Stochastic and Tychastic Approaches to Guaranteed ALM Problem

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Abstract

Unlike traditional valuation methods, viability theory provides tools for eradicating the risk, by determining the minimum initial capital that would meet the commitments of the investor, regardless market developments. In this study, we compare two approaches to risk assessment within a framework of asset-liability management (ALM) of a guaranteed fund. The optimal allocation of assets for such funds is determined initially by the classical portfolio insurance (thus with a statistical evaluation of risk) and then in a second step, by tools of viability theory. Although the results from the two approaches are not strictly comparable in terms of numerical point of view (as in both cases the goals are different in nature), this study offers, on a practical example of ALM management, two radically different philosophies : one is the statistical evaluation of risk, based on probabilistic models, while the other one eradicates risk, using viability theory.

Keywords : Asset-Liability Management, Portfolio Insurance, Viability Theory, Risk Assessment, Guaranteed Viability Kernel, Tychastic Systems

1 Introduction

Recently, the Asset-Liability Management (ALM) has seen its role become increasingly essential to ensure the solvability of a company, particularly following the recent financial crisis of 2008 and 2011. The ALM is a comprehensive and coordinated approach allowing a company to manage the composition of its assets to be larger than it liabilities. Choosing a management rule is a choice under contingent uncertainty, but a choice that the investor or

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the manager can make. Evaluation or eradication of losses hides behind the fact that there are uncertainties on which the investor cannot act. In this particular insurance problem, these uncertain variables are the returns, assumed to be unknown at the investment date where the management rule and, in some cases, the insurance in terms of economic capital, must be computed assuming some forecasting mechanism of future returns. At future dates, when the returns are known, this management rule may be used to compute the value and the exposure of the portfolio, and thus, to determine the shares.

Whereas economic theory is dedicated to the analysis and the computation of supply and demand adjustment laws, among which the Walras tâtonnement, in the hope of explaining the mechanisms of price formation (see for instance [3, Aubin] or chapter 5 of [6, Aubin]). In the last analysis, the choice of the prices is made by the invisible hand of the "Market", the new deity in which many economists and investors believe. But a hidden deity which leaves the investor with the task of forecasting it. In most financial scenarios, investors take into account their ignorance of the pricing mechanism. They assume instead that prices evolve under uncertainty, and that they can master this uncertainty. They still share the belief that the "Market knows best" how to regulate the prices, above all without human or political regulations. The question became to known how to master this uncertainty. For that, many of them trade the Adam Smith invisible hand against a Brownian movement, since it seems that this unfortunate hand is shaking the asset price like a particle on the surface of a liquid. It should then be enough to assume average returns and volatilities to be known for managing portfolios.

In practise, the portfolio insurance is one of principal management rules used until now. The objective of portfolio insurance is to enable investors to participate in market performance, while providing protection to maturity of a proportion of initial capital invested. In other words, portfolio insurance's main objective is to limit or eradicate the loss of portfolio value while allowing it to benefit to some extent by an increase in this market. The motivation of such a strategy is based on the simple observation that during, for example, crashes of October 1987 and October 1989, strategies such as "Buy and Hold" (which consists of laying down initially and once and for all a composition of given portfolio) could result in very significant losses. This technique can be among the strategies for dynamic management of a surplus in the ALM. In the particular case of guaranteed funds (where the portfolio insurance techniques are often used), the objective is to ensure the initial investment, excluding sales charges, provided to block the savings throughout the period of distribution with a possible promise of performance calculated in relation to the evolution of a basket of stocks, indices or funds.

CPPI (Constant proportion portfolio insurance) designs both a typical and widely used fund management technique and a capital guarantee derivative security which uses this dynamic trading strategy in order to provide participation to the performance of the underlying asset (see [33, Perold & Black]). Let us consider a risky asset (usually equities or mutual funds) and a riskless asset (cash, equivalents or Treasury bonds). With the CPPI rule, the percentage allocated to each depends on the "cushion" value (defined as the product of the difference between current portfolio value and floor value) by a multiplier coefficient. This coefficient is linked with the risk appetite of the investor : a higher number denotes a more aggressive strategy. In this paper we will refer to the CPPI trading strategy as the reference strategy to which we will compare other techniques built with the viability theory.

Moreover, the question of choosing an optimal asset structure is closely related to the issue of hedging the liabilities. In some sense, hedging and ALM are the same, or at least very similar issues. The hedging issue appears in the valuation process of so called variable annuities contracts, because in this context, the value of the contracts (the liability) equals the value of the hedging portfolio (see [37, Planchet]). But to build the hedging portfolio one need to define a management rule, and this is usually achieve in a probabilistic framework. There is many papers developing this point of view (see [22, 23, Coleman et al.]).

In this study, we propose to compare classical portfolio insurance approach with a viability theory based approach (in the context of guaranteed ALM problem). Actually, instead of applying only known mathematical and algorithmic techniques, most of them motivated by physics and not necessarily adapted to such problems, viability theory designs and develops mathematical and algorithmic methods for studying the evolution of such systems, organizations and networks of systems, under viability constraints (such as the "floor constraint in the ALM problem). The purpose of viability theory is to explain mathematical and numerical developments governed by "evolutionary system", which appear in economics, cognitive science, game theory, biology, etc. Such systems are not deterministic, but governed by uncertainty of developments subject to the constraints of viability (or intertemporal optimality) and guide these changes to targets to achieve them in finite time. This is basically to bring out the underlying feedbacks that help regulate the system and to identify mechanisms of selection for their implementation.

So, viability theory appears as a quite natural tool to build hedging portfolio, because it gives not only the initial value of the hedging portfolio but also the practical management rule to choose its composition at time t, depending on market information at this time. Moreover, it doesn't require to make arbitrary assumptions about an underlying probabilistic model, and in this sense it appears as a more robust approach.

The paper is organized as follows. In Section 2 we formalize the problem of insurance showing the state of nature in the choice of the business rule on one side and study the uncertainty on the other hand. Section 3 focuses on the risk assessment if it is based on the CPPI approach (which is a special case of portfolio insurance techniques). A numerical application is well developed. Section 4 is dedicated to the presentation of the results based this time on the viability theory for the same numerical example chosen above. A discussion comparing the two approaches results will be finally exposed.

2 Asset-Liability Management : the Insurance Problem

The objective of portfolio insurance is to enable investors to participate in market performance, while providing protection to maturity of a proportion of initial capital invested. In other words, portfolio insurance's main objective is

- either to limit
- or to eradicate

the loss of portfolio value while allowing it to benefit to some extent by an increase in the "market".

2.1 Insuring Liabilities

The *floor* is the minimum value of the portfolio which is acceptable to the investor at maturity. It defined by a function $L : t \geq 0 \mapsto L(t) \geq 0$. The *liability flow* to insure plays the role of a constraint : the floor must never be "pierced" by the value of the portfolio. The *capital to guarantee (at exercise time)* is the final value $L(T)$ of the floor to the exercise date T.

The *cushion* (analogous to the *surplus* in ALM) is the difference between the portfolio value and the floor (to be guaranteed).

The purpose of the portfolio to constructed is to offer subscribers a pension supplement, the deadline is adjusted according to the client's age when starting the product. The design of mechanisms guaranteeing the hedging of this type of guaranteed fund is the purpose of this study.

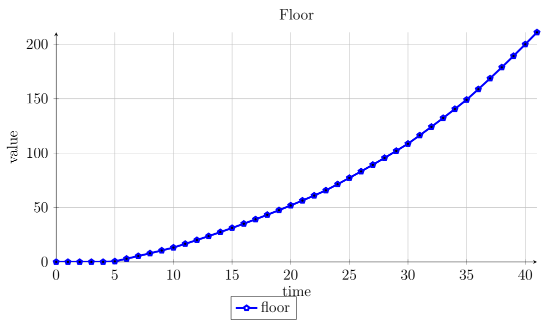
Example — We assume in this study that the employees of certain professional categories will be able to save all or part of their working time account in a guaranteed fund, providing them with the term payback as well as participation in market performance during the life of the fund. The liabilities consist of payments of annual flows of French RTT (Reduction of Working Time) on the part of investors. The fund is associated with the age of the investor in 2011 (e.g. 25 years). For this fund we have a schedule indicating specific RTT flow until retirement (see section (3), p.12).

The floor is governed by an (impulse) differential equation. It is no longer continuous, but punctuated by "jumps" at the dates of the schedule (the floor is supposed to be only lower semicontinuous).

The assumptions at the launch date of this fund are as follows :

- The value of the portfolio assets is initialized with an amount equal to the first installment of RTT of subscribers to the age associated ;
- The liability (guaranteed amount at maturity of retirement) is initialized as the amount paid the first stream, discounted from the launch date to maturity.

The example chosen in this study is illustrated by



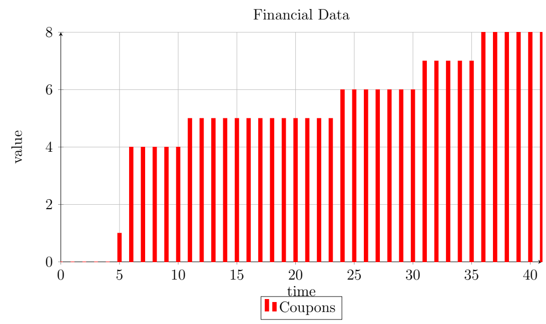


FIGURE 1 – **Floor with Variable Annuities.** The initial data consist, at this stage in the *schedule* (established by actuarial techniques) of annual flows paid by subscribers. These schedules are divided into streams according to the age group that owns the subscriber end of 2010 for example.

FIGURE 2 – **Coupons.** At each annual payment flow, the fund assets is increased by the amount of flow contributed and the liability is increased by the value of the stream between the payment date and maturity. In this figure, the amounts are represented with a negative sign, the positive ones being reserved to payments.

2.2 Assets portfolios and their exposure

The question is "to hedge" the liability flow by a portfolio made of a riskless asset and of an underlying (a "risky asset", in many examples, or any composite index or an equity index). We assume that it is self-financed.

Denote by 0 the *investment date*, by $T > 0$ the *exercise time* (or, the term, the horizon, etc.), by $t \geq 0$ the *current (or spot) time* and by $T - t \geq 0$ the *time to maturity*. We set :

$$\left\{\begin{array}{lll} L(t) & \text{the } \textit{floor} \text{ (describing liabilities)}\,; & \\ S^0(t) & \text{the } \textit{price} \text{ of the riskless asset}\,; & \\ S(t) & \text{the } \textit{price} \text{ of the underlying}\,; & \\ R^0(t) & = & \dfrac{dS^0(t)}{S^0(t)dt} \text{ the } \textit{return} \text{ of the riskless asset;} \\ R(t) & = & \dfrac{dS(t)}{S(t)dt} \text{ the } \textit{return} \text{ of the underlying;} \\ P^0(t) & \text{the number of shares of the riskless asset}\,; & \\ P(t) & \text{the number of shares of the underlying}\,; & \\ W(t) & = & P^0(t)S^0(t) + P(t)S(t) \text{ the } \textit{value of the portfolio}\,; \\ E^0(t) & = & P^0(t)S^0(t) \text{ the } \textit{liquid component} \text{ of the portfolio}\,; \\ E(t) & = & P(t)S(t) \text{ the } \textit{exposure} \text{ (risky component) of the portfolio}\,; \\ C(t_n) & = & \textit{the coupon} \text{ (impulsive cash flow)} \\ & & \text{at dates } \tau_n \text{ of the coupon schedule.} \end{array}\right. \tag{1}$$

The insurance requirement can be written in the form

$$\forall\, t \in [0, T], \quad W(t) \;\geq\; L(t) \tag{2}$$

2.3 Evaluation or Eradication of the loss ?

Two questions arise :

1. To *evaluate* or to “measure” the loss $L(t) - W(t)$;

   The *evaluation of the risk* is provided by several statistical techniques, among which
   - Sharpe ratio of the cushion in the path ;
   - Sharpe ratio of the final cushion ;
   - VaR at 95% of final cushion ;
   - Probability of loss (negative cushion) on a trajectory ;
   - Final wealth average ;
   - VaR at 95% of the final wealth ;
   - Final cushion average ;
   - Probability of negative final cushion.

2. To *eradicate* the risk by imposing that the cushions are always positive, i.e., that property (2) is always satisfied (a viability problem) The *cost of the eradication of the risk* is measured by the *minimum guaranteed investment* (MGI), or *net present value* [NPV], *economic capital*, which, in [25, Crouhy, Galai & Mark], “measures [...] risk”, p.15, “is the financial cushion [...] to absorb unexpected losses”, p. 258, “capital is

also used to absorb risk", p. 366, etc. The corresponding *minimum guaranteed cushion* (MGC) or an *insurance premium*, etc.

The initial guaranteed cushion provides a new measure of risk summarized in a single number which has an explicit meaning and is immediately usable.

2.4 Management Rules

The next question relates to the management of the portfolio. In the simple case of self-financed two assets portfolios, this is done at each instant t by choosing and acting to the exposure $E(t)$ by choosing to invest more or less underlying shares ?

A "floor management rule" tells the manager to adjust on a regular and dynamic basis the exposure to risky assets (the underlying type of shares, stock indices ...) and non-risky (bonds, money market funds ...), so to protect the capital invested.

The first step requires to look for them in a "contingent reservoir" $\widetilde{\mathcal{E}}$ of "management rules" $(t,W) \mapsto \widetilde{E}(t,W)$ available to investors (or mechanisms of regulation). In the last analysis, knowing the time and the value of the portfolio, the management rule dictates the exposure.

In this study, contingent uncertainty is described by *financial constraints on exposures*, which require that each period t

$$\forall\, t \in [0,T],\ \forall\, W \geq 0,\ \ E\ \in\ [E^{\flat}(t,W), E^{\sharp}(t,W)] \tag{3}$$

The main problem is the choice of a management rule for hedging the portfolio and thus, to assess it. This is a part of uncertainty, called *contingent uncertainty*, because we do not know what management rule to choose (the definition of contingency given by *Leibniz* in *Essais de Théodicée* : *"Contingency is a non-necessity, a characteristic attribute of freedom."*).

In this study, contingent uncertainty is described by *financial constraints on exposures*, which require that each period t

$$\forall\, t \in [0,T],\ \forall\, W \geq 0,\ \ E\ \in\ [E^{\flat}(t,W), E^{\sharp}(t,W)] \tag{4}$$

Example — If

$$\left\{ \begin{array}{cl} B(t) \geq 0 & \text{is the target allocation} \\ A(t) \geq B(t) - 1 & \text{is the maximum cash } -P^0(t)S^0(t) \end{array} \right. \tag{5}$$

we set

$$E^{\flat}(t,W)\ :=\ B(t)W(t) \text{ and } E^{\sharp}(t,W)\ :=\ (1+A(t))W(t)$$

If $A(t) = -|A(t)| < 0$, condition $-P^0(t)S^0(t) \leq A(t)W$ can be written $P^0(t)S^0(t) \geq |A(t)|W$, which means that the portfolio should *include a minimum share of the monetary value of the portfolio.*

These bounds $A(t)$ and describe more or less severe prudential constraints. The default values are $A(t) = 0$ and $B(t) = 0$ (so that the contingent reservoir is in this case the interval $[0, W]$).

Management rules can be divided into two classes, according the approach used to choose them :

1. The *direct approach* means that the choice of a management rule (regarded as a feedback) is made *a priori* or *ex ante* at initial investment date. This approach is, by far, the most often used in science : in economics, it is the law of supply and demand (Walrasian tâtonnement) ; in Artificial Intelligence, it is the Hebb law describing a *a priori* learning mechanism regardless what must be learned ; in robotics, it is usually a (most often) linear "feedback", which does not necessarily take into account either the non-linearity of mechanical systems or the constraints, etc.

   For managing liabilities, several examples have been devised :

   (a) **Fixed Mix** or **Buy and Hold**

   The fixed (or constant) mix rule states that the exposition is proportional to the value of the portfolio :

   $$\widetilde{E}(t, W) \ := \ mW \text{ where } m > 0 \text{ is "risky part" of the portfolio}$$

   (b) **CPPI** (*Constant Proportional Portfolio Insurance*)

   The CPPI management rule states that the exposition is proportional to the cushion :

   $$\widetilde{E}(t, W) \ := \ m(W - L(t)) \text{ where } m > 0 \text{ is called the (cushion) "multiplier"} \quad (6)$$

   See [34, Perold and Sharpe], [30, Hill et al.].

   Attractive for their simplicity and the fact that the choice of the management rule amounts to choosing the risky part of the multiplier m, we know that these rules do not guarantee anything, since the floor may be pierced in some cases. It is the same of its many variants which introduced time dependent multipliers $m(t)$ or even more sophisticated functions, *for the simple reason that their choice is not dictated by the problem* : eliminate the possibility of piercing the floor.

   The choice of these methods leaves open the choice of the

   – ***multiplier*** : Even if they are chosen to optimize given criteria, they may forbid to find the decision rule eradicating losses among them.
   – ***initial cushion*** required to manage dynamically the portfolio is not calculated in practice for implementing the initial value of the portfolio or its expectation, but just estimated by statistical methods. In the example treated in this paper, it is equal to the initial value of the floor.

For these two reasons, it happens that the value of the portfolio pierces the floor for some realizations of the underlying prices.

For example, crashes of October 1987 and October 1989, strategies such as "Buy and Hold" (which consists of laying down initially and once and for all a composition of given portfolio) could result in very significant losses. However, the "Buy and Hold" technique is a common strategy for dynamic management of a cushion (or surplus) in the asset-liability management (ALM). In the particular case, for example, guaranteed funds (where the portfolio insurance techniques are often used), the objective is to ensure the initial investment, excluding sales charges, provided to block the savings throughout the period of distribution with a possible promise of performance calculated in relation to the evolution of a basket of stocks, indices or funds.

In their papers [20, Boulier & Kanniganti], [24, Cont & Tankov], among several authors, point out the fact that the CPPI does not eradicate the risk : *"Yet the possibility of going below the floor, known as "gap risk," is widely recognized by CPPI managers : there is a nonzero probability that, during a sudden downside move, the fund manager will not have time to readjust the portfolio, which then crashes through the floor. In this case, the issuer has to refund the difference, at maturity, between the actual portfolio value and the guaranteed amount N. It is therefore important for the issuer of the CPPI note to quantify and manage this "gap risk."*

2. The *inverse approach* requires that the construction of the management rule *is a solution to the problem, eradication of the loss, instead of being a datum.*

   Viability theory allows us to calculate the VPPI (Viabilist Portfolio Performance and Insurance) management rule $(t, W) \mapsto E^{\heartsuit}(t, W)$ for solving this eradication problem.

   Unfortunately, the VPPI management rule $(t, W) \mapsto E^{\heartsuit}(t, W)$ does not enjoy the simplicity of the CPPI management rule (defined by a simple explicit linear analytical formula), but is tabulated by the VPPI software that computes the values of the portfolio and its exposure to each date (from we can deduce an *ad hoc* multiplier if needed as performance measures). It is not explicit, but concealed in the dark and invisible computer memory.

The mathematical tradition of the era that preceded the advent of computers in the middle of last century required mathematical results to be expressed in explicit analytical mathematical formulas needed to calculate it numerically "by hand" through the various tables of "special functions". A treat for the mathematicians, but very often at exorbitant price of much too restrictive assumptions. This tradition of "the search for the lost formula" is no longer justified since it is possible to develop suitable algorithms and software for obtaining numerical information in the absence of explicit formulas. Computation is the ultimate objective, and that only should matter.

2.5 Uncertainties

Choosing a management rule is a choice under contingent uncertainty, but a choice that the investor or the manager can make.

Evaluation or eradication of losses hides behind the fact that there are uncertainties on which the investor cannot act. In this particular insurance problem, these uncertain variables are the returns, assumed to be unknown. Whereas economic theory is dedicated to the analysis and the computation of supply and demand adjustment laws, among which the *Walras tâtonnement*, in the hope of explaining the mechanisms of price formation (see for instance [2, Aubin] or chapter 5 of [4, Aubin]). In the last analysis, the choice of the prices is made by the invisible hand of the "Market", the new deity in which many economists and investors believe. But a hidden deity which leaves the investor with the task of forecasting it.

In most financial scenarios, investors take into account their ignorance of the pricing mechanism. They assume instead that prices *evolve under uncertainty*, and that they can master this uncertainty. They still share the belief that the "Market knows best" how to regulate the prices, above all without human or political regulations. The question became to known how to master this uncertainty. For that, many of them trade the Adam Smith invisible hand against a Brownian movement, since it seems that this unfortunate hand is shaking the asset price like a particle on the surface of a liquid. It should then be enough to assume average returns and volatilities to be known for managing portfolios.

Hence the design of the management rule takes into account non only the floor, but also some "measure" of this kind of uncertainty :

1. either we choose an *a priori* management rule and we exchange the Adam Smith invisible hand on the formation of asset prices against stochastic uncertainty for deriving management rules of the portfolio evaluating the losses through a battery of measures mentioned earlier ;
2. or we built *a posteriori* management rule allowing us to eradicate the loss against stochastic uncertainty : the value of the portfolio is "always" larger or equal to the liabilities whatever the returns ranging over a "tychastic reservoir" in which returns appear unexpectedly.

The problem is no longer to assess the probability of risky returns realizations, but to *determine the subset in which they can emerge*. Prediction models or extrapolation techniques no longer consist in determining trends and volatilities, but in the tychastic case, the *lower bounds of risky returns defining the future tychastic reservoir*.

2.5.1 Stochastic Uncertainty

Stochastic uncertainty on the returns is described by a space Ω, filtration $\mathcal{F}_t$, the probability $\mathbb{P}$, a Brownian process $B(t)$, a drift $\gamma(R)$ and a volatility : $dS(t) = S(t)(\gamma(R(t))dt +$

$\sigma(R(t))dB(t))$.

1. The random events are not explicitly identified. The set Ω is not described explicitly (one can always choose the space of all evolutions). Only the drift and volatility are assumed to be explicitly known ;
2. Stochastic uncertainty does not study the "package of evolutions" (depending on $\omega \in \Omega$), but *functionals over this package*, such as the different moments and their statistical consequences (averages, variance, etc.) used as evaluation of risk (they deal with the dual of the space of evolutions and on spaces of functionals on these evolutions) ;
3. Required properties are valid for *"almost all" constant* ω.

We quote a few references among so many ones : [16, Benaim & El Karoui], [19, Bouchaud], [32, Neftci], [17, 18, Bensoussan,Crouhy & Galai] and, in the discrete case, [43, Zabczyk].

2.5.2 Tychastic Uncertainty

In this study, tyches are returns of the underlying on which the investor has no influence. The uncertainty is described by the tychastic map defined by

$$\mathcal{R}(t) \ := \ \left\{ R \in \mathbb{R} \text{ such that } R \ \geq \ R^{\flat}(t) \right\}$$

where $R^{\flat}(t)$ are the *lower bounds on returns* (instead of assuming that $dS(t) = S(t)(\gamma(R(t))dt + \sigma(R(t))dB(t)))$.

1. Tyches are identified (returns of the underlying, in our case) which can then *be used* in dynamic management systems when they are actually observed and known at each date during the evolution ;
2. For this reason, the results are computed in the worst case (*eradication of risk* instead of its *statistical evaluation*) ;
3. required properties are valid for *"all" evolutions* of tyches $t \mapsto R(t) \in \mathcal{R}(t)$.

   It requires the knowledge at investment date of the tychastic map described in this study by the function $t \mapsto R^{\flat}(t)$ of "lower bounds of underlying returns". The investor is supposed to provide the lower bounds $R^{\flat}(t)$ of the returns obtained, for example, by prediction or extrapolation method.

There are a myriad of ways for forecasting the lower bonds of the risky returns, from chartists to the most sophisticated econometric methods. For instance, these lower bounds can also be determined by stochastic methods, as we shall do in this study for the sake of comparison.

Otherwise, one can use the *VIMADES Extrapolator* (based on Laurent Schwartz distributions and on [13, Aubin & Haddad]) which bypasses the use of a "volatilimeter" by extrapolating each historical (past) time series of upper bounds ($HIGH$) and lower bounds (LOW) of the underlying prices provided by brokerage firms from which we can forecast the lower bounds of future returns of the underlying (see [6, Aubin, Chen, Dordan & Saint-Pierre])

Contingent uncertainty "offsets"tychastic uncertainty : In fact, the minimum guaranteed investment decreases when the contingent reservoir increase and increases when the "tychastic reservoir" $\mathcal{R}(t)$ increases, that is to say, in this study, when the lower bound of the underlying return decreases.

2.5.3 Nature and Measures of Insurance

In the stochastic case, the risk is "measured" by real numbers through statistical evaluation (values at risk, for example). These numbers are *abstractions*, which differ according to the methods and techniques used.

If the tychastic case, the risk is "measured" by numbers, providing at each date t the "minimum guaranteed investment " $W^{\heartsuit}(t)$, ensuring that the floor will *never* be pierced later. The "guaranteed cash flow" $t \mapsto W^{\heartsuit}(t)$ is the one associating with each date the minimum guaranteed investment.

There are parallels and differences between stochastic and tychastic approaches. Regarding viability and capturability issues, the stochastic case is a (too) special particular case of the tychastic one viability (see [6, Aubin, Chen, Dordan & Saint-Pierre] and [5, Aubin, Bayen & Saint-Pierre]).

This study provides the hedging of the same floor under the same data : annualized rate for discounting liabilities, annualized risk-free rate (changes in the cash portion of the assets), annualized return on the risky asset (stock index) and annualized volatility of the risky asset.

1. in section 3, p.12 the standard geometric Brownian movement for evaluating the losses whereas [24, Cont & Tankov] uses Lévy processes for optimizing the choice of the multipliers ;
2. in section 4, p.17, the mean return and the volatility are used to derive the lower bounds of future returns needed for computing the VPPI through the tychastic approach.

3 Risk Assessment With the CPPI Approach : Illustration

In the CPPI approach, the proportion of the portfolio invested in the risky asset varies in proportion to the amount invested in risk-free asset, to ensure at each date the floor level.

Mathematically, we introduce the (cushion) multiplier m defining the CPPI management rule (6), p.8.

In practice, the multiplier should be adjusted for market movements, while staying within a target area.

We note that the calibration of the rate used for discounting liabilities may be based on the results of Monte Carlo simulations of the portfolio (as seen in the sequel).

The mechanism of the cash flows will be assumed to be the next one. At each annual payment flow RTT :

- fund assets increased by the amount of flow contributed
- The liability is increased by the value of the stream between the payment date and maturity.

In addition, at each month :

- The allocation of the fund's assets between the risky asset (equity) and risk-free asset (cash) is adjusted by applying the CPPI strategy ;
- The liability of the fund is capitalized from the previous month (using the same rate as used for discounting).

3.1 Assets modelling

Regardless of the management CPPI, the portfolio's exposure is assumed to evolve following a stochastic process like Black-Scholes (with a monthly discretization).

Thus, at each date t (monthly), the exposure $E(t)$ evolves according to the following process :

$$E(t) \;=\; E(t-1)e^{\left[(\mu_m - \sigma_m^2/2) + \sigma_m W(t)\right]}$$

The parameters μ_m and σ_m are respectively, the monthly performance and the monthly volatility of the equity index. Monthly values of the performance of the index and its volatility are deduced from the annualized values μ and σ by the usual conversion formulas.

3.2 Liabilities Modelling

The initial liability of the various funds is taken equal to the value of the first flow paid, discounted from the date of payment to the due date (retirement).

The monthly rate τ_m for discounting liabilities and which affects the level of security (thus also the process of dynamic reallocation) may be specifically calibrated or be deducted from the other model parameters.

$$L(t_0) \;:=\; \frac{F(t_0)}{(1+\tau_m)^{n_m}}$$

where : $L(t_0)$: initial liabilities of the funds (or liabilities of the launch date) $F(t_0)$: the first flow paid n_m : the number of months to maturity (between the date of payment and the date of retirement)

A monthly frequency (time between each annual payment flow), the previous liability is capitalized according to the expression :

$$L(t) \ = \ (1 + \tau_m)L(t - 1)$$

At each annual date of flows payment, the new flows paid (discounted between the payment date and maturity) is added over the previous liabilities :

$$L(t) \ = \ (1 + \tau_m)L(t - 1) + \frac{F(t)}{(1 + \tau_m)^{n_m}}$$

The new liability (after the flow payment) is equal to the sum between the liability of the previous month multiplied by $(1 + \tau_m)$ and also the flow paid discounted between the date of payment and the due date.

3.3 Calibration of the market parameters

The various market parameters of the model to calibrate the following :
- annualized risk-free rate (changes in the cash portion of the assets) : r ;
- annualized return on the risky asset (stock index) : μ ;
- annualized volatility of the risky asset : σ ;
- annualized rate for discounting liabilities : τ_a (can be deduced from τ_m by the usual conversion formulas).

Levels of performance and volatility of equity index can be calibrated from historical data (depending on geographical area).

At this stage of the study, we propose the following choices :
- annualized risk-free rate (changes in the cash portion of the assets) : $r = 1.5\%$;
- annualized return on the risky asset (stock index) : $\mu = 5.5\%$;
- annualized volatility of the risky asset : $\sigma = 20\%$.

Finally, for the annual discount liabilities, we propose to retain for now the risk-free rate r. This choice ensure a negligible probability of negative final cushion (results of simulations).

Another approach (less conservative), however, could be to use a weighted average of the risk-free rate and the annual performance of the index action, following the initial allocation of portfolio assets :

Where is the percentage of the portfolio initially invested in the risky asset.

3.4 Calibration management parameters

Between each flow payment date, the asset allocation of the fund is adjusted monthly according to a strategy to cushion type CPPI. As seen above, this is to provide, at each month t, the amount $E(t)$ placed in the risky asset equal to a multiple m of the cushion of the portfolio at that date. This adjustment of the exposure is (in theory) instantaneously at time t, without enough time to the market to fluctuate in the interim.

In the "classic" CPPI management we so have, we denote by m the (cushion) multiplier.

An additional constraint was added in relation to the CPPI classic management, it follows from the fact that the portfolio assets (and also the cushion) is likely to be increased in frequency by the payment of annual flows.

Specifically the business rule is : where is the value of the asset portfolio at the time t. Monthly this constraint reflects the inability to invest in shares in excess of the asset portfolio. We also note that the annual payments of subscribers, whose effect is to increase the cushion to reduce the risk that the assets are monetized in case of sudden fall of the markets.

The parameters specific to the management that need to be calibrated are the following :

- the multiplier m ;
- The frequency between two adjustments of the exhibition : t.

For this study we limited ourselves, for illustration, in case of a monthly rebalancing of the exhibition. A weekly or even daily, can improve the management of the guarantee taking into account the ability to a brutal market "break" between two dates of adjustment. However, in practice, a compromise must be struck between the precision of the management of the cushion and the limitation of transaction costs (these costs are reduced if the operations of purchase / sale is via futures contracts on the equity index reference).

3.5 Risk Assessment

The choice of the multiplier m is done by using Monte Carlo simulations of the monthly management of the fund : for different values of m (i.e. from 0.5 to 20 with an increment of 0.1), we simulate 1000 trajectories of the portfolio (assets and liabilities) by integrating the timing of flows for the age group considered.

The CPPI approach can be based on risk assessment techniques by seeking a measure of the potential of a summary financial position by a sufficiently significant number (quantifying the risk associated with this position). In portfolio management, two measures of risk have been introduced :

- Measures of dispersion around a particular value : it is essentially the variance introduced by [31, Markowitz] ;
- Measures of "security" introduced for example by [39, Roy]. They are based mainly on the probability control of a return below a certain level.

It is this kind of concern that is causing the recent development of the theory of risk measures. A measure recently imposed, through its adoption by the regulatory bodies in both the worlds of finance than in the insurance : it is the "Value at Risk" (VaR). This represents the value of a quantile for a given probability distribution (with a confidence level set).

We present graphs of the levels of criteria based on the cushion, depending on the choice of m (1 000 simulations).

- Sharpe ratio of the final cushion (ratio between the mean and the standard deviation of the cushion at the final date) ;
- VaR at 95% of final cushion
- Probability of cushion negative during the life of the fund ;
- Final cushion average ;
- Probability of negative final cushion

The choice of $m = 2.5$ seems a good compromise between reducing the risk and performance for investors : first performance indicators (Sharpe ratio of the final cushion, final cushion average) are stabilized from the value $m = 2.5$ and also the likelihood of a negative cushion (on the paths or in final) are practically equal to zero from the same value (see Figures (3), (4) and (5), p.18).

As an illustration, we show in the graphs below, for the optimal value of the multiplier, the shape of the average path of portfolio assets and collateral and the 1000 cushions simulated trajectories (see Figure (17), p.24). Graphs (6), p.18 and (7), p.18 show in turn that the probability of having a negative cushion increases (either during the life of the fund or at maturity) by the increase of m. This is entirely consistent with our expectations as m reflected the degree of exposure to the risky asset and therefore the risk assumed by the investor. At this point, note that the irregularities that appear on some curves beyond $m = 2.5$ (in Figure (5), p.18 for example) are explained by the fact that the number of simulations (1000) is not enough high to eliminate the defect around the trend.

For this value of $m = 2.5$, we present the average trajectory of the portfolio value (during the projection period) and the floor (guarantee level) above which the portfolio value must be at each date (see Figure (8), p.18). An important note to point out here, always about the CPPI approach : some of the projected market scenarios can lead to a portfolio which is at some point below the floor (guarantee) : the cushion becomes negative. This is illustrated in Figure (4), p.17, which highlights the impact of the multiplier choice on the "Value at Risk" (VaR) of the cushion at maturity. We see that, the higher the multiplier m , the lower the VaR, and thus the larger the risk of piercing the floor (or guarantee) at maturity. This is also confirmed by Figure (7), p.18 which gives the probability of a negative cushion at maturity based on the level of m.

In a probabilistic approach such as CPPI, the risk that the portfolio can be found at some point below the floor is never completely eradicated : it is only controlled to a certain level of confidence (the confidence level is adjusted via the choice of the multiplier)

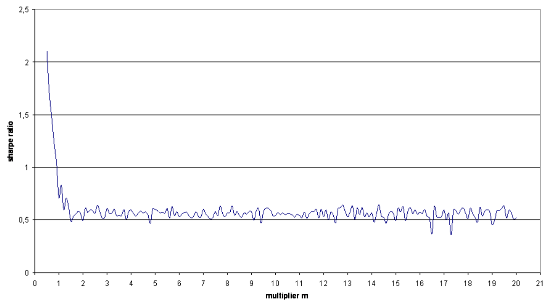


FIGURE 3 – Sharpe Ratio of final cushion based on m.

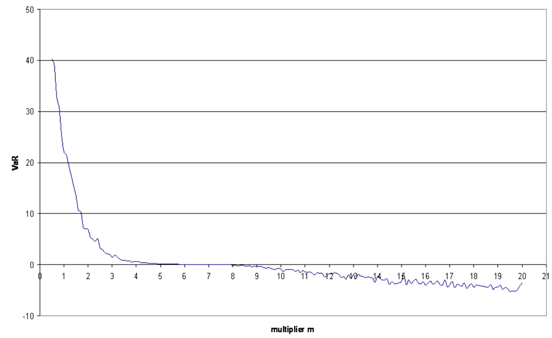


FIGURE 4 – VaR at 95% of final cushion in terms of m.

The strategy of portfolio insurance is one of the most interesting strategies in practice. A major drawback of this approach is the fact that it limits the choice of the business rule for investors : it reduces the problem to the one managing of the constant proportion. This potential lack of flexibility in terms of choice of system components to optimize (either in terms of the objective function or in terms of constraints), lead us to explore other techniques that can handle more general cases.

4 Risk Assessment With the VPPI Approach : Illustration

"Fixed Mix" and CPPI (Constant Proportional Portfolio Insurance) management rules are not fit for that insurance purpose since there are cases when the liability floor is pierced when they are used. The

1. initial investment (cushion) ;
2. fixed weights or multipliers ;

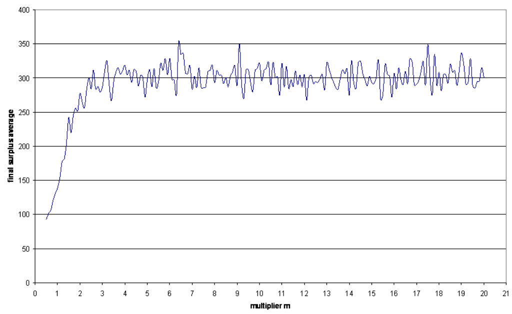


FIGURE 5 – Final cushion average in terms of m.

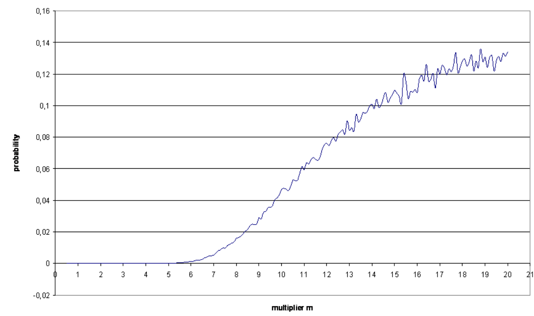


FIGURE 6 – Probability of negative cushion during the life of the fund (in terms of m).

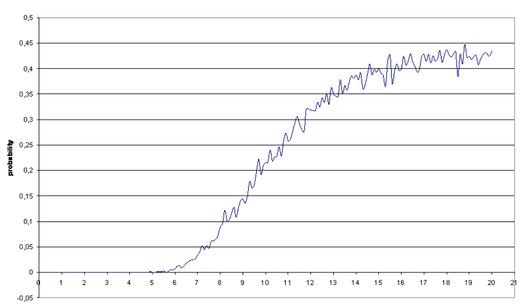


FIGURE 7 – Probability of final negative cushion with respect to m.

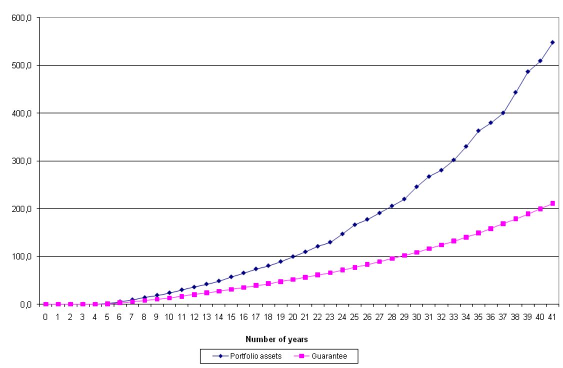


FIGURE 8 – Average trajectory of the portfolio (assets) and floor for $m = 2.5$.

have to be estimated by statistical methods optimizing several criteria.

Eradicating the risk in the sense of satisfying property (2), p.6 is a very simple *viability problem*, which has prompted the emergence of concepts and mathematical and algorithmic results gathered under the name of "viability theory".

It is quite natural to use them for defining and computing a dynamical ALM cushion management mechanism *ensuring that, at each date, the value of the portfolio is "always" exceeding liabilities.*

4.1 Portfolio insurance : a viability problem

To *insure* or *guarantee* a liability by asset-liability management can be formulated in the following way : to design a management rule of the portfolio that *guarantees* (or *insures*) that the net value of the portfolio is always higher than the liability (i.e., above the floor).

$$\forall\, t \in [0,T], \;\; W(t) \; \geq \; L(t) \tag{7}$$

Given the known liabilities at each date, the problem is to "dynamically manage the assets" in terms of

1. the dynamics governing the evolution of the portfolio based on *underlying returns* (and *transaction values* when the portfolio is not self-financed) ;
2. various *constraints* characterizing the "financial product".

The VPPI software provides an automatic report starting with a synoptic of the results represented in the following table summarizing the principal characteristic features of the portfolio :

1. At investment date, the ***insurance***

minimum guaranteed investment (MGI)	3.0
minimum guaranteed cushion (MGC)	3.0

2. At each date, when the return is known, the ***Management of the portfolio***

actualized exercise value	5.98
cumulated prediction penalties	0
liquidating dividend (in %)	0.54
net liquidating dividend (in %)	0.54

The robot-trader VPPI calculates at each date the minimum guaranteed investment (MGI) which ensures that the floor will never be pierced.

The Mobile Horizon Minimum Guaranteed Investment (MGI) provides at each date not only the insurance up to the exercise time, but to all shorter horizons ranging between this date and the exercise time : the shorter the insurance duration, the least costly the insurance.

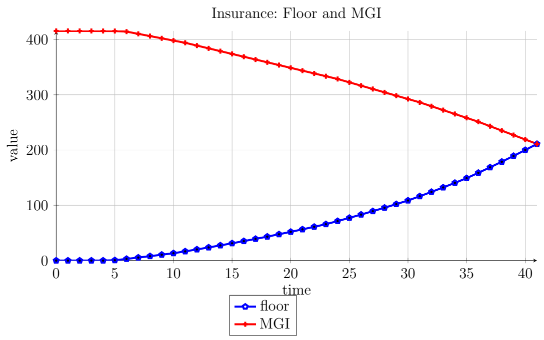


Figure 9 – **Floor and MGI.** The bottom curve represents the floor $L(t)$ that should never be pierced by the portfolio value. The top curve $W^{\heartsuit}(t)$ represents the minimum guaranteed investment, which depends only on the floor, the riskless return, constant mean return $\mu = 5.5\%$ and constant annual volatility 20%.

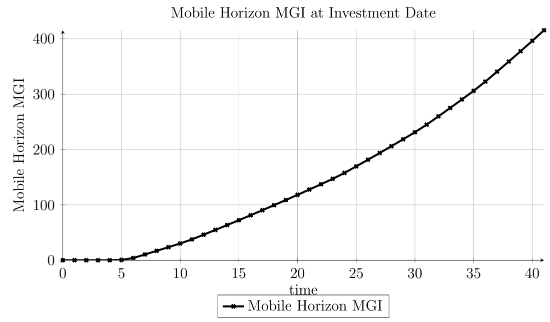


Figure 10 – **Mobile MGI.** The mobile horizon insurance costs are displayed in a table displaying at each date (row) and at each column (insurance duration) the amount of the minimum guaranteed investment (MGI). This figure represents the MGI from investment dates to increasing mobile exercise dates. Naturally, the amount the MGI increases the exercise date. Inverting this map, the investor can associates with his investment the exercise time for which he is (really) insured.

The VPPI management module calculates "opportunistically" the values of the portfolio depending on the effective realizations of the underlying prices. This is the case when these returns are higher than their forecasted lower bounds.

The management module determines each day the portfolio's exposure, from which we compute the number of shares of the underlying :

We thus derive the evolution of the cushions and the *a posteriori* multipliers (exposure over cushion ratio) :

4.2 The Viability Algorithm

The first step is the discretization of continuous time by discrete dates, functions by sequences and reformulate the data and concepts in this discrete framework (this step is not needed if the problem is directly formulated in discrete time, as it is often the case).

Then algorithms are used to calculate iteratively guaranteed capture basin of targets viable in an environment and the feedback rule. It uses techniques of *set-valued numerical*

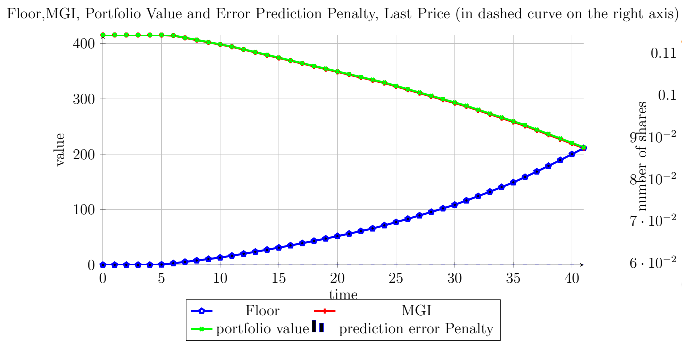


FIGURE 11 – Value of the portfolio associated with constant mean return $\mu = 5.5\%$ governed by the VPPI management rule, minimum guaranteed investment (MGI) and floor. In this example, the value of the portfolio is always, but slightly, above the MGI, as it is prescribed by the theory, so that their graphs are not distinguishable.

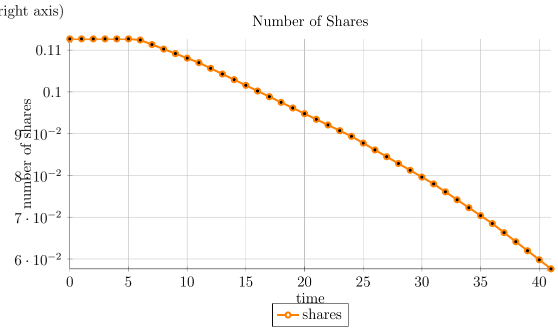


FIGURE 12 – **Shares of the Underlying.** Once the VPPI management rule is calculated and stored in the computer memory, the management module provides the number of shares of the underlying within its portfolio based on the realization of prices. The figure represents the evolution of the number of shares.

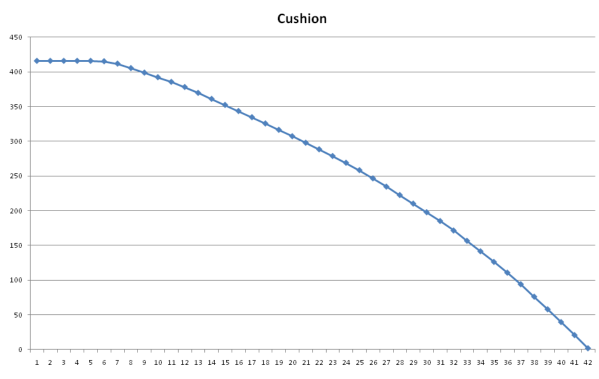


FIGURE 13 – **Evolution of Cushions.** This figure displays the value of the cushion of the evolution managed by the VPPI rule for the mean return $\mu = 5.5\%$.

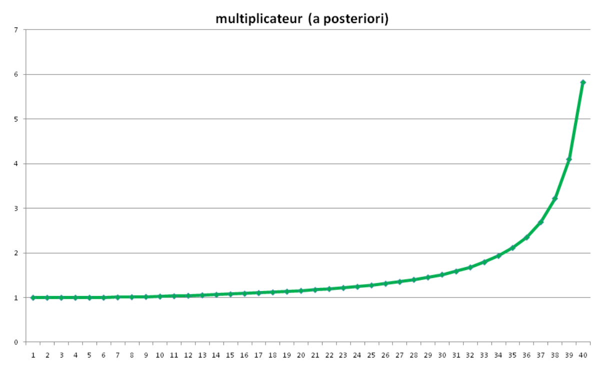
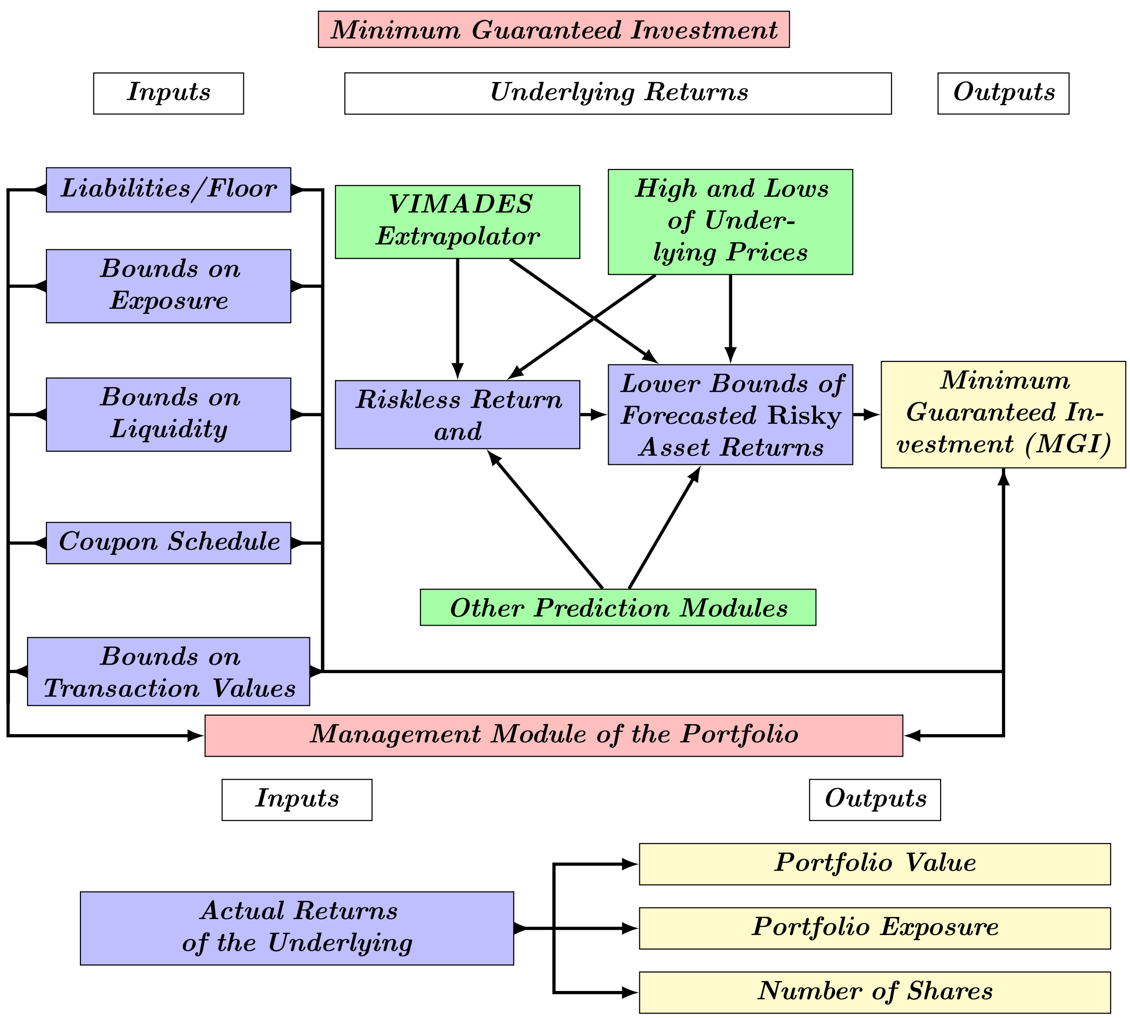


FIGURE 14 – **Evolution of *a posteriori* multipliers associated with the portfolio managed by the VPPI rule.** The cushion multipliers are the ratios of the exposition and the cushion, assumed to be constant and given in the CPPI management rule.

analysis handling discrete subsets (grids) mostly based the lattice properties of guaranteed capture basins.

These algorithms are used in the VPPI software. We describe below its flow chart. It indicates what are the inputs and outputs of the software, presented in the form of .csv files. It underlines the division of the programme into two steps :

1. **Insurance :** the *computation* of the Minimum Guaranteed Investments and Guaranteed Minimum Returns and their management rules ;
2. **Management :** the use of the management rules for managing the value of the portfolio knowing at each date the actual underlying return.



5 Conclusion : Comparisons between the VPPI and the CPPI

Even though the CPPI and VPPI approaches to the floor management of a portfolio are quite different, the first one belonging to the class of *a priori* management rules providing cushions the potential losses being statistically evaluated, the second one, *a posteriori* management rule designed for eradicating the losses. The question that the CPPI management rule leaves open is the empirical determination of the multiplier and the of the initial cushion (taken to be equal to zero), whereas the VPPI management rule "encapsulates" the cushion multipliers and computes the guaranteed minimum cushion :

1. *CPPI* : The floor, the initial investment, a prediction mechanism are given, as well as the multipliers (constant as the CPPI or not as in other rules) ;
2. *VPPI* : The MGI and the VPPI management rule are deduced from the floor and lower bounds of the future risky returns.

In both cases, however, the floor being fixed, the choice of the forecasting mechanism is open.

Parameters of and evaluation the statistical measures of the set of their solutions are needed

Whenever the CPPI management rule is chosen and "integrated" in the stochastic differential equation governing the values of the portfolio, the statistical evaluations depend naturally on this choice (see [24, Cont & Tankov], for instance).

The same is true for the VPPI management rule, the minimal guaranteed investment (or cushion) depending on the flow of the lower bounds of forecasted returns, but the losses always remaining eradicated.

We summarize in the next two figures the results obtained by the CPPI and the VPPI management rules respectively :

It may also be interesting to compare the average of the cushions of the evolutions provided by the Monte-Carlo Method used to compute the evolution obtained by the CPPI management rule and the cushion of the evolution of the portfolio managed by the VPPI rule when the return is the mean return μ. Their behavior is different. For the CPPI rule, the choice was made to avoid initial investment (zero cushion) whereas the initial cushion was computed by the VPPI rule in order that its evolution remain positive at each instant and for all returns. It thus decreases with time.

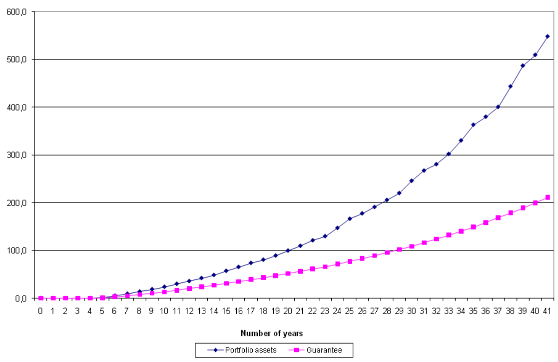


FIGURE 15 – Average trajectory of the portfolio (assets) and floor for $m = 2.5$ obtained with the CPPI Decision Rule.

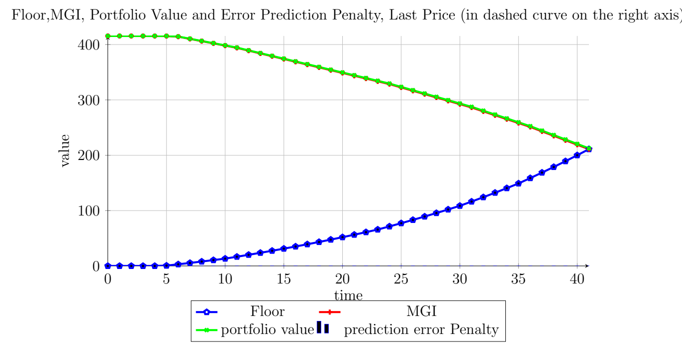


FIGURE 16 – Value of the portfolio associated with constant mean return μ governed by the VPPI management rule, minimum guaranteed investment (MGI) and floor.

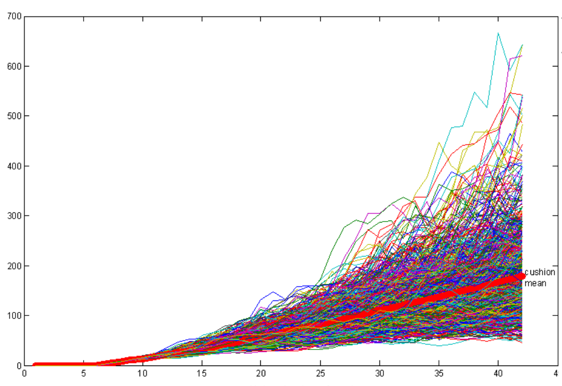


FIGURE 17 – Shapes of the 1000 simulated trajectories of the cushion ($m = 2.5$). The cushion associated with the average is displayed by a thick line.

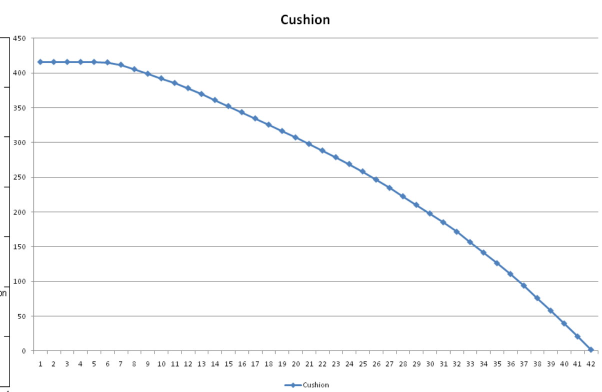


FIGURE 18 – **Evolution of Cushions.** This figure displays the value of the cushion of the evolution managed by the VPPI rule for the mean return $\mu = 5.5\%$. The cushion multipliers are the ratios of the exposition and the cushion, assumed to be constant and given in the CPPI management rule ($m = 2.5$).

Comparisons between VPPI and CPPI

	VPPI	CPPI
multipliers	computed	given
regulation	YES	given
insurance	YES (MGI)	given (loss evaluations)
prediction errors	YES (impulse management)	YES (jump processes)
Forecasting methods	Any method for predicting lower bounds of returns, e.g., Extrapolator of VIMADES	Stochastic processes

We demonstrate in this paper the interest and power of viability theory using a very specific illustration in finance. It allows, thanks to the concept of guaranteed viability kernel, to compute the minimum guaranteed investment, that is, the economic capital needed to resist to all events forecast by a given prediction model. However, a pricer makes sense only if it is associated with a specific decision rule which can be computed : in this specific case, we obtain the VPPI management rule instead of the CPPI one.

Even if the amount of the minimum guaranteed insurance appears to be too high, it provides a precious information to the investor who measures the risk of choosing a smaller investment : even if the worst is not certain, it may happen in a period of crisis. For measuring this kind of risk, the mobile horizon minimum guaranteed investment provides the duration of the guarantee of a smaller investment (computed at each date). We observe that if the reluctance to immobilize a guaranteed investment is due to the idea of allocating this amount to invest it in other assets hoping that diversification is beneficial, it is sufficient to observe the allocated investments will even be lower than the required minimum guaranteed of each of these assets, worsening the risk taken.

However, this theory devoted to the coupling between uncertain control dynamical systems and state (viability) constraints could be used to solve many insurances issues : cases when the floor may depend not only on time, but on the price of the shares, on the age, economic situation, health, choice of the optimal asset allocation in ALM problems, building of hedging rules for variable annuities contracts, financial management of pension plans, etc. So we hope this study will inform practitioners on relevant mathematic, algorithmic and software results of viability theory and lead them to consider them as a useful complement of classical stochastic approach. For that purpose, we provide a very short summary of such results in the appendix.

6 Appendix : Why Viability Theory ?

As we saw, the ALM management problem is to guarantee that the value of the portfolio satisfies the floor constraint

$$\forall\, t \in [0, T], \;\; W(t) \;\geq\; L(t) \tag{8}$$

The epigraph $\mathcal{E}p(L)$ of the floor function L is defined by

$$\mathcal{E}p(L) \ := \ \{(t, W) \text{ such that } W \ \geq \ L(t)\}$$

Hence inequality constraint (8), p.25 can be written

$$\forall\, t \in [0, T], \ \ (t, W(t)) \ \in \ \mathcal{E}p(L) \tag{9}$$

As a result, the hedging problem of a liability amounts to looking for evolutions $t \mapsto (t, W(t))$ "viable in the epigraph" $\mathcal{E}p(L)$ of the function L in the sense that

$$\forall\, t \in [0, T], \ \ (t, W(t)) \ \in \ \mathcal{E}p(L)$$

without leaving it (piercing the floor), i.e., to evolve above the graph.

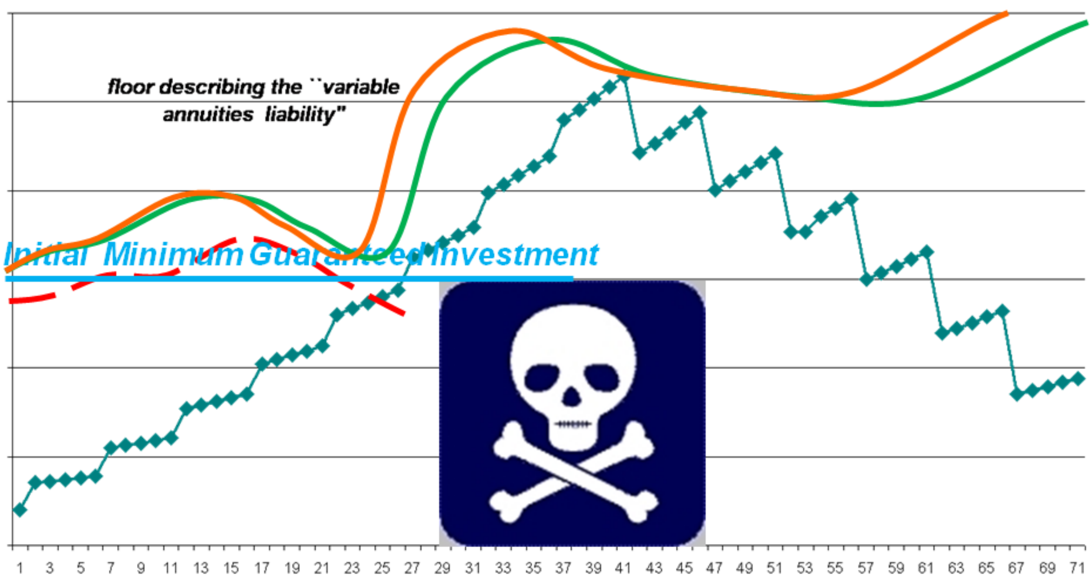


Figure 19 – **Symbolic Diagram** This diagram displays a floor (variable annuities), indicates that it should never be pierced and displays evolutions depending upon two different returns above the floor (viable in the epigraph) and one starting below which pierces the floor for at least one return. The initial MGI is the limit between these two behaviors : viable whatever the return above, unviable for at least one return below.

Instead of handling functions as in classical analysis, *viability theory manipulates subsets as in set-valued analysis* (see [12, Aubin & Frankowska] or [38, Rockafellar & Wets] for instance), and, in particular, graphs of maps and epigraphs of real-valued functions.

The second point is that the value of a self financed portfolio is governed by a (very simple) tychastic controlled system, where controls are the exposures $E(t, W) \ \in \ [0, 1]$ of the portfolio and the tyches are the returns $R(t) \geq R^{\flat}(t)$ of the underlying :

$$\begin{cases} & \forall\, t \in [0,T], \\ (i) & W'(t) \; = \; R^0(t)W(t) + E(t)(R(t) - R^0(t)) \text{ (evolutionary engine)} \\ (ii) & E(t) \; \in \; [0,1] \text{ (controls)} \\ (iii) & R(t) \; \geq \; R^\flat(t) \text{ (tyches)} \end{cases} \tag{10}$$

The study of dynamics under viability (or state) constraints is the purpose of viability theory which gathers the concepts and mathematical and algorithmic results addressing this issue (see [1, 2, Aubin] and [5, Aubin, Bayen & Saint-Pierre]).

Viability theory designs and develops mathematical and algorithmic methods for investigating the *adaptation to viability constraints of evolutions governed by complex systems under uncertainty* that are found in many domains involving living beings, from biological evolution to economics, from environmental sciences to financial markets, from control theory and robotics to cognitive sciences.

It deals with the confrontation between constraints (the epigraph of the floor or any subset $K \subset X$) and a controlled or regulated tychastic system (system (10), p.27)or $x'(t) = f(x(t), u(t), v(t))$ parameterized by controls $u \in U(x)$ and tyches $v \in V(x)$. They are examples of the viability approach to differential games extensively studied (see for instance [21, Cardaliaguet, Quincampoix & Saint-Pierre]).

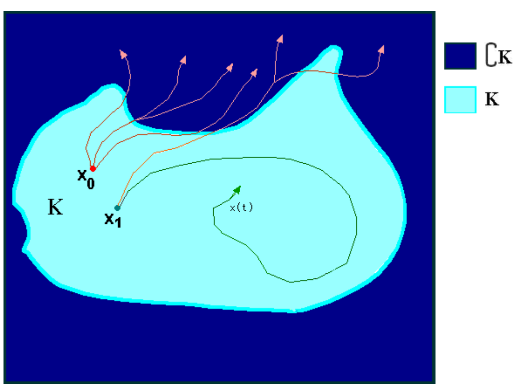


FIGURE 20 – **Environment.** This is the subset K defined by viability constraints. From x_0, all evolutions violate the constraints and fm x_1, one evolution is viable in the environment K.

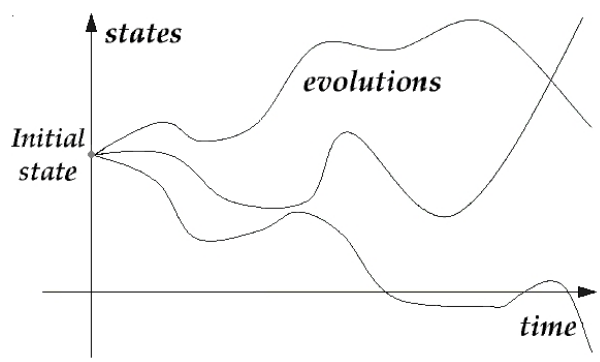


FIGURE 21 – **Evolutionary Systems.** This diagram symbolizes the fact that an evolutionary systems associates with any initial state a subset of evolutions (here, parameterized by controls and tyches).

The problem is formulated as follows : find

1. the *guaranteed viability kernel* $\text{Viab}(K) \subset K$ of K ;

2. the retroaction map associating with any $x \in \mathrm{Viab}(K)$ controls $\widetilde{u}_K(t,x)$;

such that, for any initial state $x \in \mathrm{Viab}(K)$, for any tyche $v(t) \in V(x(t))$, the evolution governed by

$$x'(t) \;=\; f(x(t), \widetilde{u}_K(t, x(t)), v(t))$$

is viable in K. The tools of Viability Theory has been designed to solve this type of problems ([5, Aubin, Bayen & Saint-Pierre] for the most recent account).

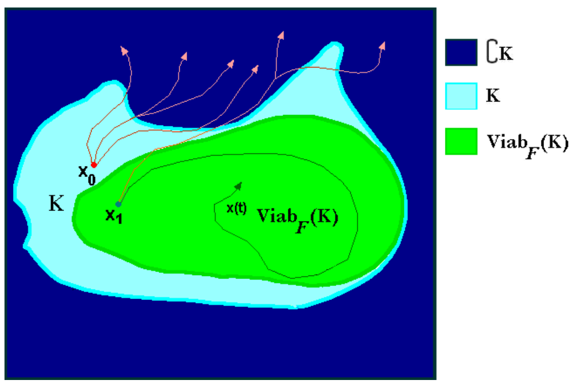


FIGURE 22 – **Guaranteed Viability Kernel.** This illustrates the subset of initial states from which all evolutions governed by the retroaction map are viable in the environment.

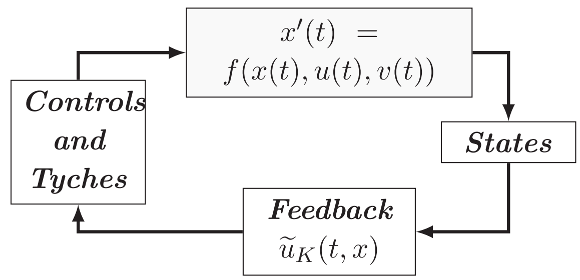


FIGURE 23 – **Feedback.**This diagram illustrates the regulated tychastic system mapping evolutions of controls and tyches to evolution of states and the feedback associating with controls states $\widetilde{u}_K(t,x)$.

The solution to this problem is given *in terms of subsets* : the guaranteed viability kernel $\mathrm{Viab}(K)$ and the graph of the retroaction map (graphical approach of maps) and uses the tools of set-valued analysis and mutational analysis (see [3, Aubin] for instance). They are not obtained through analytical formulas, but can be computed in the framework of "set-valued numerical analysis". The viability algorithms and software discovered in [40, Saint-Pierre] handle at each iteration subsets instead of vectors. They are subject to the "dimensionality curse", which limit the dimension of the problem.

Returning to the VPPI problem, we use the viability tools to characterize the MGI at each date and the feedback map associating with each state (t, W) the exposition $E^{\heartsuit}(t, W)$. We prove that the guaranteed viability kernel of the epigraph of the floor function is also an epigraph of a function, which is the MGI function.

Theorem 6.1 ***Minimum Guaranteed Investments.*** *The floor $t \mapsto L(t)$ and the lower bounds $t \mapsto R^\flat(t)$ of the returns on the underlying describing tychastic uncertainty. Then the VPPI computes at each date t*

1. *the (exposure)* management rule $E^\heartsuit(t, W) \in [0, 1]$ *(the feedback);*
2. *the* minimum guaranteed investment *(MGI) $W^\heartsuit(t)$;*
3. *and in particular the initial minimum guaranteed investment ("viability insurance") $W^\heartsuit(0)$*

such that

1. *starting at investment date* 0 *from $W_0 \geq W^\heartsuit(0)$, then* regardless the evolution of tyches $R(t) \geq R^\flat(t)$, *the value $W(t)$ of the portfolio is governed by the* management module

$$W'(t) \ = \ R^0(t)W(t) + E^\heartsuit(t, W(t))(R(t) - R^0(t)) \text{ (VPPI management module)} \quad (11)$$

   is always above the floor, and, actually, above the minimum guaranteed investment;
2. *starting at investment date* 0 *from $W_0 < W^\heartsuit(0)$, regardless the management rule $\widehat{E}(t, W) \in [E^\flat(t, W), E^\sharp(t, W)]$ (including the CPPI management rule and its variants), there exists* at least *one evolution of returns $R(t) \geq R^\flat(t)$ for which the value of the portfolio managed by*

$$W'(t) \ = \ R^0(t)W(t) + \widehat{E}(t, W(t))(R(t) - R^0(t)) \quad (12)$$

   pierces the floor.

Naturally, the method is general. If the floor $L(t, S)$ depends also on the asset price, and knowing the tychastic evolution of prices $dS(t) = R(t)S(t)dt$, the guaranteed viability kernel and its associated management rule provide the values of the portfolios replicating options (see [14, Aubin, Pujal & Saint-Pierre], [15, Aubin& Saint-Pierre]) for non self-financed portfolios. We can add many other variables and their tychastic evolution law. However, the computation (both available hardware and software) put a limit to the number of variables, so that a lot has to be done in this direction (parallel and intensive computing, etc.).

6.1 Tychastic Systems

Consider

1. a vector space $X := \mathbb{R}^d$ (interpreted as a *state space*) and a vector space $\mathcal{V} := \mathbb{R}^d$ (regarded as a *tychastic space* of tyches) ;
2. a (single-valued) map $f : X \times \mathcal{V} \mapsto X$ defining the differential equation $x'(t) = f(x(t), v(t))$ parameterized by tyches v (interpreted as a *tychastic system*) ;
3. a set-valued map $V : x \rightsquigarrow V(x)$ (interpreted as *tychastic (set-valued) map*) ;
4. a family $\widetilde{V}$ of tychastic retroactions $\widetilde{v} : (t,x) : \mathbb{R}_+ \times K \mapsto \widetilde{v}(t,x) \in V(x)$.

We associated with these data the set-valued map

$$f_{[\widetilde{V}]}(t,x) \;:=\; \bigcup_{\widetilde{v}\in\widetilde{V}} f(x, \widetilde{v}(t,x))$$

and the tychastic system

$$x'(t) \;\in\; f_{[\widetilde{V}]}(t, x(t)) \tag{13}$$

It generates the evolutionary system $\mathcal{S}_{\widetilde{V}} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ where $\mathcal{S}_{\widetilde{V}}(x)$ is the set of solutions $x(\cdot)$ of $x'(t) \;\in\; f_{[\widetilde{V}]}(t,x(t))$ such that $x(0) = x$.

Let $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$ be a subset of evolutions, For example, subsets of evolutions *viable in the environment*

$$\mathcal{V}(K) := \{x(\cdot) \text{ such that } \forall t \in [0,T],\ x(t) \;\in\; K\} \tag{14}$$

Let us consider a subset $\mathcal{H} \subset \mathcal{C}(0,\infty;\mathbb{R}^d)$ of evolutions. The *"core"* $\mathcal{S}^{\ominus 1}(\mathcal{H})$ of $\mathcal{H}$ under set-valued map$\mathcal{S}_{\widetilde{V}}$ is the set

$$\mathcal{S}_{\widetilde{V}}^{\ominus 1}(\mathcal{H}) \;:=\; \left\{x \in X \text{ such that } \mathcal{S}_{\widetilde{V}}(x) \;\subset\; \mathcal{H}\right\} \tag{15}$$

of initial states such that *all* evolutions starting from x belong to $\mathcal{H}$ and share its properties.

We obtain the following concepts of *invariance kernel of the environment* :

$$\mathrm{Inv}_f([\widetilde{V}];(K)) \;:=\; \mathcal{S}_{\widetilde{V}}^{\ominus 1}(\mathcal{V}(K)) \tag{16}$$

by taking the core under the tychastic system of the family of evolutions viable in the environment.

By defining the order relation $V_1 \;\subset\; V_2$ on set-valued maps by the inclusion relation $\mathrm{Graph}(V_1) \;\subset\; \mathrm{Graph}(V_2)$ on their graphs, we note that the application $(\widetilde{V}, K) \;\mapsto$ $\mathrm{Inv}_f([\widetilde{V}];(K))$ is decreasing with respect to the tychastic map and increasing with respect to K. It shows especially that the invariance kernel is the largest fixed point $G \subset K$ of the map $K \mapsto \mathrm{Inv}_f([\widetilde{V}];(K))$:

$$G \;\subset\; K \text{ et } G \;=\; \mathrm{Inv}_f([\widetilde{V}];(G)) \tag{17}$$

The topological properties of invariance kernels (closure and stability of the basin and kernels, etc.) are proved under the assumption that the dynamics of f and V tychastic set-valued map are Lipschitz.

Invariance theorems characterize the invariance kernels by tangential properties at the boundary relating velocities and tangent directions to the environment.

6.2 Stochastic and Tychastic Viability

The invariance kernel is an example of the core $\mathcal{S}^{\ominus 1}(\mathcal{H})$ of a subset $\mathcal{H} \subset \mathcal{C}(0, \infty; \mathbb{R}^d)$ for $\mathcal{H} = \mathcal{K}_{(K)}$ being the set of evolutions viable in K.

Let us consider *random events* $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, instead of tyches $v(\cdot) \in Q(x(\cdot))$.

A stochastic system is a specific parameterized evolutionary system described by a map $\mathbb{X} : (x, \omega) \in \mathbb{R}^d \times \Omega \mapsto \mathbb{X}(x, \omega) \in \mathcal{C}(0, \infty; \mathbb{R}^d)$ (usually denoted by $t \mapsto \mathbb{X}_\omega^x$ in the stochastic literature) where $\mathcal{C}(0, \infty; \mathbb{R}^d)$ is the space of continuous evolutions. In other words, it defines evolutions $t \mapsto \mathbb{X}(x, \omega)(t) := \mathbb{X}_\omega^x(t) \in \mathbb{R}^d$ starting at x at time 0 and parameterized by random events $\omega \in \Omega$ satisfying technical requirements (measurability, filtration, etc.) that are not relevant to involve at this stage of the exposition. The initial state x being fixed, the random variable $\omega \mapsto \mathbb{X}(x, \omega) := \mathbb{X}_\omega^x(\cdot) \in \mathcal{C}(0, +\infty; \mathbb{R}^d)$ is called a stochastic process. A subset $\mathcal{H} \subset \mathcal{C}(0, \infty; \mathbb{R}^d)$ of evolutions sharing a given property being chosen, it is natural, as we did for tychastic systems, to introduce the *stochastic core* of $\mathcal{H}$ under the stochastic system : it is the subset of initial states x from which starts a stochastic process $\omega \mapsto \mathbb{X}(x, \omega)$ such that for almost all $\omega \in \Omega$, $\mathbb{X}(x, \omega) \in \mathcal{H}$:

$$\mathrm{Stoc}_{\mathbb{X}}(\mathcal{H}) := \{x \in \mathbb{R}^d \mid \text{for almost all } \omega \in \Omega, \ \ \mathbb{X}(x, \omega) := \mathbb{X}_\omega^x(\cdot) \ \in \ \mathcal{H}\} \tag{18}$$

Regarding a stochastic process as a set-valued map $\mathbb{X}$ associating with any state x the family $\mathbb{X}(x) := \{\mathbb{X}(x, \omega)\}_{\omega \in \Omega}$, the definitions of stochastic cores (18) of subsets of evolution properties are similar in spirit to definition :

$$\mathcal{S}^{\ominus 1}(\mathcal{H}) := \{x \in \mathbb{R}^d \mid \ \text{for all } \ v(\cdot) \in Q(x(\cdot)), \ \ x_{v(\cdot)}(\cdot) \ \in \ \mathcal{H}\}$$

under a tychastic system

$$x'(t) \ = \ f(x(t), v(t)) \text{ where } v(t) \ \in \ Q(x(t))$$

Furthermore, *the parameters ω are constant in the stochastic case, whereas the tychastic uncertainty $v(\cdot)$ is dynamic* in nature and involves a state dependence, two more realistic assumptions in the domain of life sciences.

There is however a deeper similarity that we mention briefly. When the stochastic system $(x, \omega) \mapsto \mathbb{X}(x, \omega)$ is derived from a stochastic differential equation, the Strook-Varadhan Support Theorem (see [41, 42, Stroock & Varadhan]) states that there exists a tychastic system $(x, v) \mapsto \mathcal{S}(x, v)$ such that, whenever $\mathcal{H}$ is closed, the stochastic core of $\mathcal{H}$ under the stochastic system $\mathbb{X}$ and its tychastic core under the associated tychastic system $\mathcal{S}$ coincide :

$$\mathrm{Stoc}_{\mathbb{X}}(\mathcal{H}) = \mathcal{S}^{\ominus 1}(\mathcal{H})$$

We refer to [29, Doss] and [11, Aubin& Doss] for more details. It furthermore provides a characterization of stochastic viability in terms of tangent cones and general curvatures of the environments. Characterization of stochastic viability and invariance in terms of stochastic tangent cones has been carried over in [7, 8, Aubin & Da Prato] and [9, Aubin, Da Prato& Frankowska] and [26, 27, 28, Da Prato & Frankowska] in terms of distance functions among many other studies in this direction.

6.3 Regulated Tychastic Systems

We further introduce

1. a space $\mathcal{U} := \mathbb{R}^b$ (interpreted as a control space or regulon space) ;
2. a map $f : X \times \mathcal{U} \times \mathcal{V} \mapsto X$ defining the differential equation $x'(t) = f(x(t), u(t), v(t))$ parameterized by controls u and tyches v (interpreted as a controlled or regulated tychastic system) ;
3. a set-valued map $U : x \rightsquigarrow U(x)$ (interpreted as the contingent set-valued map) ;
4. a family $\widetilde{U}$ of contingent retroactions $\widetilde{u} : (t, x) : \mathbb{R}_+ \times K \mapsto \widetilde{u}(t, x) \in U(x)$.

We associate with these new data the set-valued map

$$f_{[\widetilde{u},\widetilde{V}]}(t, x) \; := \; f_{[\widetilde{V}]}(t, \widetilde{u}(t, x))$$

and the controlled (or regulated) tychastic system

$$x'(t) \; \in \; f_{[\widetilde{V}]}(t, \widetilde{u}(t, x(t))) \tag{19}$$

The *guaranteed viability kernel* is defined by

$$\text{GuarViab}_f([\widetilde{U}, \widetilde{V}]; K) \; := \; \bigcup_{\widetilde{u} \in \widetilde{U}} \text{Inv}_f([\widetilde{u}, \widetilde{V}]; K) \tag{20}$$

which depends on the set K on one hand and on the pair $[\widetilde{U}, \widetilde{V}]$ made of retroactions $\widetilde{u} \in \widetilde{U}$ defining the contingent uncertainty and $\widetilde{v} \in \widetilde{V}$ defining the tychastic uncertainty, on the other hand.

A retroaction $u^\heartsuit \in \widetilde{U}$ is a *control (or management) rule* if

$$\text{GuarViab}_f([u^\heartsuit, \widetilde{V}]; K) \; = \; \text{GuarViab}_f([\widetilde{U}, \widetilde{V}]; K) \tag{21}$$

We note that the map

$$(K, \widetilde{U}, \widetilde{V}) \mapsto \text{GuarViab}_f([\widetilde{U}, \widetilde{V}]; K)$$

is increasing respect to K and $\widetilde{U}$, on the one hand, and decreasing with respect to $\widetilde{V}$, on the other hand (for the inclusion relation sets).

If tychastic uncertainty (described by $\widetilde{V}$) increases, the guaranteed viability kernel guaranteed decreases, so it is necessary to also increase the set $\widetilde{U}$ (translating contingent uncertainty) for increasing the guaranteed viability kernel. Once the environment K is given, the map

$$(\widetilde{U}, \widetilde{V}) \mapsto \text{GuarViab}_f([\widetilde{U}, \widetilde{V}]; K)$$

leads to a new concept of "game" on set-valued maps U and V and taking values in the family of (closed) subsets of the state space. In this context, we choose here the set $\widetilde{V}$ of open loop feedbacks and the set $\widetilde{U}$ a set of closed loop feedbacks. However, many other options have been proposed and studied.

Viability theory provides mathematical and algorithmic properties of guaranteed viability kernels, including the fixed point property of the guaranteed viability kernel guarantee, simple to state but important : the *guaranteed viability kernel is the largest subset* $G \subset K$ *satisfying the fixed point property*

$$G \ = \ \text{GuarViab}_f([\widetilde{U}, \widetilde{V}]; G) \tag{22}$$

The viability algorithms provide means to compute the guaranteed viability kernels and to program them (see [40, Saint-Pierre], [21, Cardaliaguet, Quincampoix & Saint-Pierre], among many other papers).

In the same way that it is impossible to summarize in few pages the properties of stochastic differential equations for a layman in probability, it is impossible to go beyond the basic concepts and technical viability, even briefly. This is a reason why we invite the reader who would like to deepen these results to refer to the literature on viability theory.

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