(t,T,r(t))+\Phi(t)P(t,T,r(t))\tilde{\sigma}(t,r(t))\\&+D(t)P(t,T,r(t))\left[r(t)+\chi(t)+\gamma(t)+\tilde{\sigma}(t,r(t))\theta(t)\right]\end{aligned}\right\}dt\\&\quad+\left[\Phi(t)P(t,T,r(t))+D(t)P(t,T,r(t))\tilde{\sigma}(t,r(t))\right]dW_r(t)\\&\quad+\Psi(t)P(t,T,r(t))dW_1(t)+\Gamma(t)P(t,T,r(t))dW_2(t)+\mathrm{I}(t)P(t,T,r(t))dW_3(t)\end{aligned}$$

Let the drift term of $d\left[D(t)P(t,T,r(t))\right]$ equal zero and plug $\Omega(t)$ into the drift term, we have $\Phi(t)=-D(t)\theta(t)$.

A8.3 Stock, $D(t)S(t)$

$$\begin{aligned}d\left[D(t)S(t)\right]&=S(t)dD(t)+D(t)dS(t)+dD(t)dS(t)\\&=\left[\Omega(t)S(t)+\Phi(t)S(t)\sigma_S(t)\rho_{rS}+\Psi(t)S(t)\sigma_S\sqrt{1-\rho_{rS}^2}+D(t)S(t)\mu_S(t)\right]dt\\&\quad+\left[\Phi(t)S(t)+D(t)S(t)\sigma_S(t)\rho_{rS}\right]dW_r(t)+\left[\Psi(t)S(t)+D(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}\right]dW_1(t)\\&\quad+\Gamma(t)S(t)dW_2(t)+\mathrm{I}(t)S(t)dW_3(t)\end{aligned}$$

Let the drift term of $d\left[D(t)S(t)\right]$ equal zero and plug $\Omega(t)$, $\Phi(t)$ into the drift term, then $\Psi(t)$ is equal to $\frac{D(t)\left[r(t)+\chi(t)+\gamma(t)+\theta(t)\sigma_S(t)\rho_{rS}-\mu_S(t)\right]}{\sigma_S(t)\sqrt{1-\rho_{rS}^2}}$.

$$\begin{aligned} d\left[\delta(t)S(t)\right] &= \delta(t)dS(t) + S(t)d\delta(t) + d\delta(t)dS(t) \\ &= \delta(t)S(t)\left[\mu_S(t) - r(t) - \chi(t) - \gamma(t)\right]dt + \delta(t)S(t)\sigma_S(t)\rho_{rS}dW_r(t) \\ &\quad + \delta(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t) \\ &= \delta(t)S(t)\left[\mu_S(t) - r(t) - \chi(t) - \gamma(t) - \theta(t)\sigma_S(t)\rho_{rS}\right]dt + \delta(t)S(t)\sigma_S(t)\rho_{rS}d\tilde{W}_r(t) \\ &\quad + \delta(t)S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t), \text{ under } \mathbf{Q} \end{aligned}$$

We require $\mu_S(t) = r(t) + \chi(t) + \gamma(t) + \theta(t)\sigma_S(t)\rho_{rS}$ as regularity condition, such that $\delta(t)S(t)$ is a $\mathbf{Q}$-martingale. As a result, $\Psi(t) = 0$.

A8.4 Default density, $D(t)\chi(t)$

$$\begin{aligned} d\left[D(t)\chi(t)\right] &= \chi(t)dD(t) + D(t)d\chi(t) + d\chi(t)dD(t) \\ &= \left[\Omega(t)\chi(t) + \Phi(t)\sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} + \Psi(t)\sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)} + \Gamma(t)\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)} + D(t)e - D(t)f\chi(t)\right]dt \\ &\quad + \left[\Phi(t)\chi(t) + D(t)\sigma_\chi\rho_{r\chi}\sqrt{\chi(t)}\right]dW_r(t) + \left[\Psi(t)\chi(t) + D(t)\sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)}\right]dW_1(t) \\ &\quad + \left[\Gamma(t)\chi(t) + D(t)\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}\right]dW_2(t) + \mathrm{I}(t)\chi(t)dW_3(t) \end{aligned}$$

Let the drift term of $d\left[D(t)\chi(t)\right]$ equal zero and plug $\Omega(t)$, $\Phi(t)$, $\Psi(t)$ into the drift term, we have $\Gamma(t) = D(t)\left\{\frac{\theta(t)\rho_{r\chi}}{\rho'_{\chi\chi}} + \frac{\left[r(t)+\chi(t)+\gamma(t)\right]\chi(t) - e + f\chi(t)}{\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}}\right\}$.

$$\begin{aligned} d\left[\delta(t)\chi(t)\right] &= \delta(t)d\chi(t) + \chi(t)d\delta(t) + d\delta(t)d\chi(t) \\ &= \delta(t)\left\{e - \left[r(t)+\chi(t)+\gamma(t)\right]\chi(t) - f\chi(t)\right\}dt + \sigma_\chi\rho_{r\chi}\delta(t)\sqrt{\chi(t)}dW_r(t) \\ &\quad + \sigma_\chi\rho'_{S\chi}\delta(t)\sqrt{\chi(t)}dW_1(t) + \sigma_\chi\rho'_{\chi\chi}\delta(t)\sqrt{\chi(t)}dW_2(t) \\ &= \delta(t)\left\{e - \left[r(t)+\chi(t)+\gamma(t)\right]\chi(t) - f\chi(t) - \sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)}\right\}dt + \sigma_\chi\rho_{r\chi}\delta(t)\sqrt{\chi(t)}d\tilde{W}_r(t) \\ &\quad + \sigma_\chi\rho'_{S\chi}\delta(t)\sqrt{\chi(t)}dW_1(t) + \sigma_\chi\rho'_{\chi\chi}\delta(t)\sqrt{\chi(t)}dW_2(t), \text{ under } \mathbf{Q} \end{aligned}$$

We require $e = \left[r(t)+\chi(t)+\gamma(t)\right]\chi(t) + f\chi(t) + \sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)}$ as regularity condition, such that $\delta(t)\chi(t)$ is a $\mathbf{Q}$-martingale. As a result, $\Gamma(t) = 0$. In addition, we also have to consider if the Feller condition holds (though this does not guarantee $\chi(t)$ to be positive since $e$ involves several factors and is not a constant here), i.e.

$$\left[r(t)+\chi(t)+\gamma(t)\right]\chi(t) + f\chi(t) + \sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)} > \frac{1}{2}\sigma_\chi^2.$$

A8.5 Convenience yield, $D(t)\gamma(t)$

$$
\begin{aligned}
d\left[D(t)\gamma(t)\right] &= \gamma(t)dD(t)+D(t)d\gamma(t)+d\gamma(t)dD(t) \\
&= \left[\Omega(t)\gamma(t)+\Phi(t)\eta\rho_{r\gamma}+\Psi(t)\eta\rho''_{S\gamma}+\Gamma(t)\eta\rho''_{\chi\gamma}+\mathrm{I}(t)\eta\rho''_{\gamma\gamma}\right]dt+\left[\Phi(t)\gamma(t)+D(t)\eta\rho_{r\gamma}\right]dW_r(t) \\
&\quad +\left[\Psi(t)\gamma(t)+D(t)\eta\rho''_{S\gamma}\right]dW_1(t)+\left[\Gamma(t)\gamma(t)+D(t)\eta\rho''_{\chi\gamma}\right]dW_2(t) \\
&\quad +\left(\mathrm{I}(t)\gamma(t)+D(t)\eta\rho''_{\gamma\gamma}\right)dW_3(t)
\end{aligned}
$$

Let the drift term of $d\left[D(t)\gamma(t)\right]$ equal zero and plug $\Omega(t)$, $\Phi(t)$, $\Psi(t)$, $\Gamma(t)$ into the drift term, then $\mathrm{I}(t)=D(t)\left\{\dfrac{\rho_{r\gamma}\theta(t)}{\rho''_{\gamma\gamma}}+\dfrac{\left[r(t)+\chi(t)+\gamma(t)\right]\gamma(t)}{\eta\rho''_{\gamma\gamma}}\right\}$.

$$
\begin{aligned}
d\left[\delta(t)\gamma(t)\right] &= \delta(t)d\gamma(t)+\gamma(t)d\delta(t)+d\delta(t)d\gamma(t) \\
&= -\delta(t)\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]dt+\eta\rho_{r\gamma}\delta(t)dW_r(t)+\eta\rho''_{S\gamma}\delta(t)dW_1(t) \\
&\quad +\eta\rho''_{\chi\gamma}\delta(t)dW_2(t)+\eta\rho''_{\gamma\gamma}\delta(t)dW_3(t) \\
&= \delta(t)\left\{-\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]-\eta\rho_{r\gamma}\theta(t)\right\}dt+\eta\rho_{r\gamma}\delta(t)d\tilde{W}_r(t)+\eta\rho''_{S\gamma}\delta(t)dW_1(t) \\
&\quad +\eta\rho''_{\chi\gamma}\delta(t)dW_2(t)+\eta\rho''_{\gamma\gamma}\delta(t)dW_3(t), \text{ under } \mathbf{Q}
\end{aligned}
$$

We require $\eta=-\dfrac{\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\rho_{r\gamma}\theta(t)}$ as regularity condition, such that $\delta(t)\gamma(t)$ is a $\mathbf{Q}$-martingale. As a result, $\mathrm{I}(t)=0$. In addition, readers should be very careful about the initial parameter setting since $\eta$ also involves several factors and is not a constant here. In our numerical example under Longstaff et al. (2005) in Section 4.2, $\chi(t)$ becomes negative after projecting longer than 7 years and the simulation couldn't continue, in which the large $\gamma(t)$ is the reason leads to the negative value of $\chi(t)$. Further study would be to investigate the long-term behaviours of $\chi(t)$ and $\gamma(t)$ after we require the regularity conditions.

A8.6 Implementation of time discretization

In A8.6, we show the implementation of time discretization for stochastic processes in A8.1 to A8.5 (different from the example in Section 4.1).

$$
\begin{bmatrix} dB(t) \\ dP(t,T,r(t)) \\ dS(t) \\ d\chi(t) \\ d\gamma(t) \\ dD(t) \end{bmatrix}
=
\begin{bmatrix}
B(t)\left[r(t)+\chi(t)+\gamma(t)\right] & 0 & 0 & 0 & 0 \\
P(t,T,r(t))\left[r(t)+\chi(t)+\gamma(t)\right]+P_r\sigma_r\sqrt{r(t)}\theta(t) & P_r\sigma_r\sqrt{r(t)} & 0 & 0 & 0 \\
S(t)\left[r(t)+\chi(t)+\gamma(t)+\theta(t)\sigma_S(t)\rho_{rS}\right] & S(t)\sigma_S(t)\rho_{rS} & S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2} & 0 & 0 \\
\left[r(t)+\chi(t)+\gamma(t)\right]\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)} & \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)} & 0 \\
0 & -\dfrac{\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\theta(t)} & -\dfrac{\rho''_{S\gamma}\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\rho_{r\gamma}\theta(t)} & -\dfrac{\rho''_{\chi\gamma}\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\rho_{r\gamma}\theta(t)} & -\dfrac{\rho''_{\gamma\gamma}\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\rho_{r\gamma}\theta(t)} \\
-D(t)\left[r(t)+\chi(t)+\gamma(t)\right] & -D(t)\theta(t) & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} dt \\ dW_r(t) \\ dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}
$$

## A8.6.1 Euler method

$$\begin{bmatrix} B_{i+1} \\ P_{i+1} \\ S_{i+1} \\ \chi_{i+1} \\ \gamma_{i+1} \\ D_{i+1} \end{bmatrix} = \begin{bmatrix} B_i \\ P_i \\ S_i \\ \chi_i \\ \gamma_i \\ D_i \end{bmatrix} + \begin{bmatrix} B_i\left(r_i+\chi_i+\gamma_i\right) & 0 & 0 & 0 & 0 \\ P_i\left(r_i+\chi_i+\gamma_i\right)+P_{r,t_i}\sigma_r\sqrt{r_i}\theta_i & P_{r,t_i}\sigma_r\sqrt{r_i} & 0 & 0 & 0 \\ S_i\left(r_i+\chi_i+\gamma_i+\theta_i\sigma_{S,t_i}\rho_{rS}\right) & S_i\sigma_{S,t_i}\rho_{rS} & S_i\sigma_{S,t_i}\sqrt{1-\rho_{rS}^2} & 0 & 0 \\ \left(r_i+\chi_i+\gamma_i\right)\chi_i+\sigma_\chi\rho_{r\chi}\theta_i\sqrt{\chi_i} & \sigma_\chi\rho_{r\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi_i} & 0 \\ 0 & -\frac{\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\theta_i} & -\frac{\rho''_{S\gamma}\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\rho_{r\gamma}\theta_i} & -\frac{\rho''_{\chi\gamma}\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\rho_{r\gamma}\theta_i} & -\frac{\rho''_{\gamma\gamma}\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\rho_{r\gamma}\theta_i} \\ -D_i\left(r_i+\chi_i+\gamma_i\right) & -D_i\theta_i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta t_i \\ \Delta W_{r,i} \\ \Delta W_{1,i} \\ \Delta W_{2,i} \\ \Delta W_{3,i} \end{bmatrix}$$

## A8.6.2 Milstein method

$$\begin{bmatrix} B_{i+1} \\ P_{i+1} \\ S_{i+1} \\ \chi_{i+1} \\ \gamma_{i+1} \\ D_{i+1} \end{bmatrix} = \begin{bmatrix} B_i \\ P_i \\ S_i \\ \chi_i \\ \gamma_i \\ D_i \end{bmatrix} + \begin{bmatrix} B_i\left(r_i+\chi_i+\gamma_i\right) & 0 & 0 & 0 & 0 \\ P_i\left(r_i+\chi_i+\gamma_i\right)+P_{r,t_i}\sigma_r\sqrt{r_i}\theta_i & P_{r,t_i}\sigma_r\sqrt{r_i} & 0 & 0 & 0 \\ S_i\left(r_i+\chi_i+\gamma_i+\theta_i\sigma_{S,t_i}\rho_{rS}\right) & S_i\sigma_{S,t_i}\rho_{rS} & S_i\sigma_{S,t_i}\sqrt{1-\rho_{rS}^2} & 0 & 0 \\ \left(r_i+\chi_i+\gamma_i\right)\chi_i+\sigma_\chi\rho_{r\chi}\theta_i\sqrt{\chi_i} & \sigma_\chi\rho_{r\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi_i} & 0 \\ 0 & -\frac{\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\theta_i} & -\frac{\rho''_{S\gamma}\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\rho_{r\gamma}\theta_i} & -\frac{\rho''_{\chi\gamma}\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\rho_{r\gamma}\theta_i} & -\frac{\rho''_{\gamma\gamma}\gamma_i\left(r_i+\chi_i+\gamma_i\right)}{\rho_{r\gamma}\theta_i} \\ -D_i\left(r_i+\chi_i+\gamma_i\right) & -D_i\theta_i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta t_i \\ \Delta W_{r,i} \\ \Delta W_{1,i} \\ \Delta W_{2,i} \\ \Delta W_{3,i} \end{bmatrix}$$

$$+\frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\sigma_{S,t_i}^2\rho_{rS}^2 & 2\sigma_{S,t_i}^2\left(1-\rho_{rS}^2\right) & 0 & 0 \\ \sigma_\chi^2\rho_{r\chi}^2 & \sigma_\chi^2\rho'^2_{S\chi} & \sigma_\chi^2\rho'^2_{\chi\chi} & 0 \\ \frac{2\gamma_i}{\theta_i^2}\left(r_i+\chi_i+\gamma_i\right)\left(r_i+\chi_i+2\gamma_i\right) & 2\gamma_i\left(\frac{\rho''_{S\gamma}}{\rho_{r\gamma}\theta_i}\right)^2\left(r_i+\chi_i+\gamma_i\right)\left(r_i+\chi_i+2\gamma_i\right) & 2\gamma_i\left(\frac{\rho''_{\chi\gamma}}{\rho_{r\gamma}\theta_i}\right)^2\left(r_i+\chi_i+\gamma_i\right)\left(r_i+\chi_i+2\gamma_i\right) & 2\gamma_i\left(\frac{\rho''_{\gamma\gamma}}{\rho_{r\gamma}\theta_i}\right)^2\left(r_i+\chi_i+\gamma_i\right)\left(r_i+\chi_i+2\gamma_i\right) \\ 2D_i\theta_i^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\Delta W_{r,i}\right)^2-\Delta t_i \\ \left(\Delta W_{1,i}\right)^2-\Delta t_i \\ \left(\Delta W_{2,i}\right)^2-\Delta t_i \\ \left(\Delta W_{3,i}\right)^2-\Delta t_i \end{bmatrix}$$

## A8.6.2 Second Milstein method

Here are $X_t$, $W_t$, $a\left(t,X_t\right)$, and $b\left(t,X_t\right)$ of simplified Second Milstein method in Section 4.2, $dX_t=a\left(t,X_t\right)dt+b\left(t,X_t\right)dW_t$.

$$X_t=\begin{bmatrix} r(t) \\ \theta(t) \\ B(t) \\ P\left(t,T,r(t)\right) \\ S(t) \\ \chi(t) \\ \gamma(t) \\ D(t) \end{bmatrix}, W_t=\begin{bmatrix} W_r(t) \\ W_1(t) \\ W_2(t) \\ W_3(t) \\ W_\theta(t) \end{bmatrix} \quad \text{(A8.1)}$$

$$a\left(t,X_t\right)=\begin{bmatrix} a_r-b_r r(t)+\theta(t)\sigma_r\sqrt{r(t)} \\ a_\theta-b_\theta\theta(t) \\ B(t)\left[r(t)+\chi(t)+\gamma(t)\right] \\ P\left(t,T,r(t)\right)\left[r(t)+\chi(t)+\gamma(t)\right]+P_r\sigma_r\sqrt{r(t)}\theta(t) \\ S(t)\left[r(t)+\chi(t)+\gamma(t)+\theta(t)\sigma_S(t)\rho_{rS}\right] \\ \left[r(t)+\chi(t)+\gamma(t)\right]\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)} \\ 0 \\ -\left[r(t)+\chi(t)+\gamma(t)\right]D(t) \end{bmatrix} \quad \text{(A8.2)}$$

$$b(t,X_t)=\begin{bmatrix} \sigma_r\sqrt{r(t)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\theta\sqrt{\theta(t)} \\ 0 & 0 & 0 & 0 & 0 \\ P_r\sigma_r\sqrt{r(t)} & 0 & 0 & 0 & 0 \\ S(t)\sigma_S(t)\rho_{rS} & S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2} & 0 & 0 & 0 \\ \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)} & 0 & 0 \\ -\frac{\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\theta(t)} & -\frac{\rho''_{S\gamma}\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\rho_{r\gamma}\theta(t)} & -\frac{\rho''_{\chi\gamma}\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\rho_{r\gamma}\theta(t)} & -\frac{\rho''_{\gamma\gamma}\gamma(t)\left[r(t)+\chi(t)+\gamma(t)\right]}{\rho_{r\gamma}\theta(t)} & 0 \\ -\theta(t)D(t) & 0 & 0 & 0 & 0 \end{bmatrix}$$

(A8.3)

Similar to Section A6.1, We choose $a_8(t,X_t)$ as an illustrative example here.

$$a_8(t,X_t)=-\left[r(t)+\chi(t)+\gamma(t)\right]D(t),$$

$$\frac{\partial a_8(t,X_t)}{\partial x_1}=-D(t),\ \frac{\partial a_8(t,X_t)}{\partial x_6}=-D(t),\ \frac{\partial a_8(t,X_t)}{\partial x_7}=-D(t),\ \frac{\partial a_8(t,X_t)}{\partial x_8}=-\left[r(t)+\chi(t)+\gamma(t)\right]$$

$$\frac{\partial^2 a_8(t,X_t)}{\partial x_1\partial x_8}=\frac{\partial^2 a_8(t,X_t)}{\partial x_8\partial x_1}=-1,\ \frac{\partial^2 a_8(t,X_t)}{\partial x_6\partial x_8}=\frac{\partial^2 a_8(t,X_t)}{\partial x_8\partial x_6}=-1,\ \frac{\partial^2 a_8(t,X_t)}{\partial x_7\partial x_8}=\frac{\partial^2 a_8(t,X_t)}{\partial x_8\partial x_7}=-1$$

$$\begin{aligned} L^0a_8(t,X_t)&=\frac{\partial a_8(t,X_t)}{\partial t}+\sum_{i=1}^{8}a_i(t,X_t)\frac{\partial a_8(t,X_t)}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{8}\Sigma_{t,ij}\frac{\partial^2 a_8(t,X_t)}{\partial x_i\partial x_j} \\ &=-a_1(t,X_t)D(t)-a_6(t,X_t)D(t)-a_7(t,X_t)D(t)-a_8(t,X_t)\left[r(t)+\chi(t)+\gamma(t)\right] \\ &\quad-\Sigma_{t,18}-\Sigma_{t,68}-\Sigma_{t,78} \end{aligned}$$

$$\begin{aligned} L^ka_8(t,X_t)&=\sum_{i=1}^{d}b_{ik}(t,X_t)\frac{\partial a_8(t,X_t)}{\partial x_i} \\ &=-b_{1k}(t,X_t)D(t)-b_{6k}(t,X_t)D(t)-b_{7k}(t,X_t)D(t)-b_{8k}(t,X_t)\left[r(t)+\chi(t)+\gamma(t)\right] \end{aligned}$$

## 9. APPENDIX 9: ONE MORE REQUIRED REGULARITY CONDITION FOR THE DIFFUSION TERM IN STOCK PRICE

In this section, we explain the one more required regularity for the diffusion term in stock price in detail. Recall that the dynamics of stock price after requiring regularity condition for $\mu_S(t)$ is as follows.

$$dS(t)=S(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}\right]dt+S(t)\sigma_S(t)\rho_{rS}dW_r(t)+S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t) \quad \text{(A9.1)}$$

By Itô formula,

$$\begin{aligned} d\left[\ln S(t)\right]&=\frac{\partial\left[\ln S(t)\right]}{\partial S(t)}dS(t)+\frac{1}{2}\frac{\partial^2\left[\ln S(t)\right]}{\partial\left[S(t)\right]^2}dS(t)dS(t) \\ &=\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}-\frac{1}{2}\sigma_S^2(t)\right]dt+\sigma_S(t)\rho_{rS}dW_r(t)+\sigma_S(t)\sqrt{1-\rho_{rS}^2}dW_1(t) \end{aligned}.$$

Integrate both sides of $d\left[\ln S(t)\right]$, we have

$$\ln S(t)=\ln S(0)+\int_0^t\left[r(s)+\theta(s)\sigma_S(s)\rho_{rS}-\frac{1}{2}\sigma_S^2(s)\right]ds+\int_0^t\sigma_S(s)\rho_{rS}dW_r(s)+\int_0^t\sigma_S(s)\sqrt{1-\rho_{rS}^2}dW_1(s). \quad \text{(A9.2)}$$

From Appendix 4, we have $\ln D(t)=\ln D(0)+\int_0^t\left[-r(s)-\frac{1}{2}\theta^2(s)\right]ds-\int_0^t\theta(s)dW_r(s)$.

Then,

$$\begin{aligned}\ln\left[D(t)S(t)\right]&=\ln D(t)+\ln S(t)\\&=\ln D(0)+\ln S(0)+\int_0^t\left[\theta(s)\sigma_S(s)\rho_{rS}-\frac{1}{2}\sigma_S^2(s)-\frac{1}{2}\theta^2(s)\right]ds.\\&\quad+\int_0^t\left[\sigma_S(s)\rho_{rS}-\theta(s)\right]dW_r(t)+\int_0^t\sigma_S(s)\sqrt{1-\rho_{rS}^2}\,dW_1(s)\end{aligned}$$

The first identity comes from that $\ln\left[D(t)S(t)\right]$ and $\ln D(t)+\ln S(t)$ are two random variables with the same characteristic function.

Take exponential both side of $\ln\left[D(t)S(t)\right]$, we have

$$\begin{aligned}D(t)S(t)=D(0)S(0)&\exp\left\{\int_0^t\left[\theta(s)\sigma_S(s)\rho_{rS}-\frac{1}{2}\sigma_S^2(s)-\frac{1}{2}\theta^2(s)\right]ds\right\}\\&\times\exp\left\{\int_0^t\left[\sigma_S(s)\rho_{rS}-\theta(s)\right]dW_r(t)+\int_0^t\sigma_S(s)\sqrt{1-\rho_{rS}^2}\,dW_1(s)\right\}\end{aligned}. \quad \text{(A9.3)}$$

If $\theta(t)$ and $\sigma_S(t)$ are constants, then $D(t)S(t)$ is a martingale since $W_r(t)$ and $W_1(t)$ are independent with $\exp\left\{\sigma W_i(t)-\frac{1}{2}\sigma^2 t\right\}$ being a martingale for $i=r,1$.[2]

However, $\theta(t)$ and $\sigma_S(t)$ are not constants. Let $\theta(t)\sigma_S(t)\rho_{rS}-\frac{1}{2}\sigma_S^2(t)-\frac{1}{2}\theta^2(t)$ equal zero, then both $D(t)S(t)$ and $\ln\left[D(t)S(t)\right]$ are $\mathbf{P}$ -martingales.[3] Then, $\sigma_S(t)=\rho_{rS}\theta(t)\pm\theta(t)\sqrt{\rho_{rS}^2-1}$ and $\sigma_S(t)$ is a complex number if $\rho_{rS}\neq 1$ (so that $|\rho_{rS}|<1$). In our numerical example in Section 4.2, we choose $\rho_{rS}$ equalling 1 then $\sigma_S(t)$ is equal to $\theta(t)$. After plugging in $\rho_{rS}=1$ and $\sigma_S(t)=\theta(t)$ into $d\left[\ln S(t)\right]$, we have

$$d\left[\ln S(t)\right]=\left[r(t)+\frac{1}{2}\theta^2(t)\right]dt+\theta(t)dW_r(t).$$

In addition, $d\left[\ln D(t)\right]=\left[-r(t)-\frac{1}{2}\theta^2(t)\right]dt-\theta(t)dW_r(t)$. Comparing $d\left[\ln S(t)\right]$ with $d\left[\ln D(t)\right]$, we could see that the drift and the diffusion terms of $d\left[\ln S(t)\right]$ and $d\left[\ln D(t)\right]$ offset each other.

---

[2] See for example, Shreve (2004) Chapter 3.6 Theorem 3.6.1.

[3] Observe that after this required condition with the dynamics of $\ln\left[D(t)S(t)\right]$, we could derive that $D(t)S(t)$ is still a $\mathbf{P}$-martingale by Itô formula.

## 10. Appendix 10: Interest rate in physical world

In this section, we provide some analyses for the process of interest rate in physical world. In Section 4, the process of interest rate in $\mathbf{P}$-measure is

$$dr(t)=\left[a_r-b_r r(t)+\theta(t)\sigma_r\sqrt{r(t)}\right]dt+\sigma_r\sqrt{r(t)}dW_r(t).$$

We show that the mean and variance of the interest rate process $r(t)$ behave like the mean and variance of a CIR process asymptotically.

First, we could see that the mean and variance of the market price of risk $\theta(t)$ are $e^{-b_\theta t}\theta(0)+\frac{a_\theta}{b_\theta}\left(1-e^{-b_\theta t}\right)$ and $\frac{\sigma_\theta^2}{b_\theta}\theta(0)\left(e^{-b_\theta t}-e^{-2b_\theta t}\right)+\frac{a_\theta\sigma_\theta^2}{2b_\theta^2}\left(1-2e^{-b_\theta t}+e^{-2b_\theta t}\right)$ respectively.[4] When $t$ goes larger, the mean and variance of $\theta(t)$ asymptotically become $\frac{a_\theta}{b_\theta}$ and $\frac{a_\theta\sigma_\theta^2}{2b_\theta^2}$ respectively. We show the calculation details as follows.

i. $\theta(t)$

$$d\theta(t)=\left[a_\theta-b_\theta\theta(t)\right]dt+\sigma_\theta\sqrt{\theta(t)}dW_\theta(t)$$

Let $g(t,x)=e^{b_\theta t}x$, then $g_t(t,x)=b_\theta e^{b_\theta t}x$, $g_x(t,x)=e^{b_\theta t}$, and $g_{xx}(t,x)=0$.

$$\begin{aligned}d\left[e^{b_\theta t}\theta(t)\right]&=g_t(t,\theta(t))dt+g_x(t,\theta(t))d\theta(t)+\frac{1}{2}g_{xx}(t,\theta(t))d\theta(t)d\theta(t)\\&=b_\theta e^{b_\theta t}\theta(t)dt+e^{b_\theta t}\left[a_\theta-b_\theta\theta(t)\right]dt+e^{b_\theta t}\sigma_\theta\sqrt{\theta(t)}dW_\theta(t)\\&=a_\theta e^{b_\theta t}dt+\sigma_\theta e^{b_\theta t}\sqrt{\theta(t)}dW_\theta(t)\end{aligned}$$

Integrate both sides of $d\left[e^{b_\theta t}\theta(t)\right]$,

$$\int_0^t d\left[e^{b_\theta s}\theta(s)\right]=\int_0^t a_\theta e^{b_\theta s}ds+\int_0^t\sigma_\theta e^{b_\theta s}\sqrt{\theta(s)}dW_\theta(s)$$

$$e^{b_\theta t}\theta(t)-\theta(0)=a_\theta\int_0^t e^{b_\theta s}ds+\sigma_\theta\int_0^t e^{b_\theta s}\sqrt{\theta(s)}dW_\theta(s)$$

$$\begin{aligned}e^{b_\theta t}\theta(t)&=\theta(0)+a_\theta\int_0^t e^{b_\theta s}ds+\sigma_\theta\int_0^t e^{b_\theta s}\sqrt{\theta(s)}dW_\theta(s)\\&=\theta(0)+\frac{a_\theta}{b_\theta}\left(e^{b_\theta t}-1\right)+\sigma_\theta\int_0^t e^{b_\theta s}\sqrt{\theta(s)}dW_\theta(s)\end{aligned}$$

First, we calculate the mean of $\theta(t)$.

Take expectation both side of $e^{b_\theta t}\theta(t)$ with the expectation of an Itô intergral is zero, then $E\left[e^{b_\theta t}\theta(t)\right]=\theta(0)+\frac{a_\theta}{b_\theta}\left(e^{b_\theta t}-1\right)$.

As a result, $E\left[\theta(t)\right]=e^{-b_\theta t}\theta(0)+\frac{a_\theta}{b_\theta}\left(1-e^{-b_\theta t}\right)$.

Particularly, $E\left[\theta(t)\right]\to\frac{a_\theta}{b_\theta}$ as $t\to\infty$.

---

[4] See, for example, Shreve (2004) Chapter 4.4 Example 4.4.11.

Secondly, we calculate the variance of $\theta(t)$.

Given a random variable $X$, we know that $Var(X)=E(X^2)-[E(X)]^2$.

Let $h(t,x)=x^2$, then $h_t(t,x)=0$, $h_x(t,x)=2x$, and $h_{xx}(t,x)=2$.

$$
\begin{aligned}
d\left\{\left[e^{b_\theta t}\theta(t)\right]^2\right\} &= h_t(t,\theta(t))dt + h_x(t,\theta(t))d\left[e^{b_\theta t}\theta(t)\right] + \frac{1}{2}h_{xx}(t,\theta(t))d\left[e^{b_\theta t}\theta(t)\right]d\left[e^{b_\theta t}\theta(t)\right] \\
&= 2e^{b_\theta t}\theta(t)\left[a_\theta e^{b_\theta t}dt + \sigma_\theta e^{b_\theta t}\sqrt{\theta(t)}dW_\theta(t)\right] + \sigma_\theta^2 e^{2b_\theta t}\theta(t)dt \\
&= \left(2a_\theta+\sigma_\theta^2\right)e^{2b_\theta t}\theta(t)dt + 2\sigma_\theta e^{2b_\theta t}\theta^{\frac{3}{2}}(t)dW_\theta(t)
\end{aligned}
$$

Integrate both sides of $d\left\{\left[e^{b_\theta t}\theta(t)\right]^2\right\}$,

$$
\int_0^t d\left\{\left[e^{b_\theta s}\theta(s)\right]^2\right\} = \int_0^t \left(2a_\theta+\sigma_\theta^2\right)e^{2b_\theta s}\theta(s)ds + \int_0^t 2\sigma_\theta e^{2b_\theta s}\theta^{\frac{3}{2}}(s)dW_\theta(s).
$$

$$
\left[e^{b_\theta t}\theta(t)\right]^2 - \theta^2(0) = \left(2a_\theta+\sigma_\theta^2\right)\int_0^t e^{2b_\theta s}\theta(s)ds + 2\sigma_\theta\int_0^t e^{2b_\theta s}\theta^{\frac{3}{2}}(s)dW_\theta(s)
$$

$$
e^{2b_\theta t}\theta^2(t) = \theta^2(0) + \left(2a_\theta+\sigma_\theta^2\right)\int_0^t e^{2b_\theta s}\theta(s)ds + 2\sigma_\theta\int_0^t e^{2b_\theta s}\theta^{\frac{3}{2}}(s)dW_\theta(s)
$$

Suppose we could exchange integrals by the Fubini-Tonelli Theorem,
and then take expectation both sides of $e^{2b_\theta t}\theta^2(t)$ with the expectation of an Itô intergral is zero.

$$
\begin{aligned}
E\left[e^{2b_\theta t}\theta^2(t)\right] &= \theta^2(0) + \left(2a_\theta+\sigma_\theta^2\right)\int_0^t e^{2b_\theta s}E\left[\theta(s)\right]ds \\
&= \theta^2(0) + \left(2a_\theta+\sigma_\theta^2\right)\int_0^t \left[e^{b_\theta s}\theta(0) + \frac{a_\theta}{b_\theta}\left(e^{2b_\theta s}-e^{b_\theta s}\right)\right]ds \\
&= \theta^2(0) + \left(2a_\theta+\sigma_\theta^2\right)\left[\frac{1}{b_\theta}e^{b_\theta s}\theta(0) + \frac{a_\theta}{b_\theta}\left(\frac{1}{2b_\theta}e^{2b_\theta s} - \frac{1}{b_\theta}e^{b_\theta s}\right)\right]\Bigg|_0^t \\
&= \theta^2(0) + \frac{2a_\theta+\sigma_\theta^2}{b_\theta}\left[\theta(0)-\frac{a_\theta}{b_\theta}\right]\left(e^{b_\theta t}-1\right) + \frac{a_\theta\left(2a_\theta+\sigma_\theta^2\right)}{2b_\theta^2}\left(e^{2b_\theta t}-1\right)
\end{aligned}
$$

As a result, $E\left[\theta^2(t)\right] = e^{-2b_\theta t}\theta^2(0) + \frac{2a_\theta+\sigma_\theta^2}{b_\theta}\left[\theta(0)-\frac{a_\theta}{b_\theta}\right]\left(e^{-b_\theta t}-e^{-2b_\theta t}\right) + \frac{a_\theta\left(2a_\theta+\sigma_\theta^2\right)}{2b_\theta^2}\left(1-e^{-2b_\theta t}\right).$

$$
Var\left[\theta(t)\right] = E\left[\theta^2(t)\right] - E\left[\theta(t)\right]^2 = \frac{\sigma_\theta^2}{b_\theta}\theta(0)\left(e^{-b_\theta t}-e^{-2b_\theta t}\right) + \frac{a_\theta\sigma_\theta^2}{2b_\theta^2}\left(1-2e^{-b_\theta t}+e^{-2b_\theta t}\right)
$$

Particularly, $Var\left[\theta(t)\right] \to \frac{a_\theta\sigma_\theta^2}{2b_\theta^2}$ as $t\to\infty$.

Next, we calculate the mean and variance of interest rate process $r(t)$ as follows.

$ii.\ r(t)$

First, we calculate $d\left[e^{b_r t}r(t)\right]$ under risk-neutral world then tranfer into physical world.

$$\begin{cases} dr(t)=\left[a_r-b_r r(t)\right]dt+\sigma_r\sqrt{r(t)}d\tilde{W}_r(t) \\ \qquad d\tilde{W}_r(t)=\theta(t)dt+dW_r(t) \end{cases}$$

Similar to previous calculations in $d\left[e^{b_\theta t}\theta(t)\right]$,

$$d\left[e^{b_r t}r(t)\right]=a_r e^{b_r t}dt+\sigma_r e^{b_r t}\sqrt{r(t)}d\tilde{W}_r(t)=e^{b_r t}\left[a_r+\theta(t)\sigma_r\sqrt{r(t)}\right]dt+\sigma_r e^{b_r t}\sqrt{r(t)}dW_r(t)$$

Integrate both sides of $d\left[e^{b_r t}r(t)\right]$,

$$\int_0^t d\left[e^{b_r s}r(s)\right]=\int_0^t e^{b_r s}\left[a_r+\theta(s)\sigma_r\sqrt{r(s)}\right]ds+\int_0^t \sigma_r e^{b_r s}\sqrt{r(s)}dW_r(s)$$

$$e^{b_r t}r(t)-r(0)=\int_0^t e^{b_r s}\left[a_r+\theta(s)\sigma_r\sqrt{r(s)}\right]ds+\int_0^t \sigma_r e^{b_r s}\sqrt{r(s)}dW_r(s)$$

$$e^{b_r t}r(t)=r(0)+a_r\int_0^t e^{b_r s}ds+\sigma_r\int_0^t\left[e^{b_r s}\theta(s)\sqrt{r(s)}\right]ds+\int_0^t \sigma_r e^{b_r s}\sqrt{r(s)}dW_r(s)$$

$$=r(0)+\frac{a_r}{b_r}\left(e^{b_r t}-1\right)+\sigma_r\int_0^t\left[e^{b_r s}\theta(s)\sqrt{r(s)}\right]ds+\int_0^t \sigma_r e^{b_r s}\sqrt{r(s)}dW_r(s)$$

$$r(t)=e^{-b_r t}r(0)+\frac{a_r}{b_r}\left(1-e^{-b_r t}\right)+e^{-b_r t}\sigma_r\int_0^t\left[e^{b_r s}\theta(s)\sqrt{r(s)}\right]ds+e^{-b_r t}\int_0^t \sigma_r e^{b_r s}\sqrt{r(t)}dW_r(s)$$

Suppose we could exchange integrals by the Fubini-Tonelli Theorem,
and then take expectation both sides of $r(t)$ with the expectation of an Itô intergral is zero.

$$E\left[r(t)\right]=e^{-b_r t}r(0)+\frac{a_r}{b_r}\left(1-e^{-b_r t}\right)+e^{-b_r t}\sigma_r\int_0^t e^{b_r s}E\left[\theta(s)\sqrt{r(s)}\right]ds$$

Then, $E\left[r(t)\right]\to\frac{a_r}{b_r}$ as $t\to\infty$.

Similar to previous calculations in $d\left\{\left[e^{b_\theta t}\theta(t)\right]^2\right\}$,

$$d\left\{\left[e^{b_r t}r(t)\right]^2\right\}=\left(2a_r+\sigma_r^2\right)e^{2b_r t}r(t)dt+2\sigma_r e^{2b_r t}r^{\frac{3}{2}}(t)d\tilde{W}_r(t)$$

$$=\left(2a_r+\sigma_r^2\right)e^{2b_r t}r(t)dt+2\sigma_r e^{2b_r t}r^{\frac{3}{2}}(t)\theta(t)dt+2\sigma_r e^{2b_r t}r^{\frac{3}{2}}(t)dW_r(t).$$

Integrate both sides of $d\left\{\left[e^{b_r t}r(t)\right]^2\right\}$,

$$\int_0^t d\left\{\left[e^{b_r t}r(s)\right]^2\right\}=\int_0^t\left(2a_r+\sigma_r^2\right)e^{2b_r s}r(s)ds+\int_0^t 2\sigma_r e^{2b_r s}r^{\frac{3}{2}}(s)\theta(s)ds+\int_0^t 2\sigma_r e^{2b_r s}r^{\frac{3}{2}}(s)dW_r(s).$$

Then,

$$\left[e^{b_r t} r(t)\right]^2 - r^2(0) = \left(2a_r + \sigma_r^2\right)\int_0^t e^{2b_r s} r(s)\,ds + 2\sigma_r \int_0^t e^{2b_r s} r^{\frac{3}{2}}(s)\theta(s)\,ds + 2\sigma_r \int_0^t e^{2b_r s} r^{\frac{3}{2}}(s)\,dW_r(s).$$

$$\left[e^{b_r t} r(t)\right]^2 = r^2(0) + \left(2a_r + \sigma_r^2\right)\int_0^t e^{2b_r s} r(s)\,ds + 2\sigma_r \int_0^t e^{2b_r s} r^{\frac{3}{2}}(s)\theta(s)\,ds + 2\sigma_r \int_0^t e^{2b_r s} r^{\frac{3}{2}}(s)\,dW_r(s)$$

Suppose we could exchange integrals by the Fubini-Tonelli Theorem,

and then take expectation both sides of $\left[e^{b_r t} r(t)\right]^2$ with the expectation of an Itô intergral is zero.

$$E\left[e^{2b_r t} r^2(t)\right] = r^2(0) + \left(2a_r + \sigma_r^2\right)\int_0^t e^{2b_r s} E\left[r(s)\right] ds + 2\sigma_r \int_0^t e^{2b_r s} E\left[r^{\frac{3}{2}}(s)\theta(s)\right] ds$$

$$= r^2(0) + \left(2a_r + \sigma_r^2\right)\int_0^t e^{2b_r s}\left\{e^{-b_r s} r(0) + \frac{a_r}{b_r}\left(1 - e^{-b_r s}\right) + e^{-b_r s}\sigma_r \int_0^s e^{b_r u} E\left[\theta(u)\sqrt{r(u)}\right] du\right\} ds$$

$$+ 2\sigma_r \int_0^t e^{2b_r s} E\left[r^{\frac{3}{2}}(s)\theta(s)\right] ds$$

$$= r^2(0) + \frac{2a_r + \sigma_r^2}{b_r}\left[r(0) - \frac{a_r}{b_r}\right]\left(e^{b_r t} - 1\right) + \frac{a_r\left(2a_r + \sigma_r^2\right)}{2b_r^2}\left(e^{2b_r t} - 1\right)$$

$$+ \left(2a_r + \sigma_r^2\right)\sigma_r \int_0^t \left\{e^{b_r s}\int_0^s e^{b_r u} E\left[\theta(u)\sqrt{r(u)}\right] du\right\} ds + 2\sigma_r \int_0^t e^{2b_r s} E\left[r^{\frac{3}{2}}(s)\theta(s)\right] ds$$

$$e^{2b_r t} E\left[r^2(t)\right] = r^2(0) + \frac{2a_r + \sigma_r^2}{b_r}\left[r(0) - \frac{a_r}{b_r}\right]\left(e^{b_r t} - 1\right) + \frac{a_r\left(2a_r + \sigma_r^2\right)}{2b_r^2}\left(e^{2b_r t} - 1\right)$$

$$+ \left(2a_r + \sigma_r^2\right)\sigma_r \int_0^t \left\{e^{b_r s}\int_0^s e^{b_r u} E\left[\theta(u)\sqrt{r(u)}\right] du\right\} ds + 2\sigma_r \int_0^t e^{2b_r s} E\left[r^{\frac{3}{2}}(s)\theta(s)\right] ds$$

$$E\left[r^2(t)\right] = e^{-2b_r t} r^2(0) + \frac{2a_r + \sigma_r^2}{b_r}\left[r(0) - \frac{a_r}{b_r}\right]\left(e^{-b_r t} - e^{-2b_r t}\right) + \frac{a_r\left(2a_r + \sigma_r^2\right)}{2b_r^2}\left(1 - e^{-2b_r t}\right)$$

$$+ e^{-2b_r t}\left(2a_r + \sigma_r^2\right)\sigma_r \int_0^t \left\{e^{b_r s}\int_0^s e^{b_r u} E\left[\theta(u)\sqrt{r(u)}\right] du\right\} ds + e^{-2b_r t} 2\sigma_r \int_0^t e^{2b_r s} E\left[r^{\frac{3}{2}}(s)\theta(s)\right] ds$$

$$Var\left[r(t)\right] = E\left[r^2(t)\right] - E\left[r(t)\right]^2$$

$$= \frac{\sigma_r^2}{b_r} r(0)\left(e^{-b_r t} - e^{-2b_r t}\right) + \frac{a_r \sigma_r^2}{2b_r^2}\left(1 - 2e^{-b_r t} + e^{-2b_r t}\right)$$

$$+ e^{-2b_r t}\left(2a_r + \sigma_r^2\right)\sigma_r \int_0^t \left\{e^{b_r s}\int_0^s e^{b_r u} E\left[\theta(u)\sqrt{r(u)}\right] du\right\} ds + e^{-2b_r t} 2\sigma_r \int_0^t e^{2b_r s} E\left[r^{\frac{3}{2}}(s)\theta(s)\right] ds$$

$$- e^{-2b_r t}\sigma_r^2\left\{\int_0^t e^{b_r s} E\left[\theta(s)\sqrt{r(s)}\right] ds\right\}^2 - 2e^{-2b_r t} r(0)\sigma_r \int_0^t e^{b_r s} E\left[\theta(s)\sqrt{r(s)}\right] ds$$

$$- \frac{2a_r}{b_r}\left(e^{-b_r t} - e^{-2b_r t}\right)\sigma_r \int_0^t e^{b_r s} E\left[\theta(s)\sqrt{r(s)}\right] ds$$

Then, $Var\left[r(t)\right] \to \dfrac{a_r \sigma_r^2}{2b_r^2}$ as $t \to \infty$.