

Modelling cycle dependence in credit insurance

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Abstract

Business and credit cycles have an impact on credit insurance as they do on other businesses. Nevertheless, a credit insurer can adapt faster than other businesses can. In particular, it limits the consequences of a down-turning cycle. This paper proposes a model of estimating future losses of a credit insurance portfolio. The model takes into account both cycles (regimes) and the capacity of the credit insurer to take less risk in case of a downturn and inversely in case of a cycle upturn.

Keywords: credit insurance, cycles, Markov chain, rating transition matrix, multifactor Merton model.

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Introduction

Credit insurance is concerned by business or credit cycle fluctuations, but in an unusual way. Commercial businesses need a credit insurer if they think their clients won't be able to pay their invoices on time, or risk to become insolvent shortly. So they take an insurance policy which guarantees them that a part or the whole invoice amount will be reimbursed by the insurer in case the client couldn't pay. The credit insurer thus takes on the risk of insolvency or protracted default¹. This makes the credit insurer very sensitive to credit and eventually business cycle.

Many papers aim to find out whether these cycles coincide, and it seems there is no definitive answer on this question (see for example [14] and references therein). Both cycles seem to follow some macro-economic variables, and GDP variations seem to explain partly the credit cycle movements. Anyway the difference between these two cycles will not be important in what follows. We can work with both of them as long as we distinguish upturns from downturns and that in downturns there are (significantly) more defaults than in upturn periods. For us the important issue will be being able to distinguish the point where the number of defaults changes significantly, where the default regime switches.

When the cycle goes down, the number of insolvent firms or those not being able to pay their invoices increases dramatically. Hence the insurer should reimburse huge amounts of money and risks to be insolvent itself. It would probably be the case if credit insurers could not limit their losses. They can do this by diminishing their exposure towards firms (clients of the insured business) whose creditworthiness is decreasing. In this case the insurer prevents the insured firm whose granted amount decreased that its client may not be solvent anymore and that it should itself reduce the amount of commercial exchanges with the client. On the other hand, when the insurer thinks that economic conditions are more favourable, it takes more risks, and increases its exposures.

The large number of firms defaulting at the same time at the beginning of a crises impact the credit insurer quite considerably (if the insurer has not correctly predicted its starting point). However once the number of defaults increases and the crises is real, the credit insurer can lower its exposures towards the riskier firms - there is an arbitrage between immediate losses of money or credibility for future wealth - and diminish its losses. The insurer has the power to increase or decrease the risk it is bearing, which is more difficult for banks for example.

For all these reasons it is important for a credit insurer to predict the state of the economy and for the actuarial teams to compute a risk capital which is depending on the state of the economy.

Our paper presents a way to better adjust to cycles and introduce lowering or increasing exposures. In the first section we will present the one-period model where all parameters are fixed during the period. In the second section we will introduce a two-period modelling, which allows to adjust to the cycle phases. In the last section after apply the model to a credit insurance portfolio and give the results we have.

¹There is default if the payment period lasts longer than it was initially agreed on by the insured and its client

1 Single-period modelling

The model we will introduce in this section is the cornerstone of the two-period model we propose. Let us start by describing this model, so that the new steps we want to introduce and their usefulness may be clearer later on.

1.1 Default modelling

The current defaulting model is a multifactor Merton model. It is a Merton-like model (see [16], [4], [9], [8]) since one client, that we will call onward *buyer*, will default if a latent value called *ability to pay* and noted Z , falls below a certain threshold, d . In the true Merton model the latent value is the value of the assets of the firm, and it will default if the asset value falls below the liabilities amount. The probability of default for buyer n is then:

$$p_n = \mathbb{P}(Z_n \leq d_n)$$

The parameter estimated here is not the default threshold, because we are working with a latent variable, but the default probabilities p_n . In our case we will assume they are given and are exogenous to our main concern. The default modelling is a multifactor one because the latent variable Z_n is modelled as the sum of systemic risk and buyer individual risk. See [5], [6], [18], [19], [10] for further information on one-factor, multi-factor models and associated copulae. In our case, we have:

$$Z_n = \varrho_n {}^t w_n R + \sqrt{1 - \varrho_n^2} \varepsilon_n$$

where

- R is the systemic risk vector following a multivariate Gaussian distribution $N(0, \Sigma)$, with Σ the covariance matrix;
- ε_n is the idiosyncratic (individual) risk of buyer n and follows a standard Gaussian distribution $N(0, 1)$;
- ε_n and R are independent from one another;
- w_n is the vector of weights of buyer n for factors in R ;
- ϱ_n describes the correlation of buyer n to systemic risk (economy); the bigger it is, the higher the correlation to systemic risk, thus the higher the correlation to other firms also.

In the multifactor model the parameters needed to be estimated are the covariance matrix Σ , ϱ_n and the weights w_n . In practice the estimation of Σ and ϱ_n seem to be the hardest part but it will not be the object of our paper. We should also notice that since it is easier to manipulate standard Gaussian variables, and Z_n with the above definition is not a standard one, we will use instead this definition of Z

$$Z_n = \varrho_n \frac{{}^t w_n R}{\| \langle {}^t w_n M \rangle \|} + \sqrt{1 - \varrho_n^2} \varepsilon_n$$

where M is such that $\Sigma = {}^tMM$ (Cholesky decomposition).

1.2 Loss modelling

A defaulting buyer will produce a loss. This loss will be equal to the insured amount which is defined as minimum value among the invoice amount and exposure the insurer has on the buyer ². Since the insured amount is not known until the default occurs, it is modelled through *UGDs*, *Usage Given Default*, defined with the following formula:

$$UGD_n = \frac{\text{Insured amount}_n}{\text{Current Exposure}_n}$$

UGD_n is another parameter the insurer should model or estimate. Given UGD_n , the insurer estimates the loss from buyer n to be equal to $UGD_n \times \text{Current Exposure}_n$ in case buyer n defaults. ³

2 A new modelling approach

There are a large number of ways to model default probabilities and UGDs. We will not expose here those modelling techniques. Nevertheless we want to emphasize the importance of a consistent modelling since it is crucial for the estimation of losses-to-come and consistent reserving.

The current modelling is a single-period, which means that the whole parameters as well as the variables considered are defined over one period as well. The ability-to-pay is the one-period ability-to-pay. If there is default, there is one default in the period, but we don't know exactly when it occurs, so we assume all defaults occur at the end of the period. In practice the period we work on is the year.

We want to introduce in the modelling the possibility for the insurer to manage exposures during the year. Indeed it can lower exposures for buyers whose creditworthiness decreases and even cancel them. This former case is modelled through the change in the UGD, whereas the later is taken in account in the default probabilities since in case of default of the buyer after the guarantee is cancelled, the insurer pays nothing; it is like the default never occurred.

The contract (exposure) management takes all its importance during tough periods, when economic conditions influence the creditworthiness of firms and make them insolvent or incapable of paying invoices at the due date.

This means that estimated parameters, *i.e* should be different during high and low cycle periods because the contract management will be different.

Our aim is to take account of this by introducing a half-period step into the model, which gives us a two-period model. This two-period model can easily be transposed in a multi-period model.

²The maximum amount of money the insurer guarantees the insured in case of default of his client, the buyer n

³This will not be the final loss because other contract clauses, as reinsurance or deductibles for example, will be taken into account. We will not consider those clauses in this paper.

2.1 Two-period modelling

We will describe here the general idea of the model and deduce afterwards the parameters to be estimated.

2.1.1 Basic description and parameters to estimate

Parameters to be applied are predicted for the first period. Default probabilities are the parameters we are more interested in for now. They withhold information about the phase of the cycle we are in during the period.

At the end of the period we compute the number of insolvencies and given a criterion applied to this number, we say if it is more probable for the first period to be in a high or low phase cycle. We will give more details about this criterion in the section below.

We then compute the losses related to insolvency defaults and protracted defaults. The creditworthiness of buyers may change at the end of the period, and their grades may change consequently. Whereas in the one-period model buyers could change rating class only at the end of the period.

At the beginning of the second period we thus have a new portfolio since buyers might have changed grading classes and some of them have defaulted so they have exited the portfolio.

The exposures might also have changed from the beginning of the first period if the creditworthiness is lower. In the single-period model this was not possible.

The cycle phase of the second period may be high or low; it depends on the *a posteriori* phase of the cycle in the first period. The dependence is modelled by a Markov chain, i.e. probabilities of transition between high or low cycle phases. Examples in the literature are many, for example [2], where they work with business cycles.

The losses of the second semester are then added to those of the first.

The parameters we need to estimate for recession and expansion periods are then the following, for high or low cycle phases:

- Grades transition matrices, with insolvency probability defaults in the last column, noted P_l and P_h ;
- A vector of protracted default probabilities for each grade, π_l and π_h ;
- A vector with UGDs for each grade, UGD_l and UGD_h ;
- A vector (or matrix) of coefficients indicating the variation of exposures, c_l and c_h .

The parameters for the first semester should be predicted, especially insolvency default probabilities. The grades transition matrix should be given too.

The estimation of those parameters will not be the object of this article, but finding consistent estimators for those parameters would be of great interest in a future work. A consistent estimation approach would be a Hidden Markov Chain (ou Regime-Switching Markov chain) (see [11], [2], [?]). Some other papers which present estimation techniques for conditional (on business, credit cycles or other factors) are [3], [7], [13],[14],[17], [15].

2.1.2 Mathematical computation of the losses

Let N be the total number of buyers in the portfolio.

Let L_n be the exposure of buyer n at the beginning of the period. Let G_n be the grade of buyer n at the beginning of the period, $G_n \in \{1, \dots, J\}$.

Losses of the first period The total loss at the end of the first period will be:

$$\text{Loss}_1 = \sum_{n=1}^N L_n \times \text{UGD}(G_n) \times [Z_n < d(G_n)]$$

Remark: In the formula we have UGD_{G_n} because the UGDs are estimated by grades and probabilities of default too. Practically, we compute $d(G_n)$ for each grade as a standard Gaussian quantile $d(G_n) = \Phi^{-1}(p_{G_n})$ ⁴.

Changes in the portfolio during the first period Let def be the number of defaults in the first period, $\text{def} = \sum_{n=1}^N [Z_n < d(G_n)]$. If the default rate $\frac{\text{def}}{N} = \sum_{n=1}^N \frac{[Z_n < d(G_n)]}{N}$ satisfies a certain criterion, then E_l , otherwise $E = h$. We will develop this part in 2.2.

Concerning transitions between grades the same principles as for defaults apply: default thresholds are computed using the transition matrices P and the assumption on Z being a standard Gaussian.

The probability for a buyer n to go from $G_n = i$ to $j \in \{1, \dots, J\}$ is the following:

$$\mathbb{P}(d_{i,j+1} < Z < d_{ij}) = \Phi(d_{ij}) - \Phi(d_{i,j+1}) = p_{ij}$$

with $d_{i,j+1} < d_{ij}$.

The thresholds d_{ij} of going from grade i to grade j are computed:

$$\forall j \in \{J, \dots, 1\}, d_{ij} = \Phi^{-1} \left(\sum_{k=j}^{J-1} p_{ik} + p_i \right) \quad (1)$$

where J is the number of grades and $J = 10$ in our case.

At the end of the first period the structure of the portfolio would have changed: buyers having defaulted have exited the portfolio, and the others may have change grades, which implies that their characteristics as default probabilities change for the second period.

⁴ Φ denotes the cumulative distribution function of a standard normal distribution, as is Z_n .

Losses in the second period E and E_2 are the cycle phases in the first and second period. They take values in $\{l, h\}$ where l stand for low cycle and h for high cycle phase.

The transition matrix on the second semester will be given conditionally on the cycle phase in first period.

$$P_2 = \begin{cases} P_h & \text{with probability } \mathbb{P}(E_2 = h|E) \\ P_l & \text{with probability } \mathbb{P}(E_2 = l|E) \end{cases}$$

The protracted default probabilities will be randomly chosen, conditionally on the first period, so that:

$$\pi_2 = \begin{cases} \pi_h & \text{with probability } \mathbb{P}(E_2 = h|E) \\ \pi_l & \text{with probability } \mathbb{P}(E_2 = l|E) \end{cases}$$

We compute new default threshold with formula 1 for high and low cycle phases, and then we just have to choose between the two.

The losses of the second period are given by the formula:

$$\text{Loss}_2 = \sum_{n=1}^{N-\text{def}} L_n^2 \times \text{UGD}(E, G_n^2) \times [Z_n^2 < d^2(G_n^2)]$$

where L_n^2 is the exposure of buyer n at the beginning of the second period. It is estimated to be equal to $c_n \times L_n$ where c_n is a coefficient indicating if the exposure of grade n has fallen or increased.

The estimation of the coefficients c_n is highly important. They can be estimated using the history of claims declarations, and seeing how the exposure of buyers having defaulted has evolved during the year before the default. We would expect that in high cycles the exposures go up and when the cycle is low the exposures fall. Ideally we could estimate a matrix C of coefficients indicating the evolution of the exposure for buyers going from one grade to another. However in order to have more observations and a more robust estimation we choose to give a vector of coefficients, indicating the evolution of exposures of buyers relatively to their grade at the beginning of the period.

The total loss of the year would then be $\text{Loss} = \text{Loss}_1 + \text{Loss}_2$.

2.2 Hypothesis

If the default rate at the end of the period is "high enough", namely $\sum_{n=1}^N \frac{[Z_n < d(G_n)]}{N} > \text{def}^*$ than E_l is more probable than $E = h$. Otherwise $E = h$ is more probable than E_l .

Proposition 1 *The default rate for a given R , noted $\frac{\text{def}}{N}$, is a good estimator of the mean of conditional probabilities of all buyers in the portfolio $\frac{1}{N} \sum_{n=1}^N p^{n|R}$.*

Proof

We will use the Kolmogorov's theorem, see Appendix.

We may apply the Kolmogorov's theorem to the "sequence" of random variables representing conditional default, i.e. $X_n = [Z_n < d_n | R] = \left[\epsilon_n < \frac{d_n - \varrho_n R}{\sqrt{1 - \varrho_n^2}} \middle| R \right]$ for $n = 1, \dots, N$. Indeed those variables are independent (ϵ_n) pour $n = 1, \dots, N$. We choose $a_n = n$ for $n = 1, \dots, +\infty$.

The first hypothesis is satisfied.

$$\mathbb{E}([Z_n < d_n | R])^2 = \mathbb{V}([Z_n < d_n | R]) + (\mathbb{E}([Z_n < d_n | R]))^2 \quad (2)$$

$$= p^{n|R}(1 - p^{n|R}) + (p^{n|R})^2 = p^{n|R} < 1 \quad (3)$$

The second hypothesis is satisfied since we have:

$$\sum_{n=1}^{+\infty} \frac{\mathbb{V}([Z_n < d_n | R])}{n^2} = \sum_{n=1}^{+\infty} \frac{p^{n|R}(1 - p^{n|R})}{n^2} < \sum_{n=1}^{+\infty} \frac{1}{4n^2} < +\infty \quad (4)$$

So

$$\frac{\sum_{n=1}^N [Z_n < d_n | R] - \mathbb{E}\left(\sum_{n=1}^N [Z_n < d_n | R]\right)}{N} \longrightarrow 0 \text{ a.s.}$$

Then, using the following :

$$\frac{1}{N} \mathbb{E}\left(\sum_{n=1}^N [Z_n < d_n | R]\right) = \sum_{n=1}^N \mathbb{P}(Z_n < d_n | R) \quad (5)$$

$$= \frac{1}{N} \sum_{n=1}^N p^{n|R} \quad (6)$$

$$\frac{1}{N} \sum_{n=1}^N [Z_n < d_n | R] - \frac{1}{N} \sum_{n=1}^N p^{n|R} \longrightarrow 0 \text{ a.s.} \quad (7)$$

Thus the observed default rate for a given R is a converging estimator of the conditional expectation of the mean of the Bernoulli random variables representing defaults. This estimator is unbiased and efficient.

End of proof

Remark: The sum of the individual conditional probabilities of default will always be greater in low than in high cycle.

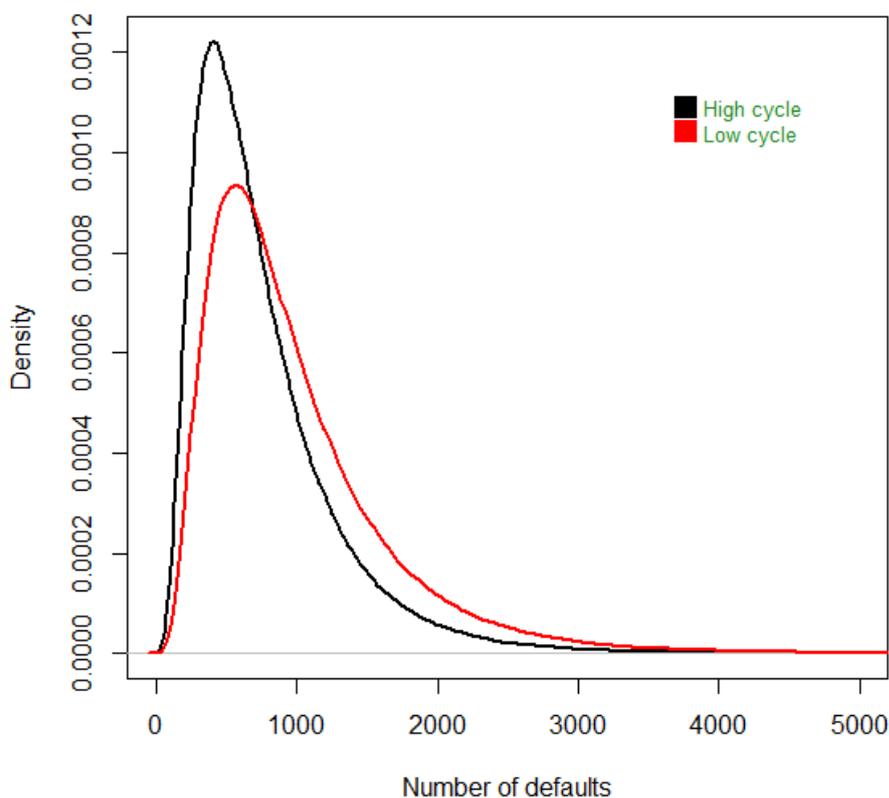
This is due to the fact that if $p_n^r > p_n^e$ then $\Phi^{-1}(p_n^r) = d_n^r > \Phi^{-1}(p_n^e) = d_n^e$.

Then

$$\frac{1}{N} \sum_{n=1}^N \mathbb{P}\left(\epsilon_n < \frac{d_n^r - \varrho_n w_n R}{\sqrt{1 - \varrho^2}} \middle| R\right) > \frac{1}{N} \sum_{n=1}^N \mathbb{P}\left(\epsilon_n < \frac{d_n^e - \varrho_n w_n R}{\sqrt{1 - \varrho^2}} \middle| R\right)$$

So conditionally to R , the number of defaults will always be greater in low cycle than in high cycle. Sometimes, for some R observations the difference between the two conditional probabilities will not be very high, and some other times the change between the two will be a lot greater. Thus the R vector implies a certain correlation structure between the buyers, which is always greater in recession than in expansion. The following is the plot of approximations of the density functions of defaults, in black in low and in red in high cycle phase.

Figure 1: Default densities approximations



We can see that the two density functions intersect at a certain point. This means that for this point, def^* the probability of having this number of defaults in low cycle is the same as the one in high cycle. For default numbers lower than def^* , $\text{def} < \text{def}^*$, the probability they are coming from a high cycle phase is greater than the probability they come from low cycle phase, and the other way around for $\text{def} > \text{def}^*$. We may also compute a posteriori probabilities of being in high or low cycle.

The data we entry in the model is made of default probabilities which contain inherent information on being in a high or low cycle phase. With the previous rule

we "decode" this information. After applying the rule we will know if these probabilities are more alike those in high or low cycles. In practice this is deduced from the number of cases where we "fall" into low cycle phases, noted x_l and in high cycle phases, x_h . If $x_h > x_l$, the probabilities entering the model "describe" an economy state closer to high cycle phase rather than to low cycle.

Remark: We want to emphasize the fact that the more disjoint the two densities are, the better we recognize a cycle phase just by computing the number of defaults in it. This is why if just business of credit cycle phases cannot be distinguished well regarding the number of defaults, relevant macro-variables should be tested.

3 Algorithm and simulations

There is no closed formula for the loss quantiles and we should use Monte Carlo simulations to find them. In this section we describe the algorithm and simulations necessary to have the loss distribution numerically.

We have computed all default thresholds.

$$pd_2 = pd_h \times \mathbb{P}(E_2 = h|E) + pd_l \times \mathbb{P}(E_2 = l|E).$$

For each simulation $t = \{1, \dots, T\}$

1. Simulate R^t following a multivariate Gaussian distribution $N(0, \Sigma)$.
2. Simulate N independent standard Gaussian variables ε_n^t for $n \in 1 \dots N$
3. Abilities to pay for each buyer are computed

$$Z_n^t = \varrho_n \frac{{}^t w_n R^t}{\| \langle {}^t w_n M \rangle \|_2} + \sqrt{1 - \varrho_n^2} \varepsilon_n^t$$

4. For each buyer n if $Z_n^t < d(G_n)$ then buyer n is insolvent and exits the portfolio.
5. For each buyer n if $d^{G_n} < Z_n^t < D^{(G_n)}$ buyer n has a protracted default.
6. Loss equals to the sum of losses for protracted and insolvency defaults.
7. For each buyer n not having defaulted if $d(G_n)_{j+1} < Z_n^t < d(G_n)_j$, the buyer n goes from rating G_n to rating j for $j \in \{1, \dots, J\}$
8. def is equal to the number of insolvencies.
9. \mathcal{P}_t is the new portfolio at the end of the period.

If $def < def^*$ then $E = h$, otherwise E_l .

Let u be a random number from a uniform distribution on $[0, 1]$.

If $u < \mathbb{P}(E_2 = h|E)$ than $E_2 = h$, otherwise $E_2 = l$.

1. Simulate $R^{t'}$ following a multivariate Gaussian distribution $N(0, \Sigma)$.
2. $N - \text{def}$ standard Gaussian variables $\varepsilon_n^{t'}$ for $n \in 1 \dots N - \text{def}$
3. Abilities to pay for each buyer are computed

$$Z_n^{t'} = \varrho_n \frac{{}^t w_n R^{t'}}{\| \langle {}^t w_n M \rangle \|_2} + \sqrt{1 - \varrho_n^2} \varepsilon_n^{t'}$$

4. For each buyer n if $Z_n^{t'} < d_{(G_n)}^2$ then buyer n is insolvent.
5. For each buyer n if $d_{(G_n)}^2 < Z_n^{t'} < D_{(G_n)}^2$ buyer n has a protracted default.
6. Loss equals to the sum of losses for protracted and insolvency defaults and the loss of the first period.

4 Data and results

4.1 Data and some details on the estimation of parameters

We estimated the parameters on the monthly rating history of a credit insurance portfolio. The portfolio was made of firms in France between 2005 and 2011. There were almost 2 million firms in our database and computations related to the estimation of parameters needed machines of at least 16Gb of RAM. Simulations of losses needed servers of 32 machines computing simultaneously in order to have some results after a reasonable delay.

We chose to use business cycle as the leading variable of our application here. So we estimated six-months transition matrices in recession and expansion times. We had quarterly data of GDP variation. If the GDP decreased in one quarter than we assumed it was recession time, otherwise it was expansion. If a quarter is recession, resp. expansion, than every month in it is a recession, resp. expansion, month. We computed a generator matrix like in [15], and then six-months recession and expansion transition matrices.

We computed similarly UGDs and the coefficients giving the exposure variation in recession and expansion.

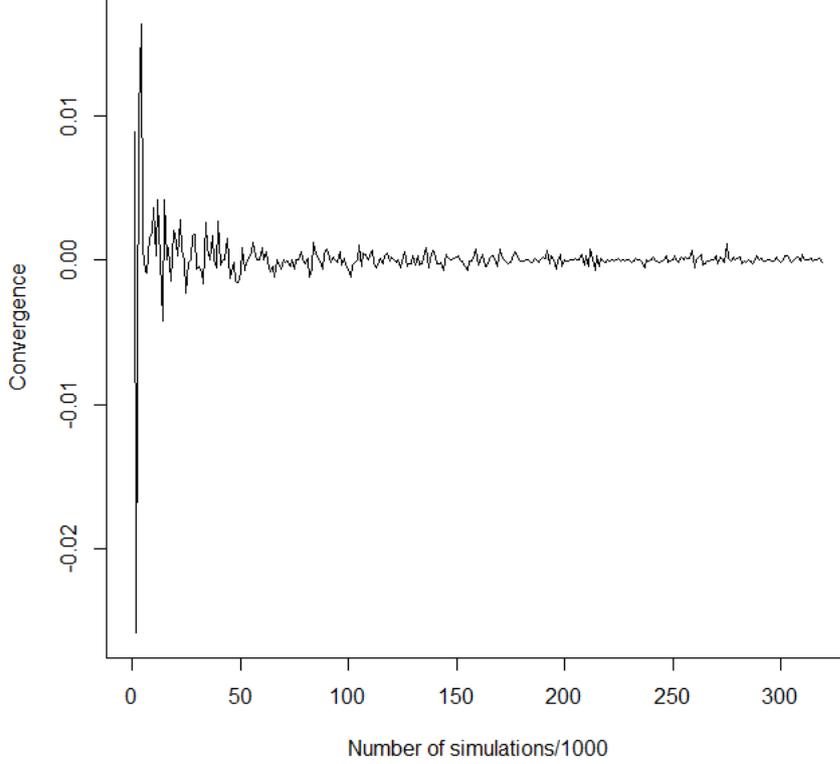
The graphic in Figure 2 gives an idea of the convergence for quantiles at 0.99.⁵

4.2 Results

The following annual loss densities are obtained by simulating the two-period model. We compared the single-period model, with high cycle hypothesis, noted *1 period H*,

⁵In the Solvency II directive it is recommended to compute quantiles at 0.995 for the economic capital estimation, but here for practical issues, since it requires less simulations to converge we chose to use the quantile at 0.99.

Figure 2: Convergence of quantiles at 0.99



and midway between high and low cycle, noted *1 period TTC*, single period along with low cycle hypothesis, noted *1 period L*, with the above two-period modelling where the first period is more likely to be in a high cycle phase, noted *2 period H*, and the last case where we suppose the first period is more likely to be a low cycle phase, noted *2 period L*.

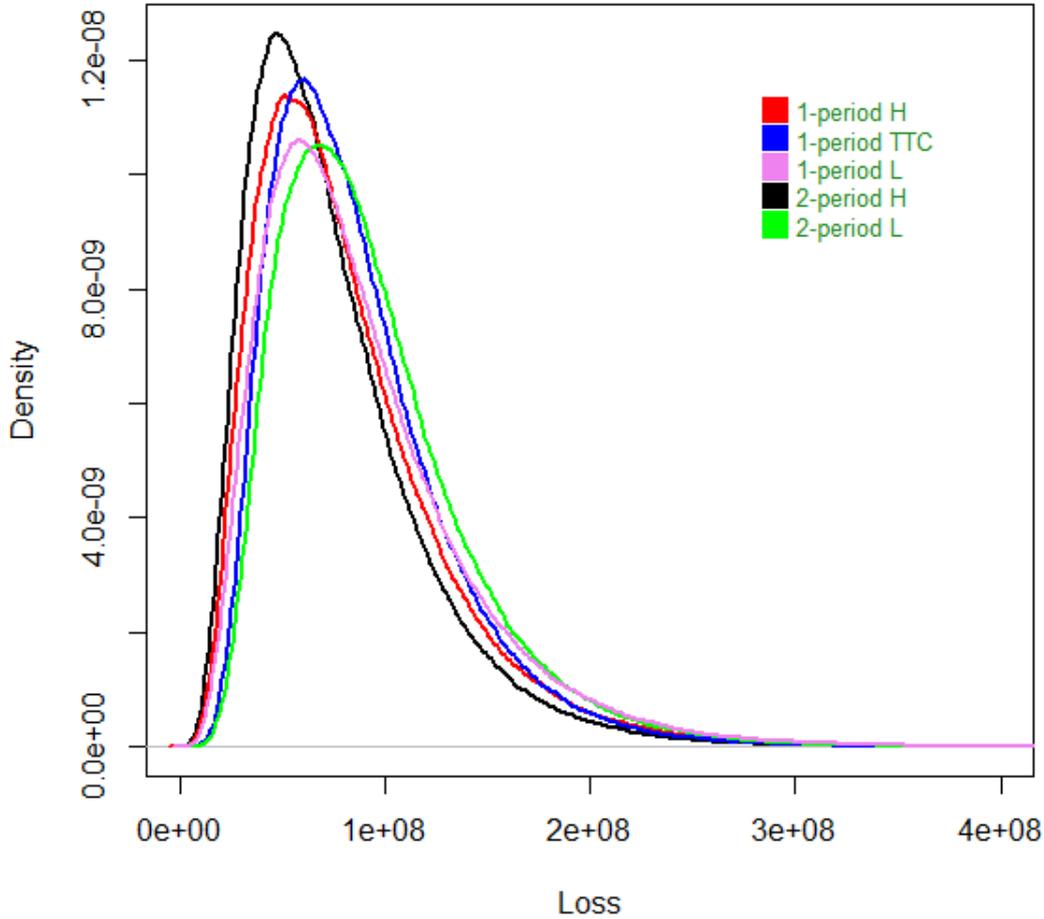
The variations of economic capital at 0.99, *i.e.* Loss quantile at 0.99 - Expected loss, are:

$\frac{\text{CE 2 period L} - \text{CE 1 period L}}{\text{CE 1 period L}}$	-9.3%
$\frac{\text{CE 2 period H} - \text{CE 1 period H}}{\text{CE 1 period H}}$	-1.8%
$\frac{\text{CE 2 period L} - \text{CE 1 period TTC}}{\text{CE 1 period TTC}}$	-3.9%
$\frac{\text{CE 2 period H} - \text{CE 1 period TTC}}{\text{CE 1 period TTC}}$	-9.6%

First we observe that the Economic Capital computed with the two-period model is always lower than the one where losses are modelled with the single-period model.

In the case where the cycle is expected to be high during at least the first semester, modelling losses with the two-period model will decrease the Economic

Figure 3: Loss densities



Capital by 1.8% compared to the same hypothesis under the single-period model. This is due to the fact that in the second semester, there is a chance that the cycle is low and in that case the exposures of the insurer will decrease. When compared to the case of a single-period model where the cycle has the same probability of being high or low (we call it *through-the-cycle* later on), then the Economic Capital with the single period model goes down by 9.6%. This is coherent with what we expected.

When the cycle is expected to be low during the first semester, the two-period model Economic Capital is lower by 9.3% compared to the one of the single-period where the cycle is low during the whole year. When we compare to the single-period Economic Capital under the through-the-cycle assumption, it is lower by 3.9%. It is coherent to have a smaller decrease under the through-the-cycle assumption compared to the case where the cycle is low the whole year.

Conclusion

We presented in this paper a model considering business or credit cycles. This model is especially suited to credit insurance because it predicts that guarantees the insurer offers can lower when the cycle is low and otherwise increase. The numerical application seems to tell that the economic capital computed with our two-period model is lower than the one with one single period, which may be an appreciated feature. It also allows the insurer to better adapt its reserves level to the business cycle and do less estimation errors than in a one period model. Indeed the more flexible model allows to have reserves that better fit to the real needs.

The major difficulty in applying this model is finding the best discriminatory variable, which will help distinguish between cycle phases, and thus different actions taken by the insurer. After finding this variable, Markov transition matrices should be estimated, in a consistent way. Applying the theory of Hidden Markov Chains to find a method to estimate these transition matrices will require further work.

5 Appendix

5.1 Kolmogorov's theorem

Theorem 1 *Kolmogorov*

Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables such that:

- for all $n \geq 1$, $\mathbb{E}(X_n^2) < +\infty$
- There exists a sequence $(a_n)_{n \geq 1}$ of positive numbers which grows to $+\infty$ such that

$$\sum_{n=1}^{+\infty} \frac{\mathbb{V}(X_n)}{a_n^2} < +\infty.$$

Then

$$\frac{\sum_{n=1}^N X_n - \mathbb{E}\left(\sum_{n=1}^N X_n\right)}{a_n} \xrightarrow{N \rightarrow +\infty} 0$$

If

$$a_n^{-1} \mathbb{E}\left(\sum_{n=1}^N X_n\right) \rightarrow m$$

then $\frac{\sum_{n=1}^N X_k}{a_n} \rightarrow m$.

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