Extreme disturbances on the drift of anticipated mortality
Application to annuity plans

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Abstract

The purpose of this paper is to propose a realistic and operational model to quantify the systematic risk of mortality included in an engagement of retirement. The model presented is built on the basis of Lee-Carter model. The prospective stochastic tables thus built make it possible to project the evolution of the random mortality rates in the future and to quantify the systematic risk of mortality.

KEYWORDS: Prospective tables, extrapolation, adjustment, life annuities, stochastic mortality.

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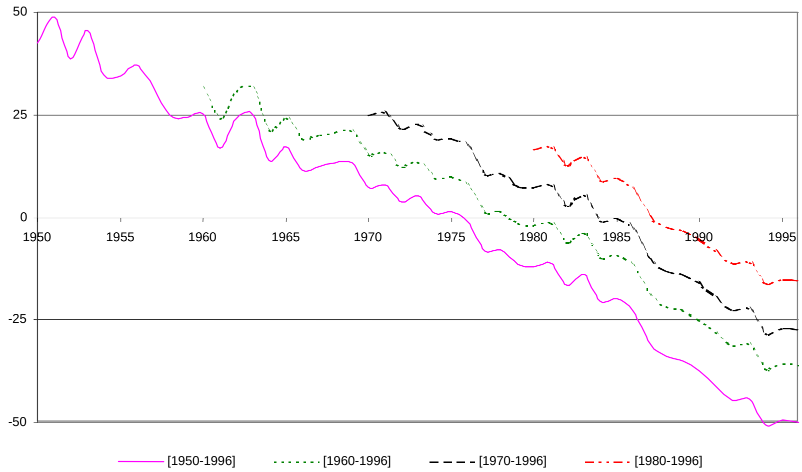
1. Introduction

Through construction of a prospective mortality table by the method of Lee & Carter (1992) or one of its derivations (cf. Planchet & Thérond (2006)), we are led to estimate the future tendency through the modelling of coefficient $\left(k_t, t \geq t_M\right)$. This model suggests an obvious linear tendency in general. Two risks are attached to this model. First we note that if the form of tendency is robust, i.e., notably in relation to the selection of observation period, its level is just as illustrated in Figure 1 which takes the tendency estimated from different periods of observation on French national data.

Figure 1 – Estimated Tendencies in Lee-Carter as a function of observation period



The consideration of this risk is introduced in Planchet [2007].

Even if the tendency is correctly positioned, the fluctuations around this tendency and notably shocks downwards, make a long-term life insurance plan (principally life annuities in the course of service or postponed) carrying a systematic risk which it is necessary to quantify.

In fact, it is about the sampling of an annuity plan or only about the valuation of its undertaking; this point is particularly important. In the project Solvability 2, the tests of quantitative impacts performed by the underwriters with a view to calibrating the future standard expression of valuation of the requirement of equity capitals insert the risk of longevity through a shock downwards on the mortality rates of the *best estimate* table.

In an approach of internal model type, it will be necessary, in any rigour, to model shocks on the levels of future mortality especially since these are generally systematic and cannot be mutualised by effect of portfolio size.

This observation leads us to search for the modelling of tendency which, while respecting generally linear paces, takes into account important asymmetrical shocks such as those we can observe from historical data. We propose such approach here and apply it to the measure of the systematic risk carried by a life annuity plan. In practice, the hypothesis of deterministic mortality is therefore replaced with a hypothesis of stochastic mortality – uncertainty on the table translating the volatility linked to derivations in comparison with reference tendency.

It is proved that this risk is potentially not negligible, and hence must be taken into account in the IFRS accounting standards as well as in the context of the prudent rules of Solvability 2. The present article supplements analyses introduced in Planchet et al. (2006) and Planchet & Juillard (2007), where the reader can refer for the methodological details of basic models.

This paper articulates in two parts. In the first part, the model of tendency is specified and then is estimated on French national data. In the second part, on the basis of this model, the analysis of the risk of a life annuity plan in the course of service is proposed.

2. The model of mortality

2.1. Presentation

The model kept to construct the prospective tables is a model proposed for the instantaneous rate of mortality in Lee-Carter which is the following: stochastically adapted by the model of Lee & Carter (1992). Recall that

$$\ln \mu_{xt} = \alpha_x + \beta_x k_t + \varepsilon_{xt} ,$$

with common notations (x indicates age and t the generation) and by assuming random variables ε_{xt}, independent and identically distributed, disposed historically $t_m \le t \le t_M$. The question of the adjustment of the parameters of the model is not studied here. The interested reader can refer to numerous references cited in Planchet & Thérond (2006).

Once having adjusted the surface of mortality on the past data, it remains to model series (k_t) to extrapolate the future rates; for that, we use the simplest modelling that we can imagine, a linear regression by assuming a known tendency:

$$k_t^* = at + b + \gamma_t ,$$

with (γ_t) a white noise.

Nevertheless, while in common approaches, this noise is assumed Gaussian or to follow an ARIMA process, we assume here that it is represented by a conditional model, which differentiates three proportions among the residuals of the adjustment of the empirical (k_t) :

- the proportion p_- of the smallest residuals,
- the proportion p_+ of the largest residuals; and
- the $1-(p_- + p_+)$ centered residuals, weak in absolute values.

Considering the small amount of available data (about fifty observations) and due to the fact that we are interested here in big variations in comparison with tendency (weak deviation without practical impact), we fix arbitrarily $p_- = p_+ = 0,429$. Moreover, we suppose that the positive deviation γ_t^+ in relation to the tendency value follows a Pareto distribution with parameters (m_+, α_+) and the deviation γ_t^+ is Pareto distributed with parameters (m_-, α_-).

We have the following parameterization of Pareto distribution:

$$S_{m,\alpha}(x) = \left(\frac{x}{m}\right)^{-\alpha}, \; x \geq m .$$

The division of three portions of identical size proposed here is principally justified due to the fact that we are interested in important derivations in comparison with tendency. These derivations do not have to be sensibly impacted by the choice of the beginning of the portion. The choice of distribution of Pareto is in the same logic, an idea to find a model compatible with observations generating an important part of systematic risk.

The estimation of the parameters of the model is performed in two steps. First of all we calculate estimator $\hat{a}$ and $\hat{b}$ which allows us to construct the surface projected by the reference by simply using $k_t^* = \hat{a}t + \hat{b}$ via an adjustment of least square ordinaries. To simplify the expressions, we let $\tau = t - t_m + 1$ and $T = t_M - t_m + 1$, which leads to expressions:

$$\hat{a} = \frac{\frac{1}{T}\sum \tau k_\tau - \frac{T+1}{2}\bar{k}}{T^2 - 1},$$

and

$$\hat{b} = \bar{k} - \frac{T+1}{2}\hat{a} ,$$

with $\bar{k} = \frac{1}{T}\sum k_\tau = \frac{1}{T}\sum k_t$.

The first step of estimation is thus not achieved in the probabilistic context, but via the simple criterion of least square. In the second step, we estimate the parameters of derivations m_+, m_-, α_- and α_+ by maximum likelihood method:

$$\hat{m}_+ = \min\left\{\gamma_{(i)}; i \geq n \times p_+\right\}, \; \hat{m}_- = -\max\left\{\gamma_{(i)}; i \leq n \times p_-\right\}$$

$$\hat{\alpha}_+ = \frac{[n \times p_+]}{\sum_{i=1}^{n_+} \ln\left(\frac{\gamma_i}{\hat{m}_+}\right)}, \; \hat{\alpha}_- = \frac{[n \times p_-]}{\sum_{i=1}^{n_-} \ln\left(\frac{-\gamma_i}{\hat{m}_-}\right)},$$

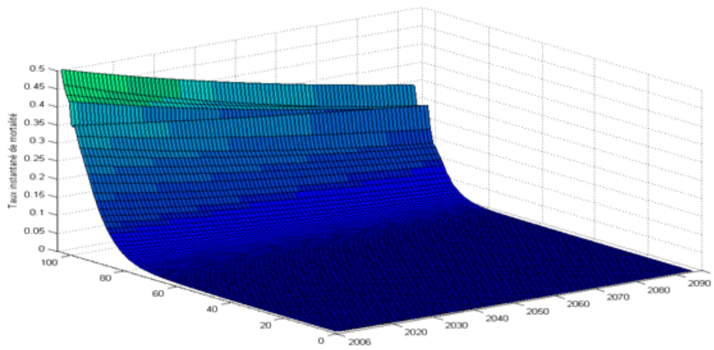
where $\left(\gamma_{(i)}\right)$ is the order statistic associated to (γ_i).

The quality of adjustment is assessed by performing Chi-2 tests on conditional distribution of positive derivations (resp. negatives).

2.2. Numerical application

The prospective table used in this study is the one used in Planchet et al. (2006); it is constructed from the tables of moment provided by INED[1] in Mesle & Vallin (2002) and leads to the surface of Lee-Carter mortality represented in Figure 2.

Figure 2 – Surface of mortality adjusted by Lee-Carter



The parameters of the stochastic model are summarized in the table below.

Negative Residuals		Positive Residuals	
p-	42.9 %	p+	42.9 %
M-	1.4385	M+	1.3878
α-	1.13	α+	1.1105

3. Application in a regime of life annuities

3.1. The portfolio

In the following, for numerical applications we will use a portfolio consisting of 374 female annuitants with an average age of 63.8 at the end of the experiment. The annual mean income comes to €5.5 k. With the provision of bank rate of 2.5 %, initial policy reserve comes to €37.9 M with the determined prospective table *supra*. For the rest of the article, we will denote:

- L_0 the total of policy reserve at initial date,
- $\tilde{F}_t$ the cash flow of benefit(random) to be paid at date t,
- i the discount rate (discrete) of policy reserve,
- **J** the sample space of individuals,
- $x(j)$ age of individual j at 0, and
- r_j the total annual payments.

[1]. The tables are available on the website: http://www.ined.fr/

3.2. Problems

The problems are described in detail in Planchet et al. (2006). We should note that here we are interested in the distribution of the total undertaking of the system:

$$\Lambda = \sum_{t=1}^{\infty} \tilde{F}_t \left(1+i\right)^{-t} = \sum_{t=1}^{\infty} \frac{1}{\left(1+i\right)^t} \sum_{j \in \mathbf{J}} r_j \times \mathbf{1}_{]t;\infty[}\left(T_{x(j)}\right),$$

Which is a random variable such that $E\left(\Lambda\right) = L_0$, with $L_0 = \sum_{t=1}^{\infty} F_t \left(1+i\right)^{-t}$.

When future mortality is assumed known (say, "determinist" approach afterwards), the analysis of the distribution of Λ comes down to measure the fluctuations of sampling. In the context of a stochastic mortality, this analysis provides us a means to quantify the systematic risk which is added to the basic risk. We should therefore be able to measure such risk associated to the selection of drift. More importantly, from now on we can note that even when the parameters of the Pareto distribution are strictly less than 1, the engagement remains integrable, as this variable has an upper bound (by the table where all individuals have a life limit of 120).

The adopted method consists of simulating the survival duration of the annuitants, $T_{x(j)}$, $j \in J$, to calculate the realizations $\lambda_1, \ldots, \lambda_n$ of Λ and then to determine the function of empirical distribution of the undertaking. The reserve L_0 is approximated by $\overline{\lambda} = \frac{1}{N}\sum_{n=1}^{N} \lambda_n$. The variance of undertaking is estimated by $\frac{1}{N-1}\sum_{n=1}^{N}\left(\lambda_n - \mathrm{L}_0\right)^2$; We calculate equivalently the coefficient of empirical variation:

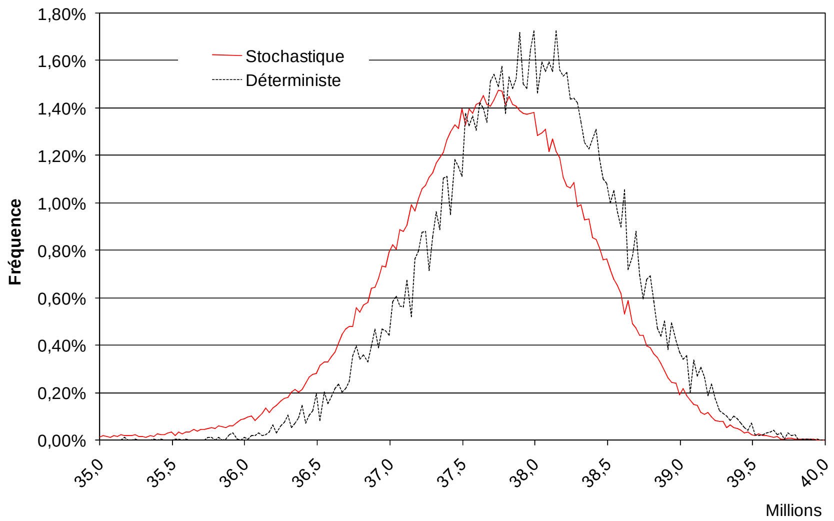
$$cv = \frac{\sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left(\lambda_n - \mathrm{L}_0\right)^2}}{\frac{1}{N}\sum_{n=1}^{N}\lambda_n},$$

which provides an indicator of the dispersion of undertaking and to a certain extent its hazards.

3.3. Results

The empirical distribution of undertaking, represented here with the reference distribution in deterministic case (with 20,000 samples) is illustrated in Figure 3.

Figure 3 – Empirical Distribution of undertaking



The detailed results are presented in Table 1.

Table 1 – Descriptive statistics of undertaking

	Deterministic	**Stochastic**
Expectation	37 937 707	37 380 862
Standard Deviation	626 918	2 418 408
Lower Bound of Confidence Interval	36 625 000	34 295 073
Upper Bound of Confidence Interval	39 075 000	38 945 073
Coefficient of variation	1.65 %	6.47 %

Even for a portfolio of small size, the impact of stochastic mortality on the structure of undertaking is important. We state two effects:

– the coefficient of variation of engagement is multiplied by 3 with respect to the situation of not taking account the systematic risk;

– the mean undertaking diminishes, due to the impact of shocks increasing with the decrement rates.

We therefore compare the consequences in terms of risk management: the *best estimate* vision of undertaking is seen decreasing, but the presence of a systematic risk leads to calibrate a risk margin taking into account the strongest volatility. In total, it is not certain that the sum changes significantly, but its decomposition (in terms of expectation and risk margin) in prudent logic (Solvability 2) and accounting (IFRS phase 2) is seen in a sensible way.

However as show in Planchet et al. (2006) the size of the portfolio is an important parameter to take into account. In fact, if the absolute level of systematic risk does not depend on the size of the portfolio, it also does not depend on mutualisable risk. The part of variance explained by the stochastic component of mortality therefore increases with the size of the portfolio; to measure this effect, we construct a fictitious portfolio by retorting the basic portfolio n times. By observing that we thus acquire a decomposition of complete undertaking Λ in sum of n i.i.d. variables $\Lambda^{(1)},\ldots,\Lambda^{(n)}$, we find that:

$$\mathrm{Var}\left[\mathrm{E}\left(\Lambda|\Pi\right)\right]=n^2\,\mathrm{Var}\left[\mathrm{E}\left(\Lambda^{(1)}|\Pi\right)\right],$$

and

$$\mathrm{E}\left[\mathrm{Var}\left(\Lambda|\Pi\right)\right]=n\,\mathrm{E}\left[\mathrm{Var}\left(\Lambda^{(1)}|\Pi\right)\right],$$

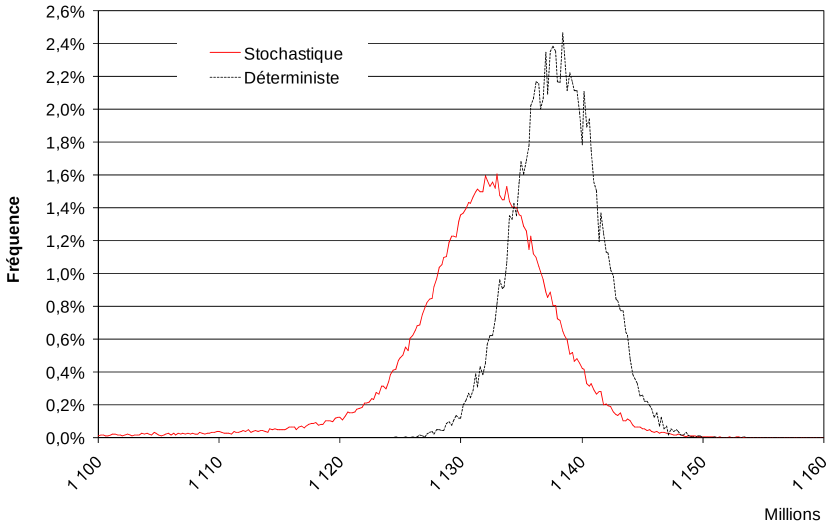
which leads to:

$$\omega_n=\left(1+\frac{1}{n}\left(\frac{1}{\omega}-1\right)\right)^{-1}$$

with $\omega=\dfrac{\mathrm{Var}\left[\mathrm{E}\left(\Lambda|\Pi\right)\right]}{\mathrm{Var}\left[\Lambda\right]}$.

If we multiply the size of portfolio by 30, we obtain the results shown in Figure 4.

Figure 4 – Empirical distribution of undertaking (size x30)



The detailed results are presented in Table 2.

Table 2 – Descriptive statistics of undertaking (triple portfolio)

	Deterministic	Stochastic
Expectation	1 138 008 113	1 121 529 390
Standard deviation	5 592 212	69 931 571
Lower bound of confidence interval	1 131 130 658	1 032 811 610
Upper bound of confidence interval	1 144 480 658	1 141 811 610
Coefficient of variation	0,30 %	6,24 %

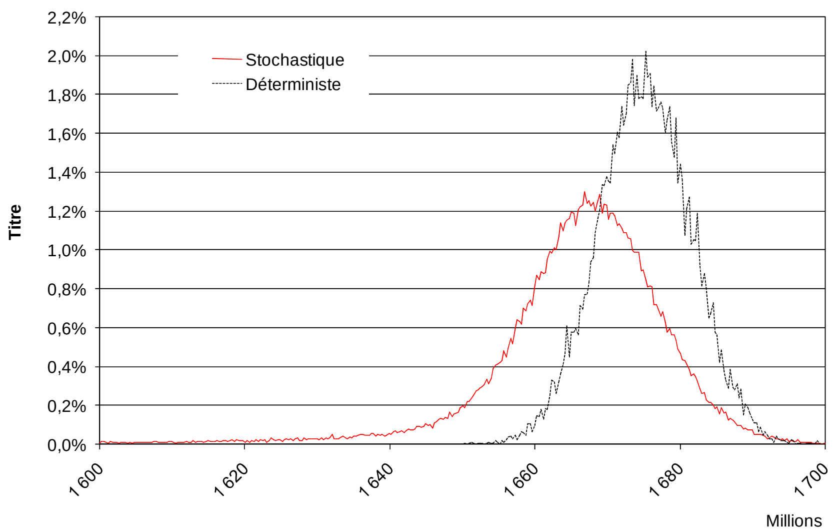
We consider that stochastic mortality multiplies the coefficient of variation by 20 by reintroducing a risk source.

In the method *value-at-risk* (VaR), we find that the 75 % percentile of the distribution of undertaking is €1,135 M in the stochastic case, which is slightly less than the value acquired in the deterministic case, i.e. €1,140 M. In other words, the consideration of the risk of drift leads here (following a VaR approach for the reserve calculation) to lower marginally the sum of reserves by 0.17 %.

The consideration of the risk of drift also has very significant effect for the consequence of augmentation in the valuation of undertaking which, at the level of 95 % confidence, crossing the interval [-0.7 %; 0.7 %] near the interval [-7.9 %; 1.8 %].

Moreover stochastic undertaking draws its volatility of incomes which are going to be served for a long time. And, the more important the duration of income is, the more important the coefficient of actualization is. By fixing the rate of actualization null, we obtain the results shown in Figure 5.

Figure 5 – Empirical Distribution of undertaking (technically null rate)



The detailed results are shown in Table 3.

Table 3 – Descriptive Statistics of undertaking (null actualisation)

	Deterministic	**Stochastic**
Expectation	1 675 256 646	1 647 334 981
Standard Deviation	6 389 153	122 294 843
Lower bound of confidence interval	1 662 554 600	1 465 849 300
Upper bound of confidence interval	1 687 754 600	1 685 449 300
Coefficient of variation	0.38 %	7.42 %

By taking a technically null rate, we augment the coefficient of variation of the deterministic undertaking of 27% and the coefficient of variation of the stochastic undertaking of 19%. So the consideration of stochastic mortality leads to greater sensibility of undertaking in the rate of actualization.

4. Conclusion

We propose a model used in calculating the reserve of life annuities, which inserts uncertainty explicitly in determining the long-term tendency through adjustment of this tendency.

If the level of reserve is not sensibly impacted by this change, the structure of reserve changes: the *best estimate* is seen again in decrease and risk margin in increase due to this systematic risk. This model seems to us in fact better considering the risk carried by the regime of income by allowing a more appropriate segmentation of the sum of undertaking between different risk sources.

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