EARLY DEFAULT,

ABSOLUTE PRIORITY RULE VIOLATIONS,

AND THE PRICING OF FIXED-INCOME SECURITIES

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First version : March 1995

Current version : August 25, 1995

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A first version of this paper has been presented at the Isaac Newton Institute, Cambridge University and at the 1995 Annual Meeting of the French Finance Association in Bordeaux. We thank François Degeorge, Philippe Henrotte and especially Denis Talay for helpful comments. The usual caveat applies. Comments are welcome.

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Abstract

This paper develops a corporate bond valuation model which takes into account both early default and interest-rate risk. The bankruptcy triggering mechanism is directly related to the payoff received by bondholders when early bankruptcy is forced upon. More specifically, the default barrier is simply defined as a fixed quantity discounted at the riskless rate up to the maturity date of the risky corporate bond. As soon as this barrier is crossed, bondholders receive an exogenously specified fraction of the remaining assets. Deviations from the absolute priority rule are also captured. Because it accounts for gaussian interest rate uncertainty, default risk and deviations from the absolute priority rule, this model is capable of producing quite diverse shapes for the term structure of yield spreads. Interest rate sensitivity and duration measures are also derived and analyzed. To sum up, the model delivers a quite complete picture of corporate spreads and their determinants.

1. INTRODUCTION

For corporate bondholders, default by the bond issuer is a possibility that cannot be ignored. Expectations of possible future losses are reflected in current bond yields. For instance, bonds issued by so-called "fallen angels" do command a higher yield than an otherwise identical treasury bond [1]. This extra-yield, the corporate default spread, rewards the corporate bondholder for carrying the risk of not being repaid. Even though the contingent claim analysis has delivered lots of insights into the modelling of default, corporate bonds turn out to be more difficult to price than equivalent treasury bonds.

In their seminal papers, Black and Scholes [1973] and Merton [1973] have modeled corporate liabilities as options on the total value of the firm. Merton [1974] and its refinements (Lee [1981], Pitts and Selby [1983]) analyze default spreads of pure discount corporate bonds and the risk structure of interest rates. In these models, default is assumed to occur when the bond matures and when the firm exhausts its assets. The term structure of interest rates is flat and deterministic. This contingent claim approach has been extensively used to price more complex securities : Black and Cox [1976] examine the effects of some indenture provisions (subordination arrangements, safety covenants). Ingersoll [1977] and Brennan and Schwartz [1980] value callable and convertible bonds. Geske [1977] shows how coupon bonds can be viewed as portfolios of compound options.

Although many theoretical papers have studied the pricing of corporate fixed-income

1. As shown by Grinblatt [1994], default is not the only reason that yields on corporate bonds may differ from yields on Treasury bonds. The relative liquidities of the two instruments is another reason. Indeed, Treasury securities are more actively traded and they are more frequently used in the implementation of hedging policies for both government and corporate bond markets. This observation induces Duffie and Singleton [1995] to introduce a "liquidity convenience" yield in their econometric modeling of term structures of defaultable bonds. In what follows, we ignore this issue.

securities, empirical investigation remains scarce. Jones, Mason and Rosenfeld [1984] show that Merton's model with non stochastic interest rate is unable to generate corporate spreads compatible with those observed in practice. Ogden [1987] obtains two results. On the one hand, the two traditional default risk measures (the corporate asset volatility and the leverage) explain about 78 percent of the variation in the agency ratings of corporate bonds. On the other hand, default premia are inversely related to firm size, which the previous contingent claim models do not obtain. To avoid some complicated bond features, Sarig and Warga [1989] use pure discount bonds to test the relationship between risk premia and time-to-maturity. The results appear to fit the predictions of theoretical models more closely : the term structure of risky spreads is downward sloping for highly leveraged firms, humped for medium leveraged firms and upward sloping for low leveraged firms.

The traditional contingent claim approach is also characterized by another weakness. Indeed, it assumes that creditors receive their full payments before shareholders seize any portion of the remaining assets. However, it is now a rather well documented fact that strict priority is rarely enforced in financially distressed corporations. Franks and Torous [1989, 1994], Eberhart, Moore and Roenfeldt [1990] and Weiss [1990] indicate that the absolute priority rule is enforced in only 25% of corporate bankruptcy cases. There is also a strong evidence that bond and equity markets anticipate violations to the strict priority rule.

The simplified approach of corporate default is thus not really satisfying : actual credit spreads are too large to be accounted for, even when excessive volatility and leverage levels are used. Three refinements should be incorporated to better fit real world corporate spreads. First, the very notion of corporate default is somewhat fuzzy. Indeed, traditional contingent claim analysis in the spirit of Merton usually ignores the possibility of early default. The corporate default threshold is not accurately modelled. To cope with this first weakness, Black and Cox [1976] for instance assume a cutoff level whereby default can occur any time. This cutoff is introduced by considering a safety covenant to protect bondholders. Second, interest rates should no longer be considered constant. The stochastic movements of the term structure of interest rates play obviously a crucial role. The assumption of constant rates is embarrassing when one deals with the pricing of interest rate sensitive instruments. Moreover, to price corporate bonds properly, the intertwined effects of interest rate uncertainty and corporate default cannot be ignored. Finally, violations of the strict priority rule should be modelled to better reflect the bargaining game between stakeholders upon default.

More recently, contributions have proposed modelling frameworks where the above mentioned issues are taken into account (Jarrow and Turnbull [1992], Kim, Ramaswamy and Sundaresan [1993], Lando [1994], Longstaff and Schwartz [1994], Nielsen, Saà-Requejo and Santa Clara [1993]) [2]. All of these contributions introduce default risk and interest rate risk. They all consider quite general default triggering mechanisms. Nielsen et alii [1993] define default as the first time when the value of corporate assets is lower than a stochastic level. This level is stochastically driven by both the term structure uncertainty and the corporate assets uncertainty. Longstaff and Schwartz [1994] and Kim, Ramaswamy and Sundaresan [1993] look at default along the lines of Black and Cox [1976]. Financial distress in their models occurs when the value of assets reach a constant or deterministic barrier. Despite this difference in the definition of the default barrier, all contributions explicitly model the deviation from the strict priority rule. Upon bankruptcy, bondholders receive an exogenously given number of riskless bonds. More specifically, their payoff upon default is limited to the

2. Leland [1994a, 1994b] derives closed-form solutions for the pricing of risky debts with an endogenous bankruptcy-triggering value. He assumes a constant default boundary, non stochastic interest rates and time-homogeneous debt cash-flows.

product of an exogenously specified number and the value at the time default intervenes of an equivalent (namely same maturity to go) riskless bond. As noted by Longstaff and Schwartz [1994] such a modelling has the advantage of being consistent with the usual practice whereby claimants are given new securities rather than cash [3].

This paper develops an analysis of corporate spreads along the same lines. The main objective is to provide a simple framework yielding a computationaly efficient "closed form" solution. To achieve such a goal this paper combines aspects of the previous contributions. Indeed, instead of relying upon a totally ad hoc threshold for default, this paper relates it to the payoff that claimants receive upon bankruptcy. More specifically, the default barrier is simply defined as a fixed quantity discounted at the riskless rate up to the maturity date of the risky corporate bond. As soon as this barrier is crossed, bondholders receive an exogenously specified fraction of the remaining assets. Thus deviations from the strict priority rule are easily captured. As a result, the barrier is stochastic as in Nielsen et alii [1993]. But, on top of it, it overcomes one of the weaknesses of Nielsen et alii' s contribution. Indeed, nothing in Nielsen et alii's paper prevent bondholders from receiving upon bankruptcy more than assets permit. Because in their model the payoff upon bankruptcy is exogenously specified (i.e. independent of the level of the stochastic barrier and of the value of the assets) everything goes as if an external guarantor were providing the bondholders with an implicit put. Consequently, the pricing of corporate spreads is affected by the presence of this implicit put. By relating the payoff upon bankruptcy to the level of the default barrier we avoid this

3. The real issue here is whether the new securities given to bondholders are riskless or default prone. Longstaff and Schwartz [1994], Nielsen and alii [1993] consider that the defaulted bonds are exchanged for equivalent riskless bonds. To put it differently, this choice implies that the defaultable term structure of interest rates collapses to the default-free term structure of interest rates upon default. Duffie and Singleton [1995] recognize that the issue of recovery under default is quite intricate. They propose a recovery payout which is proportional to the value of a non-defaulted corporate bond.

problem. Another difficulty arises in both contributions by Longstaff and Schwartz [1994] and Nielsen et alii [1993]. Indeed, when the corporate bond reaches maturity the corporation may find itself in a solvent position (according to the threshold) but nevertheless with assets insufficient to match the face value of the bond [4]. We also avoid the limitation of having a constant default boundary as in Longstaff and Schwartz [1994] and Kim, Ramaswamy and Sundaresan [1993].

The paper is organized as follows. In section 2, we set up the modelling framework and the basic assumptions. Section 3 is devoted to the derivation of a closed form solution for pricing pure discount corporate bonds. This solution is analyzed and the intuition underlying its structure is given. In section 4, a closed form solution to the valuation of corporate spreads is given and its properties are interpreted. In section 5, the interest rate sensitivity of corporate bonds is computed. This elasticity measure enables one to assess the impact of potential corporate default on the properties of the corporate bond. A conclusion summarizes the main findings and suggests some avenues of further research.

2. THE MODEL

2.1 The assumptions

In this section we develop a continuous-time valuation model for corporate debt which accounts both at the same time for interest rate risk and default risk. This model finds its roots

4. For example, assume a fixed default threshold of $50 and a promised repayment of $100. If it turns out that the value of corporate assets until maturity has always been above the threshold (no early default) and that the final value of these assets is $80 at maturity, the firm is "threshold-solvent" but unable to repay the $100.

in the contributions of Black and Cox [1976], Longstaff and Schwartz [1994] and Nielsen et alii [1993]. More specifically, our framework for modelling default is a contingent claim framework as seminally proposed by Black and Scholes [1973] and Merton [1973]. To be precise we now state the various building assumptions.

- **Assumption 1 : complete financial markets**

Financial markets are assumed to be complete and frictionless. Trading takes place continuously. Under this hypothesis, Harrison and Kreps [1979] have shown that there exists a unique probability measure Q - the risk neutral probability - under which the continuously discounted price of any security is a Q-martingale.

- **Assumption 2 : gaussian interest rate uncertainty**

Interest rates are assumed to be normally distributed and to follow a gaussian process. This general continuous time model enables us to consider several different cases : for instance, Merton [1973], Vasicek [1977], El Karoui and Rochet [1989], Jamshidian [1991] and Heath, Jarrow and Morton [1992]. In such a framework, the short term riskless interest rate r_t at time t follows a gaussian diffusion process and the volatility structure is a deterministic function. The only drawback is that negative interest rates are not precluded in such a gaussian environment. Nevertheless, it should be noted that for reasonable values of the parameter the probability for the short term riskless interest rate low is quite low.
The process followed by r_t is governed under the risk neutral probability Q by the following stochastic differential equation :

$$dr_t = a(t)\left[b(t) - r_t\right]dt + \sigma(t)\, dW_t \quad (1)$$

for some deterministic functions $a(t)$, $b(t)$ and $\sigma(t)$. $\sigma(t)$ is the instantaneous standard deviation of the riskless interest rate r_t. W_t is a standard Wiener process.

According to (1), we can write at time t under the probability Q the dynamics of return of the default-free zero coupon bond $P(t,T)$ maturing at T as:

$$\frac{dP(t,T)}{P(t,T)} = r_t\, dt - \sigma_P(t,T)\, dW_t \quad (2)$$

where $\sigma_P(t,T)$ is a deterministic function.

When it is necessary to specify the three functions $a(t)$, $b(t)$ and $\sigma(t)$ in equation (1) for numerical computations, we assume a Vasicek [1977] representation for the dynamics of the term structure of interest rates :

$$dr_t = a(b - r_t)\, dt + \sigma\, dW_t \quad (3)$$

where a, b and σ are positive constants. b is the long-run mean of the riskless interest rate r_t, a is the speed of adjustment towards that mean. In such a framework, the volatility $\sigma_P(t,T)$ of the zero-coupon bond is given by :

$$\sigma_P(t,T) = \frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right) \quad (4)$$

The value of $P(t,T)$ is given by Vasicek [1977] :

$$P(t,T) = A(T-t) \cdot \exp\left[-B(T-t) \cdot r_t\right] \quad (5)$$

where :

$$B(T-t) = \frac{1 - e^{-a(T-t)}}{a} \quad (6)$$

$$A(T-t) = \exp\left[\left(\frac{\sigma^2}{2a^2} - b\right)(T-t) + \left(b - \frac{\sigma^2}{a^2}\right) B(T-t) + \frac{\sigma^2}{4a^2} B(2(T-t)) \right] \tag{7}$$

- **Assumption 3 : corporate asset process**

  Let V_t denote the total value of the assets of the firm at time t. Under Q, we assume that V_t is governed by the following process :

$$\frac{dV_t}{V_t} = r_t dt + \sigma_V \left[\rho \, dW_t + \sqrt{1-\rho^2} dZ_t\right] \tag{8}$$

  where σ_V represents the instantaneous standard deviation of the return on corporate assets and ρ the correlation coefficient between the riskless interest rate and the value of corporate assets. Z_t is a standard Wiener process, independent of W_t.

- **Assumption 4 : Modigliani-Miller theorem**

  The total value of the firm V_t is independent of the capital structure decision of the firm. In other words we assume that the standard Modigliani-Miller proposition does hold. Cash outflows such as coupon or principal payments are assumed to be financed by issuing new securities.

2.2 Definition of bankruptcy

Bondholders are assumed to be protected by a safety covenant which allow them to trigger early bankruptcy. A safety covenant is a contractual mechanism which gives bondholders the right to bankrupt or force a reorganization of the firm if its performance does not match some prespecified benchmark. Let $v(t)$ be this benchmark. $v(t)$ denotes the threshold level at which bankruptcy occurs at time t. As soon as the value of corporate assets V_t falls below $v(t)$, the safety covenant protects the bondholders and bankruptcy or workout is forced upon. We assume that $v(t)$ is exogenously specified and takes the following form:

$$v(t) = \alpha \cdot F \cdot P(t, T) \tag{9}$$

where F denotes the face value of the corporate bond and $0 \leq \alpha \leq 1$.

To use Duffie and Singleton [1995] terminology, the default time is assumed to be accessible (predictable). In other words, the very first time the market value of the firm's assets hits the barrier is not a "sudden surprise". Another route suggested by Madan and Unal [1993] is to consider the default time as potentially inaccessible. For instance, default can be modelled as a Poisson arrival. Default occurs the first time when some Poisson process jumps from *0* to *1*. Such a framework is convenient when the objective is for example to disentangle cashflow based insolvencies from market value based insolvencies [5]. Under our set of assumptions, this distinction is not necessary. Indeed, in a frictionless world cashflow based insolvencies can only occur when the situation is viewed as hopeless by stakeholders from a corporate net worth standpoint [6].

Our default threshold $v(t)$ can be viewed as an extension of the barrier of Black and Cox [1976] to a stochastic interest rate environment. This specification has three obvious advantages. First, because of the stochasticity of interest rates, the barrier $v(t)$ is also stochastic. The second benefit is obvious when the bondholders'payoffs upon bankruptcy are defined. Franks and Torous [1993] indicate that, regardless of the reorganization form, each

5. Sudden cashflow shortages could be captured by the occurrence of the jump process.
6. To put it in Leland's [1994] words, "current cash flows could be negative, but if equity value remains, the firm need not be forced into bankruptcy".

creditor receives a bundle of securities in exchange for his original claim in a distressed firm. As can be seen in the next section, it is easy to relate the level of these payoffs, namely the fraction of an equivalent riskless zero-coupon bond, to the level of the barrier. Finally, the shape of the barrier enables us to draw comparisons with some of the previous literature. The Mertonian case corresponds to a value for α equal to zero. The completely risk-free situation is given by a very stringent covenant where α equal to *1*. In that case, bondholders are sure to receive what they were promised in the first place. For values of α lying strictly between *0* and *1*, intermediate cases of early default are considered. Other things being equal, the closer α gets to *0*, the less protective is the safety covenant for early default.

2.3 Bondholders payoffs

When bankruptcy occurs, namely when V_t crosses $v(t)$ for the first time, corporate bondholders receive a fraction of the corporate assets which is exogenously specified and represents the write-down that is applied to the value that should be received by bondholders if the strict priority rule were enforced. Let f_1 $(0 \leq f_1 \leq 1)$ denote this fraction when default occurs before maturity and f_2 $(0 \leq f_2 \leq 1)$ when default occurs at maturity. In the limit case where $f_1 = f_2 = 1$, the strict priority rule is enforced and shareholders receive nothing. Fons [1994] indicates that the Moody's recovery rates for senior secured, senior unsecured, senior subordinated, subordinated and junior subordinated debts are respectively 64.59%, 48.38%, 39.79%, 30.00% and 16.33%. Franks and Torous [1993] show that these recovery rates, based on a sample of 37 firms that formally reorganized under Chapter 11 between 1983 and 1990, for secured debt, bank debt, senior debt, junior debt and preferred stock are respectively 80.1%, 86.4%, 47.0%, 28.9% and 42.5%.

This write-down of creditor claims is usually the outcome of a bargaining process which results in shifts of gains and losses among corporate claimants relative to their contractual rights (Milgrom and Roberts [1992, p.503]). Franks and Torous [1991] for instance report that, over 41 Chapter 11 bankruptcies, junior claimants managed to extract $878 million that should have normally been received by senior claimants. Common stockholders gained a third of those $878 million although they had no valid claim on them. Franks and Torous [1991] also report the same kind of pattern in the case of 47 workouts. Franks and Torous [1993] find that on a sample of 37 firms that reorganize under Chapter 11 deviations from absolute priority for bank debt, secured debt; senior debt, junior debt, preferred stock and equity were respectively: -0.96%, -1.67%, -1.44%, 0.94%, 0.80% and 2.28%. For 45 firms that restructure their debt informally, the deviations for bank debt, senior debt, junior debt, preferred stock and equity were respectively : -3.54%, -3.44%, -0.95%, -1.39% and 9.51%.

In any case, these observations suggest that senior claimants prefer to be sure to receive a slice of a larger pie than all of a much smaller one. The objective of the paper is not to model explicitly the bargaining process between the creditors and the firm and its outcome (forced bankruptcy and liquidation, Chapter 11, informal debt workouts ...). Deviations from the absolute priority rule are viewed as implicit bond covenants which are anticipated by market players (see for instance Eberhart and Senbet [1993]). The fact that f_1 and f_2 could take different values indicates that the nature of the bargaining process upon bankruptcy may be different before or at maturity.

3. THE VALUATION OF RISKY ZERO-COUPON BONDS

The corporation at time $t=0$ issues two classes of securities : a single homogeneous debt consisting of a zero-coupon bond (with face value F and maturity T) and the residual claim (equity). In what follows, we derive the time t value D_t of the corporate zero-coupon bond with maturity T and face value F. To clarify the valuation procedure, we first look at the cash flows which corporate claimants are entitled to under the various scenarios. Bondholders have a claim on the following potential cash-flows :

- **No default before maturity**

Under this first scenario, the value V_u of total assets has always remained above the default barrier $v(u)$ with $t \leq u \leq T$ and bankruptcy can only occur at maturity. In that case, bondholders receive a fraction f_2 of the remaining assets.

Let $T_{V,v}$ denote the first passage time of the process V_u through the barrier $v(u)$:

$$T_{V,v} = inf\{u \geq t, V_u = v(u) = \alpha \cdot F \cdot P(u, T)\}$$

The payoffs D_T at maturity are thus equal to :

$$F \cdot \mathbf{1}_{T_{V,v} \geq T, V_T \geq F} + f_2 \cdot V_T \cdot \mathbf{1}_{T_{V,v} \geq T, V_T < F}$$

where the indicator function $\mathbf{1}_B$ for B is the real-valued random variable defined by :

$$\mathbf{1}_B(\omega) = \begin{cases} 1 & \forall \omega \in B \\ 0 & \text{otherwise} \end{cases}$$

- **Default before maturity**

Under this second scenario, bondholders receive an amount :

$$D_{T_{V,v}} = f_1 \cdot V_{T_{V,v}}$$
$$= f_1 \cdot \alpha F \cdot P(T_{V,v}, T) \quad \text{if} \quad T_{V,v} < T$$

To sum up, the total cash-flow picture of the firm at maturity reads as follows :

- **Equityholders :**

$$(1-f_1) \cdot \alpha F \cdot \mathbf{1}_{T_{V,v}<T} + (V_T - F) \cdot \mathbf{1}_{T_{V,v} \geq T,\, V_T \geq F} + (1-f_2) \cdot V_T \cdot \mathbf{1}_{T_{V,v} \geq T,\, V_T < F}$$

- **Bondholders :**

$$f_1 \cdot \alpha F \cdot \mathbf{1}_{T_{V,v}<T} + F \cdot \mathbf{1}_{T_{V,v} \geq T,\, V_T \geq F} + f_2 \cdot V_T \cdot \mathbf{1}_{T_{V,v} \geq T,\, V_T < F}$$

We are now in a position to price the risky zero-coupon bond issued by the firm. To do so we use the risk-neutral pricing technique. The price as of time t of the risky zero-coupon is thus given by the discounted value of future expected cash-flows under the risk neutral probability Q :

$$D_t = E^Q\left[e^{-\int_t^T r_u du} \cdot \left\{ f_1 \alpha F \cdot \mathbf{1}_{T_{V,v}<T} + F \cdot \mathbf{1}_{T_{V,v} \geq T,\, V_T \geq F} + f_2 V_T \cdot \mathbf{1}_{T_{V,v} \geq T,\, V_T < F} \right\} \right] \tag{10}$$

The first term within the expectations operator corresponds to the discount factor. The second term captures the payoff conditional upon forced bankruptcy before maturity. The third term represents the best case scenario for bondholders, namely solvency. The last term gives the final cash-flow to bondholders conditional upon both no premature forced bankruptcy and final insolvency (with assets at maturity higher than the threshold value but less than the face value of the bond).

After some computations (see *Appendix 1*), expression (10) collapses to the following closed form solution which gives the price of a corporate zero-coupon bonds :

$$D_t = FP(t,T)\cdot\Big\{1-\big(-\tfrac{1}{l_t}N[-d_1]+N[-d_2]\big)+\big(-\tfrac{1}{q_t}N[-d_5]+\tfrac{q_t}{l_t}N[-d_6]\big) \\ -(1-f_1)\big(\tfrac{1}{l_t}N[-d_3]+\tfrac{q_t}{l_t}N[-d_4]\big) \\ -(1-f_2)\big(\tfrac{1}{l_t}(N[d_3]-N[d_1])+\tfrac{q_t}{l_t}(N[d_4]-N[d_6])\big)\Big\} \tag{11}$$

$$\text{where}\quad \begin{cases} l_t = \dfrac{FP(t,T)}{V_t} \\ q_t = \dfrac{v(t)}{V_t} = \dfrac{\alpha FP(t,T)}{V_t}\end{cases}$$

$$\text{and}\quad \begin{cases} d_1 = \dfrac{-\ln l_t+\Sigma(t,T)^2/2}{\Sigma(t,T)} = d_2+\Sigma(t,T) \\ d_3 = \dfrac{-\ln q_t+\Sigma(t,T)^2/2}{\Sigma(t,T)} = d_4+\Sigma(t,T) \\ d_5 = \dfrac{-\ln q_t^2/l_t+\Sigma(t,T)^2/2}{\Sigma(t,T)} = d_6+\Sigma(t,T) \\ \Sigma(t,T) = \left[\int_t^T\Big((\rho\sigma_V+\sigma_P(u,T))^2+(1-\rho^2)\sigma_V^2\Big)du\right]^{\frac{1}{2}} \\ N(\,) = \text{the cumulative standard normal distribution}\end{cases}$$

The value of the risky bond involves two ratios, namely l_t and q_t. The first one is the classical Merton's quasi-debt ratio l_t. It is not equal to the true debt to asset ratio because the numerator (i.e. the face value of corporate debt) is discounted at the riskless rate. As a result, it is an upward biased estimate of the real debt to asset ratio. This quasi debt to asset ratio can also be given another interpretation. Indeed, it is nothing but the forward price of assets, namely the value of assets that prevails under the Q-economy. The quantity q_t can be defined as the bankruptcy or early default ratio. It is simply the ratio of the current default threshold to the current value of the firm [7]. As soon as q_t is equal to one, bankruptcy is forced upon.

Let $P_E(l_t)$ denote the price as of time t of the following european put of maturity T:

$$P_E(l_t) = -\frac{1}{l_t}N[-d_1] + N[-d_2]$$

The above formula (11) can be rewritten as:

$$\begin{aligned} D_t = FP(t,T) \cdot \Bigg\{ & 1 - P_E(l_t) + \frac{q_t}{l_t} P_E\left(\frac{q_t^2}{l_t}\right) \\ & - (1-f_1)\frac{1}{l_t}(N[-d_3] + q_t N[-d_4]) \\ & - (1-f_2)\frac{1}{l_t}((N[d_3] - N[d_1]) + q_t(N[d_4] - N[d_6])) \Bigg\} \end{aligned} \qquad (12)$$

This last expression (12) lends itself to a rather intuitive interpretation. Indeed, the risky corporate zero-coupon bond can be decomposed into five basic components. The first term corresponds to an otherwise identical riskless zero-coupon bond. The second term is the usual put-to-default at maturity as derived in both Black and Scholes [1973] and Merton [1973]. The third term, a long position on a european put, appears because of the possibility of an early default triggered by the safety covenant. As such it contributes to mitigate the effect of the previous traditional put-to-default. This interpretation is even more convincing when one considers the case whereby the absolute priority rule is strictly enforced ($f_1 = f_2 = 1$). The last two terms disappear. In the polar case $\alpha = 1$, $q_t = l_t$ and the two put options cancel out. The bondholder's situation has become riskless. As soon as the value of corporate assets reach

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7. The inverse of q_t is equivalent to Leland's ratio [1994a, b] V/V_B. The differences however are that, in Leland, the default threshold is constant and endogenous and interest rates are non stochastic.

the present value of liabilities discounted at the riskfree rate, early bankruptcy is forced upon. The bondholder is then sure to receive the face value F at maturity. The last two terms in equation (12) materialize the effect of the deviations from the strict priority rule. This effect is negative. A discount is applied : when $f_1 < 1$ or $f_2 < 1$, bondholders are somehow "spoliated" because of the non enforcement of the strict priority rule. Each term measures the impact of partially removing shareholders from their residual claimant positions. On top of being economically sensible, the formula (12) is also computationaly efficient as it only involves normal univariate distributions.

4. THE VALUATION OF CORPORATE SPREADS

In this section we derive the term structure of default spreads. For the sake of simplicity and without loss of generality, we let $t = 0$ and $F = 1$. We denote Y_0 the yield of a corporate zero-coupon bond whose maturity is T :

$$Y_0 = -\frac{1}{T} \ln D_0 \tag{13}$$

By using the closed form solution (12) of the previous section, it obtains that :

$$\begin{aligned} Y_0 = -\frac{1}{T} \cdot \ln P(0, T) \cdot \Bigg\{ & 1 - P_E(l_0) + \frac{q_0}{l_0} P_E\left(\frac{q_0^2}{l_0}\right) \\ & - (1 - f_1) \frac{1}{l_0} \left(N[-d_3] + q_0 N[-d_4]\right) \\ & - (1 - f_2) \frac{1}{l_0} \left((N[d_3] - N[d_1]) + q_0 (N[d_4] - N[d_6]) \right) \Bigg\} \end{aligned} \tag{14}$$

The corporate spread S_0 is defined as the difference between the yield Y_0 and the yield of an otherwise equivalent riskless zero-coupon bond. The corporate spread is thus given by the following expression:

$$S_0 = Y_0 + \frac{1}{T}\ln P(0, T) \tag{15}$$

or equivalently :

$$\begin{aligned} S_0 = -\frac{1}{T}\cdot \ln \Bigg\{ & 1 - P_E(l_0) + \frac{q_0}{l_0} P_E\left(\frac{q_0^2}{l_0}\right) \\ & - (1-f_1)\frac{1}{l_0}\left(N[-d_3] + q_0 N[-d_4]\right) \\ & - (1-f_2)\frac{1}{l_0}\left(\left(N[d_3] - N[d_1]\right) + q_0\left(N[d_4] - N[d_6]\right)\right) \Bigg\} \end{aligned} \tag{16}$$

Under a flat term structure of interest rates, no safety covenant and no deviations from the absolute priority rule, expression (16) boils down to Merton's formula for corporate spreads. If the safety covenant and the deviations from the strict priority rule are skipped but the stochasticity of interest rates is preserved, expression (16) is similar to the one derived by Decamps [1994]. As expected from the closed form solution (12), the term structure of corporate spreads is affected by the presence of the safety covenant and the violations of the absolute priority rule. Two immediate implications can be drawn from the wider set of parameters influencing the term structure of corporate spreads as given in (16). First, larger corporate spreads than those derived by Merton are to be expected. Spreads predicted by our model will thus be closer to those observed in practice. Second, it is reasonable to guess that corporate spreads will exhibit more complex properties than those derived in the previous literature.

To confirm these implications, we now turn to a numerical implementation of our model. The numerical computations are done with the following basic parameter values. For the interest rate process, we fix : $a = 0.2$, $b = 0.06$, $\sigma = 0.02$ and $r_0 = 0.05$. The corporate asset standard deviation is set to $\sigma_V = 0.2$ and the correlation coefficient to $\rho = -0.25$ [8]. The coefficient α which determines the level of the default barrier is set equal to $\alpha = 0.9$ in most of the simulations. Equivalently, the early default ratio is set to $q_0 = 0.9 \cdot l_0$. It is worthwhile pointing out that, as expected, our model yields larger corporate spreads than Merton's model as witnessed in Table 1. Indeed, Merton's model[9] corresponds to an early default ratio q_0 equal to zero and no deviation from the absolute priority rule ($f_1 = f_2 = 1.0$).

Figure 1 illustrates the relationship between the level of the corporate spread and the time-to-maturity of the bond for various leverage levels. Two different leverages are examined: constant quasi-debt ratio l_0 (Figure 1a) and constant face value to asset ratio (Figure 1b). These pictures resemble those drawn by Merton [1974] : the term structure is downward sloping for highly leveraged firms, humped for medium leveraged firms and upward sloping for low leveraged firms. These results match the empirical results by Sarig and Warga [1989].

Figure 2 relates the corporate spread level to the time-to-maturity of the bond for various values of the early default ratio. The main objective of this figure is to assess the impact of the early bankruptcy threshold α. Table 1 is a useful complement in that respect since it also displays the case where there is no violation of the strict priority rule. When any such

8. The correlation coefficient ρ is negative because an unexpected increase of interest rates implies an unexpected decrease in asset prices.
9. More precisely, its extension to stochastic interest rate. See Decamps [1994].

deviations are absent ($f = 1$), the spread level is a decreasing function of the degree of protection provided by the safety covenant. As soon as violations of the absolute priority rule are possible, the situation becomes more complex. Indeed, long term bonds exhibit a different pattern from short term bonds. For long term bonds, the lower the early default ratio (i.e. the less protective the safety covenant), the larger the corporate spread. For short term bonds, this result does not carry over and an indeterminate relationship holds (see also Table 1, Panel A). Bondholders are confronted with what could be dubbed a "bankruptcy dilemma". According to the pricing equation (12), they simultaneously hold a long position due to the early default covenant and a short position due to the violations of the strict priority rule. These conflicting positions combine with the solvency situation of the firm, as measured by l_0, to deliver an ambiguous picture of the spread - early default ratio relationship. In any case, firms that have issued long term bonds and are l_0 - insolvent ($l_0 > 1$) are characterized by spreads inversely related to the level of q_0. In such cases, bondholders discount negatively the absence of any safety covenant. In that respect, it is interesting to underline that, other things being equal, the less l_0 - solvent the firm, the larger the corporate spread.

Violations of the strict priority rule have also a strong influence on the level of corporate spreads as displayed by Table 1 and Figure 4. The less bondholders receive upon bankruptcy, the larger the corporate spreads. Figure 2 and Figure 4 also suggest that the corporate spread is more elastic to changes in f_1 and/or f_2 than to changes in q_0.

Figure 5 to Figure 7 capture the effect of corporate asset volatility on the level of corporate spreads. Merton [1974] showed that the corporate spread is an increasing function of corporate volatility. Decamps [1994] proved that this result was a direct product of the constant interest rate assumption. According to Decamps [1994], when interest rates and the value of the firm's assets are negatively correlated, the corporate spread is a single peaked function (first decreasing then increasing) of the volatility of the firm. Figure 6 indicates that neither Merton's result nor Decamps's proposition extend to our setting (see for instance Panel A and Panel C).

Figure 7 describes the impact of the correlation coefficient ρ on the level of corporate spreads as a function of the firm's asset volatility. For positive values of the correlation coefficient ρ, the relationship is monotonically increasing while for negative values of the same coefficient this is no longer true. As a matter of fact, corporate volatility is measured by the quantity $\Sigma(t, T)$ which is the volatility of the ratio $V_t/P(t, T)$. As already mentioned above, this last quantity is the forward price of corporate assets, namely the relevant "underlying" asset for pricing the corporate zero-coupon bond under the Q-economy. Expression (11) immediately shows that $\Sigma(t, T)$ is a monotonically increasing function of the asset volatility σ_V if ρ is positive, while it is first decreasing then increasing if ρ is negative.

5. The Interest Rate Elasticity of Corporate Bonds

Our valuation model has interesting implications for both portfolio and asset - liability management (ALM). Indeed, bond portfolios and the balance sheets of many institutions are interest rate sensitive. To protect these latter against unexpected movements in the term structure of interest rates, the investor needs to accurately evaluate his risk exposure. Interest rate elasticity and duration measures are now commonplace. Nevertheless, most of them are quite restrictive and only apply under a specific set of assumptions. Corporate default is for instance rarely taken into account. This is unfortunate and produces biased estimates of the true elasticity of a corporate bond (see Ambarish and Subrahmanyam [1989] and Chance [1990]). In that respect our model corrects this deficiencies. Moreover the simple structure of our corporate bond pricing formula makes it easy to compute the relevant elasticity measure.

Let η_t denote the interest elasticity measure of the corporate bond :

$$\eta_t = \frac{1}{D_t} \cdot \frac{\partial D_t}{\partial r_t}$$

After some computations detailed in *Appendix 2*, the following expression obtains:

$$\eta_t = -B(T-t) + \left[\frac{\rho\sigma_V}{\sigma} + B(T-t)\right] \cdot \frac{V_t}{D_t} \cdot \Omega_t \tag{17}$$

with :

$$\Omega_t = N(-d_1) - \frac{l_t}{q_t} N(-d_5)$$
$$-(1-f_1)\ \{N(-d_3) - \frac{2Z(d_3)}{\Sigma(t,T)}\}$$
$$-(1-f_2)\ \{N(-d_1) - N(-d_3) - \frac{Z(d_1)}{\Sigma(t,T)} + \frac{Z(d_3)}{\Sigma(t,T)} - \frac{q_t Z(d_6)}{\Sigma(t,T)} + \frac{q_t Z(d_4)}{\Sigma(t,T)}\}$$

where $Z(\ )$ is the standard normal distribution.

From expression (17) several polar cases can be recovered. The first one deals with the situation where there is no violation of the absolute priority rule and where the safety covenant is fully protective (i.e. $\alpha = 1$). Under these assumptions, the interest rate elasticity of the corporate bond is given by Vasicek's formula for the interest rate elasticity of a riskless zero-coupon bond :

$$\eta_t = -B(T-t) \tag{18}$$

In the second case, no safety covenant is attached to the corporate zero-coupon bond (i.e. $\alpha = 0$). There is, however, still no deviations from the strict priority rule ($f_1 = f_2 = 1$). As a result, expression (17) collapses to Merton's formula extended by Decamps [1994] :

$$\eta_t = -B(T-t) + \left[\frac{\rho\sigma_V}{\sigma} + B(T-t)\right] \cdot \frac{V_t}{D_t} \cdot N(-d_1) \tag{19}$$

A careful inspection of expression (19) enables a better understanding of the more complex expression (17). Indeed, expression (19) is basically composed of three terms which can be disentangled as follows. As already mentioned above, the first term corresponds to the interest rate elasticity of a riskless zero-coupon bond. The second term within brackets is equivalent to the interest rate elasticity gap between the firm's assets and the default free zero-coupon bond. This is so because the ratio $\rho\sigma_V/\sigma$ measures the interest rate elasticity of the firm's assets (see equation (8)). At this point it is worthwhile mentioning the crucial role that the correlation coefficient ρ plays in the determination of the overall interest rate elasticity of the corporate bond [10]. Other things being equal, a negative ρ entails a higher interest rate elasticity. The third term of (19) can be defined as a probability adjusted ratio.

Expression (17) is obviously more complex than expression (19). The full Ω_t term reflects the impacts of both the safety covenant and the violations of the absolute priority rule.

10.See Rabinovitch [1989] for a similar effect when pricing stock options under stochastic interest rates.

When it comes to immunization or related techniques, practitioners use the duration tool (especially Macaulay duration) more frequently than the interest rate elasticity itself. Duration-matching methods are for instance quite popular. The accuracy of these methods, however, crucially depends on the correct measurement of duration. To provide results that are closer to the current managerial practices, we now define the effective duration of the corporate zero-coupon bond. This duration Λ_t is defined as the maturity of the default-free bond which exhibits the same interest rate elasticity as the risky corporate zero-coupon bond, namely :

$$-\frac{\sigma_P(t, \Lambda_t)}{\sigma} = \eta_t \tag{20}$$

In a Vasicek framework, expression (20) can be rewritten, as of time $t = 0$:

$$\Lambda_0 = -\frac{\ln(1 + a\eta_0)}{a} \tag{21}$$

Table 2 and Figure 8 give some numerical insights into the behavior of the duration of the corporate zero-coupon bond. Table 2 is quite informative about the influence of the early default ratio on the effective duration. Indeed, other things being equal, the closer q_0 gets to zero, the smaller is the effective duration. This result makes sense if one remembers that a safety covenant is equivalent for bondholders to a long position on a put option. If this long position is progressively lifted, a gradual decrease in the duration of the corporate bond is rather intuitive. Table 2 also delivers that the effective duration is usually smaller than the maturity, i.e. the Macaulay duration. But this result is not general and does not carry over to short term bonds. In Panel A, the duration of a one year to maturity corporate bond is, for instance, 3.73 years (when $l_0 = 1.1$ and $q_0 = 0$). This result may seem quite counterintuitive. In a constant interest rate environment, Leland [1994] obtains that effective duration is always less than Macaulay duration. This is not true here. This point is confirmed by Figure 8. In the three panels, effective duration is greater than Macaulay duration for short term bonds. Moreover, this holds true whatever the intensity of the deviations from the strict priority rule [11]. This lengthening of the effective duration finds its roots in the interest rate elasticity gap between the firm's assets and the default free bond. This gap is one of the determinants of the overall interest rate elasticity of the corporate bond. For short maturities, the quantity $B(T-t)$ is small. Moreover, Figure 8 displays results in the case of a negative correlation coefficient ρ. Other things being equal, this entails a stronger elasticity η_t which in turn translates into a higher duration. For a positive correlation coefficient ρ, Leland's result would be recovered.

Finally it is worthwhile pointing out that more stringent deviations from the strict priority rule contribute to shorten the effective duration.

6. CONCLUSION

In this paper, we have developed a corporate bond valuation model which avoids some of the shortcomings of the previous literature. This model bears some analogies to the work of Longstaff and Schwartz [1994] and Nielsen et alii [1993]. The bankruptcy triggering mechanism is however somewhat different in that it is directly related to the payoff received by bondholders when early bankruptcy is forced upon. As a result, a fairly simple closed-form

11. In the specific case where $q_0 = l_0$ and $f_1 = f_2 = 1$ (Panel C), effective duration is equal to Macaulay duration (45 degree line). But in this scenario the risky bond becomes a riskless bond.

solution for the pricing of corporate bonds is obtained. Because it accounts for stochastic interest rates, default risk and deviations from the absolute priority rule, this model is capable of producing quite diverse shapes for the term structure of corporate spreads. A detailed analysis of corporate spreads has been proposed. This analysis and a thorough numerical investigation show that corporate spreads exhibit complex relationships with the parameters of the model. Interest rate elasticity and duration measures have also been derived. A more accurate picture of the interest risk exposure of corporate bonds has been provided. Among other things, effective duration has been shown to be significantly different from the traditional Macaulay duration.

Some additional work remains obviously to be done. Extensions to the pricing of coupon-paying bonds is a natural candidate. Last but not least a detailed empirical investigation comparing the theoretical spreads predicted by this paper and actual credit spreads has to be carried out.

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APPENDIX 1

A. Preliminary

By integrating between *0* and *t* the processes defined respectively by equation (2) and equation (8), we obtain :

$$P(t,T) = P(0,T) \cdot \exp\left[\int_0^t r_u du - \frac{1}{2}\int_0^t \sigma_P^2(u,T)\,du - \int_0^t \sigma_P(u,T)\,dW_u\right]$$

$$V_t = V_0 \cdot \exp\left[\int_0^t r_u du - \frac{1}{2}\int_0^t \sigma_V^2\,du + \sigma_V \int_0^t (\rho\,dW_u + \sqrt{1-\rho^2}dZ_u)\right]$$

We introduce the two following processes l_t and q_t :

$$\begin{cases} l_t = \dfrac{FP(t,T)}{V_t} \\ q_t = \dfrac{\alpha FP(t,T)}{V_t} \end{cases} \quad \text{with} \quad \begin{cases} l_0 = \dfrac{FP(0,T)}{V_0} \\ q_0 = \dfrac{\alpha FP(0,T)}{V_0} \end{cases}$$

or,

$$l_t = l_0 \cdot \exp\left[\frac{1}{2}\Sigma(0,t)^2 - \int_0^t \sigma_P(u,T)\,(\rho\sigma_V + \sigma_P(u,T))\,du \right.$$
$$\left. - \int_0^t (\rho\sigma_V + \sigma_P(u,T))\,dW_u - \int_0^t \sigma_V\sqrt{1-\rho^2}dZ_u\right]$$

where :

$$\Sigma(0,t)^2 = \int_0^t \left[(\rho\sigma_V + \sigma_P(u,T))^2 + (1-\rho^2)\sigma_V^2\right]du$$

B. First passage time

Let W_t be a standard brownian motion. We define the stopping time $T_{W,h}$ by :

$$T_{W,h} = inf\{\, t \geq 0,\ W_t = h(t)\,\}$$

From Harrison [1985] and Conze and Viswanathan [1991], one can easily show the following results when the frontier is of the form $h(t) = a + bt$:

$$P[W_T \geq x, T_{W,h} \geq T] = N\left[\frac{-x}{\sqrt{T}}\right] - e^{-2ab} N\left[\frac{2a - x}{\sqrt{T}}\right]$$

Taking $x = a + bT$ gives :

$$P[T_{W,h} < T] = N\left[\frac{a + bT}{\sqrt{T}}\right] + e^{-2ab} N\left[\frac{a - bT}{\sqrt{T}}\right]$$

C. Valuation of the risky zero-coupon bond

The time $t = 0$ price D_0 of the risky zero-coupon bond is defined by equation (10) :

$$D_0 = E^Q\left[e^{-\int_0^T r_u du} \cdot \left\{f_1 \alpha F \cdot \mathbf{1}_{T_{V,v} < T} + F \cdot \mathbf{1}_{T_{V,v} \geq T, V_T \geq F} + f_2 V_T \cdot \mathbf{1}_{T_{V,v} \geq T, V_T < F}\right\}\right]$$

where $v(t) = \alpha F \cdot P(t, T)$

Remarks :

- $\{V_T \geq F\} \Leftrightarrow \{l_T \leq 1\}$ because $P(T, T) = 1$
- $\{V_t \geq \alpha FP(t, T)\} \Leftrightarrow \{l_t \leq \frac{l_0}{q_0}\}$

  The first stopping time $T_{V,v}$ of the process V_t on the barrier $v(t) = \alpha FP(t, T)$ and the first stopping time $T_{l,s}$ of the process l_t on the barrier $s(t) = \frac{l_0}{q_0}$ have the same law.

Equation (10) thus becomes :

$$D_0 = E^Q\left[e^{-\int_0^T r_u du} \cdot F \cdot \left\{f_1 \frac{q_0}{l_0} \cdot \mathbf{1}_{T_{l,s} < T} + \mathbf{1}_{T_{l,s} \geq T, l_T \leq 1} + f_2 \cdot \frac{1}{l_T} \cdot \mathbf{1}_{T_{l,s} \geq T, l_T > 1}\right\}\right]$$

C.1 Computation of $\varepsilon_1 = E^Q\left[e^{-\int_0^T r_u du} \cdot f_1 \frac{q_0}{l_0} F \cdot \mathbf{1}_{T_{l,s} < T}\right]$

We define the probability $\tilde{Q}$ by its Radon-Nikodym derivative :

$$\frac{d\tilde{Q}}{dQ} = \frac{e^{-\int_0^T r_u du}}{P(0, T)}$$

Under $\tilde{Q}$, ε_1 becomes :

$$\varepsilon_1 = f_1 \frac{q_0}{l_0} F P(0, T) \cdot E^{\tilde{Q}}\left[\mathbf{1}_{T_{l,s} < T}\right]$$

We can define also the following $\tilde{Q}$-brownian motion (Girsanov theorem) :

$$\tilde{W}_t = W_t + \int_0^t \sigma_P(u, T)\, du$$

The process l_t is thus given by :

$$l_t = l_0 \cdot \exp\left[\frac{1}{2}\Sigma(0, t)^2 - \int_0^t (\rho\sigma_V + \sigma_P(u, T))\, d\tilde{W}_u - \int_0^t \sigma_V \sqrt{1-\rho^2} dZ_u\right]$$

By defining the process :

$$\tilde{X}_t = \int_0^t (\rho\sigma_V + \sigma_P(u, T))\, d\tilde{W}_u + \int_0^t \sigma_V \sqrt{1-\rho^2} d\tilde{Z}_u$$

one can write :

$$E^{\tilde{Q}}\left[\mathbf{1}_{T_{l,s} < T}\right] = E^{\tilde{Q}}\left[\mathbf{1}_{T_{\tilde{X},\tilde{x}} < T}\right]$$

where :

$$\tilde{x}(t) = \ln q_0 + \frac{1}{2}\Sigma(0, t)^2$$

Moreover :

$$\langle \tilde{X}_t \rangle = \Sigma(0, t)^2$$

With the following change of variable :

$$\begin{cases} \tau = k(t) = \Sigma(0,t)^2 \\ \tau^* = k(0) = 0 \\ \tau^{**} = k(T) \end{cases}$$

according to Karatzas and Shreve [1991, p.174], one can show that $\tilde{B}_\tau = \tilde{X}_{k^{-1}(\tau)}$ is a $\tilde{Q}$-brownian motion.

Moreover :

$$\begin{aligned} E^{\tilde{Q}}\left[\mathbf{1}_{T_{\tilde{X},\tilde{x}}<T}\right] &= \tilde{Q}\left[inf_{0\le t\le T}\{\tilde{X}_t-\tilde{x}(t)\}<0\right] \\ &= \tilde{Q}\left[inf_{k^{-1}(\tau^*)\le k^{-1}(\tau)\le k^{-1}(\tau^{**})}\{\tilde{X}_{k^{-1}(\tau)}-\tilde{x}(k^{-1}(\tau))\}<0\right] \\ &= \tilde{Q}\left[inf_{k^{-1}(\tau^*)\le k^{-1}(\tau)\le k^{-1}(\tau^{**})}\{\tilde{B}_\tau-\tilde{b}(\tau)\}<0\right] \\ &= \tilde{Q}\left[inf_{\tau^*\le\tau\le\tau^{**}}\{\tilde{B}_\tau-\tilde{b}(\tau)\}<0\right] \\ &= E^{\tilde{Q}}\left[\mathbf{1}_{T_{\tilde{B},\tilde{b}}<\tau^{**}}\right] \end{aligned}$$

because k^{-1} is an increasing function and where :

$$\tilde{b}(t) = \tilde{x}(k^{-1}(t)) = \ln q_0 + \frac{t}{2}$$

With the previous results based on first passages times, we obtain :

$$\varepsilon_1 = f_1\, FP(0,T)\cdot\left\{\frac{1}{l_0}N\left[\frac{\ln q_0-\Sigma(0,T)^2/2}{\Sigma(0,T)}\right]+\frac{q_0}{l_0}N\left[\frac{\ln q_0+\Sigma(0,T)^2/2}{\Sigma(0,T)}\right]\right\}$$

C.2 Computation of $\varepsilon_2 = E^Q\left[e^{-\int_0^T r_u du}\cdot F\cdot\mathbf{1}_{T_{l,s}\ge T,\, l_T\le 1}\right]$

Under the previous probability $\tilde{Q}$, we have :

$$E^Q\left[e^{-\int_0^T r_u du}\cdot F\cdot\mathbf{1}_{T_{l,s}\ge T,\, l_T\le 1}\right] = FP(0,T)\cdot E^{\tilde{Q}}\left[\mathbf{1}_{T_{l,s}\ge T,\, l_T\le 1}\right]$$

With the same notations :

$$E^{\tilde{Q}}\left[\mathbf{1}_{T_{l,s}\geq T,\, l_T\leq 1}\right] = E^{\tilde{Q}}\left[\mathbf{1}_{\tilde{X}_T\geq \tilde{x}^*}\cdot \mathbf{1}_{T_{\tilde{X},\tilde{x}}\geq T}\right]$$

where :

$$\begin{cases} \tilde{x}^* = \ln l_0 + \frac{1}{2}\Sigma(0,T)^2 \\ \tilde{x}(t) = \ln q_0 + \frac{1}{2}\Sigma(0,t)^2 \end{cases}$$

One can write :

$$E^{\tilde{Q}}\left[\mathbf{1}_{\tilde{X}_T\geq \tilde{x}^*}\cdot \mathbf{1}_{T_{\tilde{X},\tilde{x}}\geq T}\right] = E^{\tilde{Q}}\left[\mathbf{1}_{\tilde{X}_{k^{-1}(\tau^{**})}\geq \tilde{x}^*}\cdot \mathbf{1}_{T_{\tilde{X},\tilde{x}}\geq k^{-1}(\tau^{**})}\right]$$

$$= E^{\tilde{Q}}\left[\mathbf{1}_{\tilde{B}_{\tau^{**}}\geq \tilde{x}^*}\cdot \mathbf{1}_{T_{\tilde{B},\tilde{b}}\geq \tau^{**}}\right]$$

Then we obtain :

$$\varepsilon_2 = FP(0,T)\cdot\left\{N\left[-\frac{\ln l_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)}\right] - \frac{1}{q_0}N\left[\frac{\ln q_0^2/l_0 - \Sigma(0,T)^2/2}{\Sigma(0,T)}\right]\right\}$$

C.3 Computation of $\varepsilon_3 = E^{Q}\left[e^{-\int_0^T r_u du}\cdot f_2 F\cdot \frac{1}{l_T}\cdot \mathbf{1}_{T_{l,s}\geq T,\, l_T>1}\right]$

A similar proof can be developed thanks to Girsanov theorem. By introducing the probability $\bar{Q}$ defined by its Radon-Nikodym derivative :

$$\frac{d\bar{Q}}{dQ} = \exp\left[-\frac{1}{2}\int_0^T \sigma_V^2 du + \int_0^T \sigma_V(\rho dW_u + \sqrt{1-\rho^2}dZ_u)\right]$$

we have :

$$E^{Q}\left[e^{-\int_0^T r_u du}\cdot f_2 F\cdot \frac{1}{l_T}\cdot \mathbf{1}_{T_{l,s}\geq T,\, l_T>1}\right] = f_2 F\frac{1}{l_0}\cdot E^{\bar{Q}}\left[\mathbf{1}_{T_{l,s}\geq T,\, l_T>1}\right]$$

$$= f_2 F\frac{1}{l_0}\cdot\left\{E^{\bar{Q}}\left[\mathbf{1}_{T_{l,s}\geq T}\right] - E^{\bar{Q}}\left[\mathbf{1}_{T_{l,s}\geq T,\, l_T\leq 1}\right]\right\}$$

We can also define the following $\bar{Q}$-brownian motions :

$$\begin{cases} \bar{W}_t = W_t - \int_0^t \rho\sigma_V du \\ \bar{Z}_t = Z_t - \int_0^t \sqrt{1-\rho^2}\sigma_V du \end{cases}$$

The process l_t is then given by :

$$l_t = l_0 \cdot \exp\left[-\frac{1}{2}\Sigma(0,t)^2 - \int_0^t (\rho\sigma_V + \sigma_P(u,T))\,d\bar{W}_u - \int_0^t \sigma_V\sqrt{1-\rho^2}\,d\bar{Z}_u\right]$$

We can go through the same computations by writing :

$$\bar{X}_t = \int_0^t (\rho\sigma_V + \sigma_P(u,T))\,d\bar{W}_u + \int_0^t \sigma_V\sqrt{1-\rho^2}\,d\bar{Z}_u$$

with the following parameters :

$$\begin{cases} \bar{x}^* = \ln l_0 - \frac{1}{2}\Sigma(0,T)^2 \\ \bar{x}(t) = \ln q_0 - \frac{1}{2}\Sigma(0,T)^2 \end{cases}$$

Finally, we obtain :

$$\varepsilon_3 = \frac{f_2 FP(0,T)}{l_0}\left\{ N\left[\frac{-\ln q_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)}\right] - N\left[\frac{-\ln l_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)}\right]\right\}$$
$$- \frac{f_2 FP(0,T)\, q_0}{l_0}\left\{ N\left[\frac{\ln q_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)}\right] - N\left[\frac{\ln q_0^2/l_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)}\right]\right\}$$

C.4 Conclusion

$$D_0 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

or :

$$D_0 = FP(0,T)\cdot\Big\{1 - \big(-\tfrac{1}{l_0}N[-d_1] + N[-d_2]\big) + \big(-\tfrac{1}{q_0}N[-d_5] + \tfrac{q_0}{l_0}N[-d_6]\big)$$
$$- (1-f_1)\big(\tfrac{1}{l_0}N[-d_3] + \tfrac{q_0}{l_0}N[-d_4]\big)$$
$$- (1-f_2)\big(\tfrac{1}{l_0}(N[-d_1] - N[-d_3]) + \tfrac{q_0}{l_0}(N[-d_6] - N[-d_4])\big)\Big\}$$

where :

$$\begin{cases} d_1 = \dfrac{-\ln l_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)} = d_2 + \Sigma(0,T) \\ d_3 = \dfrac{-\ln q_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)} = d_4 + \Sigma(0,T) \\ d_5 = \dfrac{-\ln q_0^2/l_0 + \Sigma(0,T)^2/2}{\Sigma(0,T)} = d_6 + \Sigma(0,T) \end{cases}$$

APPENDIX 2

We define the short term interest rate sensitivity of the corporate bond by :

$$\eta_t = \frac{1}{D_t} \cdot \frac{\partial D_t}{\partial r_t}$$

Applying Itô's lemma to $D_t = D\,[t, V_t, P\,(t, T)\,]$ gives :

$$dD_t = f[t, V_t, P\,(t, T)\,]\,dt$$
$$+ \left[\rho\, \sigma_V V_t \frac{\partial D_t}{\partial V_t} - \sigma_P\,(t, T)\, P\,(t, T)\, \frac{\partial D_t}{\partial P\,(t, T)}\right] dW_t$$
$$+ \sqrt{1 - \rho^2}\, \sigma_V V_t \frac{\partial D_t}{\partial V_t} dZ_t$$

and :

$$\eta_t = \frac{1}{\sigma D_t} \cdot \left[\rho \sigma_V V_t \frac{\partial D_t}{\partial V_t} - \sigma_P\,(t, T)\, P\,(t, T)\, \frac{\partial D_t}{\partial P\,(t, T)}\right]$$

The two partial derivatives of D_t are given by :

$$\frac{\partial D_t}{\partial V_t} = N(-d_1) - \frac{l_t}{q_t} N(-d_5)$$
$$- (1 - f_1)\, \{N(-d_3) - \frac{2Z(-d_3)}{\Sigma\,(t, T)}\}$$
$$- (1 - f_2)\, \{N(-d_1) - N(-d_3) - \frac{Z(-d_1)}{\Sigma\,(t, T)} + \frac{Z(-d_3)}{\Sigma\,(t, T)} - \frac{q_t Z(-d_6)}{\Sigma\,(t, T)} + \frac{q_t Z(-d_4)}{\Sigma\,(t, T)}\}$$

and :

$$\frac{\partial D_t}{\partial P\,(t, T)} = F - FN(-d_2) + \frac{q_t}{l_t} FN(-d_6) - (1 - f_1) \frac{q_t}{l_t} F \left\{N(-d_4) + \frac{Z(-d_4)}{\Sigma\,(t, T)} + \frac{Z(-d_3)}{q_t \Sigma\,(t, T)}\right\}$$
$$- (1 - f_2) \frac{q_t}{l_t} F \left\{ (N(-d_6) - N(-d_4)) + \left(\frac{Z(-d_1)}{q_t \Sigma\,(t, T)} + \frac{Z(-d_3)}{q_t \Sigma\,(t, T)}\right) \right.$$
$$\left. + \left(\frac{Z(-d_6)}{\Sigma\,(t, T)} - \frac{Z(-d_4)}{\Sigma\,(t, T)}\right) \right\}$$

where $Z(\,)$ is the standard normal distribution.

Because :

$$P(t,T)\frac{\partial D_t}{\partial P(t,T)} = D_t - V_t\frac{\partial D_t}{\partial V_t}$$

we obtain finally :

$$\eta_t = -\frac{\sigma_P(t,T)}{\sigma} + \frac{V_t}{\sigma D_t}[\rho\sigma_V + \sigma_P(t,T)]\frac{\partial D_t}{\partial V_t}$$

By noticing that $\sigma_P(t,T) = \sigma \cdot B(T-t)$ where $(-B(T-t))$ represents the sensitivity of a default free zero-coupon bond, we have :

$$\eta_t = -B(T-t) + \left[\frac{\rho\sigma_V}{\sigma} + B(T-t)\right]\cdot\frac{V_t}{D_t}\cdot$$

$$\left[N(-d_1) - \frac{l_t}{q_t}N(-d_5) - (1-f_1)\ \{N(-d_3) - \frac{2Z(d_3)}{\Sigma(t,T)}\}\right.$$

$$\left. -(1-f_2)\ \{N(-d_1) - N(-d_3) - \frac{Z(d_1)}{\Sigma(t,T)} + \frac{Z(d_3)}{\Sigma(t,T)} - \frac{q_t Z(d_6)}{\Sigma(t,T)} + \frac{q_t Z(d_4)}{\Sigma(t,T)}\}\right]$$

Table 1
Yield spreads as a function of initial quasi-debt ratio l_0 for different early default ratios q_0. The parameters used are : $\sigma_V = 0.2$, $f_1 = f_2 = f$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$. Spreads are expressed in basis points.

Panel A : T = 2

	$q_0 = l_0$		$q_0 = 0.9 . l_0$		$q_0 = 0.8 . l_0$		$q_0 = 0$	
l_0	f = 1.0	f = 0.8	f = 1.0	f = 0.8	f = 1.0	f = 0.8	f = 1.0	f = 0.8
0.4	0	2	0	1	0	1	0	1
0.6	0	86	21	67	23	64	23	63
0.8	0	493	148	452	184	424	188	418
1.0	0	1116	389	1191	550	1144	586	1113
1.2	0	1598	622	1972	1007	2018	1150	1941
1.4	0	1818	768	2513	1400	2816	1773	2733

Panel B : T = 5

	$q_0 = l_0$		$q_0 = 0.9 . l_0$		$q_0 = 0.8 . l_0$		$q_0 = 0$	
l_0	f = 1.0	f = 0.8	f = 1.0	f = 0.8	f = 1.0	f = 0.8	f = 1.0	f = 0.8
0.4	0	22	7	20	8	19	9	19
0.6	0	127	44	129	61	127	67	124
0.8	0	290	109	327	168	334	196	329
1.0	0	446	179	546	298	588	379	589
1.2	0	560	233	735	419	836	588	866
1.4	0	627	270	870	516	1044	805	1135

Panel C : T = 10

	$q_0 = l_0$		$q_0 = 0.9 . l_0$		$q_0 = 0.8 . l_0$		$q_0 = 0$	
l_0	f = 1.0	f = 0.8	f = 1.0	f = 0.8	f = 1.0	f = 0.8	f = 1.0	f = 0.8
0.4	0	43	16	47	24	48	29	48
0.6	0	109	43	128	71	137	93	141
0.8	0	173	72	216	125	242	180	258
1.0	0	223	95	293	174	342	278	383
1.2	0	259	112	353	215	428	378	506
1.4	0	282	124	396	245	495	477	624

Table 2
Duration Λ_0 as a function of time-to-maturity for different early default ratios q_0. The parameters used are : $\sigma_V = 0.2$, $f_1 = f_2 = 0.8$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

Panel A : $l_0 = 1.1$

	$q_0 = l_0$		$q_0 = 0.9 \, . \, l_0$		$q_0 = 0$	
T (yrs)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)
1	2.76	176.0	4.20	320.0	3.73	273.0
5	4.47	- 10.6	4.14	- 17.2	3.79	- 24.2
10	7.90	- 21.0	6.83	- 31.7	4.80	- 52.0
15	10.87	- 27.5	9.21	- 38.6	5.41	- 63.9
20	13.14	- 34.3	11.02	- 44.9	5.75	- 71.3

Panel B : $l_0 = 0.8$

	$q_0 = l_0$		$q_0 = 0.9 \, . \, l_0$		$q_0 = 0$	
T (yrs)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)
1	2.02	102.0	1.82	82.0	1.76	76.0
5	4.43	- 11.4	4.27	- 14.6	4.20	- 16.0
10	7.57	- 24.3	6.77	- 32.3	5.77	- 42.3
15	10.23	- 31.8	8.84	- 41.1	6.49	- 56.7
20	12.23	- 38.9	10.40	- 48.0	6.78	- 66.1

Panel C : $l_0 = 0.4$

	$q_0 = l_0$		$q_0 = 0.9 \, . \, l_0$		$q_0 = 0$	
T (yrs)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)	Λ_0 (yrs)	$(\Lambda_0 - T) / T$ (%)
1	1.00	0.0	1.00	0.0	1.00	0.0
5	4.90	- 2.0	4.90	- 2.0	4.91	-1.8
10	8.64	- 13.6	8.45	- 15.5	8.35	- 16.5
15	11.11	- 25.9	10.42	- 30.5	9.74	- 35.1
20	12.63	- 36.9	11.46	- 42.7	9.91	- 50.5

Figure 1a

Yield spreads as a function of time-to-maturity for different quasi-debt ratios l_0. The parameters used are : $\sigma_V = 0.2$, $f_1 = f_2 = 0.8$, $q_0 = 0.9\ l_0$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

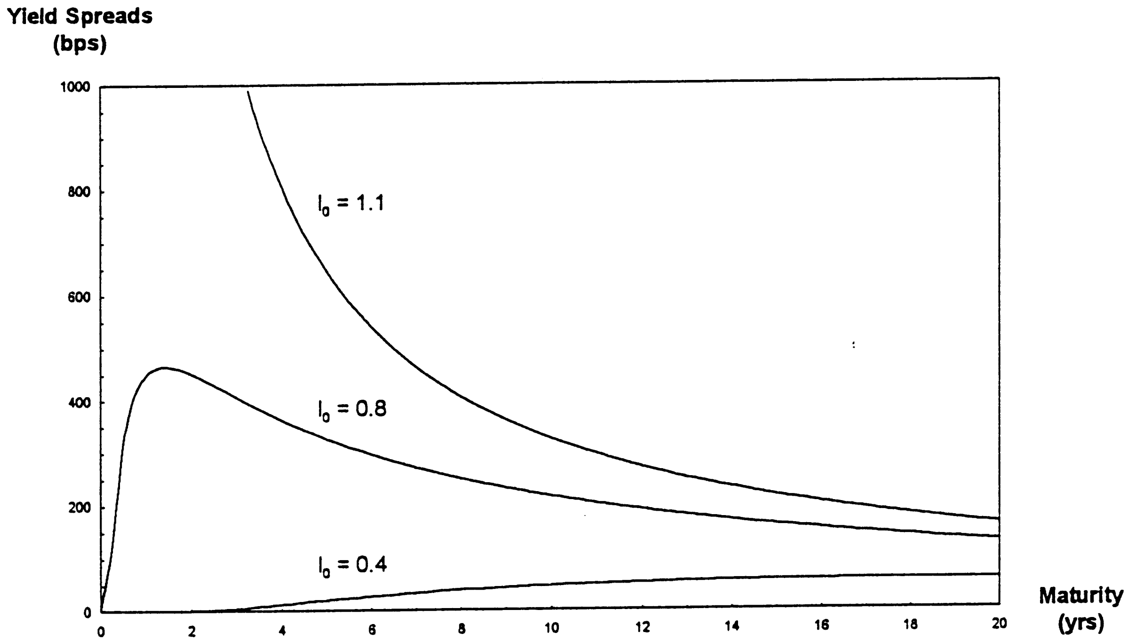


Figure 1b
Yield spreads as a function of time-to-maturity for different quasi-debt ratios l_0. The parameters used are : $\sigma_V = 0.2$, $f_1 = f_2 = 0.8$, $q_0 = 0.9\ l_0$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

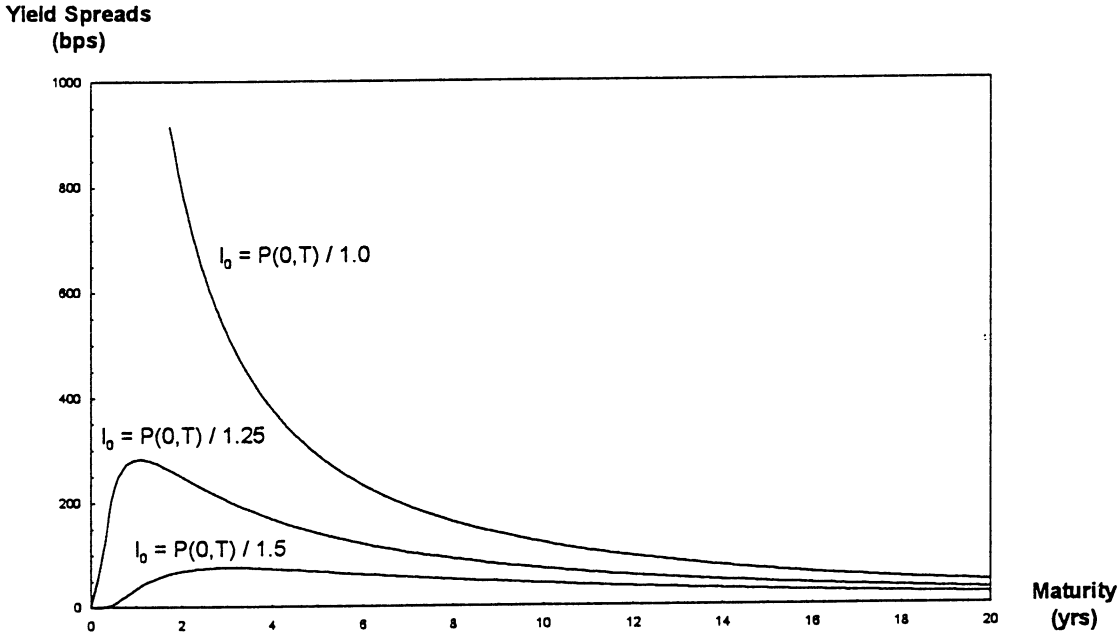
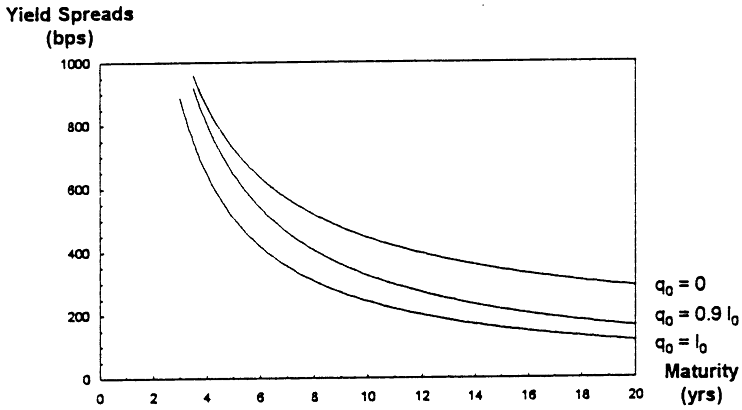


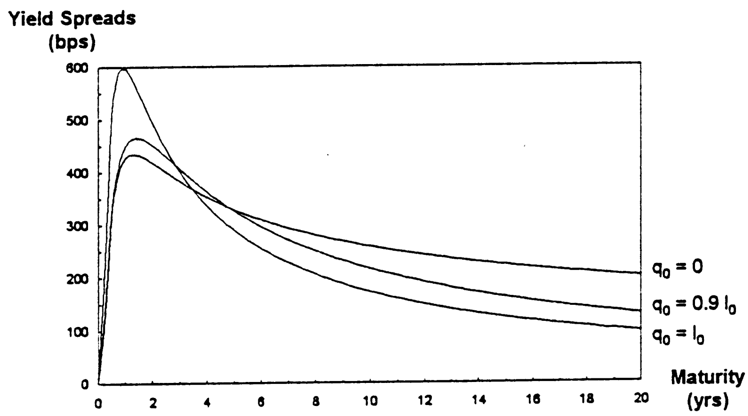
Figure 2

Yield spreads as a function of time-to-maturity for different early default ratios q_0. The parameters used are : $\sigma_V = 0.2$, $f_1 = f_2 = 0.8$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

Panel A : $l_0 = 1.1$



Panel B : $l_0 = 0.8$



Panel C : $l_0 = 0.4$

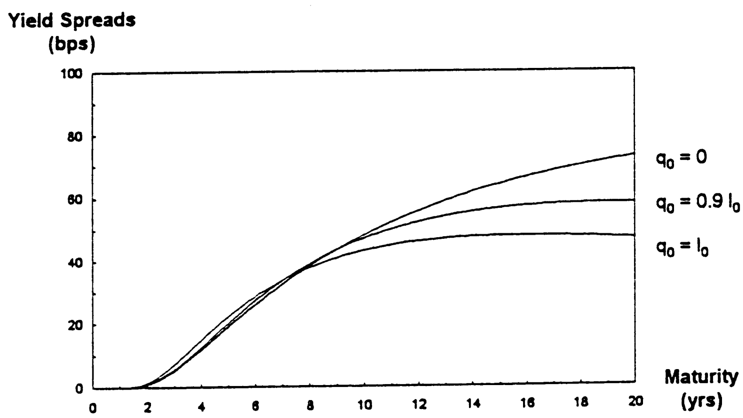


Figure 3

Yield spreads as a function of time-to-maturity and early default ratio q_0.
The parameters used are : $l_0 = 0.8$, $\sigma_V = 0.2$, $f_1 = f_2 = 0.8$, $q_0 = 0.9\, l_0$,
$a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

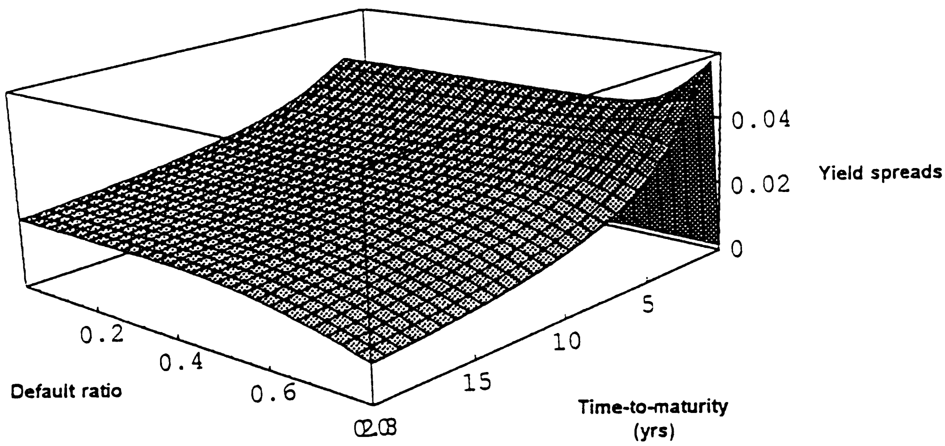
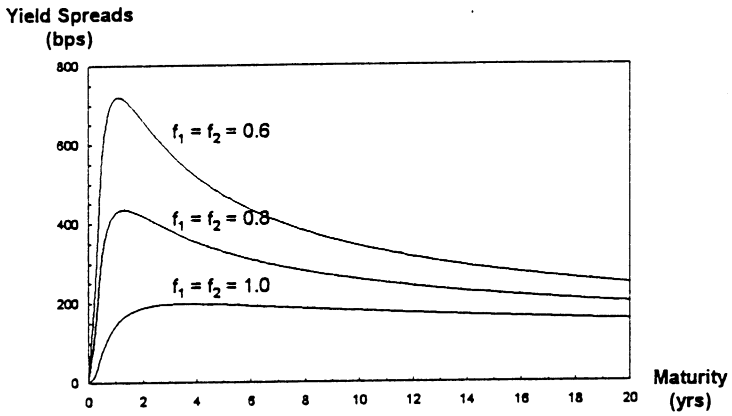


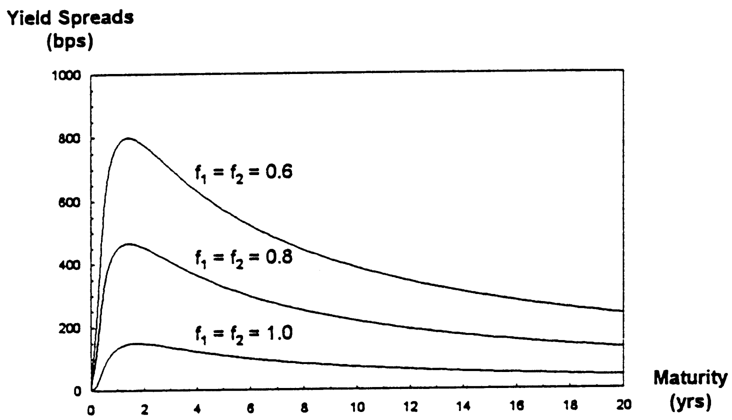
Figure 4

Yield spreads as a function of time-to-maturity for different deviations from the absolute priority rule. The parameters used are : $l_0 = 0.8$, $\sigma_V = 0.2$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

Panel A : $q_0 = 0.0$



Panel B : $q_0 = 0.9\ l_0$



Panel C : $q_0 = l_0$

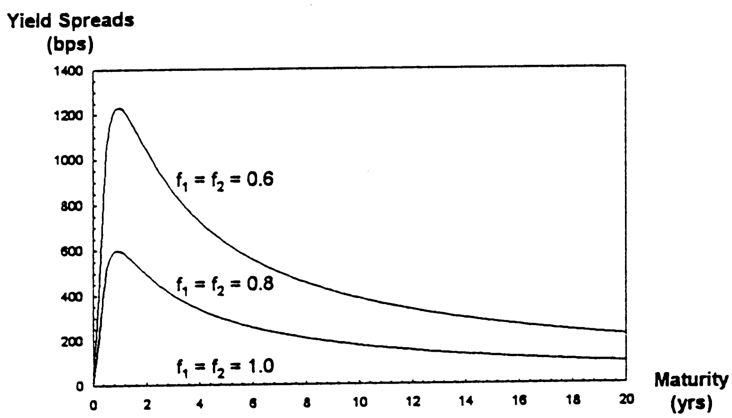


Figure 5

Yield spreads as a function of time-to-maturity for different asset volatility levels σ_V. The parameters used are : $q_0 = 0.9\, l_0$, $f_1 = f_2 = 0.8$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

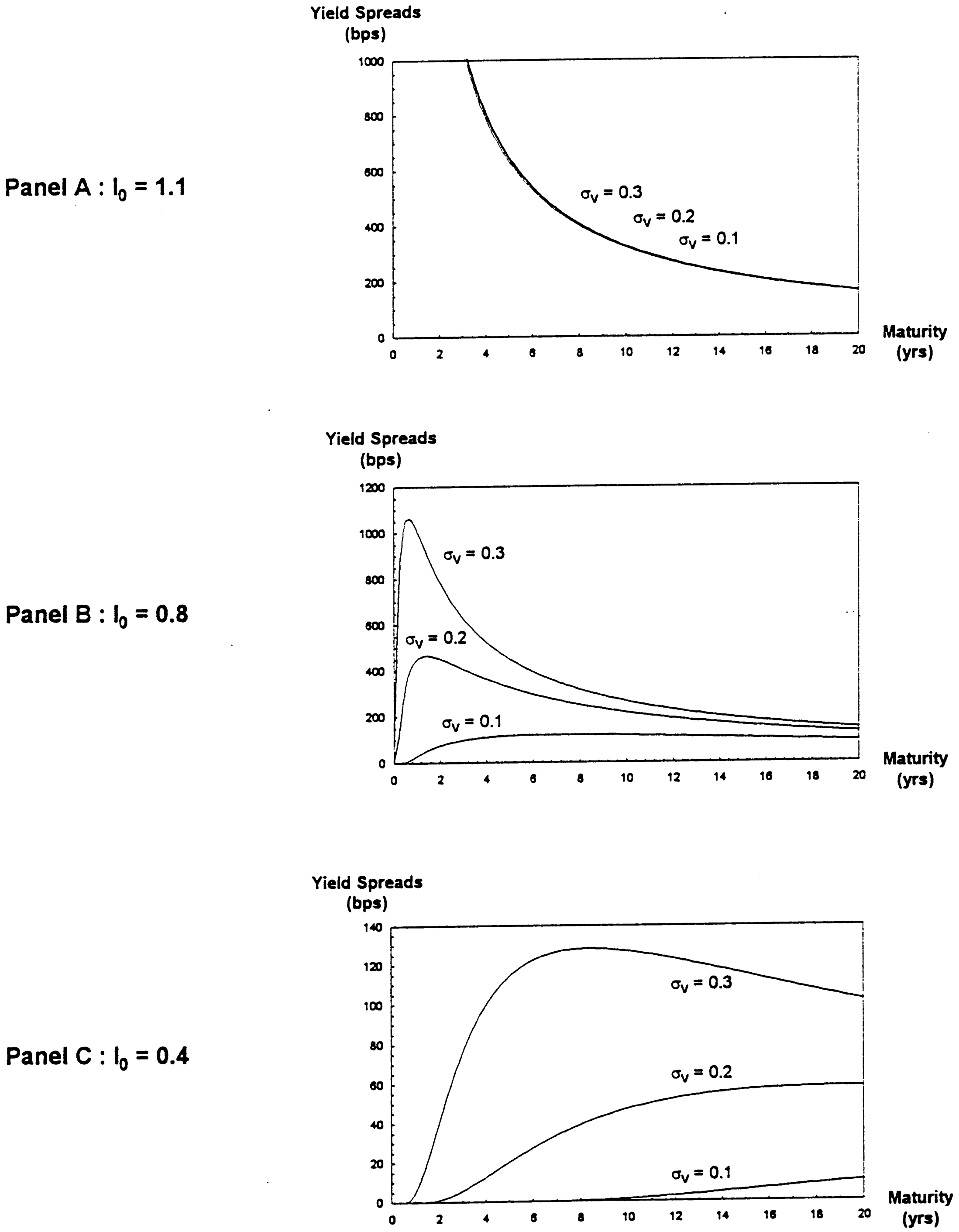
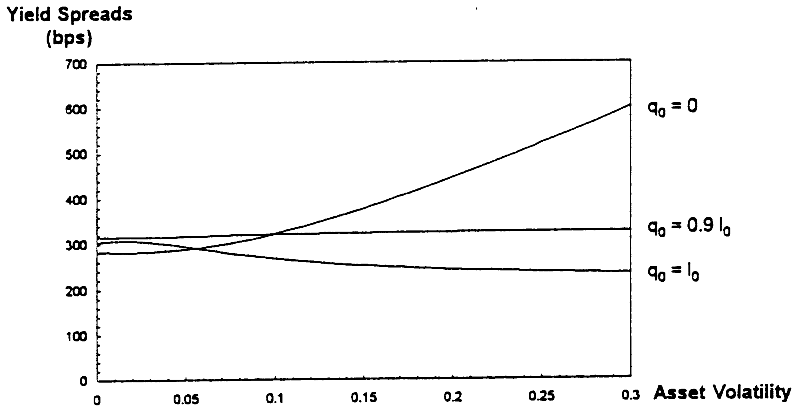
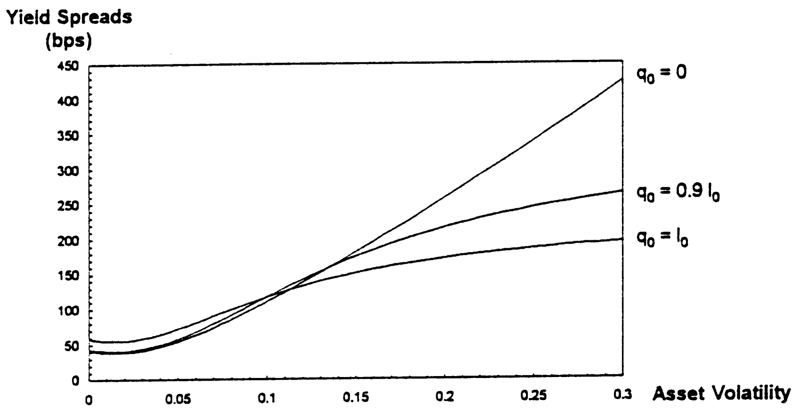


Figure 6
Yield spreads as a function of asset volatility for different early default ratios q_0. The parameters used are : $f_1 = f_2 = 0.8$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$, $T = 10$.

Panel A : $l_0 = 1.1$



Panel B : $l_0 = 0.8$



Panel C : $l_0 = 0.4$

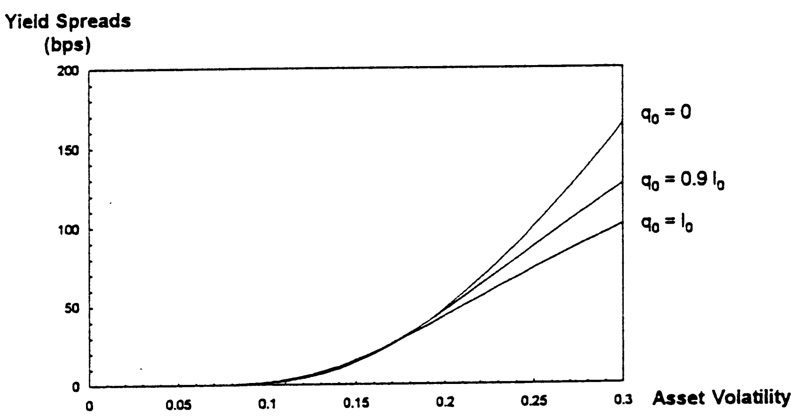
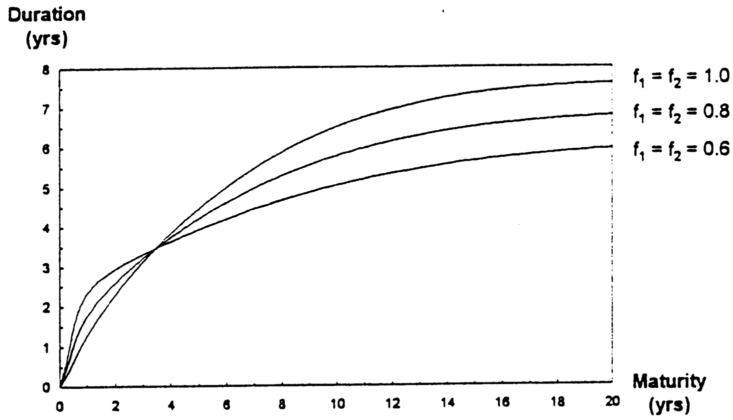


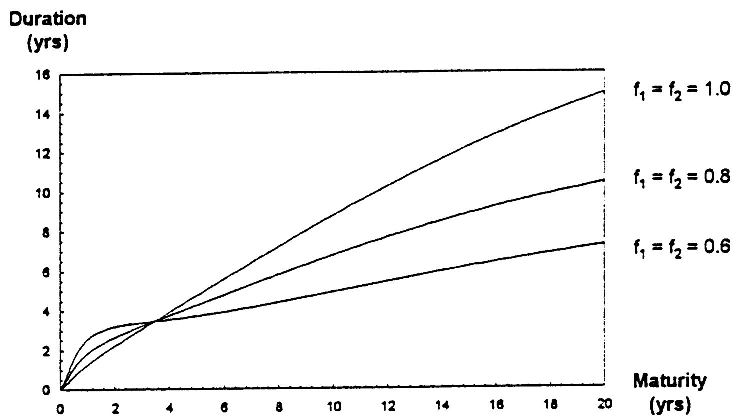
Figure 8

Duration as a function of time-to-maturity for different deviations from the absolute priority rule. The parameters used are : $l_0 = 0.8$, $\sigma_V = 0.2$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $\rho = -0.25$.

Panel A : $q_0 = 0.0$



Panel B : $q_0 = 0.9\ l_0$



Panel C : $q_0 = l_0$

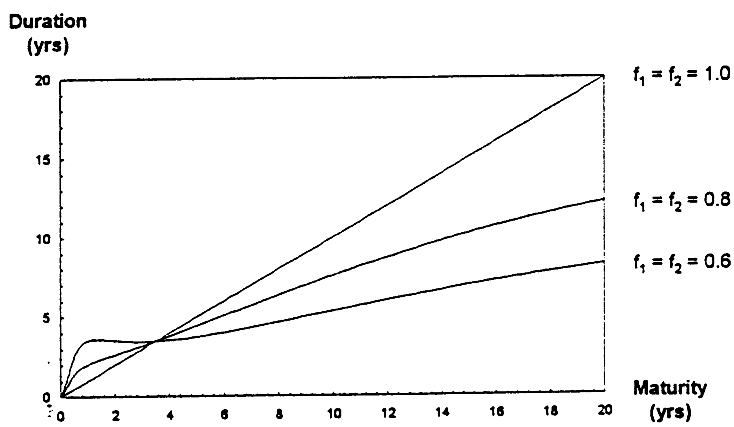
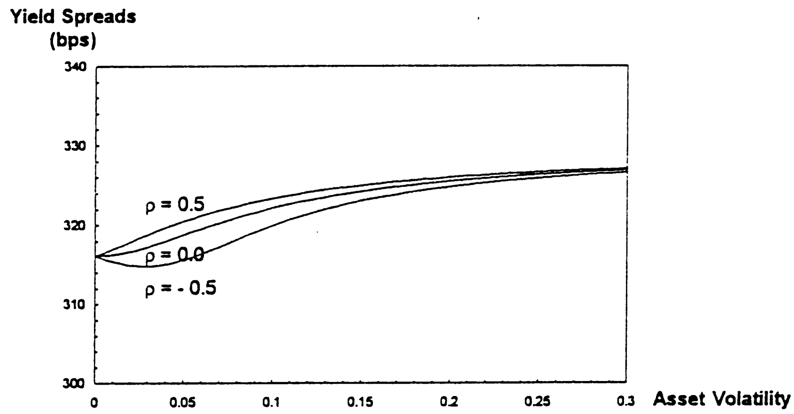
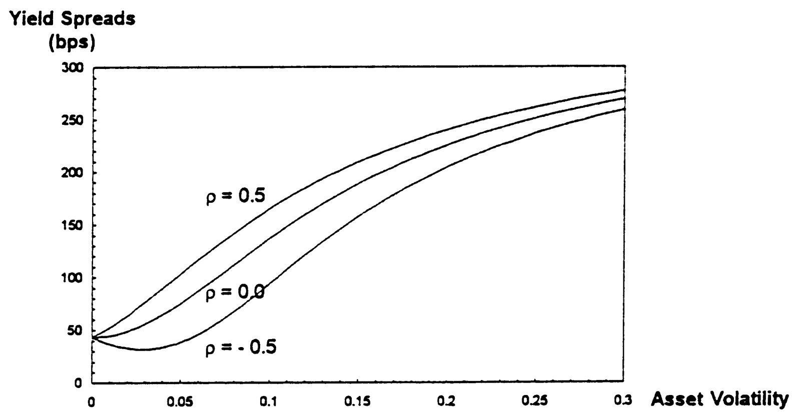


Figure 7
Yield spreads as a function of asset volatility for different correlation coefficients. The parameters used are : $q_0 = 0.9\ l_0$, $f_1 = f_2 = 0.8$, $a = 0.2$, $b = 0.06$, $\sigma = 0.02$, $r_0 = 0.05$, $T = 10$.

Panel A : $l_0 = 1.1$



Panel B : $l_0 = 0.8$



Panel C : $l_0 = 0.4$