Measuring uncertainty of solvency coverage ratio in ORSA for Non-Life Insurance

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Abstract. We apply a simple model to project the Solvency Capital Requirement (SCR) over several years, using an ORSA perspective, in order to assess the probability of achieving a solvency coverage ratio. To do so, we rely on a simplified framework proposed in Guibert [10] which provides a detailed explanation of the SCR. Then, we take into account temporal dynamics for liabilities, premiums and asset returns. Here, we consider guarantees in non-life insurance. This context, when simplified, allows us to use a lognormal distribution to approximate the distribution of the liabilities. It leads to a simple and tractable model for measuring the uncertainty of the solvency ratio in an ORSA perspective.

Keywords. *ORSA, Risk Appetite, Solvency Capital Requirement projection, Non-Life Insurance, Semi-analytical formula.*

1 Introduction

With the introduction of the *Own Risk Solvency Assessment* (ORSA) under Pillar 2 of Solvency II (see Planchet and Juillard [12]), regulators require insurance companies to prove their ability to meet the regulatory margin requirements not only on the date of inventory but also prospectively, under the horizons of their strategic plans (see CEIOPS [5]). Accordingly, an insurer must be able to project the main characteristics of its balance sheet over a period of 3 to 5 years depend-

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ing on the duration of the strategic plan in addition to accounting for new business plans written over the period.

In particular, to prove the ability to cover the Solvency Capital Requirement (SCR), the insurer must be able to estimate the uncertainty associated with the future value of its assets and its liabilities in order to predict the probability of failure to cover the regulatory margin over the required duration.

This estimation is potentially tricky as it requires forecasting the main balance sheet items and regulatory ratios over the period chosen in order to approximate the distribution of the solvency coverage ratio at each date. Implementing a model at a level of detail similar to that used for inventory calculations is unsuitable here because of its complexity and its lack of potential robustness. The issues associated with a multi-year model approach are already described for example in Diers [6] and a more general framework for internal models and their potential use for multi-year calculus can be found in e.g., Liebwein [11]. The main drawback of these approaches is that they may potentially lead to very complex and intractable models. At this point, ORSA aims to provide for an insurance company a global view of the main risks it underwrites. As a consequence, the quantitative part of the ORSA framework must be easy to use and updated on a regular basis.

Therefore, we turn to more holistic approaches that model the risks through some state variables. To achieve this, one can use the general ruin theory (see for example Asmussen and Albrecher [2] for a complete overview of this topic). Thus, We choose to use a model inspired by the ruin theory taking into account financial risk, underwriting risk, and business risk.

In this paper, we propose an extension of Guibert et al. [10] by introducing a time dynamic to the aggregate model in a non-life insurance context.

The article is organized as follows: first, we describe in Section 2 a generic model which accounts for interactions between three types of risks – financial risk, underwriting risk, and business risk associated with the uncertainty of future premiums perceived in order to project the future SCR. In Section 3, we set the temporal dynamics for the key drivers of the balance sheet which can reasonably be used for non-life insurance and we deduce in Section 4, the way to value the solvency coverage ratios for future year. A numerical illustration is then given in

Section 5. A separate Section (Section 6) is devoted to the possible extensions of the model in an ORSA context to reflect the provision of premium imposed by the standard model of Solvency II and to allow for multiple lines of business. The article ends by presenting adjustments to be made when using such a model for life and health insurance. These adjustments are only shown in theory; their development will be the subject of ensuing studies.

2 General Structure and Notations

Under a discrete time model, the periods are indexed by $t = 0,\ldots,T$ to a time horizon of *T*. We take similar notations from Guibert et al. [10] in this paper, where *t* refers to the end of the period [*t*, *t*+1[(i.e. the time is considered to be discrete):

- A_t denotes the market value of the theoretical assets invested in asset S_t of return R_t; all financial interest is credited at the end of the period. The risk free discount rate used is assumed to be constant and is denoted as *r*.
- L_t denotes the technical reserves and is equal to the sum of the best estimate BEL_t and the risk margin RM_t, so $L_t = BEL_t + RM_t$. D_t denotes the duration of the liabilities at time *t*.
- P_t and C_t represent the premiums earned and the claims paid, respectively. For simplicity, we assume that any fluctuations in these values occur at the end of the period.
- β_t denotes the combined ratio at time *t*, such that for premiums collected P_t we associate a cost of $\beta_t \times P_t$.
- SCR_t denotes the margin requirement amount set by Solvency II and therefore equals the negative value of the 0.5% quantile of the one-year forecasted net assets.

Finally, we observe that the following variable plays a central role

$$\chi_t = \frac{C_t + L_t - P_t}{1 + R_t}.$$

We call χ_t the *discounted net liability* at time *t*.

The risk margin must be computed according to the Solvency II rules on the cost of capital: so we choose a cost of capital of $\alpha = 6\%$. For an analysis of the issues associated to account the risk margin for SCR calculation in a non-life framework, see Robert [14].

This relies on the balance sheet analysis proposed by Guibert et al. [10] observing that the basic relationship

$$SCR = VaR\left(\frac{C_1 + L_1}{1+R_1}, 99.5\%\right) - L_0,$$

which can be rewritten at any given instant, taking future premiums into account, as

$$SCR_t = VaR_t\left(\frac{C_{t+1} + L_{t+1} - P_{t+1}}{1+R_{t+1}}, 99.5\%\right) - L_t, \qquad (1)$$

with $L_t = BEL_t + RM_t$.

The subscript t on the value at risk (VaR), expectations and variances indicates that the values are conditional on information available at time t. Furthermore, we need to make two assumptions to account for the risk margin. First, by retaining the assumption of proportionality between the Solvency Capital Requirement (SCR) and the best estimate (see Guibert et al. [10]), we observe that

$$\chi_{t+1} = \frac{C_{t+1} + L_{t+1} - P_{t+1}}{1+R_{t+1}} = \frac{C_{t+1} + h_t \times BEL_{t+1} - P_{t+1}}{1+R_{t+1}},$$

with $h_t = 1 + \alpha \times \frac{SCR_t}{BEL_t} \times D_{t+1}$. Second, this term is approximated in practice[2] by

$$h_t = 1 + \alpha \times \frac{SCR_t}{BEL_t} \times D_t,$$

to remove the randomness introduced by the presence of duration D_{t+1} in the definition of h_t. Hence, the Equation (1) defining SCR becomes

$$SCR_t = VaR_t\left(\frac{C_{t+1} + h_t \times BEL_{t+1} - P_{t+1}}{1+R_{t+1}}, 99.5\%\right) - BEL_t - \alpha \times SCR_t \times D_t,$$

and therefore,

[2] This justifies index t and not t+1.

$$SCR_t = \frac{VaR_t\left(\frac{C_{t+1}+\left(1+\alpha\times\frac{SCR_t}{BEL_t}\times D_t\right)\times BEL_{t+1}-P_{t+1}}{1+R_{t+1}},99.5\%\right)-BEL_t}{1+\alpha\times D_t}. \quad (2)$$

This expression is not easy to handle as it is an implicit equation of SCR_t which can only be solved numerically, as will be shown later.

To start with, the behavior of the various elements accounted for in this projection, such as the value of assets, value of liabilities, claims, premiums, etc., must first be specified.

3 Defining the Dynamics of the Model

The calculations are carried out in two stages. First, we specify the dynamics for the four risk factors appearing in the model: premiums, best estimate (past pricing risk), combined ratio (future pricing risk and risk of expenses), and assets return. Next, we deduce the expressions of other variables of interest, such as claims and asset value.

3.1 Risk Factor Dynamics

We assume that, conditionally on the information available at time *t*, the variables BEL_{t+1}, P_{t+1}, β_{t+1} and S_{t+1} (the asset with yield $\frac{S_{t+1}}{S_t}=1+R_{t+1}$) follow lognormal random walks,

$$P_{t+1}=P_t\times X_p \text{ with } X_p\sim\mathcal{LN}\left(\mu_p-\frac{\sigma_p^2}{2},\sigma_p^2\right),$$

$$\beta_t\sim\mathcal{LN}\left(\ln(\beta)-\frac{\sigma_\beta^2}{2},\sigma_\beta^2\right),$$

$$BEL_{t+1}-\beta_{t+1}\times P_{t+1}=BEL_t\times X_l \text{ with } X_l\sim\mathcal{LN}\left(\mu_l-\frac{\sigma_l^2}{2},\sigma_l^2\right),$$

$$S_{t+1}=S_t\times X_a \text{ with } X_a\sim\mathcal{LN}\left(\mu_a-\frac{\sigma_a^2}{2},\sigma_a^2\right),$$

where β is a fixed target combined ratio and the parameters σ_β, (μ_p, σ_p), (μ_l, σ_l) and (μ_a, σ_a) describe the distribution of the independent lognormal risk factors.

The rate of change of reserves μ_l must be affected by discounting and the level of claims paid. We propose to retain the simple relation of $\mu_l = r + \ln(1-\varphi)$ with φ a constant decrease in claims payable between two dates to take into account these effects. Based on this notation, the equation defining BEL_{t+1} is

$$BEL_t \times e^{-\frac{\sigma_l^2}{2} + \sigma_l \times \varepsilon_{t+1,l}} = e^{-r} \times \left(BEL_{t+1} - \beta_{t+1} \times P_{t+1}\right) + \varphi \times BEL_t \times e^{-\frac{\sigma_l^2}{2} + \sigma_l \times \varepsilon_{t+1,l}},$$

or

$$BEL_{t+1} - \beta_{t+1} \times P_{t+1} = BEL_t \times e^{r - \frac{\sigma_l^2}{2} + \sigma_l \times \varepsilon_{t+1,l}} - \varphi \times BEL_t \times e^{r - \frac{\sigma_l^2}{2} + \sigma_l \times \varepsilon_{t+1,l}}, \qquad (3)$$

with $\varepsilon_{t+1,\beta}$, $\varepsilon_{t+1,p}$, $\varepsilon_{t+1,l}$ and $\varepsilon_{t+1,a}$ are independent Gaussian white noises.

The Equation (3) reflects the consumption of reserves in the run-off – the variation trend of the best estimate is caused by the effect of reduced discounting of the claims paid. By identifying the two terms in the equation, we find that the claims are equal to

$$C_{t+1} = e^{r - \frac{\sigma_l^2}{2} + \sigma_l \times \varepsilon_{t+1,l}} \times \varphi \times BEL_t = \theta \times \left(BEL_{t+1} - \beta_{t+1} \times P_{t+1}\right),$$

with $\theta = \frac{\varphi}{1-\varphi}$. On the basis of these assumptions, and in the absence of new premiums, the martingale property of the process of the best estimates holds,

$$BEL_t = E_t\left(e^{-r} \times \left(C_{t+1} + BEL_{t+1} - \beta_{t+1} \times P_{t+1}\right)\right).$$

3.2 Evolution Equations of Other Factors

Once the processes describing the evolution of various balance sheet items are defined, we need to analyze the distribution of the discounted net liability $\chi_t = \frac{C_t + L_t - P_t}{1 + R_t}$. In the absence of a risk margin, we get an expression of the form

$$\chi_{t+1} = \frac{(1+\theta)\times BEL_{t+1} - (1+\theta\times\beta_{t+1})\times P_{t+1}}{1+R_{t+1}},$$

with θ as defined in 3.1,which helps determine the distribution so as to calculate the SCR using the relation (2) presented in Section 2. In the presence of a risk margin, we use

$$\chi_{t+1} = \frac{(h_t+\theta)\times BEL_{t+1} - (1+\theta\times\beta_{t+1})\times P_{t+1}}{1+R_{t+1}}, \qquad (4)$$

with h_t as defined in Section 2. Since it leads to simpler calculation and in order to to avoid circularity dependence between the both quantities, the QIS5 (see CEIOPS [5]) computes the SCR without taking into account the risk margin effect. But this simplification leads to an overestimation of the SCR. As exact calculus can here be achieved, we choose to take the risk margin into account to avoid a bias in our model.

Once SCR_t is determined, all other interest variables are easily obtained:

– The value of assets: $A_t = A_{t-1}\times(1+R_t) - C_t + P_t$,

– The value of liabilities: $L_t = BEL_t + RM_t = BEL_t + \alpha\times D_t\times SCR_t$.

Then, the solvency coverage ratio of the regulatory margin is calculated by

$$s_t = \frac{A_t - L_t}{SCR_t}.$$

However, we must first determine the distribution of χ_{t+1} conditional on the information available at time *t*. This will be discussed in the next Section.

4 Calculating the Distribution of the Net Discounted Liability

The random variable χ_{t+1} is a weighted sum of lognormal variables in the numerator over a lognormal variable in the denominator (see Equation (4)). The form of this distribution is not particularly simple due to the numerator and we can estimate this random variable by a lognormal distribution whose parameters are obtained by the method of moments (Fenton-Wilkinson approximation described in Fenton [9]). Hence, the ratio is approximated by a lognormal distribution as the denominator is assumed to be lognormal.

This approximation of the net discounted liability leads to an implicit equation of SCR_t easily resolvable compared to Equation (2).

4.1 Distribution of the Net Liability

The *net liability* χ_{t+1}^N, corresponding to the numerator of χ_{t+1}, is the sum of two lognormal variables minus a another lognormal variable

$$\begin{aligned}\chi_{t+1}^N &= \left(h_t+\theta\right)\times BEL_{t+1}-\left(1+\theta\times\beta_{t+1}\right)\times P_{t+1} \\ &= \left(h_t+\theta\right)\times BEL_t\times e^{\mu_l-\frac{\sigma_l^2}{2}+\sigma_l\times\varepsilon_{t+1,l}}+h_t\times\beta_{t+1}\times P_{t+1}-P_{t+1}\end{aligned}.$$

Upon dividing by P_{t+1}, we have

$$\frac{\chi_{t+1}^N}{P_t}=\left(h_t+\theta\right)\times\frac{BEL_t}{P_t}\times e^{\mu_l-\mu_p-\frac{\sigma_l^2}{2}+\frac{\sigma_p^2}{2}+\sigma_l\times\varepsilon_{t+1,l}-\sigma_p\times\varepsilon_{t+1,p}}+h_t\times\beta_{t+1}-1,$$

and we deduce than $\frac{\chi_{t+1}^N}{P_t}$ can be represented as $\frac{X+Y-1}{Z}$ with X, Y and Z lognormal variables. Consequently, studying if χ_{t+1}^N is lognormal can be reduced to studying if the variable $X+Y-1$ is lognormal, i.e., searching the range of parameters where the lognormal approximation is appropriate.

A large amount of literature is dedicated towards the approximation of the sum of lognormal distributions (see Fenton [9], El Faouzi and Maurin [8] or Schwartz and Yeh [16]). However, in situations where volatility is not too significant (see Dufresne [7]), we can use the lognormal approximation. It is well known that, for higher standard deviation values, this approximation tends to underestimate the mean and overestimate the variance of the sum of lognormal distributions. This approximation can also be applied when taking the difference of lognormal distributions, but it is only valid for situations where $X+Y\gg 1$, i.e., the premiums are relatively small in comparison to stock commitments to ensure a positive numerator with high probability[3].

[3] In the case of a negative numerator with high probability (e.g. new product), this approximation can be applied by considering the opposite of the numerator.

The validation procedure of $X+Y-1$; with $X \sim \mathcal{LN}\left(\mu_X - \frac{\sigma_X^2}{2}, \sigma_X^2\right)$, $Y \sim \mathcal{LN}\left(\mu_Y - \frac{\sigma_Y^2}{2}, \sigma_Y^2\right)$, as a lognormal variable is carried out by Jarque-Bera (Bera and Jarque [3]) and Anderson-Darling tests (Anderson and Darling [1]). $X+Y-1$ is said to be approximately lognormal if the p-value is greater than 5%. Naturally this lognormal approximation depends on the values of the parameters μ_X, μ_Y, σ_X and σ_Y. By denoting $C = \sqrt{\sigma_X^2 + \sigma_Y^2}$ which is a simple criterion to handle, we compute the set of parameters where the lognormal approximation is justified. On **Fig. 1.** Limit of the, we provide the upper limit of the criterion C as a function of μ_X and μ_Y for $X+Y-1$ to be log normally distributed.

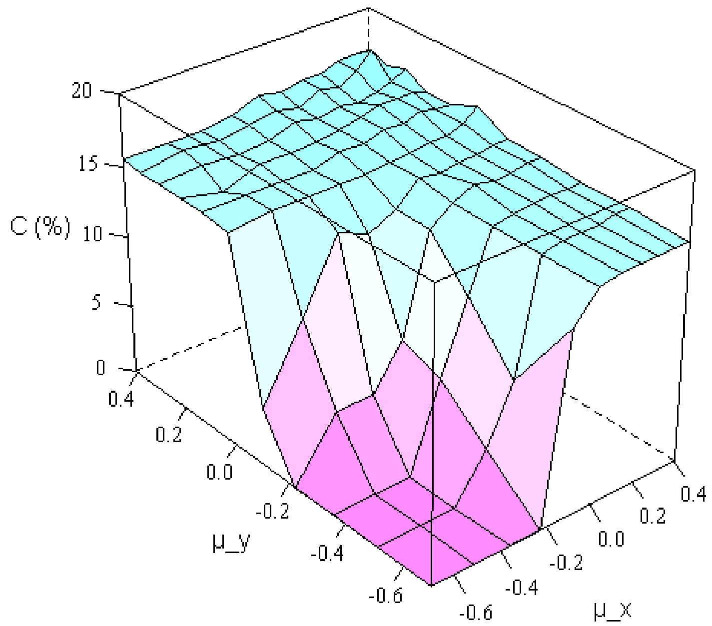

Fig. 1. Limit of the set of admissible parameters for lognormal approximation

Since Z is also lognormal, the random variable $\frac{X+Y-1}{Z}$ has an approximate lognormal distribution provided that the parameters satisfy numerically the condition illustrated on **Fig. 1.** Limit of the set of admissible parameters for lognormal approximationIn practice, this approximation should be validated on a case by case basis following example of validation process given in Section 5.2.

4.2 Approximation of the Net Discounting Liability

In approximating the variable $C_{t+1} + BEL_{t+1} - P_{t+1}$ conditionally to the information available at time t by a lognormal distribution attained by the method of moments, we find that the solvency coverage ratio χ_{t+1} can be approximated by a lognormal distribution. Therefore, we have an explicit expression of the quantile of χ_{t+1}, as a lognormal distribution of X with parameters (μ,σ), we have

$$x_p = VaR_p(X) = \exp\left(\mu + \sigma \times \Phi^{-1}(p)\right),$$

where Φ is the distribution function of the standard normal distribution.

Calculate this quantile comes down to calculating the parameters (μ,σ) of the lognormal approximation of the solvency coverage ratio χ_{t+1}. As the mean m and variance v^2 of the lognormal distribution X are given respectively by

$$m = \exp\left(\mu + \frac{\sigma^2}{2}\right) \text{ and } v^2 = \left(e^{\sigma^2} - 1\right) \times m^2 \text{ (see Saporta [15])},$$

and using the method of moments, we get an explicit expression of SCR_t in terms of BEL_t, β_t, P_t and $1+R_t$.

To do so, we observe that the variance of the underlying normal distribution is calculated simply by using the coefficient of variation of the lognormal distribution

$$\sigma^2 = \ln\left(1+\omega^2\right), \tag{5}$$

where $\omega = \frac{v}{m}$. Once this parameter is known, the expectation of the underlying normal distribution is calculated using $\mu = \ln(m) - \frac{\sigma^2}{2}$ or $\mu = \ln\left(\frac{m}{\sqrt{1+\omega^2}}\right)$.

The calculation of the conditional expectation and the conditional variance of the variable $(h_t + \theta) \times BEL_{t+1} - (1 + \theta \times \beta_{t+1}) \times P_{t+1}$ is described in Appendix 8.1 and then we use a lognormal approximation with the followings parameters

$$\sigma_t^2 = \ln\left(1+\omega_t^2\right),\ \mu_t = \ln\left(\frac{(h_t+\theta) \times BEL_t \times e^{\mu_l} - (1 - h_t \times \beta) \times P_t \times e^{\mu_p}}{\sqrt{1+\omega_t^2}}\right).$$

Finally, the distribution of the net discounted liability

$$\chi_{t+1} = \frac{(h_t + \theta) \times BEL_{t+1} - (1 + \theta \times \beta_{t+1}) \times P_{t+1}}{1 + R_{t+1}},$$

conditional on information at time *t*, is lognormal with the parameters of the underlying normal distribution

$$\mu_t(\chi) = \mu_t - \mu_a + \frac{\sigma_a^2}{2},\ \sigma_t^2(\chi) = \sigma_t^2 + \sigma_a^2.$$

We finally derive from Equation (2) the following implicit equation of SCR_t

$$SCR_t = \frac{1}{1 + \alpha \times D_t}\left(\exp\left(\mu_t(\chi) + \sigma_t(\chi) \times \Phi^{-1}(99.5\%)\right) - BEL_t\right). \quad (5)$$

Actually, $\mu_t(\chi)$ and $\sigma_t^2(\chi)$ are dependent on SCR_t because χ depends on h_t as defined in Section 2. This equation can only be solved numerically using some standard root-finding algorithms.

The value of assets is determined using $A_t = A_{t-1} \times (1 + R_t) - C_t + P_t$ and finally, the amount of liabilities is given by $L_t = BEL_t + RM_t = BEL_t + \alpha \times D_t \times SCR_t$.

5 A Simple Example of Implementation in ORSA

In this Section we illustrate the above model in the context of establishing an ORSA process. The purpose of this example is to show that the model fits naturally within this framework and provides the quantitative requirements.

This Section presents the application of the model to a non-life insurance company, in a more general context of an ORSA[4] process (as defined in Solvency II). This process consists of the following step:

- Step 1: Define the risk appetite of the company, the risk metrics and the horizon used to manage the company,
- Step 2: Compute these metrics and check if the risk appetite constraints are satisfied on the first projection year: this is a necessary condition to meet the risk appetite constrains,

[4] A description of the general ORSA structure can be found in Planchet and Juillard [12].

- Step 3: Defined operational risk limits in accordance with the step 2 results,
- Step 4: Check if the risk appetite constrains are verified on the duration of the strategic plan,
- Step 5: Perform a sensitive analysis to assess the robustness of the strategic plan.

The model is programmed with the R software (R Development Core Team [13]) and the code is available upon request. We work with a sample size of 5000.

5.1 Description of the Company

We consider, for our example, an insurance company selling a single health contract with the following general structure:

- average combined ratio of $\beta_0 = 100\%$,
- average premium sales of $C_0 =$ €75 million,
- asset allocation consisting of 20% stock and 80% bonds (1 year treasury bonds),
- initial SCR coverage ratio of $s_0 = 204\%$,
- the strategic plan is to maintain the current risk profile of the structure (i.e. stable sales and allocation),
- to simplify our example below, we assume that the duration of liabilities is stable over time $D_0 = 2$ (this assumption is particularly appropriate for health care contracts). This value of D_0 is based on a statistical analysis of a French insurance portfolio (medical expenses).

Based on a statistical analysis of its portfolio, the company confirmed lognormal characteristics of its risks, whose respective parameters are as follows:

- mean and volatility of the premiums: $\mu_p = 0,\ \sigma_p = 1\%$,
- volatility and decrease speed of claims: $\sigma_l = 10\%, \varphi = 80\%$,
- volatility and target value of the combined ratio: $\sigma_\beta = 2\%, \beta = 100\%$,
- mean and volatility of the asset: $\mu_a = 3{,}6\%,\ \sigma_a = 6{,}3\%$.

Upstream of the numerical application and after setting a base scenario in the following Section, we validate the lognormal distribution of the key risk driver χ_{t+1}. After that, we describe the ORSA process by using the model as described above.

5.2 Validation of the Lognormal Approximation

After checking the parameters satisfy the test defined in Section 4.1, we assess the fit of the lognormal approximation of the conditional distribution of χ_{t+1} by comparing the approximated lognormal distribution with the empirical distribution obtained by simulation. That is, we use simulations to compute the empirical distribution of (see Section 3.2)

$$\chi_{t+1} = \frac{(h_t + \theta) \times BEL_{t+1} - (1 + \theta \times \beta_{t+1}) \times P_{t+1}}{1 + R_{t+1}}.$$

The resulting empirical distribution of χ_1 is plotted in **Fig. 2**

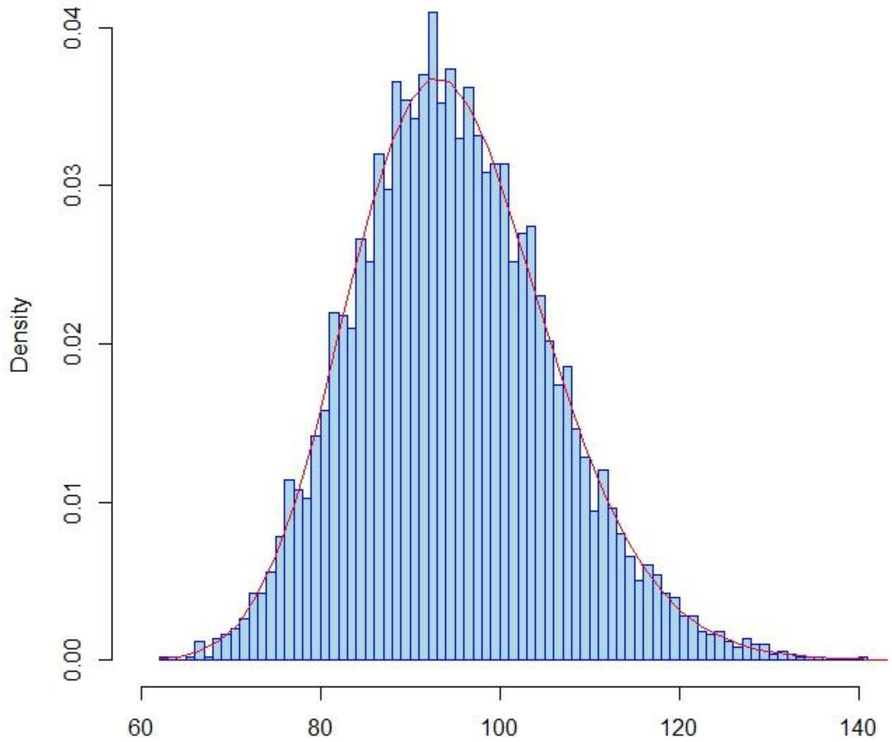

Fig. 2. Comparison of the histogram of the simulated and the fitted density of the net discounted liability

Thus, the fit seems graphically (**Fig. 2**) correct. Using the Jarque-Bera test (Bera and Jarque [3]), generally intended for large sample sizes, to match $\ln(\chi_1)$ to a normal distribution, we get a p-value of 35%. Therefore, the lognormal approximation seems acceptable.

The base scenario gives the following results (Table 1) of solvency coverage ratio for years 0 through 5

Table 1. Statistics of the evolution of solvency coverage ratio

Projection year	Min	1st Quartile	Median	Mean	3rd Quartile	Max
0	2.037	2.037	2.037	2.037	2.037	2.037
1	0.5236	1.7850	2.0600	2.0540	2.3200	3.4800
2	-0.1324	1.7330	2.0990	2.1240	2.5100	4.3880
3	-0.2213	1.7040	2.1690	2.1940	2.6700	5.2240
4	-0.1557	1.6790	2.2420	2.2650	2.8160	5.5960
5	-0.6712	1.6800	2.3060	2.3450	2.9630	6.1900

5.3 Choice of Risk Appetite

As part of the risk appetite process, the Board of Directors aims to control two indicators: the solvency and the profitability of its stockholders' equity. The interpretation of this risk appetite is set out below:

- to present 95% of the time an SCR coverage ratio of 130% over 5 years (which is the duration of the strategic plan),
- to present 80% of the time a loss on return on equity (measured by $W_t = 1 - \frac{A_t - L_t}{A_{t-1} - L_{t-1}}$) of 13.5% over at least 1 year.

On the basis of the projection model proposed, the empirical correspondence between the company's strategic plan and its policy for risk appetite is initially verified by its structure:

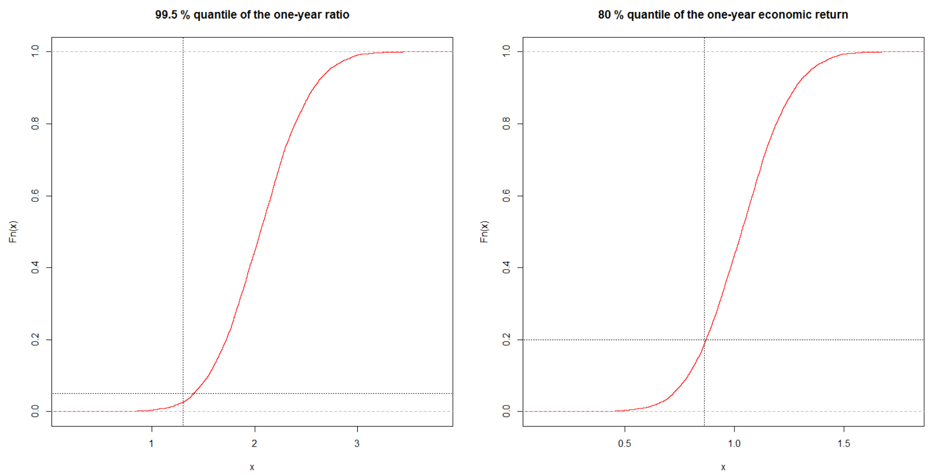

Fig. 3. Empirical distribution function of the one-year solvency ratio and the equity return

The graphs above (**Fig. 3**) represent the empirical cumulative distribution functions of the one-year solvency coverage ratio and the one-year return of equity. We also plot the risk appetite constrains with the vertical lines at 130% and 86.5% (the limits on SCR coverage ratio and return of equity) and the horizon lines at 5% and 20%, which are the limit on the probability. The graphs show that while the strategic plan can meet the risk appetite constraints applied by the company since the empirical distributions are lower the intersection points. This first step helps avoiding performing calculation on the duration of the strategic plan and save computing time, but it does not provide operational limits as it consists of points rather than interval allocations. To address this issue, two solutions are available:

- to test several types of arbitrarily fixed allocations,
- to define the set of allocations able to meet the risk appetite as defined by company policy.

Testing arbitrarily a set of strategic allocations is not advisable in a risk appetite context since this process aims to seek optimal strategies while remaining within the feasible risk tolerance set by the company. The company therefore has to define the set of acceptable allocations representing the 5% quantile of a one-year solvency coverage ratio over the set of all possible allocations. **Fig. 4** represents the set of strategic allocations (defined by the percentage of the portfolio invested in bonds) and the amounts of future premiums for which the SCR coverage ratio exceeds 130% in 95% of cases.

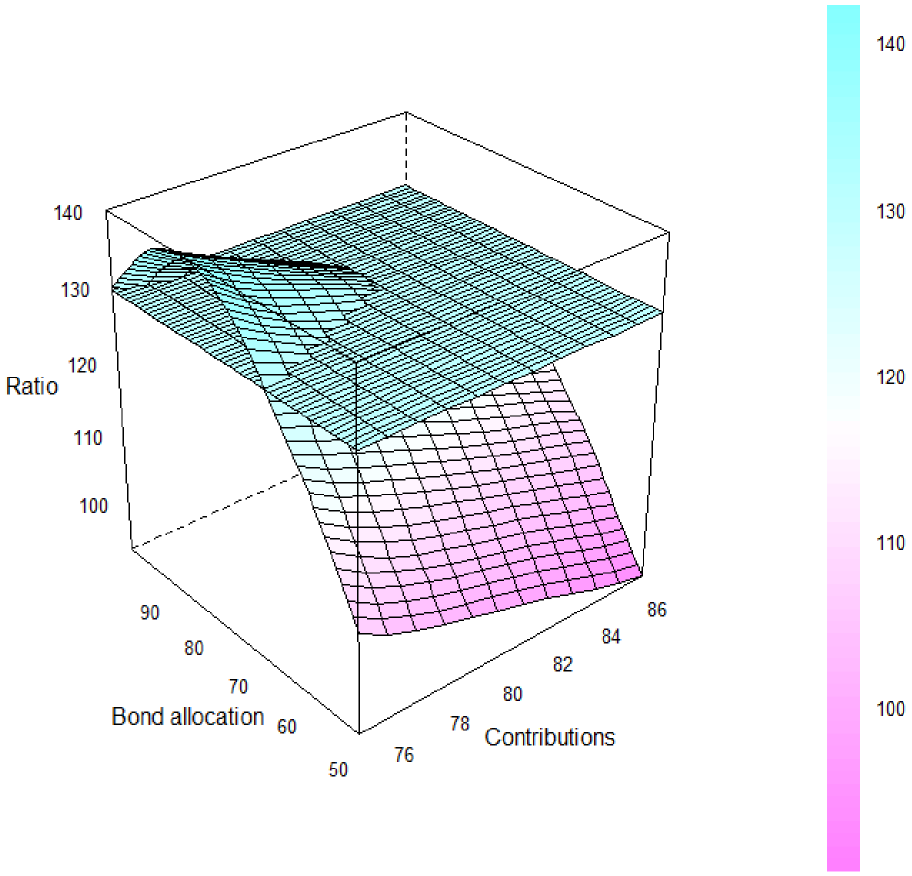

Fig. 4. Evolution based on the strategic allocation (Bond allocation in % and Contributions in € million) of the constraint on SCR

The **Fig. 4** shows that:

- the greater the company's volume of premiums, the lower its share allocation (this explains the concept of capital transfer between risks),
- certain amounts of premiums appear to be a minimum share allocation constraint as well as an inability to comply with SCR constraints.

At first, the company analyzes the restrictions on the return on investment of stockholders' equity and then it defines the investment limits. The **Fig. 5** shows the set of strategic allocations (defined by the percentage of the portfolio invested in bonds) and the amounts of future premiums for which the return on equity exceeds 86.5% in 80% of cases.

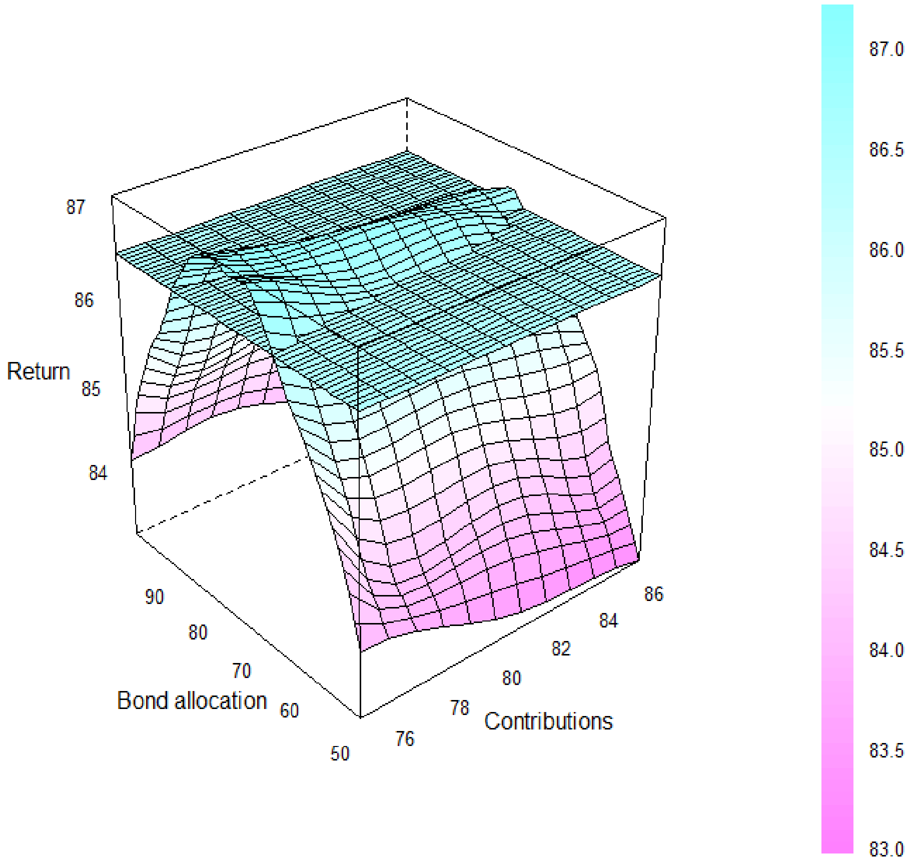

Fig. 5. Evolution of strategic allocation (Bond allocation in % and Contributions in € million) of performance constraint

The graph above (**Fig. 5**) shows that the "return" dimension (i.e. the evolution of the return on investment) defines the minimum limits of risky investments, while the “solvency” dimension (i.e. the evolution of the solvency coverage ratio described in **Fig. 4**) mainly shows the maximum investment constraints.

At this stage, the company must first express its risk preference – it must decide whether it prefers to allocate its risk to assets or to liabilities. Generally, the fact that risk’s liabilities are related to business development lead to concentrate the risk capital on liabilities. Thus, the company initially chooses the operational limits concerning pricing risk, marketing between €75m and €78m of premiums over the 5 upcoming years (or an increase of about 5%). Secondly the company adapts the limits on assets as the risk appetite constrains are verified.

Given the changes in risk based on the premium amounts, the constraints related to the target allocations are based on a target premium of €78m.

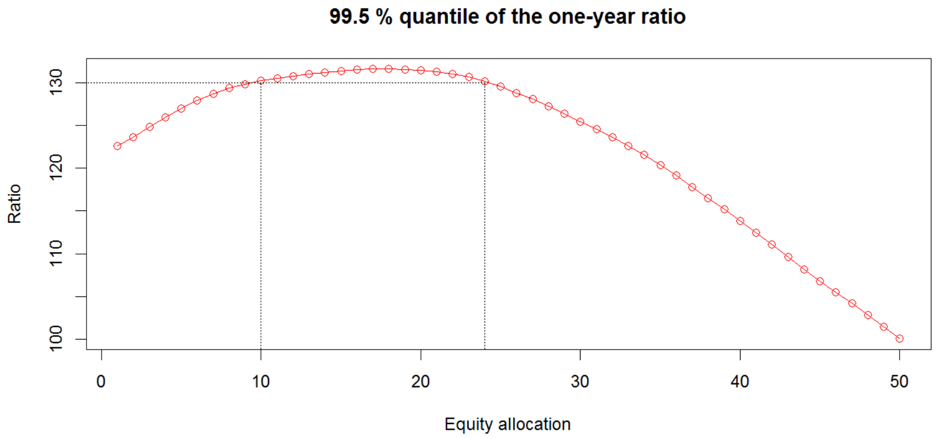

Fig. 6. Operational limit shares caused by a constraint on the SCR

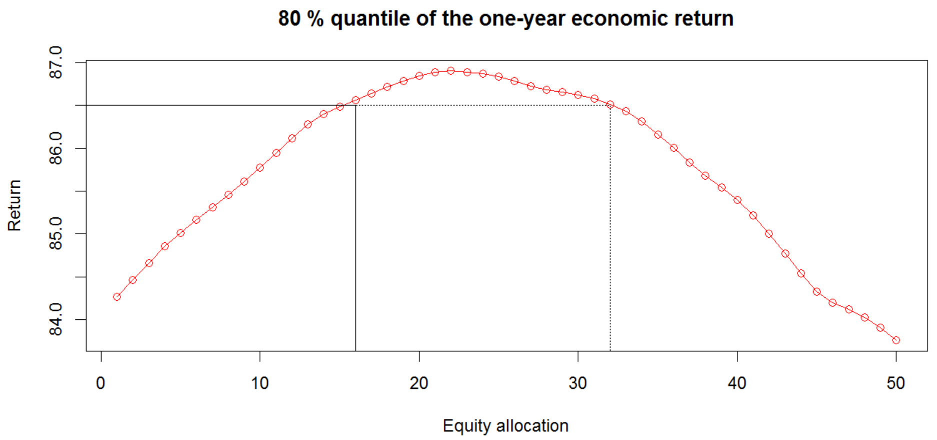

Fig. 7. Operational limit shares caused by a constraint on the yield

Fig. 6 and **Fig. 7** show how the SCR coverage ratio at 95% and the return on equity at 80% evolve according to the equity allocation chosen with €78m of premiums and give the range of allocation percentages that satisfy the risk appetite constrains on the first year. We deduce that keeping within the one year SCR constraint for a €78m sale of premiums involves retaining an equity allocation between 10% and 24%, and that a compliance with the constraint relative to the one year return of investment requires retaining an equity allocation between 16% and 32%. The company decided to set the following operational limits:

- earning between €75 to €78 million of premiums,
- allocating equity between 16% and 24%.

In order to finish the risk appetite process, confirming compliance with SCR constraints over the duration of the strategic plan (i.e. 5 years) is necessary. Then, the model is revived by retaining the upper bound risk limits as well as a 5 year forecast as the strategic plan. As, we consider an annual 95% probability, the quantile level for the year “n” is 0.95^n.

Table 2. Evolution of the quantile of the solvency coverage ratio

Projection year	Quantile level	Solvency coverage ratio
1	95%	1.31
2	90%	1.3
3	85%	1.33
4	81%	1.41
5	77%	1.52

Table 2 shows that the operational limits can meet the risk appetite over the entire duration of the strategic plan. The constraint on the return of investment is verified as it lasts 1 year.

We should also note that keeping within the constraints over the entire duration of the strategic plan is not acquired beforehand; the operational limits are set after analyzing one year results. Thus, a (relatively small) breach can occur and would involve reviewing operational limits. The development of the distribution of the coverage ratio over 5 years is presented in **Fig. 8**.

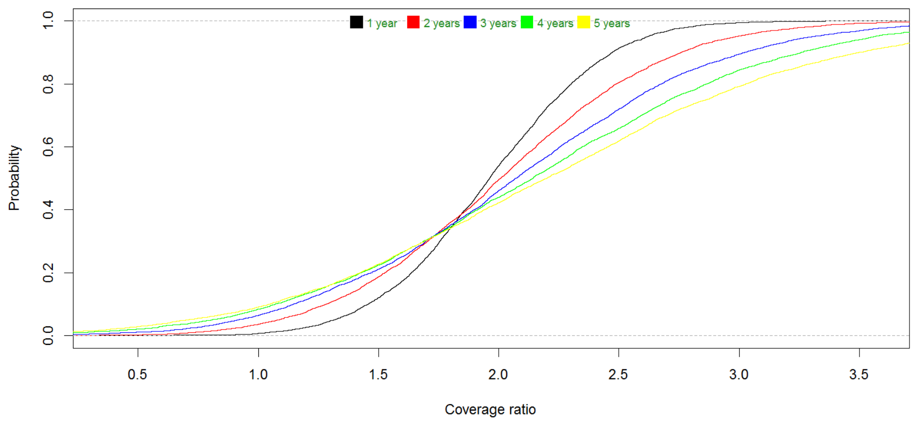

Fig. 8. Analysis of compliance with the risk appetite over the duration of the strategic plan

5.4 Sensitivity analysis

The last stage of the ORSA process consists in making a sensitivity analysis of the model to parameters so that the insurance company can identify the key drivers of insolvency. Note that it is necessary to keep in mind two essential points:

- this sensitivity analysis has first to be done on the exogenous risk factors, because this risks are well- known (therefore the analyze of the premium's uncertainty is not presented afterwards),
- the purpose of this sensitivity analysis has to be done on the risk capacity and not on the risk appetite (the risk appetite is naturally unchecked when risks increase).

As a general rule, this analyze can be done in the following way:

- by taking into account the intrinsic volatility of the estimators relative to the main risk drivers,
- on the basis of the stress tests. This second method has the advantage to allow us to use the expert judgment.

In a risk management perspective, we consider that the expert statements should be taking into account. The main risks of non-life insurance being the premium and reserve risks, we choose to analyze the evolution of the minimum coverage ratio (over the following five years):

- on the basis of stress tests which assess the impact of an increase of the reserve and combined ratio volatilities,

– on the basis of statistical test which assess the robustness of the coverage ratio in case combined ratio value increase to the upper 95% confidence intervals bounds.

We summarize the major points of this analysis into the Table 3 (reminder the reference situation points out a minimum coverage ratio over the following five years of 130 percents).

Table 3. Sensitivity analysis

Parametring of the test	minimum coverage ratio over 5 years
increase of σ_β by 10%	1.18
increase of σ_β by 20%	1.06
increase of σ_β by 25%	0.99
increase of σ_I by 10%	1.32
increase of σ_I by 20%	1.32
increase of σ_I by 50%	1.30
increase of β to the upper 95% confidence intervals bounds	0.99

The Table 3 points out the robustness of the ORSA's results : the risk capacity is only outnumbered at extreme ends. The insurance company can have confidence in the capacity of the strategic plan to reflect its risk appetite.

6 Extensions

Below, we describe two possible extensions of the model by considering in Section 6.1 the inclusion of a premium reserve, and in Section 6.2 the presence of multiple lines of business.

6.1 Inclusion of a Provision of Premiums

The QIS5 (see CEIOPS [5]) provides a breakdown of the best estimate between:

– a best estimate of claims, relating to claims which already occurred as of the date of stocktaking,

– and a best estimate of premiums, relating to possible future claims arising from contracts in the portfolio at the date of inventory. The expected future premiums that these contracts will issue must therefore be considered.

The accounting rules for future premiums are relatively complex in Solvency II. Indeed, the contract boundaries depend on the process of determining each insurer's rates. Therefore, they lead to a large heterogeneity of situations even under identical risks. In the Pillar 1 calculations, and in the context of non-life insurance used here, we take a one year horizon into account for future premiums. As in the model presented earlier in the paper, the situation is projected over several years and the only impact of not strictly complying with the rule to determine future premiums included in calculating the one year margin requirement is a lag of time in collecting the premium considered. Consequently, the absolute level of margin requirements derived from the model may be biased, but its variation is not and it is the variations of this value (and of the solvency coverage ratio) that we seek to describe.

That said, adding future premiums under Solvency II may be introduced simply by modifying the dynamics of BEL_t, defined in Section 3.1, as follows

$$BEL_{t+1} = BEL_t \times X_l + \beta_{t+1} \times P_{t+1} + \delta \times e^{\mu} \times \left(\beta_{t+1} - 1\right) \times P_{t+1},$$

with δ the contract renewal rate and $\mu < \mu_p$ the annual price adjustment rate. This adjustment only has a limited effect on the model's results when the combined ratio is close to one.

6.2 Considering Multiple Lines of Business

In practice, one have to consider multiple lines of business leveraged by general assets, i.e., to distinguish between BEL_t^j and P_t^j for $j = 1, \ldots, n$. In such a case, we have

$$\chi_{t+1} = \frac{\sum_{j=1}^{n}\left(C_{t+1}^j + BEL_{t+1}^j - P_{t+1}^j\right) + RM_{t+1}}{1 + R_{t+1}}.$$

Therefore, we can once again use a lognormal approximation of the conditional distribution of χ_{t+1} by matching the first two moments of the numerator. The dependency between lines of business is measured by the correlation coefficients between the underlying normal distributions. This approach allows us to measure the effect of change of premiums on the mix product and to identify arbitrages in the underwriting policy.

Denoting by ω_t the coefficient of variation of the variable $\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_{t+1}^j-\left(1+\beta_{t+1}^j\times\theta^j\right)\times P_{t+1}^j\right)$ and recalling that V_t represents the conditional variance

$$\omega_t=\frac{\sqrt{V_t\left(\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_{t+1}^j-\left(1+\beta_{t+1}^j\times\theta^j\right)\times P_{t+1}^j\right)\right)}}{\left(\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_t^j\times e^{\mu_l^j}-\left(1-h_t\times\beta^j\right)\times P_t^j\times e^{\mu_p^j}\right)\right)},$$

the parameters of the lognormal approximation is then derived as

$$\sigma_t^2=\ln\left(1+\omega_t^2\right),\ \mu_t=\ln\left(\frac{\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_t^j\times e^{\mu_l^j}-\left(1-h_t\times\beta^j\right)\times P_t^j\times e^{\mu_p^j}\right)}{\sqrt{1+\omega_t^2}}\right).$$

As in the case of a single line of business, the distribution of χ_{t+1}, conditional on the information available at time *t*, is approximated by a lognormal distribution with parameters

$$\mu_t\left(\chi\right)=\mu_t-\mu_a+\frac{\sigma_a^2}{2},\ \sigma_t^2\left(\chi\right)=\sigma_t^2+\sigma_a^2.$$

Conforming to the QIS5 requirements, the risk margin is generally calculated by taking the SCR into account using the coefficient $h_t=\alpha\times\frac{SCR_t}{BEL_t}\times D_t$. The calculation of the variance used to evaluate ω_t is presented in Appendix 8.2. We also observe numerically (by broadening the scope of the example presented in Section 5) that the lognormal approximation of the variable χ_{t+1} is valid.

7 Conclusion

In this paper, we present a simple model to determine the distribution of the solvency coverage ratio by considering the key risk drivers (reserves, premiums and financial risk. This model allows taking into account margin requirement over the duration of the strategic plan.

Such a model used in ORSA allows measuring the impact of management choices (related to the level of premiums, asset allocations, product mix, etc.) on covering the insurer solvency coverage ratio. The model enables the company to assess the likelihood of non-coverage over a given horizon. In other words, the failure to comply with a minimum coverage threshold would comply internal governance. It is therefore a valuable tool for decision making. This model, which measures the effect of future production on the margin requirements and margin coverage, enters the Pillar 1 framework of Solvency II. It enables risk management to assess the regulatory solvency requirements. This approach can in particular be deployed along with the standard model as an internal model and has the advantage of being inexpensive in terms of time computation, therefore making it suitable for daily use.

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8 Appendix

The calculation of the moments of the variable χ_{t+1} is presented in this Section for one or more lines of business.

8.1 For one line of business

Upstream of the presentation of the multiple- lines of business, we start by introducing the calculation of the moments of the variable χ_{t+1} in the presence of only one line of business.

Thanks to the following expression

$$\begin{aligned}(h_t+\theta)\times BEL_{t+1}-(1+\theta\times\beta_{t+1})\times P_{t+1} &= (h_t+\theta)\times BEL_t\times e^{\mu_l-\frac{\sigma_l^2}{2}+\sigma_l\times\varepsilon_{t+1,l}} \\ &\quad -(1-h_t\times\beta_{t+1})\times P_{t+1}\end{aligned},$$

we find the value of the conditional expectancy

$$\begin{aligned}E_t\left((h_t+\theta)\times BEL_{t+1}-(1+\theta\times\beta_{t+1})\times P_{t+1}\right) &= (h_t+\theta)\times BEL_t\times e^{\mu_l} \\ &\quad -(1-h_t\times\beta)\times P_t\times e^{\mu_p}\end{aligned}.$$

Also, given that $\varepsilon_{t+1,\beta}$, $\varepsilon_{t+1,p}$, $\varepsilon_{t+1,l}$ are independent, we have the conditional variance

$$\begin{aligned}&V_t\left((h_t+\theta)\times BEL_{t+1}-(1+\theta\times\beta_{t+1})\times P_{t+1}\right) \\ &= V_t\left((h_t+\theta)\times BEL_t\times e^{\mu_l-\frac{\sigma_l^2}{2}+\sigma_l\times\varepsilon_{t+1,l}}-(1-h_t\times\beta_{t+1})\times P_{t+1}\right) \\ &= (h_t+\theta)^2\times BEL_t^2\times e^{2\mu_l}\times\left(e^{\sigma_l^2}-1\right)+h_t^2\times\beta^2\times\left(e^{\sigma_\beta^2}-1\right)\times P_t^2\times e^{2\mu_p}\times\left(e^{\sigma_p^2}-1\right) \\ &\quad + h_t^2\times\beta^2\times\left(e^{\sigma_\beta^2}-1\right)\times P_t^2\times e^{2\mu_p}+P_t^2\times e^{2\mu_p}\times\left(e^{\sigma_p^2}-1\right)\times(1-h_t\times\beta)^2 \\ &= (h_t+\theta)^2\times BEL_t^2\times e^{2\mu_l}\times\left(e^{\sigma_l^2}-1\right)+h_t^2\times\beta^2\times\left(e^{\sigma_\beta^2}-1\right)\times P_t^2\times e^{2\mu_p+\sigma_p^2} \\ &\quad +P_t^2\times e^{2\mu_p}\times\left(e^{\sigma_p^2}-1\right)\times(1-h_t\times\beta)^2\end{aligned},$$

and thus, we deduce of (5) the coefficient of variation of $(h_t+\theta)\times BEL_{t+1}-(1+\theta\times\beta_{t+1})\times P_{t+1}$ is

$$\omega_t = \frac{\sqrt{\begin{aligned}&\left(h_t+\theta\right)^2 \times BEL_t^2 \times e^{2\mu_l} \times \left(e^{\sigma_l^2}-1\right) + h_t^2 \times \beta^2 \times \left(e^{\sigma_\beta^2}-1\right) \times P_t^2 \times e^{2\mu_p+\sigma_p^2} \\ &+P_t^2 \times e^{2\mu_p} \times \left(e^{\sigma_p^2}-1\right) \times \left(1-h_t \times \beta\right)^2\end{aligned}}}{\left(h_t+\theta\right) \times BEL_t \times e^{\mu_l} - \left(1-h_t \times \beta\right) \times P_t \times e^{\mu_p}}.$$

8.2 For several lines of business

We continue with model containing several lines of business supposing the lognormal approximation is validated. First, we calculate the first two moments of the numerator of χ_{t+1}, and then we deduce those of χ_{t+1}. We have calculated the conditional expectation of the variable $\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right) \times BEL_{t+1}^j - \left(1+\beta_{t+1}^j \times \theta^j\right) \times P_{t+1}^j\right)$

$$\begin{aligned}&E_t\left(\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right) \times BEL_{t+1}^j - \left(1+\beta_{t+1}^j \times \theta^j\right) \times P_{t+1}^j\right)\right) \\ &= \sum_{j=1}^{n}\left(E_t\left(\left(h_t+\theta^j\right) \times BEL_{t+1}^j - \left(1+\beta_{t+1}^j \times \theta^j\right) \times P_{t+1}^j\right)\right) \\ &= \sum_{j=1}^{n}\left(\left(h_t+\theta^j\right) \times BEL_t^j \times e^{\mu_l^j} - \left(1-h_t \times \beta^j\right) \times P_t^j \times e^{\mu_p^j}\right)\end{aligned},$$

and its conditional variance

$$\begin{aligned}&V_t\left(\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right) \times BEL_{t+1}^j - \left(1+\beta_{t+1}^j \times \theta^j\right) \times P_{t+1}^j\right)\right) \\ &= \sum_{j=1}^{n}\left(V_t\left(\left(h_t+\theta^j\right) \times BEL_{t+1}^j - \left(1+\beta_{t+1}^j \times \theta^j\right) \times P_{t+1}^j\right)\right) \\ &+2\sum_{1\le i<j\le n} Cov_t\left(\left(h_t+\theta^i\right) \times BEL_{t+1}^i - \left(1+\beta_{t+1}^i \times \theta^i\right) \times P_{t+1}^i, \left(h_t+\theta^j\right) \times BEL_{t+1}^j - \left(1+\beta_{t+1}^j \times \theta^j\right) \times P_{t+1}^j\right)\end{aligned}.$$

The first component of the conditional variance is obtained simply

$$\begin{aligned}&\sum_{j=1}^{n}\left(V_t\left(\left(h_t+\theta^j\right) \times BEL_{t+1}^j - \left(1+\beta_{t+1}^j \times \theta^j\right) \times P_{t+1}^j\right)\right) \\ &= \sum_{j=1}^{n}\left(\begin{aligned}&\left(h_t+\theta^j\right)^2 \times BEL_t^{j2} \times e^{2\mu_l^j} \times \left(e^{\sigma_l^{j2}}-1\right) \\ &+P_t^{j2} \times e^{2\mu_p^j} \times \left(h_t^2 \times \beta^{j2} \times \left(e^{\left(\sigma_p^{j2}+\sigma_\beta^{j2}\right)}-1\right) + \left(e^{\sigma_p^{j2}}-1\right)\left(1-2h_t \times \beta^j\right)\right)\end{aligned}\right)\end{aligned}.$$

The second term is obtained by noting that the covariance of the two random lognormal variables $\left(Y_1, Y_2\right)$ with parameters $\left(\mu_1, \sigma_1\right)$ and $\left(\mu_2, \sigma_2\right)$, respectively, are obtained from the covariance of the underlying normal variables $\left(\varepsilon_1, \varepsilon_2\right)$

$$Cov(Y_1,Y_2)=E(Y_1)\times E(Y_2)\left(e^{Cov(\varepsilon_1,\varepsilon_2)}-1\right).$$

Denoting that the correlation coefficients ρ_l^{ij}, ρ_p^{ij} and ρ_β^{ij} are associated with the variables $\left(BEL_t^i,BEL_t^j\right)$, $\left(C_t^i,C_t^j\right)$ and $\left(\beta_t^i,\beta_t^j\right)$, respectively, we get

$$\begin{aligned}
&Cov_t\left(\left(h_t+\theta^i\right)\times BEL_{t+1}^i-\left(1+\beta_{t+1}^i\times\theta^i\right)\times P_{t+1}^i,\left(h_t+\theta^j\right)\times BEL_{t+1}^j-\left(1+\beta_{t+1}^j\times\theta^j\right)\times P_{t+1}^j\right)\\
&=\left(h_t+\theta^i\right)\left(h_t+\theta^j\right)\times BEL_t^i\times BEL_t^j\times e^{\mu_l^i+\mu_l^j}\times\left(e^{\rho_l^{ij}\sigma_l^i\sigma_l^j}-1\right)\\
&\ +h_t^2\times Cov_t\left(\beta_{t+1}^i\times P_{t+1}^i,\beta_{t+1}^j\times P_{t+1}^j\right)-h_t\times Cov_t\left(\beta_{t+1}^i\times P_{t+1}^i,P_{t+1}^j\right)\\
&\ -h_t\times Cov_t\left(\beta_{t+1}^j\times P_{t+1}^j,P_{t+1}^i\right)+Cov_t\left(P_{t+1}^i,P_{t+1}^j\right)
\end{aligned}.$$

But we have

$$Cov_t\left(P_{t+1}^i,P_{t+1}^j\right)=P_t^i\times P_t^j\times e^{\mu_p^i+\mu_p^j}\times\left(e^{\rho_p^{ij}\sigma_p^i\sigma_p^j}-1\right),$$

$$\begin{aligned}
Cov_t\left(P_{t+1}^i,\beta_{t+1}^j\times P_{t+1}^j\right)&=E_t\left(\beta_{t+1}^j\right)\times Cov_t\left(P_{t+1}^i,P_{t+1}^j\right)\\
&=\beta^j\times P_t^i\times P_t^j\times e^{\mu_p^i+\mu_p^j}\times\left(e^{\rho_p^{ij}\sigma_p^i\sigma_p^j}-1\right),
\end{aligned}$$

And

$$\begin{aligned}
&Cov_t\left(\beta_{t+1}^i\times P_{t+1}^i,\beta_{t+1}^j\times P_{t+1}^j\right)\\
&=E_t\left(\beta_{t+1}^i\right)\times E_t\left(\beta_{t+1}^j\right)\times Cov_t\left(P_{t+1}^i,P_{t+1}^j\right)+E_t\left(P_{t+1}^i\right)\times E_t\left(P_{t+1}^j\right)\times Cov_t\left(\beta_{t+1}^i,\beta_{t+1}^j\right).\\
&=\beta^i\times\beta^j\times P_t^i\times P_t^j\times e^{\mu_p^i+\mu_p^j}\left(e^{\rho_p^{ij}\sigma_p^i\sigma_p^j+\rho_\beta^{ij}\sigma_\beta^i\sigma_\beta^j}-1\right)
\end{aligned}$$

We then deduce

$$\begin{aligned}
&Cov_t\left(\left(h_t+\theta^i\right)\times BEL_{t+1}^i-\left(1+\beta_{t+1}^i\times\theta^i\right)\times P_{t+1}^i,\left(h_t+\theta^j\right)\times BEL_{t+1}^j-\left(1+\beta_{t+1}^j\times\theta^j\right)\times P_{t+1}^j\right)\\
&=\left(h_t+\theta^i\right)\left(h_t+\theta^j\right)\times BEL_t^i\times BEL_t^j\times e^{\mu_l^i+\mu_l^j}\times\left(e^{\rho_l^{ij}\sigma_l^i\sigma_l^j}-1\right)\\
&+P_t^i\times P_t^j\times e^{\mu_p^i+\mu_p^j}\begin{bmatrix}h_t^2\times\beta^i\times\beta^j\times\left(e^{\rho_p^{ij}\sigma_p^i\sigma_p^j+\rho_\beta^{ij}\sigma_\beta^i\sigma_\beta^j}-1\right)\\+\left(e^{\rho_p^{ij}\sigma_p^i\sigma_p^j}-1\right)\left(1-h_t\times\left(\beta^i+\beta^j\right)\right)\end{bmatrix}
\end{aligned}.$$

We finally infer the coefficient of variation of $\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_{t+1}^j-\left(1+\beta_{t+1}^j\times\theta^j\right)\times P_{t+1}^j\right)$

$$\omega_t = \frac{\sqrt{V_t\left(\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_{t+1}^j-\left(1+\beta_{t+1}^j\times\theta^j\right)\times P_{t+1}^j\right)\right)}}{\left(\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_t^j\times e^{\mu_l^j}-\left(1-h_t\times\beta^j\right)\times P_t^j\times e^{\mu_p^j}\right)\right)},$$

and then the parameters of the lognormal approximation

$$\sigma_t^2=\ln\left(1+\omega_t^2\right),\ \mu_t=\ln\left(\frac{\sum_{j=1}^{n}\left(\left(h_t+\theta^j\right)\times BEL_t^j\times e^{\mu_l^j}-\left(1-h_t\times\beta^j\right)\times P_t^j\times e^{\mu_p^j}\right)}{\sqrt{1+\omega_t^2}}\right).$$

As in the case of a single line of business (see Section 8.1), the distribution of χ_{t+1} conditional on the information available at time *t* is approximated by a lognormal distribution with parameters

$$\mu_t\left(\chi\right)=\mu_t-\mu_a+\frac{\sigma_a^2}{2},\ \sigma_t^2\left(\chi\right)=\sigma_t^2+\sigma_a^2.$$