Longevity Risk, Rare Event Premia and Securitization

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ABSTRACT. Longevity securitization is a financial innovation that provides annuity insurers and pension plans a capital market hedge for their longevity risks. This paper proposes longevity derivatives written on population longevity indices for older ages. Moreover, a model for analyzing and pricing longevity rare event contingent claims is introduced, following Liu et al. (2005). The equilibrium longevity risk premium has three components: the diffusion-risk and jump-risk premiums, both driven by risk aversion; and a "rare-event premium", driven exclusively by model uncertainty aversion. We show how to price longevity options using this model. Among other results, we explain the arguably high risk premium of securities linked to catastrophe risks.

1. INTRODUCTION

Longevity risk is defined as the risk of dramatic improvement in life times relative to current expected values. Longevity is a dynamic phenomenon. No matter how rare catastrophic longevity events may be, there are economic and policy changes that make management of longevity risk more important than ever. The rapidly growing populations of the elderly are putting unprecedented stresses on both public and private pension plans as well as the annuity industry. The Social Security system in the United States is facing significant future imbalances attributed to unanticipated long life. This problem already plagues the underfunded defined benefit systems of many European and Asian nations (Brown, 2000). Furthermore, current developments in defined contribution pension plans, particularly the growth of 401(k) plans, and baby boomers approaching retirement seem likely to stimulate future annuity demand. In addition, ongoing Social Security reform discussions in the United States and other nations also have the potential to increase the demand for private annuities. As demand for individual annuities increases, the need for insurers to manage their potential longevity risk increases as they write new individual annuity business (Lin and Cox, 2005).

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The increased activity in insurance-linked securities offers a capital market solution for longevity risk. According to Bauer and Kramer (2007), this is having an effect on insurance industries all over the world. The securitization of insurance risks began with catastrophe-based bonds, bringing the financial and insurance markets closer to convergence. Insurance-linked securitization, similar to mortgage-backed securities in banking, repackages risks and allows for more efficient allocation. Henri de Castries, chairman of the management board and CEO of the AXA group summarized the role of insurance securities,

> "I don't see mortgage risk in banks' balance sheets. I see them give the service, take the fees and offload the risk. We need to develop that model in the future because it will make the industry less capital intensive, both life and non-life."

That is, insurers and reinsurers could move from their traditional *risk warehousing* function toward a *risk intermediation* function allowing them to operate more efficiently as well as increase underwriting capacity by extention to capital markets, as others have suggested (Jaffee and Russell, 1997; Froot, 2001; Cowley and Cummins, 2005; Lane, 2006; Cox and Lin, 2007).

Financial innovation has led to the creation of several new classes of mortality securities in 2000s that provide opportunities to manage catastrophe death risks more efficiently. After successfully issuing the first-ever pure death-linked security in December 2003, the Swiss Re sold another three mortality bonds (Lane, 2006). Following Swiss Re, other life insurers may want to reduce their extreme mortality exposures by finding a financial market solution. For example, in November 2006, AXA issued its first catastrophe mortality deal — the Osiris bond (Lane and Beckwith, 2007).

However, capital market solutions for unanticipated longevity risk have only been explored relatively recently, first appearing in articles by Blake and Burrows (2001), Milevsky and Promislow (2001), Lin and Cox (2005) and others. Possibly inspired by the successful securitization of catastrophe mortality risks, in November 2004, the European Investment

Bank (EIB) offered the first longevity bond to provide a solution for pension plans to hedge their long-term systematic longevity risks.

However, unlike the Swiss Re mortality bonds, the EIB longevity bond did not sell. The design of the EIB bond is problematic. The EIB bond provides "ground up" protection, covering the entire annuity payment. But the plan can predict the number of survivors to some extent, especially in the early contract years. The EIB bond price includes coverage the plan doesn't need (including rates, commissions, etc.). A more attractive design might cover payments to survivors in excess of some strike level. The price would be much lower as shown in Lin and Cox (2005). The payoffs of longevity bond in Lin and Cox (2005) are based on the mortality experience of insurance companies which may not be transparent to the investors. To reduce moral hazard problem and thus transaction costs, we introduce a population longevity index for older ages constructed from publicly accessible data. Then we propose a longevity call option with cash flows linked to our mortality index.

Moreover, we notice there are only a few preliminary papers in longevity securitization modeling. Developing asset pricing theory in this area is important since it will help market participants better understand this new financial instruments. Cairns et al. (2006) provide a detailed overview and a categorization of stochastic mortality models. Most of those stochastic mortality models are short rate mortality models. Miltersen and Persson (2005) model the forward mortality intensity instead of the spot mortality intensity, taking the whole forward mortality curve as an infinite-dimensional state variable. Dahl and Møller (2006) derive risk-minimizing hedging strategies for insurance liabilities in a market without mortality derivatives. Lin and Cox (2007) price mortality bonds in an incomplete market framework with the Wang transform. Biffis (2005) employs affine jump-diffusion processes to model financial and demographic risk factors. Cox et al. (2006) use the multivariate exponential method to capture mortality correlations.

However, recent changes in mortality evolution challenge mortality projection models. We have imprecise knowledge about rare longevity events. Rogers (2002) shows that mortality operates within a complex framework and is influenced by socioeconomic factors, biological variables, government policies, environmental influences, health conditions and health behaviors. Sometimes because of limited sources or inadequate technical support, a projection model cannot reflect all aspects of an insured or a pension pool. The quality of data is also a concern for mortality projection. For example, although detailed data on old-age mortality are collected in most developed countries, they are not so commonly available for developing countries. Buettner (2002) claims that even in developed countries, the quality of age reporting deteriorates among the very old. Indeed, we have very few extreme mortality improvement cases — leaving little room to learn from experience. In dealing with mortality fluctuations, one might have reasonable faith in the model built by actuaries or financial economists. However, one cannot help but feel a tremendous amount of uncertainty about the model of rare longevity events. And if market participants are uncertainty averse, as described in Ellsberg (1961), then the uncertainty about rare events will eventually find its way into security prices in the form of a premium (Liu et al., 2005).

Liu et al. (2005) study the asset pricing implication of imprecise knowledge about rare events. Their approach to model uncertainty falls under the general literature that accounts for imprecise knowledge about the probability distribution with respect to the fundamental risks in the economy (Gilboa and Schmeidler, 1989; Epstein and Wang, 1994; Andersen et al., 2000; Chen and Epstein, 2002; Hansen and Sargent, 2001; Epstein and Miao, 2003; Routledge and Zin, 2002; Uppal and Wang, 2003; Maenhout, 2004). The equilibrium security premium in Liu et al. (2005) has three components: the diffusion- and jump-risk premiums, both standard risk-based premiums; and the "rare-event premium," driven exclusively by model uncertainty aversion. They argue that the investor is worried about model misspecification with respect to rare events, while feeling reasonably comfortable with the diffusion component of the model. They derive the equilibrium pricing kernel explicitly, and use the standard power utility function to price the diffusion and jump shocks. Moreover, they introduce a new parameter to capture the risk aversion to model uncertainty. The longevity securitization provides an ideal setting for their pricing theories because catastrophe longevity risks are apparently illiquid and rare events. Therefore, we apply Liu et al. (2005) to explicitly derive the equilibrium longevity risk premium including uncertainty aversion component and then price our proposed longevity option. Apparently our introduction of rare-event premium to insurance securitization is new. Our results nicely explain arguably high risk premia of those securities.

Our paper is organized as follows. Section 2 provides an overview of longevity risk. Section 3 discusses the capital market solution of longevity risk, points out the design problem of the EIB bond and presents an improved structure with our proposed longevity index of old ages. Section 4 first models the mortality dynamics as the combination of a geometric Brownian motion and a compound Poisson process, estimates the equilibrium longevity risk premium considering model uncertainty aversion of participants in longevity risk business and prices our longevity call option. Section 5 concludes the article.

2. Longevity Risk

Over the past half century, and especially in the most recent decades, remarkable improvements have been achieved in survival, especially at the highest ages. This progress has accelerated the growth of the population of older people and has advanced the frontier of human survival substantially beyond the extremes of longevity attained in preindustrial times. For example, average life expectancy in the world has more than doubled, rising from 26 years in 1820 to 60 years today (Bourguignon and Morrission, 2002). Since 1960, the share of the U.S. population above 65 years of age has grown substantially, from about 9 percent to 14 percent. Other developed countries have experienced even more rapid growth. For example, in many European nations, the elderly population accounts for nearly one-fifth of the total population (Lakdawalla and Philipson, 2002). Vaupel (1998) attributes the past mortality improvement to the better health conditions for the elderly, salubrious behavior, the large number of healthy immigrants into the United States in the decades before 1920, and the better childhood health at the beginning of last century.[1] However, Pope and Wimmer (1998) observe that the dramatic improvements in morbidity and mortality began well before major investments in public health or gains in modem medical technologies.

What would be the future mortality trend? The mortality improvement results from some mix of genetic, environmental, behavioral, bio-reliability, and heterogeneity forces and constraints, but the mix is not well understood. In contrast, we are warned of excess mortality from obesity, sun exposure, SARS, etc. (Hardy, 2005). Therefore, mortality may improve either moderately or dramatically (Buettner, 2002), stabilize around current level (Rogers, 2002; Hayflick, 2002) or even worsen (Rogers, 2002; Goss et al., 1998; Lin and Cox, 2007).

Longevity risk has profound impact on both public and private pension plans as well as the annuity industry. Hardy (2005) points out that life expectancy for men aged 60 is more than five years longer in 2005 than it was anticipated to be in mortality projections made in the 1980s. It is a good news for the pensioner, but is potentially catastrophic for the pension provider who failed to anticipate the longevity improvement and has to pay five more years of pension. As the population ages, the Trustees of the U.S. Social Security system forecast that, without changes, contributes will fall below benefits in 2012, and the system's trust fund will be exhausted in 2030 (Mitchell and Zeldes, 1996). On the other hand, insurance companies factor future mortality improvement into their premium, but, to the extent that it is unanticipated, longevity risk is still an enormous problem for annuity managers.

A major problem is that mortality improvement of the entire population is not a diversifiable risk. Traditional diversifiable mortality risk is the random variation around a reasonably well-known mortality probability following the law of large numbers. Mortality improvement risk, though, affects the whole portfolio and thus breaks down the risk pooling mechanism. There is some possibility of hedging for insurers, by selling life insurance to the same

[1]Conditions during childhood have lingering effects on health at advanced ages (Vaupel, 1998).

lives that are buying annuities or maintaining more balanced business between life insurance and annuities, but that could be expensive and technically infeasible (Cox and Lin, 2007). Furthermore, pension plans are not allowed to sell life insurance. Currently, the reaction to the longevity problem is twofold (Hardy, 2005). First, we are trying to produce better models for mortality prediction. In this, we are more concerned than ever before in the levels of uncertainty involved in our forecasts. However, how precisely can we predict longevity rare events? A second possibility is to turn to the capital markets to share the risk. And that is the focus of this paper.

3. Capital Market Solution for Longevity Risk

In November 2004, the European Investment Bank (EIB) offered the first pure longevity-risk linked deal — a 25-year 540 million-pound (775 million-euro) bond as part of a product designed by BNP Paribas.[2] The bond offered a longevity hedge to UK pension schemes. However, as far as we know, no one has purchased the bond. In this section, we first discuss the EIB longevity bond and we suggest an alternative hedge, an option on a longevity index for older ages.

3.1. **EIB Longevity Bond.** Here is how the EIB bond works: The bond's cash flows are based on the actual longevity experience of the English and Welsh male population aged 65 years old in November 2004, as published annually by the U.K. Office for National Statistics. The future cash flows to bondholders amount to an annuity, £50 million, multiplied by the percentage of the reference population still alive at each anniversary.

Since the number of survivor to the first anniversiary can be forecast rather accurately, the first annuity payment in November 2005 (as considered in November 2004) is essentially risk-free. The uncertainty in future estimates increases slowly so the riskiness of the future cash flows increases gradually.

[2] From *www.IPE.com* on November 8, 2004.

We see this as a problem with the EIB bond. It is capital intensive; it requires a high degree of upfront capital commitment for the degree of protection it offers. From the buyer's perspective, the EIB bond provides coverage in the early years that is expensive and unnecessary. This problem may contribute to its failure. Perhaps this is a lesson for us. This problem does not arise with options which can hedge "extreme risks" without the added cost of "ground-up" protection.

3.2. **Longevity Index Derivatives.** There is a growing number of derivatives written on indicies, which are based on underlying risks that are not traded. For example, there are derivatives written on weather variables such as rainfall and temperature. There seems to be no limit so long as a reasonable index can be developed which meets the needs of hedgers and investors. We think this will work for longevity too.

3.2.1. *A Longevity Index for Older Ages.* We propose a set of survival rates for a range of ages and survival periods as longevity indices. Other factors could work as well. Regardless of the index choice, it should be based on relatively frequent mortality studies or assessments of a reference population. The mortality assessment has to be open to the public so investors and hedgers can make their own calculations. For example, a pension plan making a hedge will need to have data in order to determine its basis risk. Basis risk arises because the populations underlying the index and the pension plan are different. Although the people are subject to the same forces of mortality, there will be random differences in there future survival rates.

The index provider will have to be accepted as an expert, unbiased provider of the mortality information. As far as we know there is no really good choice for a series of mortality assessments, but this may be changing. We learn that J.P. Morgan is developing mortality assessment technology with the aim of providing indices for trading (Loeys et al., 2007).

For this paper we are using the time series of mortality tables produced by the U.S. Census Bureau and National Center for Health Statistics. We refer to these as the U.S. Population

Tables. They are produced annually and would make an acceptable, practical basis for a mortality index, except that they are published only annually and only several years after the experience year. As of May 2007, the latest mortality table is based on 2003 mortality experience. Other data bases (or a combination) could be used.

The U.S. Population mortality data is available from the Human Mortality Database (HMD),[3] The population mortality experience of the year t is reported (in part) in the form of a table of values ℓ_x for age $x = 0, 1, \ldots, 109, 110+$ in the U.S. in year t where $t = 1946, 1947, \ldots, 2003$. The table values represent an idealized population of ℓ_0 new born lives all independent and subject to the mortality observed in the current year. The next value ℓ_1 is the number of those new born live expected to survive 1 year in the current mortality environment. In the same way, ℓ_x denotes the number of those new born lives expected to survive to age x. In the mortality study, no attempt is made to forecast future mortality. The expected values are a result of assessing current mortality experience.

The ratio

$$_kp_x = \frac{\ell_{x+k}}{\ell_x}$$

is the implied probability that a life age x will survive k years, given the current mortality. For example, $_{20}p_{75,2003}$ is the probability, determined in 2003, that a person aged 75 survives for 20 years, to age 95. These ratios are the basis for our proposed mortality index. We add a subscript to denote the year of the underlying mortality experience. The U.S. population longevity index for ages x in year t is denoted $_k\mathcal{I}_{x,t}$. For past years, the index is the ratio $_kp_{x,t}$ from the U.S. Population Tables for year t. For a future year t, the index $_k\mathcal{I}_{x,t}$ is a set of random variables indexed by x and k, modeled as a stochastic process (in section 4):

$$(1) \qquad _k\mathcal{I}_{x,t} = \begin{cases} _kp_{x,t} = \dfrac{\ell_{x+k,t}}{\ell_{x,t}} & \text{for past years } t \\ \text{Random value to be modeled} & \text{for future years } t \end{cases}.$$

[3] `http://www.mortality.org`.

The special case of $k = 1$, the one-year survival rate for age $x + i$ has a slightly different notation

$$p_{x+i,t} = \frac{\ell_{x+1,t}}{\ell_{x,t}},$$

as determined in year t. The complementary probability is the death rate, denoted $q_{x,t} = 1 - p_{x,t}$. Loeys et al. (2007) propose to use the one-year death rates as a mortality index. In contrast, we are using the k year survival rate. For a given x and k, an increase in ${}_kp_{x,t}$ from t to $t + 1$ means that in year $t + 1$ the observed probability of survival from age x to age $x + k$ is higher than the estimate of the same probability as measured in year t. Of course, as the index increases, longevity risk of annuity insurers and pension plans increases as well. Therefore, we believe that the survivor index is a better choice since it provides a direct hedge of annuity and pension plan longevity risk.

In Figure 1 we plot the longevity index $10,000_{10}p_{75,t}$ for 10,000 people aged 75 at time $t = 1946, 1947, \dots, 2003$ surviving for 10 years, surviving for 15 years $10,000_{15}p_{75,t}$ as well as surviving for 20 years $10,000_{20}p_{75,t}$. We observe a sharp rise in survival rates in the 1970s. While longevity has been steadily rising for many decades before and after 1970's, the spurt during 1970's hastened the trend. Moreover, compared to demography changes of the whole population in the 1970's, the improvement in mortality of old ages in the same period is much more dramatic than young ages. In fact, Cutler and Richardson (1998) find that greater improvement for the elderly than for the young after 1970 is a result of a decrease cardiovascular disease deaths. Since cardiovascular disease is more prominent late in life than earlier, the life expectancy gain is greater for the elderly than for the young. Thus mortality improvements due to a particular cause of death may vary by age.

3.2.2. *Longevity Option.* As an example of a longevity index option application, consider a pension plan (or an annuity provider) expected to have 10,000 pensioners aged $x = 75$ in year $T = t + 10$, 10 years from the current year t. The pension plan makes current estimates of survival rates ${}_1p_{75}$, ${}_2p_{75}$, $\dots$, in order to estimate and fund payments it will make to the

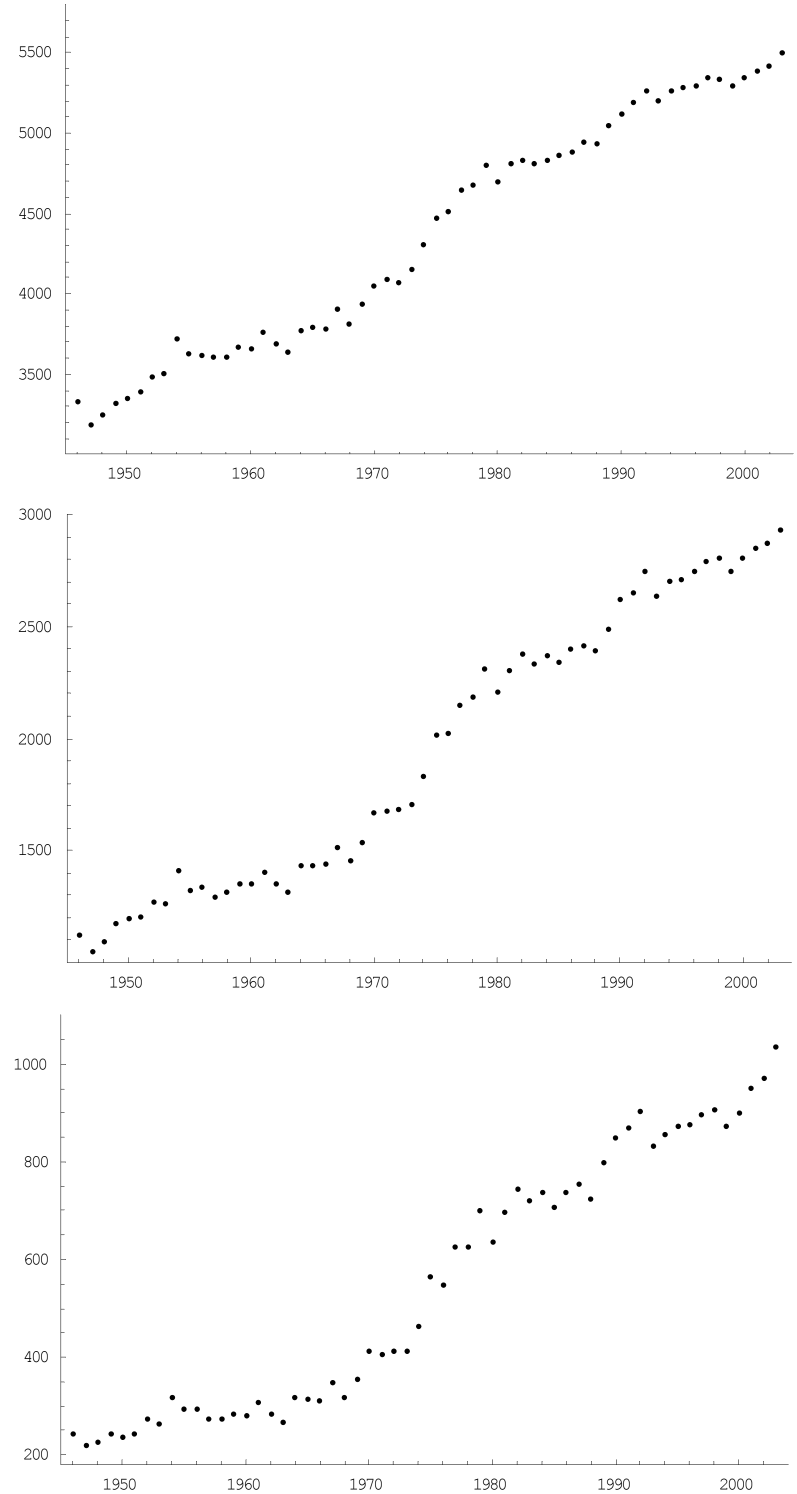

FIGURE 1. 1946 – 2003 U.S. Population Aged 75 Survival Rate per 10,000 for 10 years ${}_{10}p_{75,t}$ (upper graph), for 15 years ${}_{15}p_{75,t}$ (middle graph) and for 20 years ${}_{20}p_{75,t}$ (lower graph) where $t = 1946, 1947, ..., 2003$.

10,000 surviving beneficiaries at age 76, 77, $\ldots$. These rates are usually returned by the plan's actuary and are usually not the same as population rates. The plan expects to pay

$$10,000\,{}_1p_{75} \text{ in year } T+1$$

$$10,000\,{}_2p_{75} \text{ in year } T+2$$

$$10,000\,{}_3p_{75} \text{ in year } T+3$$

$$\vdots$$

$$10,000\,{}_kp_{75} \text{ in year } T+k$$

While the plan may have a good estimate of how many current participants will survive to age 75 in year T, there is much more uncertainty in the future survivor rates. This is where the longevity index comes in.

For simplicity assume that the plan anticipates its mortality experience will be the same as the population mortality. We don't mean that the plan participants and the lives underlying the index are the same people, just that they are subject to the some mortality forces. There will be basis risk, but the larger the plan and the more it looks like the U.S. population the smaller will be the basis risk. We are ignoring the basis risk problem in this paper.

Having fixed on age $x = 75$, let us also consider a single survival rate corresponding to $k = 15$. Currently the plan expects to pay $10,000\,{}_{15}p_{75}$ pensioners aged 90 per 10,000 at age 75 in year T. The U.S Population survivor index rate $10,000\,{}_{15}\mathcal{I}_{75,T}$ determined soon after year T will reveal how longevity has progressed. If the population index has increased, it is very likely that the plan will need to revise upward its own estimates of how many pensioners it will have in year $T+k$, per 10,000 in year T. This suggests that the plan could use a forward contract or a call option on the longevity index to hedge its risk in underestimation of future survivor rates used in funding its liabilities.

A European call option on the rate ${}_k\mathcal{I}_{x,T}$ written in year t, with strike price $\mathcal{I}$ and maturing in T pays the option owner

$$B_T = 10,000 \begin{cases} {}_k\mathcal{I}_{x,T} - \mathcal{I} & \text{if } {}_k\mathcal{I}_{x,T} > \mathcal{I} \\ 0 & \text{if } {}_k\mathcal{I}_{x,T} \leq \mathcal{I} \end{cases} . \tag{2}$$

The call option has a fixed trigger level $\mathcal{I}$ such that the issuer pays the pension plan when the population rate $10,000{}_k\mathcal{I}_{x,T}$ exceeds the strike rate $10,000\mathcal{I}$ at maturity T. The plan will choose a strike level using its own expectation ${}_kp_x$ as a guide. The population index is likely to be in the money in the same circumstances that the plan has experienced unexpected increases in longevity, since the two groups are subject to the same forces of mortality. If the index ${}_k\mathcal{I}_{x,T}$ is lower than a strike level $\mathcal{I}$ (lower population survival rate), then the pension plan will not exercise the option at time T. In this case, it is likely that the plan survival rate is lower too.

Using the population longevity index reduces the moral hazard problem since the index is transparent to all investors. However, as for this structure, there exists basis risk for the pension plan. Basis risk arises because the hedge is not exactly the same as the pension plan's risk - different groups of people are involved. However, if the plan is reasonably large and has good records, it may be able to accurately measure the correlation of its own survival rates with population survival rates and calculate an appropriate hedge.

In addition, the pension plan or the annuity provider can use a portfolio of options to choosing ages x and durations k depending on its own plan age distribution. This is similar to the Swiss Re mortality bond. The death risk of the Swiss Re deal is defined in terms of an index based on the weighted average annual population death rates of five countries (U.S., UK, France, Italy and Switzerland) with weights determining its exposure in each country.

4. Pricing Longevity Option

In this section, we describe the mortality dynamics as a jump-diffusion process. We show how to compute the longevity risk premium and option price using a method developed by Liu et al. (2005). We focus one a single contact but the method easily applies to other ages and durations. Specifically, we price a 10-year longevity call option.

4.1. Mortality Dynamics.

Here we describe the dynamics of the U.S. population longevity index for age x at time t surviving k years, ${}_k\mathcal{I}_{x,t}$, with a combination of a geometric Brownian motion and a compound Poisson process. The number of mortality jumps during the time interval $(0,t)$ is a Poisson process N_t with parameter λ. The U.S. population longevity index for age x, ${}_k\mathcal{I}_{x,t}$, follows a stochastic jump-diffusion equation, described in terms of standard Brownian motion W_t and the Poisson process N_t:

$$\frac{\mathrm{d}_k\mathcal{I}_{x,t}}{{}_k\mathcal{I}_{x,t}} = \begin{cases} (\alpha - \lambda c)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, & \text{if no Poisson event occurs at time } t; \\ (\alpha - \lambda c)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + (Y_t - 1), & \text{if a Poisson event occurs at time } t. \end{cases} \tag{3}$$

where α is the instantaneous expected change rate of the longevity index ${}_k\mathcal{I}_{x,t}$; σ is the instantaneous volatility of the index, conditional on no jumps.

The quantity $Y_t - 1$ is an impulse function producing a jump from ${}_k\mathcal{I}_{x,t}$ to ${}_k\mathcal{I}_{x,t}\ Y_t$. Let c denote the mean jump multiplier $\mathrm{E}(Y_t - 1)$; it is the expected percentage change in the longevity index if a Poisson event occurs.

The $\sigma\,\mathrm{d}W_t$ term describes the instantaneous unanticipated normal longevity index change, and the $Y_t - 1$ term describes the abnormal longevity shock size. If $\lambda = 0$, then $Y_t - 1 = 0$ the process evolves as the geometric Brownian motion model without jumps.

The longevity index ${}_k\mathcal{I}_{x,t}$ will be continuous most of the time with finite jumps of differing signs and amplitudes occurring at discrete points of time. If α, λ, c, and σ are constants, we can solve the differential equation (3) as

$$\frac{{}_k\mathcal{I}_{x,t}}{{}_k\mathcal{I}_{x,0}} = \exp\left[(\alpha - \frac{1}{2}\sigma^2 - \lambda c)t + \sigma W_t\right] Y(N_t), \tag{4}$$

where N_t is the total number of longevity jumps with parameter λt during a time interval of length t. The cumulative jump size $Y(N_t) = 1$ if $N_t = 0$ and $Y(N_t) = \prod_{j=1}^{N_t} Y_j$ for $N_t \geq 1$ where the jump sizes Y_j are independently and identically distributed.

From equation (4), we can derive the index value ${}_k\mathcal{I}_{x,t+h}$, given ${}_k\mathcal{I}_{x,t}$ resulting in

$${}_k\mathcal{I}_{x,t+h}|\mathcal{F}_t = {}_k\mathcal{I}_{x,t} \exp\left[\left(\alpha - \sigma^2/2 - \lambda c\right) h + \sigma \Delta W_t\right] \prod_{j>N_t}^{N_{t+h}} Y_j \tag{5}$$

where $\mathcal{F}_t$ is the information at time t.

We assume Y_t is log-normally distributed with parameters α_J and σ_J, that is,

$$Y_t = \exp(\alpha_J + \sigma_J u_t) = \exp(Z_t), \text{ where } u_t \sim \text{Normal}(0, 1). \tag{6}$$

Since Y_t is log-normally distributed, then the distribution of $\frac{{}_k\mathcal{I}_{x,t+h}}{{}_k\mathcal{I}_{x,t}}$ is log-normal too. Appendix A shows how to get the maximum likelihood function from equations (5) and (6).

4.2. **Equilibrium Model with Rare Events.** Consider a representative agent (*e.g.* a longevity security investor) who exhibits model uncertainty aversion in the sense of Knight (1921) and Ellsberg (1961) in addition to being risk averse. Following Liu et al. (2005), we assume the investor of the longevity security considers alternative models in terms of jump component to protect himself against possible model misspecifications.

4.2.1. *Liu et al. (2005)'s Model.* The physical probability measure associated with the reference model in equation (3) is denoted as $\mathcal{P}$ and the alternative model is defined by its probability measure $\mathcal{P}(\xi)$. So $\xi_T = \mathrm{d}\mathcal{P}(\xi)/\,\mathrm{d}\mathcal{P}$ is the Radon-Nikodym derivative with respect to $\mathcal{P}$. Specifically, Liu et al. (2005) assume that ξ_t changes the investor's probability assessment with respect to the jump component without altering his view about the diffusion component.

Therefore, their Radon-Nikodym derivative follows this stochastic jump-diffusion equation

$$\mathrm{d}\xi_t = \left(e^{a+bZ_t-b\alpha_J-b^2\sigma_J^2/2}-1\right)\xi_{t-}\,\mathrm{d}N_t-\left(e^a-1\right)\lambda\xi_{t-}\,\mathrm{d}t, \tag{7}$$

where a and b are predictable processes (fixed just before time t) and $\xi_0 = 1$. The investor can express his uncertainty toward one specific part of the jump component (the likelihood of jump arrival or jump size) or both. When $b = 0$, the investor builds a set of alternative models that are different from the reference model only in terms of the likelihood of jump arrival. When $a = 0$, he builds another set of alternative models that are different from the reference model only in terms of jump size. The jump amplitude Z_t is normally distributed with mean α_J and standard deviation σ_J as shown in equation (6). Moreover, the process $\{\xi_t, 0 \le t \le T\}$ is constructed as a martingale with mean 1. Therefore, the measure $\mathcal{P}(\xi)$ is indeed a probability measure. With this setup, the jump arrival intensity λ^ξ and the mean jump size c^ξ under the alternative measure $\mathcal{P}(\xi)$ become

$$\lambda^\xi = \lambda e^a \quad \text{and} \quad 1+c^\xi = (1+c)e^{b\sigma_J^2},$$

where λ and c are in the reference measure $\mathcal{P}$. If $a = 0$ and $b = 0$, the investor follows the reference model and does not care about model uncertainty. However, if he is risk averse to model misspecifications, the investor ventures into other models by choosing some other a and b (i.e. $a \neq 0$ and/or $b \neq 0$). The entire collection of such models defined by a and b is expressed as $\mathcal{D}$.

Choosing an alternative model $\mathcal{P}(\xi)$ affects this investor in two ways. On the one hand, to protect himself against model uncertainty with the jump component, the investor focuses on other jump models that provide the worst prospect (the first effect). On the other hand, since he understands that statistically $\mathcal{P}$ is the best representation of the existing data, he penalizes his choice of $\mathcal{P}(\xi)$ according to how far it deviates from the reference $\mathcal{P}$ (the second effect). Taking into account these two effects, Liu et al. (2005) define the investor's time-t utility function recursively as

$$U_t = \frac{s_t^{1-\gamma}}{1-\gamma}\Delta + e^{-\rho\Delta} \inf_{\mathcal{P}(\xi)\in\mathcal{D}} \left\{ \mathrm{E}_t^{\xi}(U_{t+\Delta}) + \frac{1}{\phi}\psi(\mathrm{E}_t^{\xi}(U_{t+\Delta}))\mathrm{E}_t^{\xi}\left[h\left(\ln \frac{\xi_{t+\Delta}}{\xi_t} \right) \right] \right\} \tag{8}$$

$$\text{and} \quad U_T = 0,$$

where s_t is the consumption in time t with risk aversion γ and $\rho > 0$ is a constant discount rate. The normalization factor $\psi(.)$ is the same as Maenhout (2004) for analytical tractability. The infimum over $\mathcal{P}(\xi) \in \mathcal{D}$ in equation (8) captures the first effect that implies the worse outcome than the reference model $\mathcal{P}$. The second factor in the infimum of equation (8) reflects the second effect, the penalty of "distance from the reference model". $h(.)$ is a distance function defined as

$$h(x) = x + \beta(e^x - 1),$$

where $\beta > 0$ and $x \in \mathbb{R}$. Intuitively, $h(.)$ increases as the alternative model $\mathcal{P}(\xi)$ is further away from the reference model $\mathcal{P}$. The constant parameter $\phi > 0$ captures the trade-off between "impact on future prospects" and "distance from the reference model". The higher ϕ suggests that the investor puts less weight on the penalty of choosing alternative model and thus more weight on how it would worsen his future prospect. That is, the investor with higher ϕ is more aversion to model uncertainty.

4.2.2. *Diffusion Risk Premium, Jump-Risk Premium and Rare-Event Premium.* Maximizing the utility function (8), we can explicitly solve for the three components of the equilibrium longevity risk premium:[4]

$$\begin{aligned} \text{Diffusion risk premium} &= \gamma\sigma^2, \\ \text{Jump-risk premium} &= \lambda c - \bar{\lambda}\bar{c}, \\ \text{Rare-event premium} &= \bar{\lambda}\bar{c} - \lambda^Q c^Q, \end{aligned} \tag{9}$$

[4]See Liu et al. (2005) for detailed derivation.

where

$$(10)\qquad \begin{aligned} \bar{\lambda} &= \lambda \exp\left(-\gamma\alpha_J + \frac{1}{2}\gamma^2\sigma_J^2\right), \\ \bar{c} &= (1+c)\exp\left(-\gamma\sigma_J^2\right) - 1, \\ \lambda^Q &= \lambda \exp\left(-\gamma\alpha_J + \frac{1}{2}\gamma^2\sigma_J^2 + a^* - b^*\gamma\sigma_J^2\right) \qquad \text{and} \\ c^Q &= (1+c)\exp\left((b^* - \gamma)\sigma_J^2\right) - 1. \end{aligned}$$

The variables a^* and b^* are obtained from the following nonlinear equations:

$$\begin{aligned} & a + \frac{1}{2}b^2\sigma_J^2 + 2\beta\left(e^{a+b^2\sigma_J^2} - 1\right) + \frac{\phi}{1-\gamma}\left(\left[(1+c)e^{(b-\frac{1}{2}\gamma)\sigma_J^2}\right]^{1-\gamma} - 1\right) = 0 \\ & b\left(1 + 2\beta^{a+b^2\sigma_J^2}\right) + \phi\left[(1+c)e^{(b-\frac{1}{2}\gamma)\sigma_J^2}\right]^{1-\gamma} = 0. \end{aligned}$$

The diffusion risk premium and the jump-risk premium are exclusively attributed to risk aversion coefficient γ. One additional component—rare-event premium is included when the investor exhibits model uncertainty aversion ($\phi > 0$). Therefore, in equilibrium, the total risk premium is the sum of three components in equation (9):

$$(11)\qquad \text{Total risk premium} = \gamma\sigma^2 + \lambda c - \lambda^Q c^Q.$$

4.2.3. *Option Pricing.* Liu et al. (2005) modify the famous European call option pricing formula of Black and Scholes (1973) and Merton (1976) to capture model uncertainty risk premium. We apply their model to our longevity options. Consider a European call option written on ${}_k\mathcal{I}_{x,T}$ at time t, with strike price $\mathcal{I}$, and maturing at time $T = t + \tau$. The price C is

$$(12)\qquad C = \exp(-\lambda'\tau)\sum_{j=0}^{\infty}\frac{(\lambda'\tau)^j}{j!}\text{BS}({}_k\mathcal{I}_{x,t}, \mathcal{I}, r_j, \sigma_j, \tau),$$

where $\lambda' = \lambda^Q(1+c^Q)$ and for $j = 0, 1, \ldots,$

$$r_j = r - \lambda^Q c^Q + \frac{j\ln(1+c^Q)}{\tau}, \quad \sigma_j^2 = \sigma^2 + \frac{j\sigma_J^2}{\tau}.$$

and $\text{BS}({}_k\mathcal{I}_{x,t}, \mathcal{I}, r_j, \sigma_j, \tau)$ is the standard Black-Scholes option pricing formula, as if the index dynamics were usual geometric Brownian motion, with the initial longevity index value ${}_k\mathcal{I}_{x,t}$, strike level $\mathcal{I}$, risk-free rate r_j, volatility σ_j and time to maturity τ.

4.3. **Example.** The basic idea of longevity securitization is to issue longevity securities with cash flows linked to pre-specified mortality indices. The following example details how to price our longevity call option described in Section 3.2.

4.3.1. *Parameter Estimation.* Suppose in year $t = 2003$ an option is written on the $k = 15$ year survival rate for someone age 75 in year $T = t + 10 = 2013$ with a strike level $\mathcal{I}$.

The longevity index ${}_{15}\mathcal{I}_{75,2013}$ underlies the option. On average, ${}_{15}p_{75,t}$ increased by 1.80% each year from 1946 to 2003. Since the 15-year survival probability for age 75 at time $t = 0$ is ${}_{15}\mathcal{I}_{75,2003} = 0.2935$, the pension plan may reasonable expect that the longevity index will be

$$\text{E}\left[{}_{15}\mathcal{I}_{75,2013}|\mathcal{F}_{2003}\right] = {}_{15}\mathcal{I}_{75,2003}\exp(0.018 \times 10) = 0.3513$$

at maturity $T = 2013$, given current information. But the plan is concerned about the risk that the actual index after 10 years will be substantially higher than what it anticipates today (e.g. 25% higher) so it purchases this longevity call option. In our example, we set the strike level $\mathcal{I}$ equal to 125% of the current expected value

$$\mathcal{I} = 1.25 \times 0.3513 = 0.4392.$$

The option payoff B at maturity is

$$B = 10,000 \begin{cases} {}_{15}\mathcal{I}_{75,2013} - 0.4392 & \text{if } {}_{15}\mathcal{I}_{75,2013} > 0.4392 \\ 0 & \text{if } {}_{15}\mathcal{I}_{75,2013} \leq 0.4392 \end{cases}. \tag{13}$$

Parameter	Estimate	Parameter	Estimate
α	0.018	α_J	0.058
σ	0.035	σ_J	0.041
λ	0.100		

TABLE 1. Parameter estimates based on the U.S. Population 15-year Survival Rate of Age 75, 1946 –2003.

Based on the U.S. population 15-year longevity index for age 75 from 1946 to 2003 shown in the middle graph of Figure 1, our maximum likelihood estimation result is an instantaneous mortality change rate α of 0.018. The positive sign of α due to the fact that the U.S. population mortality of older ages improved over time. The instantaneous volatility of the longevity index, conditional on no jumps, σ is equal to 0.035. However, the likelihood ratio test does not reject the model without jumps. We still believe the pension plans and annuity insurers are keenly interested in managing longevity events like the significant mortality improvement in 1970s. During the period from 1974 to 1977, the mortality improvement rate is 231% higher than that of the whole period from 1946 to 2003. Therefore, we use the annual mortality change rates and volatility from 1974 to 1977 as jump parameters α_J and σ_J. We also set the probability of a jump event each year equal to 10%. Table 1 reports the estimation results for our longevity option pricing.

4.3.2. *Longevity Risk Premium.* Liu et al. (2005) suggest risk aversion of U.S. stockholders γ generally falls between 1.5 and 3.5. We set γ equal to 2 or 3 respectively to calculate diffusion risk premium, jump-risk premium and rare-event premium of the longevity option. Given the reference model, four scenarios are considered for the representative investor's model uncertainty aversion ϕ which are similar to Liu et al. (2005). As shown in Table 2 with $\gamma = 2$, each scenario corresponds to an economy with a distinct level of model uncertainty aversion ϕ and yields a distinct composition of the diffusion-risk premium, the jump-risk premium, and the rare-event premium. For example, the rare-event premium is zero when the representative longevity security investor exhibits no aversion to model uncertainty, and increases to 0.405% per year when the uncertainty aversion coefficient becomes $\phi = 30$.

TABLE 2. The 15-year survival probability of the age cohort 75: The three components of the risk premium, historical mortality data with $\gamma = 2$

Jump parameter	Aversion ϕ	γ	Risk premium component (%) Diffusion	Jump	Rare event	Total risk premium (%)
	0	2	0.239	0.096	0	0.335
$\lambda = 0.10$	10	2	0.239	0.096	0.209	0.544
$\alpha_J = 5.8\%$	20	2	0.239	0.096	0.330	0.665
	30	2	0.239	0.096	0.405	0.740

Table 3 shows the results with risk aversion $\gamma = 3$. Let us first consider the case of zero uncertainty aversion, where risk aversion is the only source of premia. As expected, the investor who is more risk averse to longevity shocks ($\gamma = 3$) requests higher diffusion-risk premium and jump-risk premium than the one in the previous example with $\gamma = 2$. Moreover, our results explain the risk premium puzzle of reinsurance and insurance-linked securities (Froot and O'Connel, 1997; Lin and Cox, 2007). Froot and O'Connel (1997) have documented the very high average hurdle rate of the catastrophe property reinsurance business. On average, over the period 1980-1994, the price is on the order of four times the actuarial value. Lin and Cox (2007) use the Wang transform to price the Swiss Re bond. Based on the U.S. historical data, their calculated risk premium (0.39%) is much lower than that offered by the Swiss Re (1.35%).[5] From Tables 2 and 3, we can see that the rare-event premia are higher than the jump-risk premia and account for a big proportion of the total risk premium of longevity security when $\phi > 0$. Our results are also consistent with Froot and Stein (1998). Their model suggests that the hurdle rate of an investment opportunity consists two parts, the standard market-risk factor and the unhedgeable risk factor. The investor (as well as the market) has imprecise knowledge about rare longevity events and thus the longevity risk is unhedgeable. Therefore, the investor requests a rare-event premium for model uncertainty.

[5] It is not surprising that our calculated total risk premium of longevity bond is lower than that of death-linked mortality bond since the longevity process is generally less dramatic than catastrophe death events (e.g. flu epidemics).

TABLE 3. The 15-year survival probability of the age cohort 75: The three components of the risk premium, historical mortality data with $\gamma = 3$

Jump parameter	Aversion ϕ	γ	Risk premium components (%) Diffusion	Jump	Rare event	Total risk premium (%)
	0	3	0.358	0.138	0	0.496
$\lambda = 0.10$	10	3	0.358	0.138	0.186	0.682
$\alpha_J = 5.8\%$	20	3	0.358	0.138	0.296	0.792
	30	3	0.358	0.138	0.365	0.861

TABLE 4. The 15-year survival probability of the age cohort 75: The three components of the risk premium, assuming extreme mortality improvement with $\gamma = 3$

Jump parameter	Aversion ϕ	γ	Risk premium components (%) Diffusion	Jump	Rare event	Total risk premium (%)
	0	3	0.358	0.398	0	0.756
$\lambda = 0.10$	10	3	0.358	0.398	0.519	1.275
$\alpha_J = 11.6\%$	20	3	0.358	0.398	0.723	1.478
	30	3	0.358	0.398	0.798	1.554

To show the robustness of our results, we modify the key jump parameter, α_J, in the reference model considered in Table 3. In Table 3, we consider jumps that happen once every 10 years, with a mean magnitude of 5.8%, capturing the magnitude of longevity jump event in 1970s. In Table 4, we double the jump size thus jumps happen with a magnitude of 11.6%. The pricing implications of these models are reported in Table 4. As we expect, the larger magnitude of jump size implies higher risk premium. Although both reference models incorporate rare events that are very different in intensity and magnitude, the impact of model uncertainty aversion remains qualitatively similar.

4.3.3. *Longevity Call Option Pricing.* After estimating the risk premium, we are ready to price the longevity option. Liu et al. (2005) show that their model with uncertainty aversion $\phi = 20$ reaches a result consistent with empirical option prices. We believe the model uncertainty aversion is higher for longevity security than stocks or options since now we

Parameter	Estimate	Parameter	Estimate
α	0.009	α_J	0.028
σ	0.013	σ_J	0.014
λ	0.100		

TABLE 5. Parameter estimates based on the U.S. Population 10-year survival probabilities for age 75, 1946 –2003.

Parameter	Estimate	Parameter	Estimate
α	0.028	α_J	0.105
σ	0.072	σ_J	0.097
λ	0.100		

TABLE 6. Parameter Estimates Based on the U.S. Population 20-year Survival Probabilities For Age 75, 1946 –2003.

know little about why people live beyond age 80 (Vaupel, 1998) and the quality of age reporting deteriorates among the very old (Buettner, 2002). So we choose a higher model uncertainty aversion $\phi = 30$ in Table 3. This suggests a total risk premium of 0.861%.

Given the current longevity index per 10,000 pensioners at age 75 ($10,000{}_{15}\mathcal{I}_{75,2003} = 2,935$), the strike level $10,000\mathcal{I} = 4,392$, the risk-free rate $r = 0.03$ and the estimated mortality dynamic parameters in Table 1, our 10-year European-style longevity call option price equals to 35.00 based on the call option equation (12). Compared with the expected total liability of the pension plan $\mathrm{E}\left[{}_{15}\mathcal{I}_{75,2013}|\mathcal{F}_0\right] = 3,513$ in year 10, the option premium the pension plan pays is only a negligible proportion (0.996%).

The pension plan will need to purchase a portfolio of options for different obtained ages to hedge its longevity risk. Therefore, we price two more 10-year longevity call options based on the 10-year and 20-year survival probabilities of the age cohort 75 respectively. Our estimated mortality dynamic parameters for these two survival rates are shown in Tables 5 and 6.

From Tables 1, 5 and 6, we can see the 15- and 20-year longevity indices of age group 75 have greater mortality improvement but are more volatile than that of 10-year index from 1974 to 1977. Given the current 10-year longevity index per 10,000 pensioners at age 75

($10,000_{10}p_{75,2003} = 5,501$) and the 20-year index ($10,000_{20}p_{75,2003} = 1,034$), the strike level equal to 125% of the current expected level in year $T = 2013$ ($10,000\mathcal{I} = 7,524$ for 10-year index of age 75 and 1,711 for 20-year index), the risk-free rate $r = 0.03$ and the estimated mortality dynamic parameters in Table 5 and Table 6, our 10-year European-style longevity call options are priced at 70.36 (1.169% of the expected liabilities in $T = 10$) based on the 10-year survival rate of age group 75 and 29.48 for the 20-year survival probability (2.154% of the expected liabilities in $T = 10$).

5. Conclusion

This article explores a recent capital market solution for longevity risk, a topic that has attracted growing interests from scholars (Blake and Burrows, 2001; Milevsky and Promislow, 2001; Hardy, 2005; Lin and Cox, 2005; Cairns et al., 2006; Lin and Cox, 2007; Cox and Lin, 2007). Longevity securitization enables annuity insurers and pension plans to transfer longevity risk from their liability side to capital markets. So far, there has been only one public deal offered to the market. The EIB longevity bond did not sell (Lin and Cox, 2007). Its design is problematic: it provides a "ground-up" protection part of which is not needed by the pension plans; it is capital intensive and requires a high degree of upfront capital commitment for the degree of protection it offered. To address this issue, we first propose a longevity index on old ages and then suggest a longevity call option. That is, its cash flows are only linked to longevity tail risk.

A second novel aspect of this article is that it introduces to the insurance securitization literature a pricing approach to handle the aversion to model uncertainty. As for longevity risk, models with rare events are easy to build but hard to estimate with confidence. At least, we have little room to learn from the past — our available data only suggest one relatively significant mortality improvement. We believe the participants in longevity risk business treat rare longevity shocks differently from common, more frequent events. Motivated by the observation that insurance-linked securities usually provide much higher risk premia than the historical loss data suggest, we apply the equilibrium framework of Liu et al. (2005) to disentangle the risk premium linked to imprecise knowledge about rare events from the standard risk-based premiums. Uncertainty aversion toward longevity rare events nicely explains the risk premium puzzle of insurance–linked securities.

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Appendix A: Maximum Likelihood Estimation of Mortality Stochastic Model with Jumps

After taking logarithm on both sides of Equation (5), we obtain

$$Z(h) = \log {}_k\mathcal{I}_{x,t+h} - \log {}_k\mathcal{I}_{x,t} = (\alpha - \frac{1}{2}\sigma^2 - \lambda c)h + \sigma \Delta W_t + \sum_{j>N_t}^{N_{t+h}} \log(Y_j). \tag{14}$$

If the variable $\Delta N_h = N_{t+h} - N_t$ is the number of events during the period h, the variable $Z(h)|(\Delta N_h = n)$ will be normally distributed with mean $M_n = (\alpha - \frac{1}{2}\sigma^2 - \lambda c)h + n\alpha_J$ and variance $S_n^2 = \sigma^2 h + n\sigma_J^2$. From $\mathrm{E}[Y_j] = \exp(\alpha_J + \sigma_J^2/2)$, we get $c \equiv \exp(\alpha_J + \sigma_J^2/2) - 1$ since the expected value of the longevity index percentage change $c \equiv \mathrm{E}[Y_j - 1]$ if the Poisson event occurs.

The density function of $Z(h)$, $f_{Z(h)}(z)$, can be written in terms of the conditional density of $Z(h)|(\Delta N_h = n)$, denoted $f_{Z(h)}(z|\Delta N_h = n)$, which has a normal distribution:

$$\begin{aligned} f_{Z(h)}(z) &= \sum_{n=0}^{\infty} f_{Z(h)}(z|\Delta N_h = n) \Pr(\Delta N_h = n) \\ &= \sum_{n=0}^{\infty} f_{Z(h)}(z|\Delta N_h = n) \frac{e^{-\lambda h}(\lambda h)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{S_n\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-M_n}{S_n}\right)^2} \frac{e^{-\lambda h}(\lambda h)^n}{n!}. \end{aligned} \tag{15}$$

If we have a time series of Q observations of ${}_k\mathcal{I}_{x,t}$ where $t = 0, 1, 2, ..., Q-1$, there will be $Q-1$ observations of z's with time interval equal to $h = 1$. In each time interval of length $h = 1$, we assume that the probability of an event from time t to $t+h$ is λ and the probability of more than one event during such a time interval is negligible. We can estimate the parameters $\lambda, \alpha, \sigma, \alpha_J$ and σ_J by maximizing the following loglikelihood function (16) based on observations $z_1, z_2, ..., z_{Q-1}$:

$$\sum_{i=1}^{Q-1} \log f_{Z(1)}(z_i) = \sum_{i=1}^{Q-1} \log\left(\sum_{n=0}^{\infty} f_{Z(1)}\left(z_i|\Delta N_h = n\right) \Pr\left(\Delta N_h = n\right)\right) \tag{16}$$

$$= \sum_{i=1}^{Q-1} \log\left(\sum_{n=0}^{\infty} \frac{1}{S_n\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_i - M_n}{S_n}\right)^2} \frac{e^{-\lambda h}\left(\lambda h\right)^n}{n!}\right)$$

$$\approx \sum_{i=1}^{Q-1} \log\left(\sum_{n=0}^{10} \frac{1}{S_n\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_i - M_n}{S_n}\right)^2} \frac{e^{-\lambda h}\left(\lambda h\right)^n}{n!}\right),$$

where $M_n = (\alpha - \frac{1}{2}\sigma^2 - \lambda c)h + n\alpha_J$ and variance $S_n^2 = \sigma^2 h + n\sigma_J^2$. For example, when n = 0 or 1, we can get

$$M_0 = \alpha - \frac{1}{2}\sigma^2 - \lambda\left[\exp(\alpha_J + \sigma_J^2/2) - 1\right],$$

$$M_1 = \alpha - \frac{1}{2}\sigma^2 - \lambda\left[\exp(\alpha_J + \sigma_J^2/2) - 1\right] + \alpha_J,$$

$$S_0^2 = \sigma^2,$$

$$S_1^2 = \sigma^2 + \sigma_J^2.$$