STOCHASTIC DEFLATOR FOR AN ECONOMIC SCENARIO GENERATOR WITH FIVE FACTORS

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Abstract

In this paper, we implement a stochastic deflator with five economic and financial risk factors: interest rates, market price of risk, stock prices, default intensities, and convenience yields. We examine the deflator with different financial assets, such as stocks, zero-coupon bonds, vanilla options, and corporate coupon bonds. We find required regularity conditions to implement our stochastic deflator. Our numerical results show the reliability of the deflator approach in pricing financial derivatives.

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1. INTRODUCTION

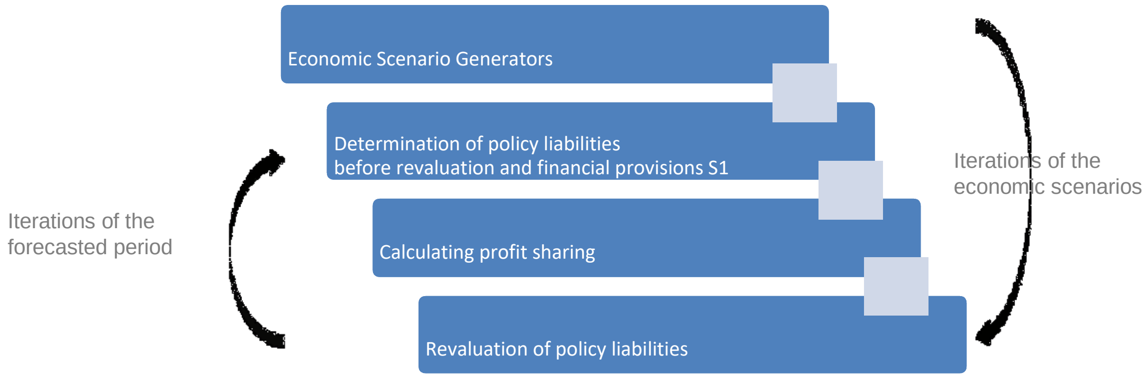
The Arrow-Debreu model of general equilibrium introduced the existence of an equilibrium in which the allocation of consumption and production is Pareto optimal with a system of prices for contingent commodities.[1] Their works have inspired tremendous research in fields of macroeconomics, financial economics, and asset pricing theory. Based on the concept of Arrow-Debreu securities, researchers had developed the fundamental theorems of asset pricing, which the second theorem tells us that an arbitrage-free market is complete if and only if the equivalent martingale measure is unique.[2]

In the case of Brownian diffusion, the Girsanov's Theorem enables us to change probability measure from a physical world to a risk-neutral world. Under risk-neutral measure, we have a closed-form solution for Black-Scholes options pricing model. However, we wouldn't always have analytical solutions for various classes of stochastic processes, which motivates us to study numerical methods for approximating solutions. In this paper, we investigate stochastic deflator approach for pricing of life insurance contracts.

Due to the complicatedness of life insurance contracts and interactions among economic and financial risk factors, a reliable tool for asset/liability management (ALM) and calculations of reserves would be demanded. In practice, "economic scenario generators" assist insurers in pricing insurance contracts and managing long-term risk[3].

The usual pricing scheme is as follows.

Fig. 1 - **Calculating the best estimate reserve for a life insurance contract**



Usually, economic scenarios are computed under a risk-neutral measure; the actualization process involving risk-free rate is quite simple, numerically speaking. However, we like to point out that many "unusual" scenarios occur (e.g. 10-year rate $\geq$ 50%) under risk-neutral measure, which increases the difficulty to justify the calibration of "reaction functions" embedded in the ALM-projection model used to compute cash flows.

For example, the lapse rate is often a function of the difference between the revalorization rate of the contract and a reference rate; the parameters are calibrated

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[1] See, for example, Arrow and Debreu (1954), Geanakoplos (1989), and Mas-Colell et al. (1995) Chapter 19.
[2] See, for example, Harrison and Kreps (1979), Harrison and Pliska (1981), Delbaen and Schachermayer (1994), and Shreve (2004) Chapter 5.4.
[3] See, for example, Varnell (2011), Laurent et al. (2016) Chapters 3,4, and 5, and Pedersen et al. (2016).

observing "usual" values of economic parameters but may become difficult later to justify for atypical values of economic risk factors. We could use a stochastic deflator to address this problem, using only scenarios under physical measure[4]. The numerical calculations become tedious due to the complexity of the deflator, which involves a risk-free rate process and a change of measure between physical and risk-neutral measure. But the benefit is that we could calculate the deflator separately and multiply the deflator with projected cash flows for pricing insurance contracts.

In this paper, we adopt the deflator approach initiated by Dastarac and Sauveplane (2010) and include the processes of default and convenience yield from Longstaff et al. (2005) to calculate prices for financial derivatives. We compare the values calculated from the deflator approach with the values suggested by analytical formulas in simple cases. We find required regularity conditions to implement our stochastic deflator. Also, our numerical results show the reliability in statistics of the deflator approach for quite simple financial derivatives. Our goal is then to use this deflator to compute best estimates for a life insurance contract.

The remainder of the paper is organized as follows. Section 2 shows the deflator approach. Section 3 discusses the implementation of time discretization. Section 4 presents the numerical results. Section 5 concludes.

2. DEFLATOR APPROACH

Before discussing and deriving the general form of deflator, we need to generate correlated Brownian motions for the stochastic processes in our model. In our model, we consider the processes of interest rates, market prices of risk, stock prices, default intensities and convenience yields. Sections 2 and 3 discuss the technical details of implementations of the deflator and time discretization. Readers who are familiar with stochastic deflator and time discretization could directly skip to numerical results in Section 4.

2.1. GENERATE CORRELATED BROWNIAN MOTIONS

Let the Brownian motion part of each process $\mathbf{W}_{ESG}$ and the correlations matrix $\mathbf{C}_{ESG}$ among interest rates, stock prices, default intensities, and convenience yields be as follows.[5]

$$\mathbf{W}_{ESG} = \begin{bmatrix} W_r \\ W_S \\ W_\chi \\ W_\gamma \end{bmatrix}, \ \mathbf{C}_{ESG} = \begin{array}{c c} & \begin{array}{cccc} W_r & W_S & W_\chi & W_\gamma \end{array} \\ \begin{array}{c} W_r \\ W_S \\ W_\chi \\ W_\gamma \end{array} & \begin{bmatrix} 1 & \rho_{rS} & \rho_{r\chi} & \rho_{r\gamma} \\ \rho_{rS} & 1 & \rho_{S\chi} & \rho_{S\gamma} \\ \rho_{r\chi} & \rho_{S\chi} & 1 & \rho_{\chi\gamma} \\ \rho_{r\gamma} & \rho_{S\gamma} & \rho_{\chi\gamma} & 1 \end{bmatrix} \end{array} \tag{1}$$

[4] See, for example, Bonnin et al. (2014), Borel-Mathurin et al. (2015), and Vedani et al. (2017).

[5] Here the correlations matrix $\mathbf{C}_{ESG}$ describes the linear correlation between each two processes of their Brownian motion parts. For discussions of dependence structure among random variables, see, for example, McNeil et al. (2005) Chapter 5 and Rachev et al. (2011) Chapter 2.6.4 .

Denote r interest rate; S stock price; χ default intensity; γ convenience yield; $W_i,\ i=r,S,\chi,\gamma$ Brownian motion part of each process; and $\rho_{jk},\ j\neq k$ correlation between each two processes. To generate correlated $W_i,\ i=r,S,\chi,\gamma$, we require four independent Brownian motions $W_i,\ i=0,1,2,3$. Following is the construction of $\mathbf{W}_{ESG}$, technical details are provided in Appendix 1 of Supplementary materials.

$$\mathbf{W}_{ESG}=\begin{bmatrix}W_r\\W_S\\W_\chi\\W_\gamma\end{bmatrix}=\begin{bmatrix}1&0&0&0\\\rho_{rS}&\sqrt{1-\rho_{rS}^2}&0&0\\\rho_{r\chi}&\rho'_{S\chi}&\rho'_{\chi\chi}&0\\\rho_{r\gamma}&\rho''_{S\gamma}&\rho''_{\chi\gamma}&\rho''_{\gamma\gamma}\end{bmatrix}\begin{bmatrix}W_0\\W_1\\W_2\\W_3\end{bmatrix}\quad(2)$$

where

$$\rho'_{S\chi}=\frac{\rho_{S\chi}-\rho_{rS}\rho_{r\chi}}{\sqrt{1-\rho_{rS}^2}},\ \rho'_{\chi\chi}=\sqrt{\frac{1-\rho_{rS}^2-\rho_{r\chi}^2-\rho_{S\chi}^2+2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}{1-\rho_{rS}^2}},\ \rho''_{S\gamma}=\frac{\rho_{S\gamma}-\rho_{rS}\rho_{r\gamma}}{\sqrt{1-\rho_{rS}^2}},$$

$$\rho''_{\chi\gamma}=\frac{\rho_{\chi\gamma}-\rho_{r\chi}\rho_{r\gamma}-\rho_{S\chi}\rho_{S\gamma}-\rho_{rS}^2\rho_{\chi\gamma}+\rho_{rS}\rho_{r\chi}\rho_{S\gamma}+\rho_{rS}\rho_{r\gamma}\rho_{S\chi}}{\sqrt{1+\rho_{rS}^4-2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi}-2\rho_{rS}^2+\rho_{rS}^2\rho_{r\chi}^2+\rho_{rS}^2\rho_{S\chi}^2-\rho_{r\chi}^2-\rho_{S\chi}^2+2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}},$$

$$\rho''_{\gamma\gamma}=\sqrt{1-\rho_{r\gamma}^2-\rho''^2_{S\gamma}-\rho''^2_{\chi\gamma}}.$$

2.2. GENERAL FORM OF DEFLATOR WITH FIVE FACTORS[6]

Let $r(t)$, $B(t)$, $P(t,T,r(t))$, $S(t)$, $\chi(t)$, $\gamma(t)$, $D(t)$ be processes of interest rate, short-term saving, zero coupon bond of no risk with maturity T, stock price, default density, convenience yield, and deflator respectively.[7] Denote $E(\cdot)$ expectation under physical measure and $E^{\mathbf{Q}}(\cdot)$ expectation under risk-neutral measure. Let a discount process $\delta(t)$ equal $e^{-\int_0^t r(s)ds}$. For a nonnegative random variable X, we would like to have $E^{\mathbf{Q}}(\delta(t)X)=E\left[D(t)X\right]$ (i.e. $D(t)=\delta(t)\frac{d\mathbf{Q}}{d\mathbf{P}}$, where $\frac{d\mathbf{Q}}{d\mathbf{P}}$ is a Radon-Nikodym derivative). We describe the dynamics of each process in the following paragraphs, in a quite general Markovian framework.

[6] For discussions of stochastic deflator in insurance, see, for example, Dastarac and Sauveplane (2010) and Caja and Planchet (2011). For a reference of stochastic calculus related to Itô's lemma and Girsanov's Theorem, see Shreve (2004) Chapters 4 and 5.

[7] Note that research studies of pricing, default and liquidity are fundamental no matter which currency we use.

2.2.1. Dynamics of each process

For simplicity, we present the dynamics of each process in matrix as follows.[8]

$$\begin{bmatrix} dr(t) \\ dB(t) \\ \dfrac{dP(t,T,r(t))}{P(t,T,r(t))} \\ \dfrac{dS(t)}{S(t)} \\ d\chi(t) \\ d\gamma(t) \end{bmatrix} = \begin{bmatrix} \alpha(t,r(t)) & \beta(t,r(t)) & 0 & 0 & 0 \\ B(t)r(t) & 0 & 0 & 0 & 0 \\ \tilde{\mu}(t,r(t)) & \tilde{\sigma}(t,r(t)) & 0 & 0 & 0 \\ \mu_S(t) & 0 & \sigma_S(t) & 0 & 0 \\ e-f\chi(t) & 0 & 0 & \sigma_\chi\sqrt{\chi(t)} & 0 \\ 0 & 0 & 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} dt \\ dW_r(t) \\ dW_S(t) \\ dW_\chi(t) \\ dW_\gamma(t) \end{bmatrix}$$

$\alpha(t,r(t))$ and $\beta(t,r(t))$ are the drift term and the diffusion term of interest rate process $r(t)$ respectively. $B(t)r(t)$ is the drift term of short-term saving process $B(t)$. $\dfrac{dP(t,T,r(t))}{P(t,T,r(t))}$ is the process of zero coupon bond of no risk with maturity T.

We would like to derive the drift term $\tilde{\mu}(t,r(t))$ and the diffusion term $\tilde{\sigma}(t,r(t))$ for $P(t,T,r(t))$, in which technical details are provided in Appendix 2 of Supplementary materials. From Appendix 2 of Supplementary materials, $\tilde{\mu}(t,r(t))$ equals $r(t)+\tilde{\sigma}(t,r(t))\theta(t)$ and $\tilde{\sigma}(t,r(t))$ equals $\dfrac{P_r\beta(t,r(t))}{P(t,r(t))}$. Here, $\theta(t)$ is the process of market price of risk under probability measure $\mathbf{Q}'$ and P_r is the first partial derivative of $P(t,T,r(t))$ with respect to $r(t)$, $\dfrac{\partial P}{\partial r}$.

$\mu_S(t)$ and $\sigma_S(t)$ are the drift term and the diffusion term respectively of stock price $S(t)$. $\chi(t)$ and $\gamma(t)$ are the processes of default density and convenience yield respectively, following the model settings in Longstaff et al. (2005).[9] $e-f\chi(t)$ and $\sigma_\chi\sqrt{\chi(t)}$ are the

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[8] One more benefit for matrix is that we could do some analyses on the coefficient matrix, e.g. eigenvalues and eigenvectors of the coefficient matrix.

[9] Different from Longstaff et al. (2005), η in our model could be negative. The regularity condition introduced later require η equalling $-\dfrac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}$. If we impose η to be positive, then $\rho_{r\gamma}$ has to be negative (positive) when $\gamma(t)$ is positive (negative) given $r(t)$ and $\theta(t)$ are positive in our numerical examples later in Section 4. The switch of sign of $\rho_{r\gamma}$ could be a further research question.

drift term and the diffusion term of $\chi(t)$ respectively. $B(t)r(t)$ is the diffusion term of $\gamma(t)$.

Based on equation (2), we could rewrite the dynamics of each process as follows.

$$\begin{bmatrix} dr(t) \\ dB(t) \\ dP(t,T,r(t)) \\ dS(t) \\ d\chi(t) \\ d\gamma(t) \end{bmatrix} = \begin{bmatrix} \alpha(t,r(t)) & \beta(t,r(t)) & 0 & 0 & 0 \\ B(t)r(t) & 0 & 0 & 0 & 0 \\ P(t,T,r(t))\tilde{\mu}(t,r(t)) & P(t,T,r(t))\tilde{\sigma}(t,r(t)) & 0 & 0 & 0 \\ S(t)\mu_S(t) & S(t)\sigma_S(t)\rho_{rS} & S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2} & 0 & 0 \\ e-f\chi(t) & \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)} & 0 \\ 0 & \eta\rho_{r\gamma} & \eta\rho''_{S\gamma} & \eta\rho''_{\chi\gamma} & \eta\rho''_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} dt \\ dW_r(t) \\ dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix} \quad (3)$$

2.2.2. General form of deflator

We are now able to derive the general form of deflator. First, let

$$dD(t) = \Omega(t)dt + \Phi(t)dW_r(t) + \Psi(t)dW_1(t) + \Gamma(t)dW_2(t) + \mathrm{I}(t)dW_3(t). \quad (4)$$

We would like to have $D(t)B(t)$, $D(t)P(t,T,r(t))$, $D(t)S(t)$, $D(t)\chi(t)$, and $D(t)\gamma(t)$ be $\mathbf{P}$-martingales.

By Itô product rule, we have $d[D(t)X(t)] = X(t)dD(t) + D(t)dX(t) + dX(t)dD(t)$ for a stochastic process $X(t)$. We derive $\Omega(t)$, $\Phi(t)$, $\Psi(t)$, $\Gamma(t)$, and $\mathrm{I}(t)$ step by step in Appendix 3 of Supplementary materials.

Let
$$\begin{cases} K_\Psi(t) = \dfrac{\Psi(t)}{D(t)} = \dfrac{r(t)+\theta(t)\sigma_S(t)\rho_{rS}-\mu_S(t)}{\sigma_S(t)\sqrt{1-\rho_{rS}^2}} \\ K_\Gamma(t) = \dfrac{\Gamma(t)}{D(t)} = \dfrac{\theta(t)\rho_{r\chi}}{\rho'_{\chi\chi}} + \dfrac{r(t)\chi(t)-e+f\chi(t)}{\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}} + \dfrac{\rho'_{S\chi}\left[\mu_S(t)-r(t)-\theta(t)\sigma_S(t)\rho_{rS}\right]}{\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}} \\ K_\mathrm{I}(t) = \dfrac{\mathrm{I}(t)}{D(t)} = \dfrac{\rho_{r\gamma}\theta(t)}{\rho''_{\gamma\gamma}} + \dfrac{r(t)\gamma(t)}{\eta\rho''_{\gamma\gamma}} - \dfrac{\rho''_{\chi\gamma}\rho_{r\chi}\theta(t)}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}} + \dfrac{\rho''_{\chi\gamma}\left[e-r(t)\chi(t)-f\chi(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_\chi\sqrt{\chi(t)}} \\ \qquad + \dfrac{\left(\rho''_{S\gamma}\rho'_{\chi\chi}-\rho''_{\chi\gamma}\rho'_{S\chi}\right)\left[\mu_S(t)-r(t)-\rho_{rS}\theta(t)\sigma_S(t)\right]}{\rho''_{\gamma\gamma}\rho'_{\chi\chi}\sigma_S(t)\sqrt{1-\rho_{rS}^2}} \end{cases}$$
, then

$$dD(t) = -D(t)r(t)dt - D(t)\theta(t)dW_r(t) + D(t)K_\Psi(t)dW_1(t) + D(t)K_\Gamma(t)dW_2(t) + D(t)K_\mathrm{I}(t)dW_3(t).$$

We have the general form of deflator $D(t)$ as follows.

$$D(t) = D(0)\mathbf{exp}\left\{-\int_0^t r(s)ds - \int_0^t \frac{1}{2}\left[\theta^2(s)+K_\Psi^2(s)+K_\Gamma^2(s)+K_\mathrm{I}^2(s)\right]ds\right\}$$
$$\times\mathbf{exp}\left[-\int_0^t \theta(s)dW_r(s) + \int_0^t K_\Psi(s)dW_1(s) + \int_0^t K_\Gamma(s)dW_2(s) + \int_0^t K_\mathrm{I}(s)dW_3(s)\right]$$

In addition, we require $\begin{cases} \mu_S(t) = r(t) + \theta(t)\sigma_S(t)\rho_{rS} \\ e = r(t)\chi(t) + f\chi(t) + \sigma_\chi \rho_{r\chi}\theta(t)\sqrt{\chi(t)} \\ \eta = -\dfrac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)} \end{cases}$ as regularity conditions, such that $\delta(t)S(t)$, $\delta(t)\chi(t)$, and $\delta(t)\gamma(t)$ are $\mathbf{Q}$-martingales (technical details are provided in Appendix 4 of Supplementary materials). As a result, $K_\Psi(t) = 0$, $K_\Gamma(t) = 0$, and $K_\mathrm{I}(t) = 0$. Then, $dD(t) = -D(t)r(t)dt - D(t)\theta(t)dW_r(t)$.[10]

We could rewrite the general form of deflator $D(t)$ as follows.[11]

$$dD(t) = -D(t)r(t)dt - D(t)\theta(t)dW_r(t) \tag{5}$$

$$D(t) = D(0)\mathbf{exp}\left[-\int_0^t r(s)ds - \frac{1}{2}\int_0^t \theta^2(s)ds - \int_0^t \theta(s)dW_r(s)\right] \tag{6}$$

[10] We will show later that one more regularity condition is required for the diffusion term of stock price in our model, but the deflator remains the same as shown in equation (5).

[11] The disappearance of $K_\Psi(t)$ term also tells us that stocks are financial derivatives. Given a rate of time value (growth) $r(t)$ and a rate of market risk $\theta(t)$, the proper expected return of a stock in our model is equal to $r(t) + \theta(t)\sigma_S(t)\rho_{rS}$. Also, the regularity conditions tell us more about the relations among interest rates, market prices of risk, stock prices, default intensities, and convenience yields in our model based on how we derive the deflator. For example, if we choose $\mathbf{exp}[\chi(t)]$ instead of $\chi(t)$ to derive the deflator, the regularity conditions will be different.

3. IMPLEMENTATION OF TIME DISCRETIZATION

From equations (3) and (5) with regularity conditions required in Section 2.2.2,

$$\begin{bmatrix} dr(t) \\ dB(t) \\ dP(t,T,r(t)) \\ dS(t) \\ d\chi(t) \\ d\gamma(t) \\ dD(t) \end{bmatrix} = \begin{bmatrix} \alpha(t,r(t)) & \beta(t,r(t)) & 0 & 0 & 0 \\ B(t)r(t) & 0 & 0 & 0 & 0 \\ P(t,T,r(t))\left[r(t)+\tilde{\sigma}(t,r(t))\theta(t)\right] & P(t,T,r(t))\tilde{\sigma}(t,r(t)) & 0 & 0 & 0 \\ S(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}\right] & S(t)\sigma_S(t)\rho_{rS} & S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2} & 0 & 0 \\ r(t)\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)} & \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)} & 0 \\ 0 & -\dfrac{\gamma(t)r(t)}{\theta(t)} & -\dfrac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}\rho''_{S\gamma} & -\dfrac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}\rho''_{\chi\gamma} & -\dfrac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}\rho''_{\gamma\gamma} \\ -D(t)r(t) & -D(t)\theta(t) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dt \\ dW_r(t) \\ dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}.$$

To implement the deflator approach, we need to discretize time steps for each process. We discuss the time discretization here. We adopt the Euler method, the Milstein method, and the simplified Second Milstein method for time discretization in our model.[12]

Denote a stochastic process $X(t)$ with its dynamics $dX(t)=b_X(t,X(t))dt+\sigma_X(t,X(t))dW_X(t)$ where $W_X(t)$ is the Brownian part of $X(t)$. We partition the time $[0,T]$ into N segments with each length equalling $(T-0)/N$, then we have a time discretization $\Pi_N=\Pi_N([0,T])$ with $0=t_0<t_1<\cdots<t_N=T$.

3.1. EULER METHOD

In Euler method, we approximate $X(t)$ by Y_t discretely, in which $Y_{i+1}=Y_i+b_X(t_i,Y_i)(t_{i+1}-t_i)+\sigma_X(t_i,Y_i)(W_{i+1}-W_i)$, $i=0,1,\ldots,N-1$, W_i is the value of a Brownian motion at time period i, and Y_0 is equal to $X(0)$. Denote $\Delta t_i=t_{i+1}-t_i=(T-0)/N$ and $\Delta W_{k,i}=W_{k,t_{i+1}}-W_{k,t_i}$, $k=r,1,2,3$.

We present the approximations of $r(t)$, $B(t)$, $P(t,T,r(t))$, $S(t)$, $\chi(t)$, $\gamma(t)$, $D(t)$ by the Euler method in matrix as follows.

$$\begin{bmatrix} r_{i+1} \\ B_{i+1} \\ P_{i+1} \\ S_{i+1} \\ \chi_{i+1} \\ \gamma_{i+1} \\ D_{i+1} \end{bmatrix} = \begin{bmatrix} r_i \\ B_i \\ P_i \\ S_i \\ \chi_i \\ \gamma_i \\ D_i \end{bmatrix} + \begin{bmatrix} \alpha_{t_i,r_i} & \beta_{t_i,r_i} & 0 & 0 & 0 \\ B_i r_i & 0 & 0 & 0 & 0 \\ P_i\left(r_i+\tilde{\sigma}_{t_i,r_i}\theta_i\right) & P_i\tilde{\sigma}_{t_i,r_i} & 0 & 0 & 0 \\ S_i\left(r_i+\theta_i\sigma_{S,t_i}\rho_{rS}\right) & S_i\sigma_{S,t_i}\rho_{rS} & S_i\sigma_{S,t_i}\sqrt{1-\rho_{rS}^2} & 0 & 0 \\ r_i\chi_i+\sigma_\chi\rho_{r\chi}\theta_i\sqrt{\chi_i} & \sigma_\chi\rho_{r\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi_i} & 0 \\ 0 & -\dfrac{\gamma_i r_i}{\theta_i} & -\dfrac{\gamma_i r_i}{\rho_{r\gamma}\theta_i}\rho''_{S\gamma} & -\dfrac{\gamma_i r_i}{\rho_{r\gamma}\theta_i}\rho''_{\chi\gamma} & -\dfrac{\gamma_i r_i}{\rho_{r\gamma}\theta_i}\rho''_{\gamma\gamma} \\ -D_i r_i & -D_i\theta_i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta t_i \\ \Delta W_{r,i} \\ \Delta W_{1,i} \\ \Delta W_{2,i} \\ \Delta W_{3,i} \end{bmatrix} \quad (7)$$

[12] For references of time discretization, see, for example, Kloeden and Platen (1992), Iacus (2009), and Glasserman (2013).

3.2. MILSTEIN METHOD

Denote $\sigma_{X'} = \dfrac{\partial \sigma_X(t,x)}{\partial x}$, we approximate $X(t)$ by Y_t discretely as

$$Y_{i+1} = Y_i + b_X(t_i, Y_i)(t_{i+1} - t_i) + \sigma_X(t_i, Y_i)(W_{i+1} - W_i) + \frac{1}{2}\sigma_X(t_i, Y_i)\sigma_{X'}(t_i, Y_i)\left[(W_{i+1} - W_i)^2 - (t_{i+1} - t_i)\right].$$

We present the approximations of $r(t)$, $B(t)$, $P(t,T,r(t))$, $S(t)$, $\chi(t)$, $\gamma(t)$, $D(t)$ by the Milstein method in matrix as follows.

$$\begin{bmatrix} r_{i+1} \\ B_{i+1} \\ P_{i+1} \\ S_{i+1} \\ \chi_{i+1} \\ \gamma_{i+1} \\ D_{i+1} \end{bmatrix} = \begin{bmatrix} r_i \\ B_i \\ P_i \\ S_i \\ \chi_i \\ \gamma_i \\ D_i \end{bmatrix} + \begin{bmatrix} \alpha_{t_i,r_i} & \beta_{t_i,r_i} & 0 & 0 & 0 \\ B_i r_i & 0 & 0 & 0 & 0 \\ P_i\left(r_i + \tilde{\sigma}_{t_i,r_i}\theta_i\right) & P_i\tilde{\sigma}_{t_i,r_i} & 0 & 0 & 0 \\ S_i\left(r_i + \theta_i\sigma_{S,t_i}\rho_{rS}\right) & S_i\sigma_{S,t_i}\rho_{rS} & S_i\sigma_{S,t_i}\sqrt{1-\rho_{rS}^2} & 0 & 0 \\ r_i\chi_i + \sigma_\chi\rho_{r\chi}\theta_i\sqrt{\chi_i} & \sigma_\chi\rho_{r\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi_i} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi_i} & 0 \\ 0 & -\dfrac{\gamma_i r_i}{\theta_i} & -\dfrac{\gamma_i r_i}{\rho_{r\gamma}\theta_i}\rho''_{S\gamma} & -\dfrac{\gamma_i r_i}{\rho_{r\gamma}\theta_i}\rho''_{\chi\gamma} & -\dfrac{\gamma_i r_i}{\rho_{r\gamma}\theta_i}\rho''_{\gamma\gamma} \\ -D_i r_i & -D_i\theta_i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta t_i \\ \Delta W_{r,i} \\ \Delta W_{1,i} \\ \Delta W_{2,i} \\ \Delta W_{3,i} \end{bmatrix}$$

$$+\frac{1}{4}\begin{bmatrix} 2\beta_{t_i,r_i}\beta_{r,t_i,r_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\tilde{\sigma}_{t_i,r_i}^2 & 0 & 0 & 0 \\ 2\sigma_{S,t_i}^2\rho_{rS}^2 & 2\sigma_{S,t_i}^2\left(1-\rho_{rS}^2\right) & 0 & 0 \\ \sigma_\chi^2\rho_{r\chi}^2 & \sigma_\chi^2\rho_{S\chi}'^2 & \sigma_\chi^2\rho_{\chi\chi}'^2 & 0 \\ 2\gamma_i\left(\dfrac{r_i}{\theta_i}\right)^2 & 2\gamma_i\left(\dfrac{\rho''_{S\gamma}r_i}{\rho_{r\gamma}\theta_i}\right)^2 & 2\gamma_i\left(\dfrac{\rho''_{\chi\gamma}r_i}{\rho_{r\gamma}\theta_i}\right)^2 & 2\gamma_i\left(\dfrac{\rho''_{\gamma\gamma}r_i}{\rho_{r\gamma}\theta_i}\right)^2 \\ 2D_i\theta_i^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\Delta W_{r,i}\right)^2 - \Delta t_i \\ \left(\Delta W_{1,i}\right)^2 - \Delta t_i \\ \left(\Delta W_{2,i}\right)^2 - \Delta t_i \\ \left(\Delta W_{3,i}\right)^2 - \Delta t_i \end{bmatrix} \quad (8)$$

3.3. SIMPLIFIED SECOND MILSTEIN METHOD

We advance to multi-dimensional case in this sub-section. Let X_t be multi-dimensional stochastic processes with the dynamics $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$, where X_t is a $d\times 1$ vector, $a(t, X_t)$ is a $d\times 1$ vector, $b(t, X_t)$ is a $d\times m$ matrix, and W_t is a $m\times 1$ vector. d is the number of different stochastic processes in X_t, and m is the number of independent Brownian motions involved in X_t.

For a continuously twice differentiable function $f(t, x_{d\times 1})$, we could write $df(t, X_t)$ by Itô formula for multi-dimensional case as follows.

$$df\left(t,X_t\right)=\left[\frac{\partial f\left(t,X_t\right)}{\partial t}+\sum_{i=1}^{d}\frac{\partial f\left(t,X_t\right)}{\partial x_i}a_i\left(t,X_t\right)+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 f\left(t,X_t\right)}{\partial x_i\partial x_j}\Sigma_{t,ij}\right]dt \\ +\sum_{i=1}^{d}\sum_{k=1}^{m}b_{ik}\left(t,X_t\right)\frac{\partial f\left(t,X_t\right)}{\partial x_i}dW_{t,k},\ \Sigma_t=b\left(t,X_t\right)b^T\left(t,X_t\right) \tag{9}$$

In equation 11, $a_i\left(t,X_t\right)$ is the element of i^{th} row of $a\left(t,X_t\right)$, $b_{ik}\left(t,X_t\right)$ is the element of $b\left(t,X_t\right)$ at its i^{th} row and k^{th} column, $b^T\left(t,X_t\right)$ is the transpose of $b\left(t,X_t\right)$, $\Sigma_{t,ij}$ is the element of Σ_t at its i^{th} row and j^{th} column, and $W_{t,k}$ is the element of k^{th} row of W_t. Next, we introduce operators L^0 and L^k and rewrite $df\left(t,X_t\right)$ for multi-dimensional case.

$$L^0=\frac{\partial}{\partial t}+\sum_{i=1}^{d}a_i\left(t,X_t\right)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d}\Sigma_{t,ij}\frac{\partial^2}{\partial x_i\partial x_j} \tag{10}$$

$$L^k=\sum_{i=1}^{d}b_{ik}\left(t,X_t\right)\frac{\partial}{\partial x_i},\ \forall k=1,\dots,m \tag{11}$$

$$df\left(t,X_t\right)=L^0 f\left(t,X_t\right)dt+\sum_{k=1}^{m}L^k f\left(t,X_t\right)dW_{t,k} \tag{12}$$

We approximate X_t by Y_t discretely by simplified Second Milstein method, where Y_t is a $d\times 1$ vector. For each $i=1,\dots,d$,

$$Y_{n+1,i}=Y_{n,i}+a_i\left(n,Y_n\right)\Delta t+\sum_{k=1}^{m}b_{ik}\left(n,Y_n\right)\Delta W_{n,k}+\frac{1}{2}L^0a_i\left(n,Y_n\right)\left(\Delta t\right)^2 \\ +\frac{1}{2}\sum_{k=1}^{m}\left[L^k a_i\left(n,Y_n\right)+L^0 b_{ik}\left(n,Y_n\right)\right]\Delta W_{n,k}\Delta t+\frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}L^j b_{ik}\left(n,Y_n\right)\left(\Delta W_{n,j}\Delta W_{n,k}-V_{jk}\right) \tag{13}$$

$Y_{n+1,i}$ is the element of i^{th} row of Y_t in the time step $n+1$. V_{jk} is an independent random variable with probabilities $\Pr\left(V_{jk}=\Delta t\right)=\Pr\left(V_{jk}=-\Delta t\right)=\frac{1}{2}$ for $j<k$, $V_{kj}=-V_{jk}$ for $j>k$, and $V_{jk}=\Delta t$ for $j=k$. The following are the X_t, $a\left(t,X_t\right)$, W_t, and $b\left(t,X_t\right)$ in our model.

$$X_t=\begin{bmatrix} r(t)\\ \theta(t)\\ B(t)\\ P\left(t,T,r(t)\right)\\ S(t)\\ \chi(t)\\ \gamma(t)\\ D(t)\end{bmatrix},\ a\left(t,X_t\right)=\begin{bmatrix}\alpha\left(t,r(t)\right)\\ a_\theta-b_\theta\theta(t)\\ B(t)r(t)\\ P\left(t,T,r(t)\right)\left[r(t)+\tilde{\sigma}\left(t,r(t)\right)\theta(t)\right]\\ S(t)\left[r(t)+\theta(t)\sigma_S(t)\rho_{rS}\right]\\ r(t)\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)}\\ 0\\ -r(t)D(t)\end{bmatrix},W_t=\begin{bmatrix}W_r(t)\\ W_1(t)\\ W_2(t)\\ W_3(t)\\ W_\theta(t)\end{bmatrix},$$

$$b(t,X_t)=\begin{bmatrix} \beta(t,r(t)) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\theta\sqrt{\theta(t)} \\ 0 & 0 & 0 & 0 & 0 \\ P(t,T,r(t))\tilde{\sigma}(t,r(t)) & 0 & 0 & 0 & 0 \\ S(t)\sigma_S(t)\rho_{rS} & S(t)\sigma_S(t)\sqrt{1-\rho_{rS}^2} & 0 & 0 & 0 \\ \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)} & \sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)} & 0 & 0 \\ -\frac{\gamma(t)r(t)}{\theta(t)} & -\frac{\rho''_{S\gamma}\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)} & -\frac{\rho''_{\chi\gamma}\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)} & -\frac{\rho''_{\gamma\gamma}\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)} & 0 \\ -\theta(t)D(t) & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. NUMERICAL RESULTS

We implement the deflator approach with three methods for time discretization and adopt CIR interest rate model for short-term saving.[13] In addition, we also incorporate parallel computing with a variance technique, antithetic sampling in our algorithm.[14] In CIR interest rate model, $dr(t)=\left[a_r-b_r r(t)\right]dt+\sigma_r\sqrt{r(t)}d\tilde{W}_r(t);\ a_r,b_r,\sigma_r>0$. The process of interest rate is defined under probability measure $\mathbf{Q}'$. To convert the process into physical measure $\mathbf{P}$, we have to consider the process of market price of risk $\theta(t)$. From Section 2.2.1, we let $d\tilde{W}_r(t)=\theta(t)dt+dW_r(t)$. Thus, we could rewrite $dr(t)$ in $\mathbf{P}$ -measure as $dr(t)=\left[a_r-b_r r(t)+\theta(t)\sigma_r\sqrt{r(t)}\right]dt+\sigma_r\sqrt{r(t)}dW_r(t)$.

Let $\theta(t)$ also be CIR process here and W_θ is an independent Brownian motion of $W_i,\ i=r,1,2,3$.[15] The dynamics of $\theta(t)$ is

$$d\theta(t)=\left[a_\theta-b_\theta\theta(t)\right]dt+\sigma_\theta\sqrt{\theta(t)}dW_\theta(t);a_\theta,b_\theta,\sigma_\theta>0.$$

In CIR interest rate model, the price of zero coupon bond of no risk with maturity T, $P(t,T,r(t))$, is equal to $e^{-r(t)C_P(t,T)-A_P(t,T)}$ where

$$C_P(t,T)=\frac{\sinh(\gamma_{CIR}(T-t))}{\gamma_{CIR}\cosh(\gamma_{CIR}(T-t))+\frac{1}{2}b_r\sinh(\gamma_{CIR}(T-t))},\ \gamma_{CIR}=\frac{1}{2}\sqrt{b_r^2+2\sigma_r^2},$$

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[13] Here we choose CIR interest rate model because the model has a closed-form formula for prices of zero-coupon bonds of no risk.

[14] The R codes are available from the authors by inquiry. For examples of computing time, user CPU time is 3.363 s, system CPU time is 0.286 s, and elapsed time is 15.747 s in 2500 simulations; user CPU time is 2828.413 s, system CPU time is 161.860 s, and elapsed time is 5127.524 s in 1000000 simulations.

[15] Here we choose $\theta(t)$ to be CIR process, so that $\theta(t)$ would be positive in any time period t.

and $A_P(t,T) = -\frac{2a_r}{\sigma_r^2}\ln\left[\frac{\gamma_{CIR}e^{\frac{1}{2}b_r(T-t)}}{\gamma_{CIR}\cosh\left(\gamma_{CIR}(T-t)\right)+\frac{1}{2}b_r\sinh\left(\gamma_{CIR}(T-t)\right)}\right]$.

Note that $\sinh u = \frac{e^u - e^{-u}}{2}$, $\cosh u = \frac{e^u + e^{-u}}{2}$, and $P(0,T,r(0)) = e^{-r(0)C_P(0,T)-A_P(0,T)}$.[16]

To calculate the option price for stock under CIR interest rate process analytically, we use the formula proposed by Kim (2002), which we leave the technical detail in Appendix 5 of Supplementary materials.[17] In addition, notice that the drift term of stock prices in Kim (2002) is a constant, which is different from our model in which $\mu_S(t) = r(t) + \theta(t)\sigma_S(t)\rho_{rS}$. Thus, our numerical results could be different from the value suggested by the formula in Kim (2002).

We present the approximations of $r(t)$, $\theta(t)$, $P(t,T,r(t))$ in matrix by the Euler method and the Milstein method as follows.

First, we rewrite the dynamics of $r(t)$, $\theta(t)$, $P(t,T,r(t))$ in matrix.

$$\begin{bmatrix} dr(t) \\ d\theta(t) \\ dP(t,T,r(t)) \end{bmatrix} = \begin{bmatrix} a_r - b_r r(t) + \theta(t)\sigma_r\sqrt{r(t)} & \sigma_r\sqrt{r(t)} & 0 \\ a_\theta - b_\theta\theta(t) & 0 & \sigma_\theta\sqrt{\theta(t)} \\ P(t,T,r(t))r(t) + P_r\sigma_r\sqrt{r(t)}\theta(t) & P_r\sigma_r\sqrt{r(t)} & 0 \end{bmatrix} \begin{bmatrix} dt \\ dW_r(t) \\ dW_\theta(t) \end{bmatrix} \quad (14)$$[18]

Then, $\begin{bmatrix} r_{i+1} \\ \theta_{i+1} \\ P_{i+1} \end{bmatrix} = \begin{bmatrix} r_i \\ \theta_i \\ P_i \end{bmatrix} + \begin{bmatrix} a_r - b_r r_i + \theta_i\sigma_r\sqrt{r_i} & \sigma_r\sqrt{r_i} & 0 \\ a_\theta - b_\theta\theta_i & 0 & \sigma_\theta\sqrt{\theta_i} \\ P_i r_i + P_{r,t_i}\sigma_r\sqrt{r_i}\theta_i & P_{r,t_i}\sigma_r\sqrt{r_i} & 0 \end{bmatrix} \begin{bmatrix} \Delta t_i \\ \Delta W_{r,i} \\ \Delta W_{\theta,i} \end{bmatrix}$ by the Euler method;

and $\begin{bmatrix} r_{i+1} \\ \theta_{i+1} \\ P_{i+1} \end{bmatrix} = \begin{bmatrix} r_i \\ \theta_i \\ P_i \end{bmatrix} + \begin{bmatrix} a_r - b_r r_i + \theta_i\sigma_r\sqrt{r_i} & \sigma_r\sqrt{r_i} & 0 \\ a_\theta - b_\theta\theta_i & 0 & \sigma_\theta\sqrt{\theta_i} \\ P_i r_i + P_{r,t_i}\sigma_r\sqrt{r_i}\theta_i & P_{r,t_i}\sigma_r\sqrt{r_i} & 0 \end{bmatrix} \begin{bmatrix} \Delta t_i \\ \Delta W_{r,i} \\ \Delta W_{\theta,i} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\Delta W_{r,i})^2 - \Delta t_i \\ (\Delta W_{\theta,i})^2 - \Delta t_i \end{bmatrix}$

by the Milstein method.

The details of implementation of simplified Second Milstein method is provided in the Appendix 6 of Supplementary materials.

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[16] See, for example, Shreve (2004) Chapter 6.

[17] For references of option pricing under stochastic interest rates, see, for example, Shreve (2004) Chapter 9, and Brigo and Mercurio (2006) Chapter 3 and Appendix B.

[18] For process of $P(t,T,r(t))$, plug $\tilde{\sigma}(t,r(t)) = \frac{P_r\beta(t,r(t))}{P(t,r(t))}$ and $\beta(t,r(t)) = \sigma_r\sqrt{r(t)}$ in equation (3). Note that $P_r = -C_P(t,T)e^{-r(t)C_P(t,T)-A_P(t,T)}$ in CIR interest rate model.

4.1. AN NUMERICAL EXAMPLE WITH CIR MODEL

The following is the settings describing the dynamics of each process in our example.[19]

$$\begin{cases} d\theta(t)=\left[0.05-0.01\theta(t)\right]dt+0.01\sqrt{\theta(t)}dW_\theta(t),\ \theta(0)=0.3 \\ dr(t)=\left[0.02-0.04r(t)\right]dt+0.01\sqrt{r(t)}d\tilde{W}_r(t),\ r(0)=0.02 \\ dS(t)=S(t)\left[r(t)+0.2\rho_{rS}\theta(t)\right]dt+0.2S(t)dW_S(t),\ S(0)=1 \\ d\chi(t)=\left[r(t)\chi(t)+0.01\rho_{r\chi}\theta(t)\sqrt{\chi(t)}\right]dt+0.01\sqrt{\chi(t)}dW_\chi(t),\ \chi(t)=0.05 \\ d\gamma(t)=-\dfrac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}dW_\gamma(t),\ \gamma(0)=0.01 \end{cases}$$

$$T=1,\ \Delta t=0.01,\ \rho_{rS}=0.6,\ \rho_{r\chi}=0.7,\ \rho_{r\gamma}=0.5,\ \rho_{S\chi}=0.1,\ \rho_{S\gamma}=0.3,\ \rho_{\chi\gamma}=0.1$$

Let $D(0)$ equal 1 in equation (10). The deflator approach tells us that for a nonnegative random variable $X(t)$, we would have $E^{\mathbf{Q}}\left[\delta(t)X(t)\right]=E\left[D(t)X(t)\right]$.

4.1.1. Zero-coupon bond of no risk with maturity T

The price of a zero-coupon bond of no risk with maturity T at time period T is equal to 1.

$$D(0)P(0,T,r(0))=P(0,T,r(0))=E^{\mathbf{Q}}\left[\delta(T)P(T,T,r(T))\right]=E\left[D(T)P(T,T,r(T))\right]=E\left[D(T)\right] \tag{15}$$

Tables 1 shows the numerical results. Figures 2, 3, 4, 5, 6, 7, 8, and 9 show the convergence of approximations to expected values, i.e. $P(0,T,r(0))$, and the differences between approximations and expected values. In general, we could see that the simplified Second Milstein method provides better approximations and converges faster than the Euler method and the Milstein method do. This could be explained by convergence order in which the simplified Second Milstein method has larger weak order of convergence.[20]

4.1.2. Corporate coupon bond

Longstaff et al. (2005) assumed the independence among interest rate, default intensity, and convenience yield. Thus, we let $\rho_{r\chi}=0$, $\rho_{r\gamma}=0$, $\rho_{S\chi}=0$, $\rho_{S\gamma}=0$, and $\rho_{\chi\gamma}=0$.[21] To accommodate the three risk factors (interest rate, default intensity, and convenience

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[19] Here we provide a numerical example for the model, in which the chosen values for model settings could be different. In our example, there are strong positive correlations between interest rates and other factors (i.e. stock prices, default densities, and convenience yields), but weak positive correlations between each two of stock prices, default densities, and convenience yields. Note that the Feller condition holds in our numerical examples, e.g. $2\times 0.05>0.01^2$.

[20] The Euler method and the Milstein method have weak order of convergence 1, and the simplified Second Milstein method has weak order of convergence 2, see, for example, Glasserman (2013) Chapter 6.

[21] We let $dW_i dW_j=0$ here, i.e. pairwise independence.

yield) with deflator, we let $dB(t)=B(t)\left[r(t)+\chi(t)+\gamma(t)\right]dt$.[22] In addition, notice again that the formula provided in Longstaff et al. (2005) is not directly applicable after we require regularity conditions in our model, which we leave technical detail of the formula in Longstaff et al. (2005) in Appendix 7 of Supplementary materials.

To implement the deflator, we look at the original definition of $CB(c,\omega,T)$.

$$\begin{aligned} CB(c,\omega,T)=E\left\{c\int_0^T \exp\left[-\int_0^t\left(r(s)+\chi(s)+\gamma(s)\right)ds\right]dt\right\}+E\left\{\exp\left[-\int_0^T\left(r(s)+\chi(s)+\gamma(s)\right)ds\right]\right\} \\ +E\left\{(1-\omega)\int_0^T \chi_t \exp\left[-\int_0^t\left(r(s)+\chi(s)+\gamma(s)\right)ds\right]dt\right\} \end{aligned} \tag{16}$$

For the time period t when a bond holder receives a coupon or a fraction of the par value of the bond (because of default), the payoff at that time period t is equal to c or $(1-\omega)$ multiply the par value of the bond respectively. Thus, we could implement the deflator as follows, the details of implementation of time discretization is provided in Appendix 8 of Supplementary materials.

$$D(0)CB(c,\omega,T)=E\left[D(T)\right]+cE\left[\int_0^T D(t)dt\right]+(1-\omega)E\left[\int_0^T \chi(t)D(t)dt\right] \qquad (17)^{23}$$

Tables 2 shows the numerical results. Figures 10, 11, 12, and 13 show the convergence of approximations and the differences between approximations and expected values.

4.2. ONE MORE REGULARITY CONDITION REQUIRED FOR THE DIFFUSION TERM IN STOCK PRICE

Up to Section 4.1, we successfully implement the deflator approach for zero-coupon bond of no risk with maturity T and corporate coupon bond. However, we notice that one more regularity condition is required for the diffusion term in stock price. As illustrative examples, Figures 14 and 15 show the instability of the deflator approach corresponding to stock price in Second Milstein method with 10000 simulations after projecting longer than 15 years.

Recall that we would have $E^{\mathbf{Q}}\left[\delta(t)X(t)\right]=E\left[D(t)X(t)\right]$ for a nonnegative random variable $X(t)$. We calculate the price of Put option of $S(T)$ with strike K equalling to 2, and expect the following equations to hold.

$$D(0)S(0)=S(0)=E^{\mathbf{Q}}\left[\delta(T)S(T)\right]=E\left[D(T)S(T)\right]=1 \tag{18}$$

$$D(0)Put(0,S(0),T,K)=Put(0,S(0),T,K)=E^{\mathbf{Q}}\left[\delta(T)\left(K-S(T)\right)^+\right]=E\left[D(T)\left(K-S(T)\right)^+\right] \tag{19}$$

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[22] Recall that $D(t)=Discount\ factor\cdot\frac{d\mathbf{Q}}{d\mathbf{P}}$, $D(t)$ could not be the same given different discount factors with the same Radon-Nikodym derivative, i.e. different discount factors imply different risk-neutral worlds.

[23] We approximate $\int_0^T D(t)dt$ by $\sum_{t_i=0}^{T}\frac{\left(D_{t_i}+D_{t_{i+1}}\right)}{2}\Delta t_i$; similarly, we approximate $\int_0^T \chi(t)D(t)dt$ by $\sum_{t_i=0}^{T}\frac{\left(\chi_{t_i}D_{t_i}+\chi_{t_{i+1}}D_{t_{i+1}}\right)}{2}\Delta t_i$.

From equations (6) and (18), we derive one more regularity condition $\sigma_S(t) = \rho_{rS}\theta(t) \pm \theta(t)\sqrt{\rho_{rS}^2 - 1}$, technical details are provided in Appendix 9 of Supplementary materials.

We could see that $\sigma_S(t)$ is a complex number if $\rho_{rS} \neq 1$ (so that $|\rho_{rS}| < 1$). We choose ρ_{rS} equalling 1 here as an example, then $\sigma_S(t)$ is equal to $\theta(t)$. Then, we could reduce the matrix form of equation (2) as follows.

$$\mathbf{W}_{ESG} = \begin{bmatrix} W_r \\ W_S \\ W_\chi \\ W_\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \rho_{r\chi} & \sqrt{1-\rho_{r\chi}^2} & 0 \\ \rho_{r\gamma} & \rho'_{\chi\gamma} & \rho'_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} W_0 \\ W_2 \\ W_3 \end{bmatrix}, \tag{20}$$

$$\text{where } \rho'_{\chi\gamma} = \frac{\rho_{\chi\gamma} - \rho_{r\chi}\rho_{r\gamma}}{\sqrt{1-\rho_{r\chi}^2}}, \ \rho'_{\gamma\gamma} = \sqrt{\frac{1-\rho_{r\chi}^2 - \rho_{r\gamma}^2 - \rho_{\chi\gamma}^2 + 2\rho_{r\chi}\rho_{r\gamma}\rho_{\chi\gamma}}{1-\rho_{r\chi}^2}}$$

From equations (3) and (5) with regularity conditions required in Section 2.2.2 and here (i.e. $\sigma_S(t) = \theta(t)$),

$$\begin{bmatrix} dr(t) \\ d\theta(t) \\ dB(t) \\ dP(t,T,r(t)) \\ dS(t) \\ d\chi(t) \\ d\gamma(t) \\ dD(t) \end{bmatrix} = \begin{bmatrix} \alpha(t,r(t)) & \beta(t,r(t)) & 0 & 0 & 0 \\ a_\theta - b_\theta\theta(t) & 0 & 0 & 0 & \sigma_\theta\sqrt{\theta(t)} \\ B(t)r(t) & 0 & 0 & 0 & 0 \\ P(t,T,r(t))\left[r(t)+\tilde{\sigma}(t,r(t))\theta(t)\right] & P(t,T,r(t))\tilde{\sigma}(t,r(t)) & 0 & 0 & 0 \\ S(t)\left[r(t)+\theta^2(t)\right] & S(t)\theta(t) & 0 & 0 & 0 \\ r(t)\chi(t)+\sigma_\chi\rho_{r\chi}\theta(t)\sqrt{\chi(t)} & \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} & \sigma_\chi\sqrt{\left(1-\rho_{r\chi}^2\right)\chi(t)} & 0 & 0 \\ 0 & -\frac{\gamma(t)r(t)}{\theta(t)} & -\frac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}\rho'_{\chi\gamma} & -\frac{\gamma(t)r(t)}{\rho_{r\gamma}\theta(t)}\rho'_{\gamma\gamma} & 0 \\ -r(t)D(t) & -D(t)\theta(t) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dt \\ dW_r(t) \\ dW_2(t) \\ dW_3(t) \\ dW_\theta(t) \end{bmatrix}.$$

Figures 16, 17, 18, and 19 show the numerical results of Second Milstein method with 10000 simulations after projecting 100 years.[24]

4.3. Discussions

Given the variance of a random variable X, $Var(X)$, the variance of $\frac{1}{n}X$ is equal to $\frac{1}{n^2}Var(X)$. Suppose the risk factors and parameters involved are constant at time period t, $D(t)$ is lognormal distributed. With the sample size being equal to n, the mean of

[24] Note that the formulas in Longstaff et al. (2005) and Kim (2002) are not applicable for longer periods, i.e. $C_{CB}(t) = \exp\left(\frac{\eta^2 t^3}{6}\right)$ in Longstaff et al. (2005) and the square root term $\sqrt{r_0 - \theta_{Kim}\left(1-e^{-\kappa_{Kim}T}\right)}$ of C_{11} are not computable when t is larger.

$D(t)$, $E\left[D(t)\right]$, equals $\frac{1}{n}\sum_{n\ trials} D(t)$; and its variance $Var\left(E\left[D(t)\right]\right)$ is equal to $\frac{1}{n}Var\left(D(t)\right)$. We could calculate its 95% confidence interval as follows.[25]

$$CI_{95\%} = E\left[D(t)\right] + \frac{Var\left(E\left[D(t)\right]\right)}{2} \pm t_{d.f.=n-1}\sqrt{\frac{Var\left(E\left[D(t)\right]\right)}{n} + \frac{\left[Var\left(E\left[D(t)\right]\right)\right]^2}{2(n-1)}} \quad (21)$$

Here $t_{d.f.=n-1}$ is the t statistics with degree of freedom equalling $n-1$. For example, in our numerical results of Second Milstein method with antithetic sampling and sample size equalling 2500, the 95% confidence interval of $E\left[D(T)\right]$ is equal to $\left[0.9714838, 0.9718523\right]$.

Suppose the weights of investment in a portfolio on stock, zero coupon bond of no risk with maturity T, and corporate coupon bond equal w_S, w_P, and w_{CB} respectively. Theoretically, the variance of the portfolio is equal to $\sum_{i=S,P,CB} w_i^2 Var(i) + 2\sum_{j,k=S,P,CB;\,j\neq k} w_j w_k Cov(j,k)$, where $Cov(j,k)$ is the covariance between j and k. Given stochastic differential equations of two normalized stochastic processes dX and dY, we could calculate $Cov(X,Y)$ by $dXdY$. The multiplication of lognormal random variables is again lognormal distributed, and the sum of lognormal random variables most likely behaves as either normal or lognormal distributions (so that we could still calculate the confidence interval).[26] As a numerical example, we let w_S, w_P, and w_{CB} be 0.15, 0.65, and 0.2 respectively. Figure 20 shows the comparison of histograms with/without antithetic sampling of the portfolio.

However, the risk factors and parameters involved are not constant. For example, $r(t)$, $\theta(t)$ and $\chi(t)$ in our numerical examples are not constant. In addition, $r(t)$ rises sharply over long time periods as we could see in Figures 21. By switching the coefficients in the drift term of $\theta(t)$, $0.01-0.05\theta(t)$, we could alleviate this situation observed in Figure 22. In Appendix 10 of Supplementary materials, we show that the mean and variance of the interest rate process $r(t)$ behave like the mean and variance of a CIR process asymptotically. In addition, we observe negative values of $D(t)$ while implementing time discretization over long time periods, which could result from discretization bias and no differentiability of Brownian motion.[27]

We provide one more example related to insurance contract. From Bonnin et al. (2014), the flow of benefits for a saving contract $\Lambda(\tau)$ at time τ is equal to $VR(\tau\wedge T)\delta(\tau\wedge T)$, in which $\tau\wedge T$ is the minimum of τ and T, $VR(t)$ is the value of a saving contract with its

[25] See, for example, Olsson (2005).
[26] See, for example, Dufresne (2004), Lo (2012), and Gulisashvili and Tankov (2016).
[27] See, for example, Glasserman (2013) Chapter 6.3.3, and Mörters and Peres (2010) Chapter 1.3.

instantaneous accumulation rate $r_s(t)$ at time t, and $\delta(t)$ is the discount factor. Figure 23 shows the expected value of $\Lambda(\tau)$ at each year τ, denote $E\left[\Lambda(\tau)|\tau\right]$. Suppose τ is uniform distributed at time interval $(0,T)$, then we could calculate the best estimated of a saving contract $BEL(0,T)$ as average of the expected value of $\Lambda(\tau)$ at each year τ, $BEL(0,T) \approx \frac{1}{T}\sum_{\tau=1}^{T} E\left[\Lambda(\tau)|\tau\right]$. In our example here, $BEL(0,T) \approx 0.1490473$.

Further study would be to investigate the situation when the diffusion term in stock price is a complex number and when the observed estimated processes are deviated from the processes with required regularity conditions.

5. Conclusion

In this paper, we derive the general form of deflator for four risk factors: interest rates, stock prices, default intensities, and convenience yields and then we find the regularity conditions for the deflator. We examine the deflator with different financial derivatives, comparing the numerical results with values calculated from closed-form formulas. We find required regularity conditions to implement our stochastic deflator. Our results indicate the reliability in statistics of the deflator for financial asset pricing.

Except the benefit that we could compute best estimate value by simply averaging the multiplication of deflator and projected cash flows, the fact that we observe data only in physical world would provide the motivation for us to use deflator for the convenience to estimate parameters of “reaction functions” in an ALM projection model as in Chapter 4 of Laurent et al. (2016).

Further work would be to compare the best estimate values of a life insurance contract by the deflator approach under physical measure and risk-neutral measure. More importantly, how to handle the situation when the diffusion term in stock price is a complex number and when the observed estimated processes are deviated from the processes with required regularity conditions would be further research directions.

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7. TABLES AND FIGURES

Tab. 1. Zero coupon bond of no risk with maturity T

# of Simulations	$E[D(T)]$	$E[D(T)P(T,T,r(T))]$	$P(0,T,r(0))$
Euler method			
2500	0.967646653771768	0.967552873529761	
5000	0.968545566976094	0.968451758793272	
10000	0.971055574970586	0.970961666405972	
100000	0.969964099445665	0.969870204097554	0.970957220487724
250000	0.970861713011257	0.970767802364827	
500000	0.970697882733399	0.970603984186568	
1000000	0.971001216710056	0.970907309007362	
Milstein method			
2500	0.969203041882130	0.969109232549717	
5000	0.968364554353440	0.968270761261484	
10000	0.972015353313569	0.971921436693164	
100000	0.969796855984494	0.969702984763468	0.970957220487724
250000	0.970693248836965	0.970599353426790	
500000	0.970367620896040	0.970273749033845	
1000000	0.970880147642130	0.970786256114490	
Second Milstein method			
2500	0.971985461477127	0.971985482351078	
5000	0.973386426354929	0.973386494993151	
10000	0.970979266245196	0.970979241795446	
100000	0.972928673921800	0.972928710489062	0.970957220487724
250000	0.970544559568911	0.970544535585851	
500000	0.970579400873414	0.970579381127470	
1000000	0.970630578417248	0.970630560272500	

Tab. 2. Corporate coupon bond

# of Simulations	Deflator	Longstaff et al. (2005)
Euler method		
2500	1.03001393884536	
5000	1.02907100520235	
10000	1.03312948158384	
100000	1.03162033904634	1.03313616115971
250000	1.03255241573140	
500000	1.03237954508927	
1000000	1.03271899062813	
Milstein method		
2500	1.02572713722014	
5000	1.02914390668280	
10000	1.03349796386179	
100000	1.03135227993609	1.03313616115971
250000	1.03247145621524	
500000	1.03214895930589	
1000000	1.03261933018597	
Second Milstein method		
2500	1.03584708542322	
5000	1.03243391095283	
10000	1.03330315641903	
100000	1.03468566679080	1.03313616115971
250000	1.03243619762781	
500000	1.03234018373030	
1000000	1.03229088096527	

Fig. 2 - **Zero coupon bond,** $E\left[D(T)\right]$

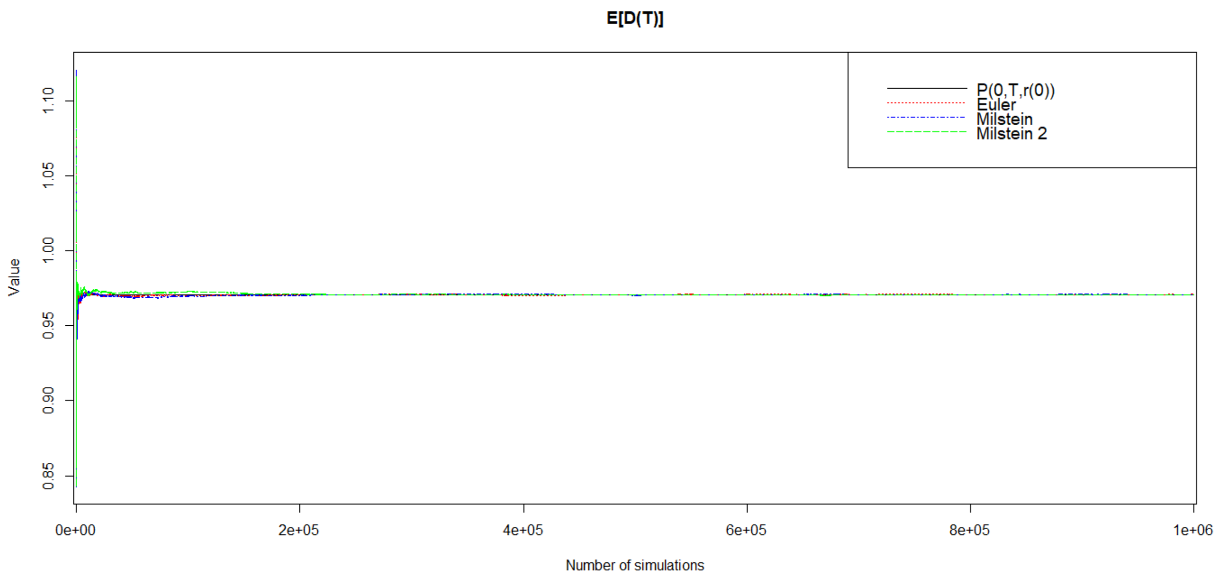

Fig. 3 - **Differences between approximation and expected value of Zero coupon bond,** $E\left[D(T)\right]$

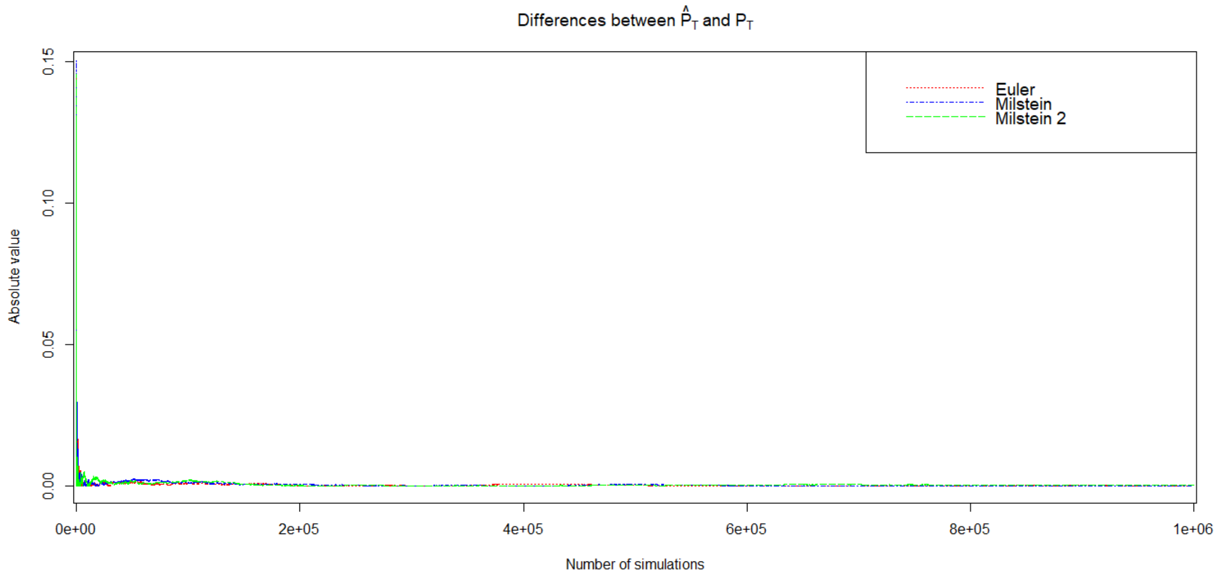

Fig. 4 - **Zero coupon bond,** $E\left[D(T)\right]$**, number of simulations less than 10000**

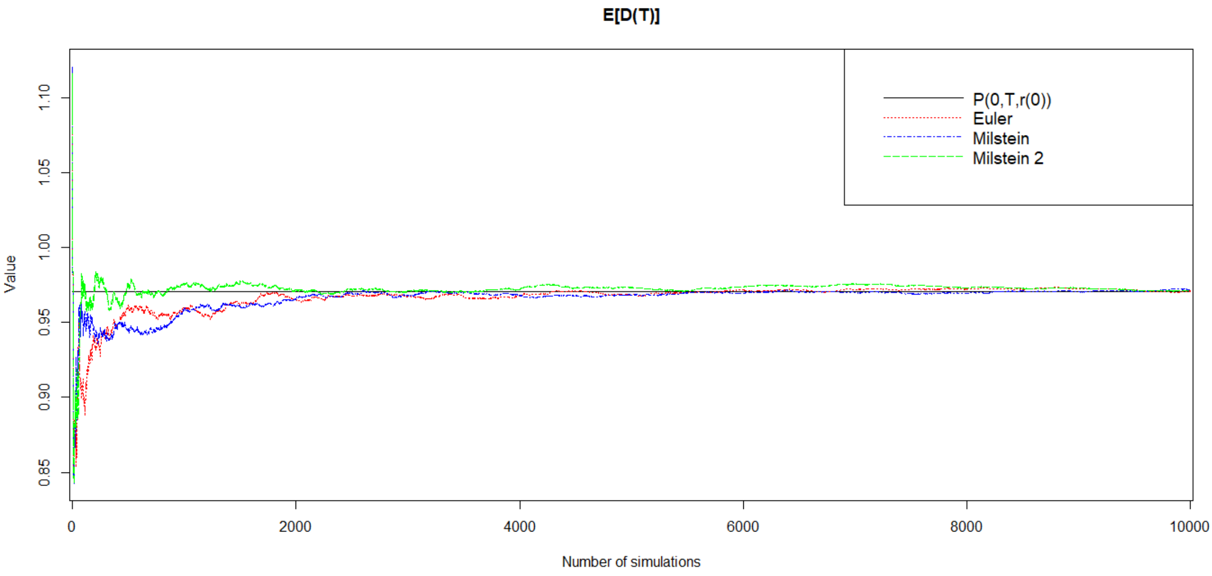

Fig. 5 - **Differences between approximation and expected value of Zero coupon bond,** $E\left[D(T)\right]$**, number of simulations less than 10000**

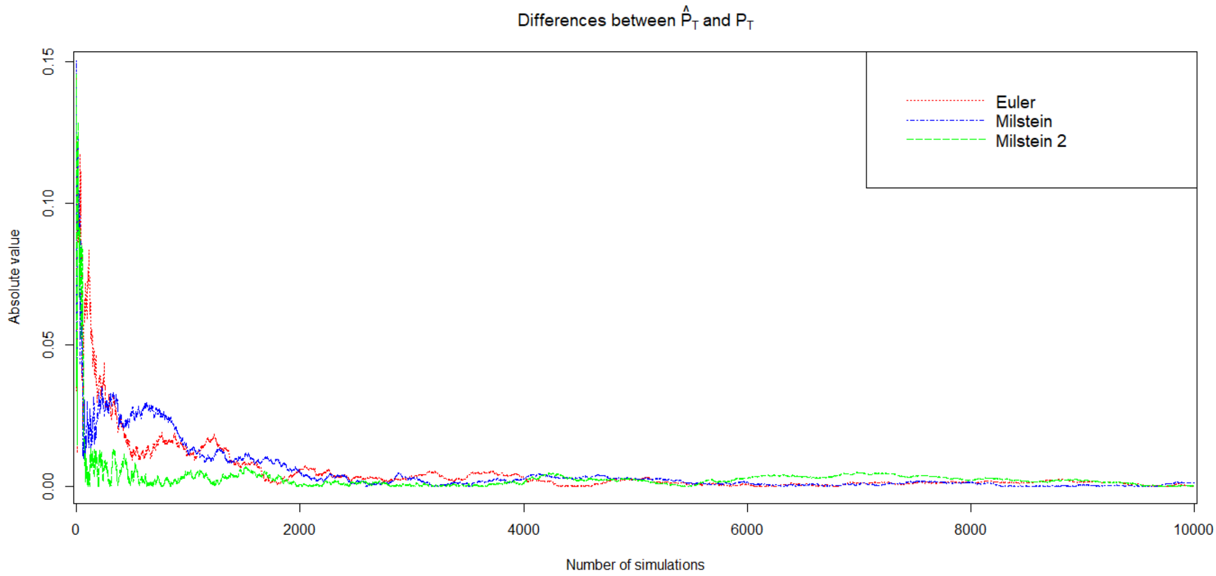

Fig. 6 - **Zero coupon bond,** $E\left[D(T)P\left(T,T,r(T)\right)\right]$

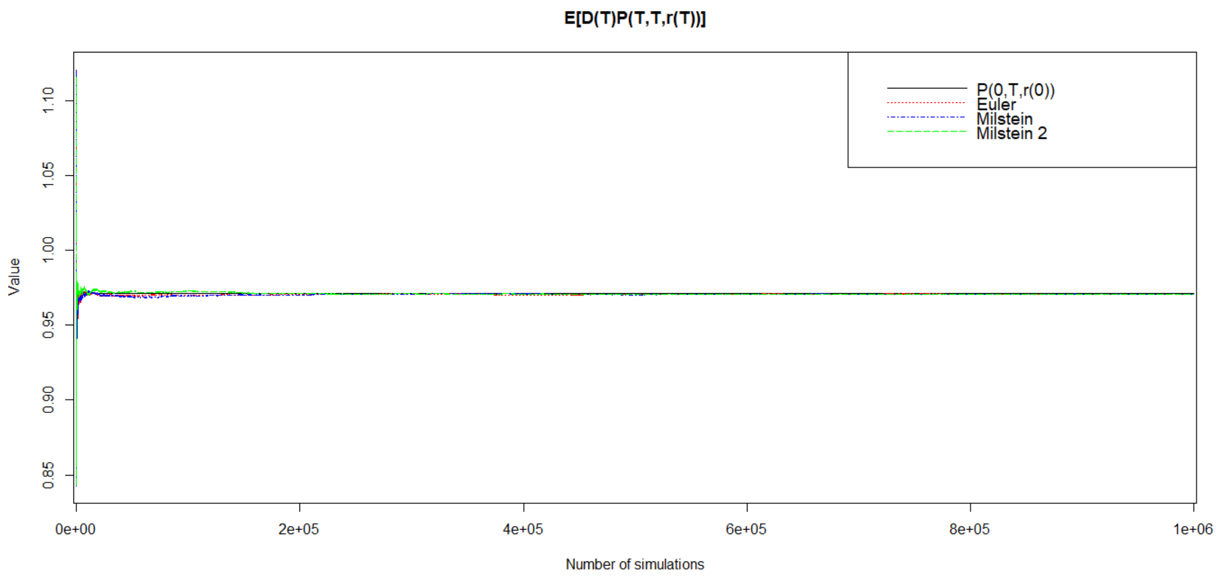

Fig. 7 - **Differences between approximation and expected value of Zero coupon bond,** $E\left[D(T)P\left(T,T,r(T)\right)\right]$

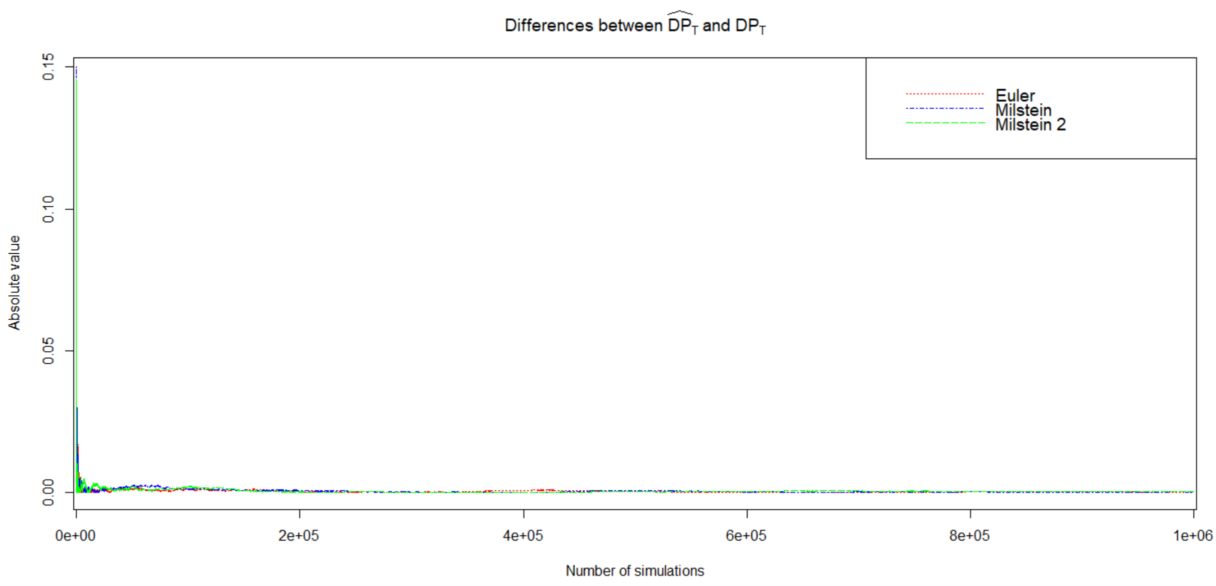

Fig. 8 - **Zero coupon bond,** $E\left[D(T)P(T,T,r(T))\right]$**, number of simulations less than 10000**

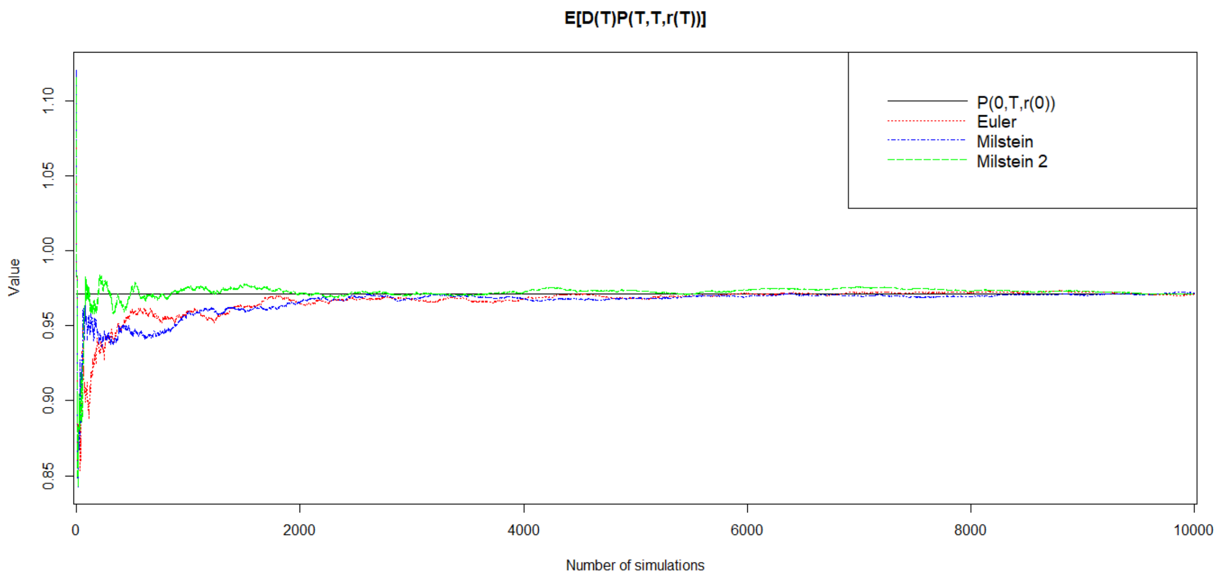

Fig. 9 - **Differences between approximation and expected value of Zero coupon bond,** $E\left[D(T)P(T,T,r(T))\right]$**, number of simulations less than 10000**

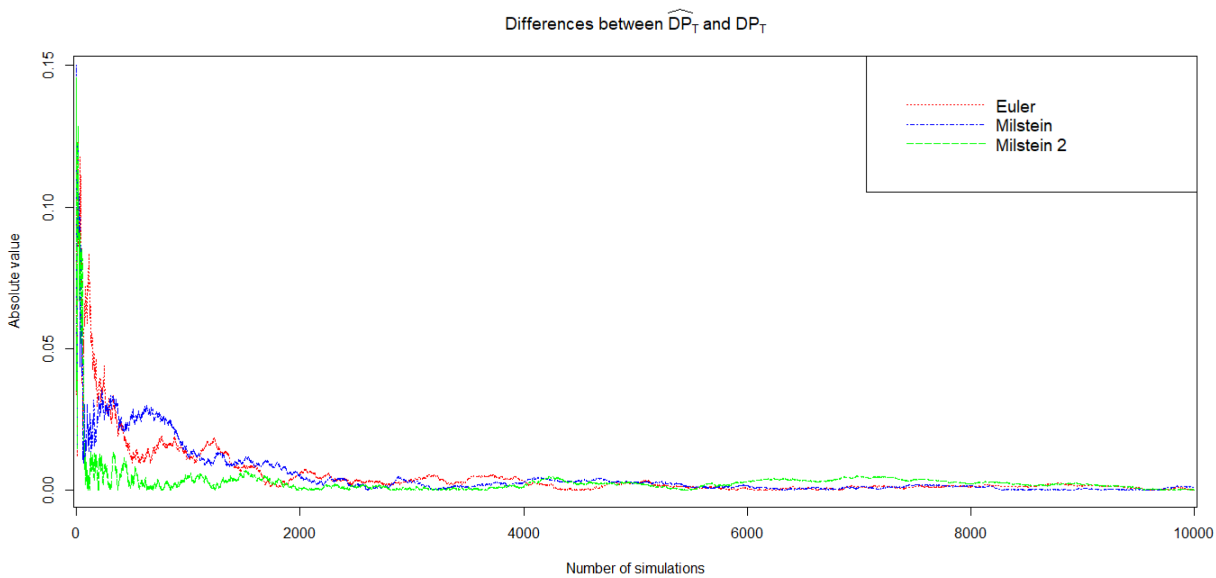

Fig. 10 - **Corporate coupon bond**

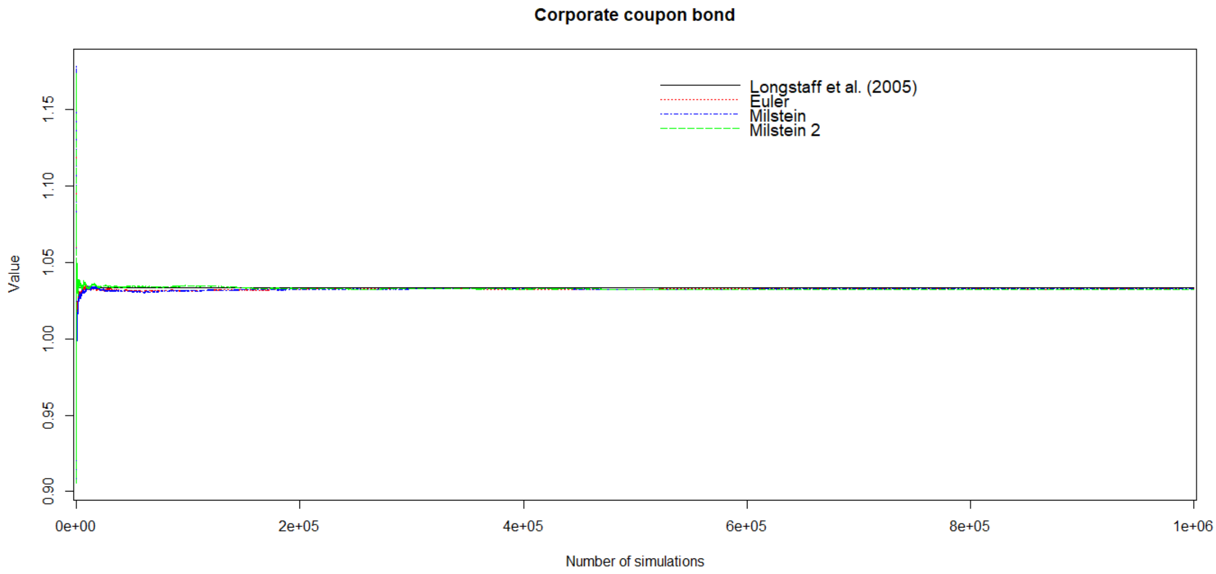

Fig. 11 - **Differences between approximation and expected value of Corporate coupon bond in Longstaff et al. (2005)**

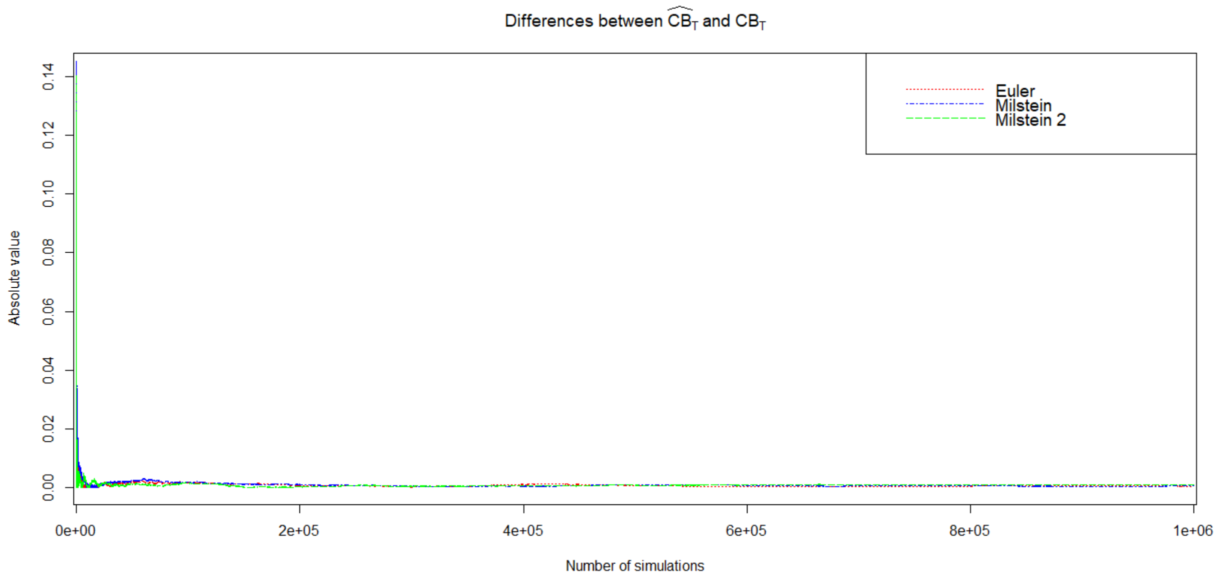

Fig. 12 - **Corporate coupon bond, number of simulations less than 10000**

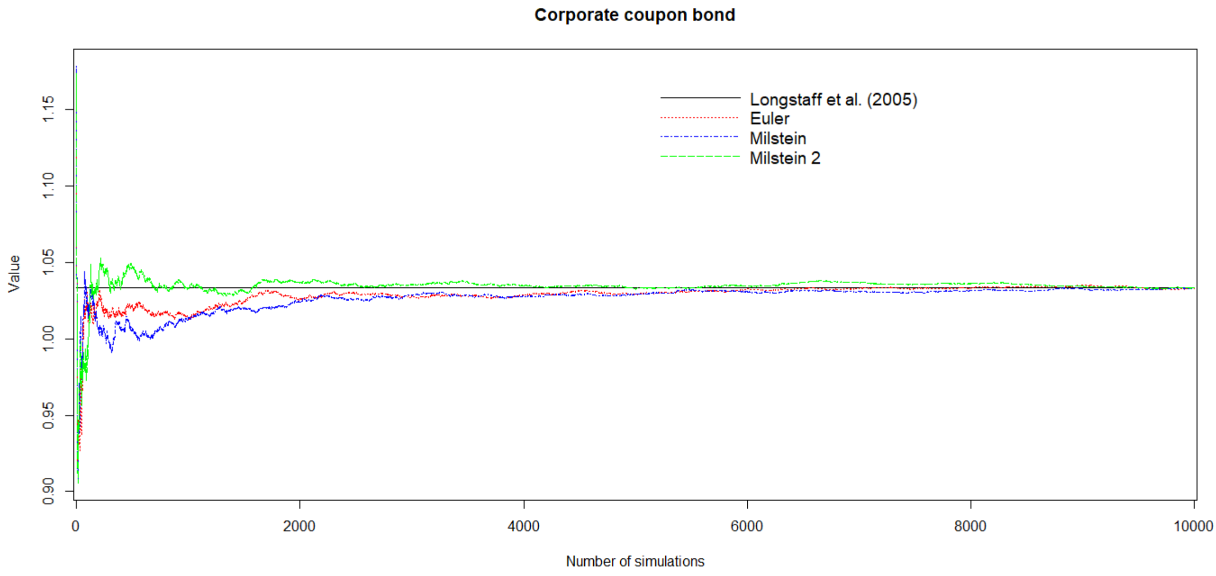

Fig. 13 - **Differences between approximation and expected value of Corporate coupon bond in Longstaff et al. (2005), number of simulations less than 10000**

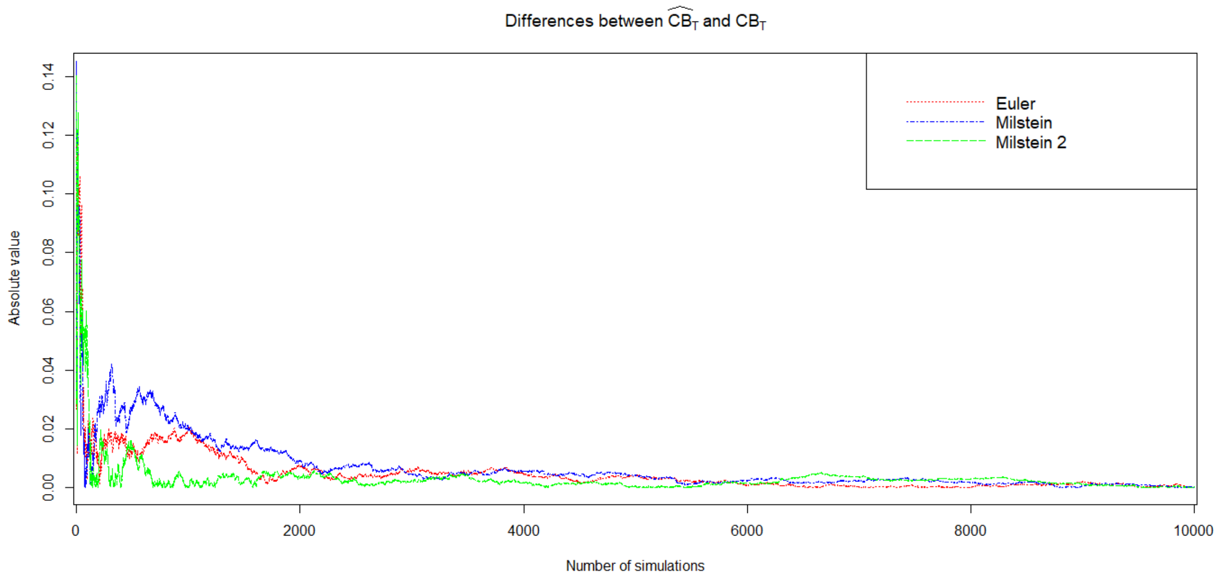

Fig. 14 - **Deflator multiplies stock over long time periods**

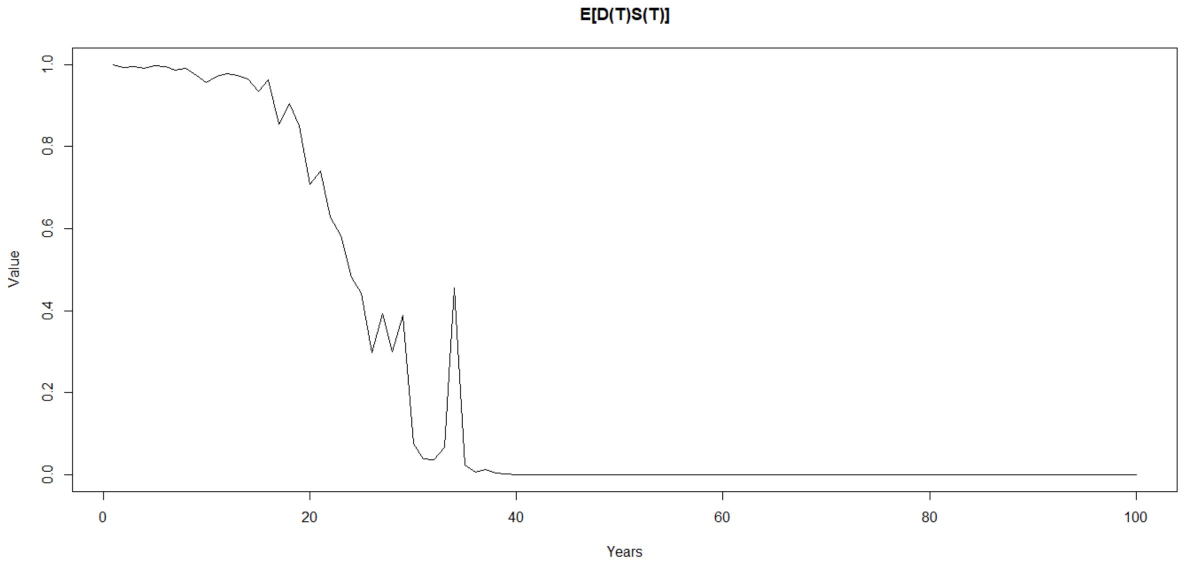

Fig. 15 - **Deflator multiplies stock over long time periods, 16 years**

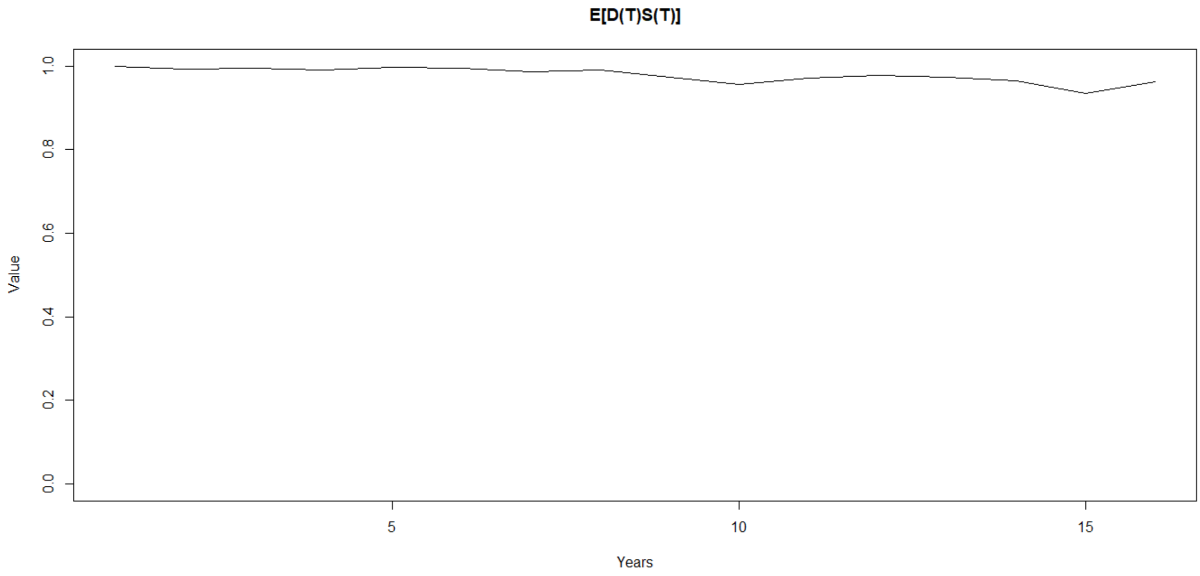

Fig. 16 - **Deflator multiplies stock over long time periods, 16 years**

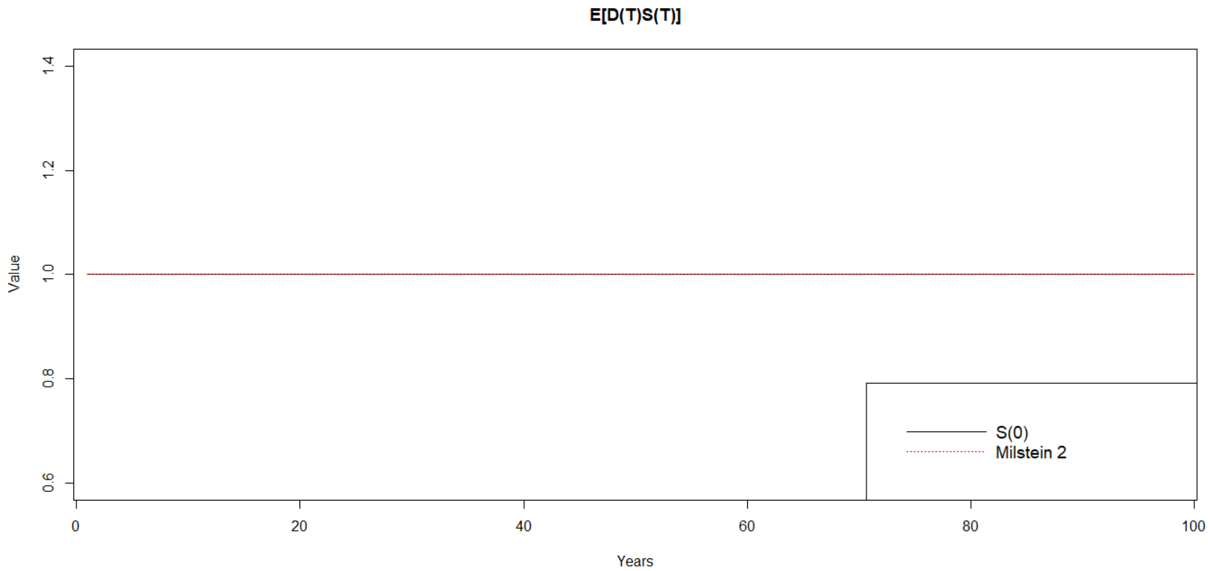

Fig. 17 - **Deflator multiplies stock over long time periods, 16 years**

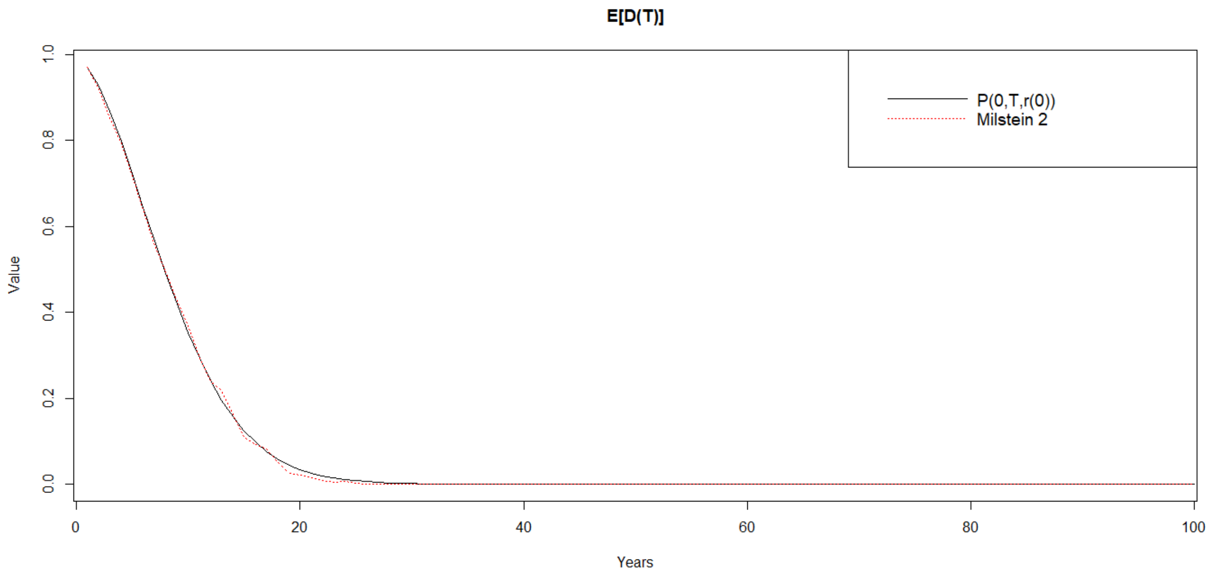

Fig. 18 - **Deflator multiplies stock over long time periods, 16 years**

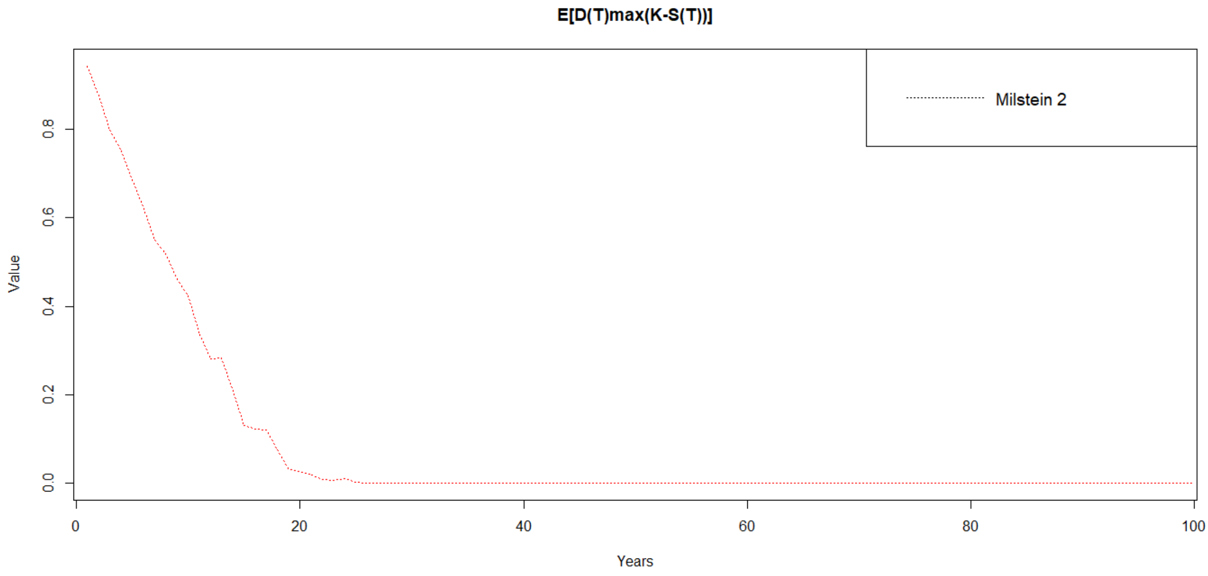

Fig. 19 - **Deflator multiplies stock over long time periods, 16 years**

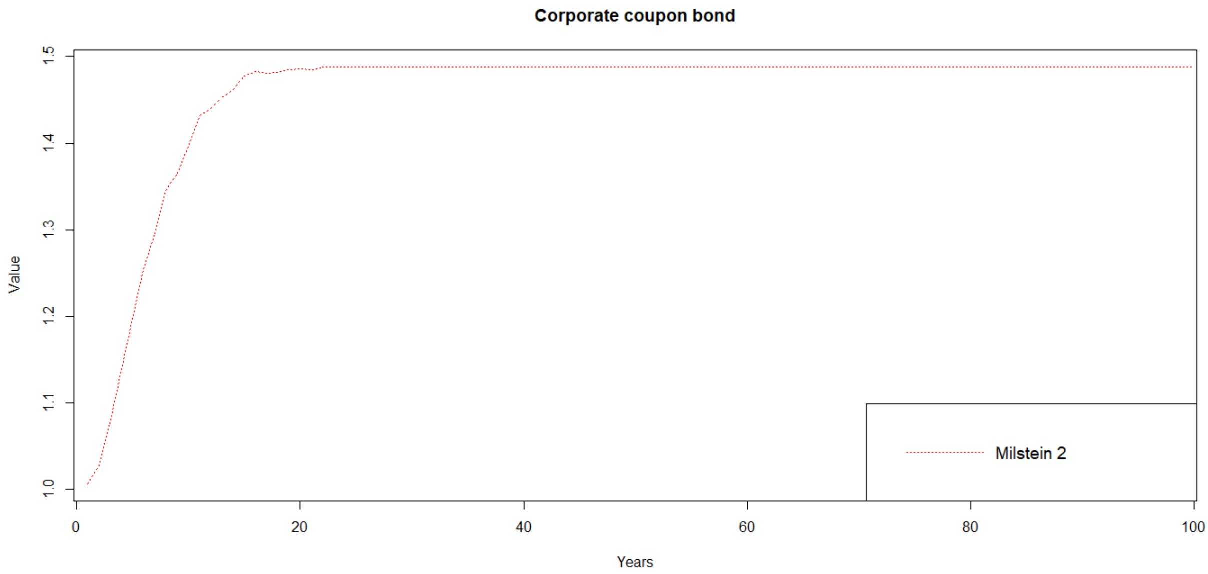

Fig. 20 - **Histogram comparison with antithetic sampling**

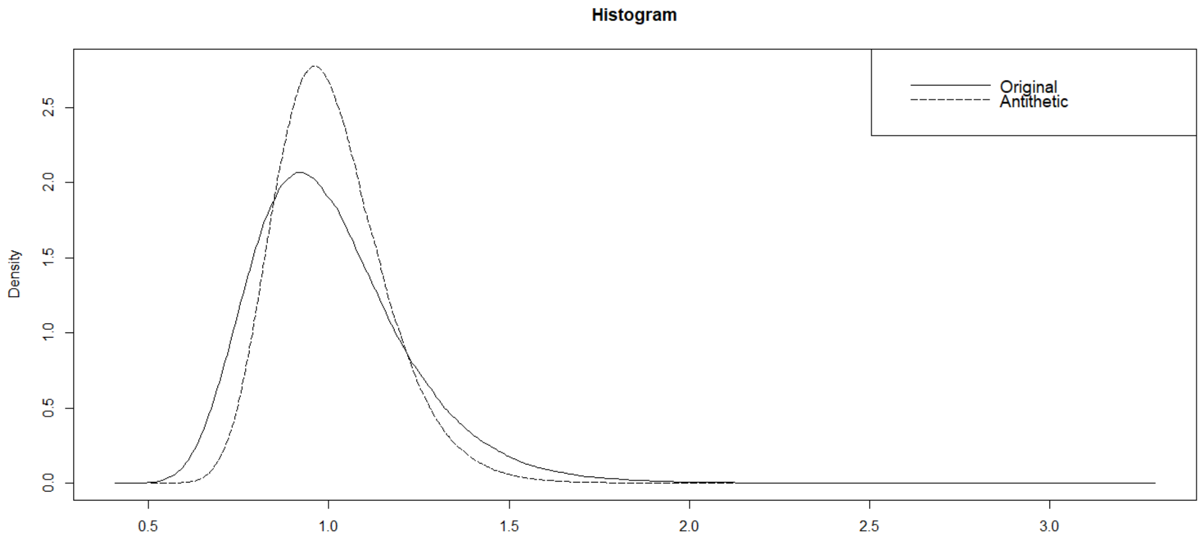

Fig. 21 - **Interest rate over long time periods**

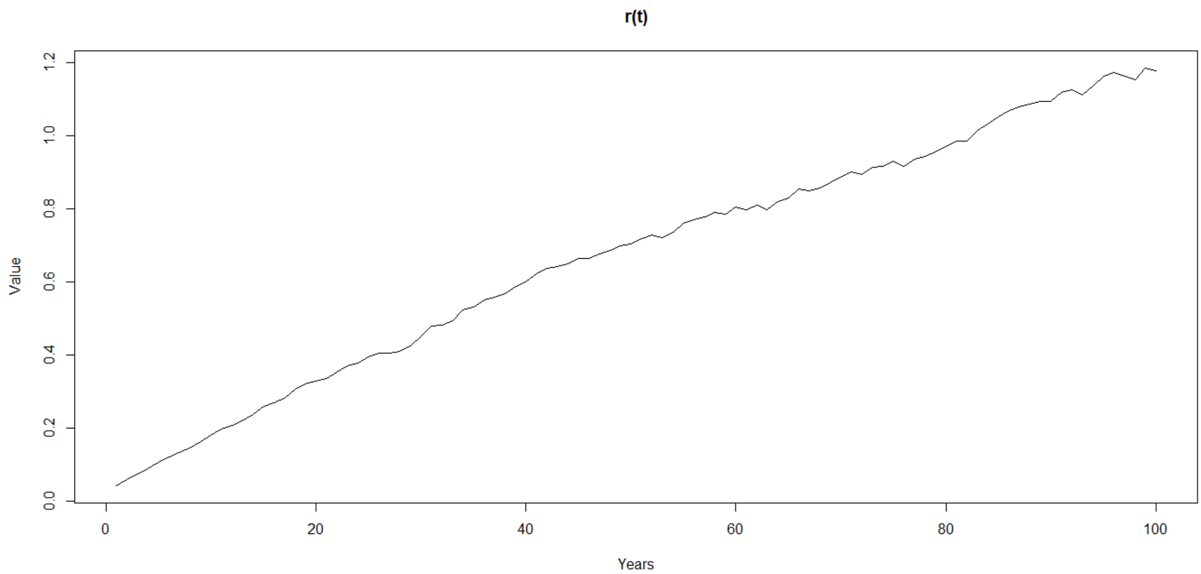

Fig. 22 - **Interest rate over long time periods, alleviated**

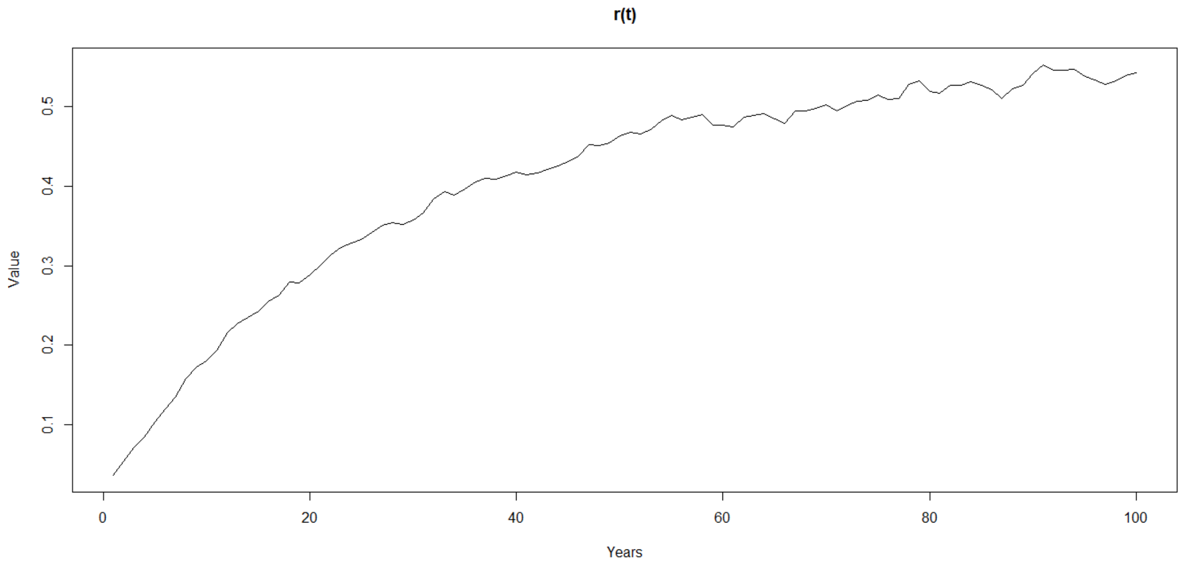

Fig. 23 - **Expected flow of benefits across time**