Solvency II and Nested Simulations – a Least-Squares Monte Carlo Approach

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Abstract

Within the European Union, risk-based funding requirements for life insurance companies are currently being revised as part of the Solvency II project. However, many insurers are struggling with the implementation, which is in part due to the inefficient methods underlying their numerical computations.

We review these methods and propose a significantly faster approach for the calculation of the required risk capital based on least-squares regression and Monte Carlo simulations akin to the well-known Least-Squares Monte Carlo method for pricing non-European derivatives introduced by Longstaff and Schwartz (2001, [20]).

Keywords: Solvency II, Nested Simulations, Least-Square Monte Carlo

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1 Introduction

Within the European Union, risk-based funding requirements for life insurance companies are currently being revised as part of the Solvency II project (for discussions regarding the necessity and the benefits of solvency regulation and Solvency II in particular, see e.g. [7], [15] and [8], [13], respectively). One key aspect of the new regulatory framework is the determination of the required risk capital for a one-year time horizon, i.e. the amount of capital the company must hold against unforeseen losses during a one-year period, based on a market-consistent valuation of assets and liabilities in a so-called *internal model*.

However, many insurers are struggling with the implementation, which, to a large extent, is due to inefficient methods underlying their numerical computations. As a consequence, many companies rely on second-best approximations within so-called *standard models*, which are usually not able to accurately reflect an insurer's risk situation and may lead to deficient outcomes (see e.g. [22], [23], [24]).

The current paper addresses this problem. We provide a mathematical framework for the calculation of the Solvency Capital Requirement (SCR) based on the MCEV principles issued by the CFO Forum ([1]) and discuss different approaches for the numerical implementation. More specifically, we examine in detail the estimation of the SCR via nested simulations which appears to be the straightforward approach in practical applications and consider analytical approximations. Moreover, we propose an alternative approach based on least-squares regression and Monte Carlo simulations akin to the well-known Least-Squares Monte Carlo method for pricing non-European derivatives introduced by Longstaff and Schwartz ([20]). While this method shows similarities to previous ideas (e.g. the grid-based methods for estimating value-at-risk by [10], [14]) and its applicability within a related problem was independently pointed out by other authors ([12]), at least its application in the insurance context, where it bears profound advantages, appears to be new. The drawbacks and advantages of the different approaches are illustrated based on numerical experiments using the participating contract model introduced in [3].

The remainder of the paper is structured as follows. Section 2 provides background information on the Solvency II requirements and gives precise definitions of the quantities of interest. We particularly illustrate the relation between these quantities and the concept of a market-consistent embedded value (MCEV). In Section 3, we introduce the mathematical framework underlying our considerations and describe the Nested Simulations Approach. In particular, we investigate the quality of the resulting estimator for the SCR. As an alternative to this computationally challenging approach, we propose an analytic approximation of the SCR in Section 4. Subsequently, Section 5 describes how Least-Squares Monte Carlo methods can be adapted to our valuation problem. It also contains some results on the convergence of the resulting estimator. In Section 6, we illustrate the different methods based on numerical experiments. Finally, Section 7 summarizes our findings and conclusions.

2 Solvency II Requirements

2.1 Required Risk Capital under Solvency II

The quantitative assessment of the solvency position of a life insurer can be split into two components, the derivation of the Available Capital (AC) and the derivation of the Solvency Capital Requirement (SCR).

2.1.1 Available Capital

The Available Capital (also called "own funds" under Solvency II) corresponds to the amount of financial resources available at $t = 0$ which can serve as a buffer against risks and absorb financial losses. It is derived from a market-consistent valuation approach as the difference between the market value of assets and the market value of liabilities.

The market-consistent valuation of assets is usually quite straightforward for the typical investment portfolio of an insurance company since market values are either readily available (mark-to-market, level 1) or can be derived from standard models with market-observable inputs (level 2). The former is not the case for the liabilities of a life insurance company. Moreover, due to the relatively complex financial structure of life insurance contracts containing embedded options and guarantees, the market-consistent valuation of liabilities generally cannot be done in closed form. Therefore, life insurance companies usually follow a mark-to-model approach that relies on simulations.[1]

To reduce the arbitrariness in the choice of the model underlying this valuation, i.e. to ensure comparability of results across companies, over the last decade, the life insurance industry developed principles for assessing the market-consistent value of a life insurance company's assets and liabilities from the shareholders' perspective. This so-called *Market-Consistent Embedded Value* (MCEV) corresponds to the present value of shareholders' interest in the earnings distributable from assets backing the life insurance business, after allowance for the aggregate risks in the life insurance portfolio. It is important to note that the MCEV does not reflect the shareholders' default put option resulting from their limited liability. More precisely, it is assumed that the shareholders would make up any deficit arising in the future with no upper limit on the amount of deficit. Consequently, the market-consistent value of insurance liabilities can be derived indirectly as the difference between the market value of assets and the MCEV.

Overall, the Available Capital (AC) derived under Solvency II principles is usually very similar to the MCEV, so that for the purpose of this paper – without loss of generality – we will assume that the two quantities coincide.[2] Therefore, at $t = 0$ we have

$$\mathrm{AC}_0 := \mathrm{MCEV}_0. \tag{1}$$

2.1.2 Solvency Capital Requirement

For deriving the SCR, the quantity of interest is the Available Capital at $t = 1$. Assuming that the profit of the first year (denoted by X_1) has not been paid to shareholders yet, it can be described by

$$\mathrm{AC}_1 := \mathrm{MCEV}_1 + X_1. \tag{2}$$

Intuitively, an insurance company is considered to be solvent under Solvency II if its Available Capital at $t = 1$ as seen from $t = 0$ is positive with a probability of at least 99.5%, i.e.

$$P\left(\mathrm{AC}_1 \geq 0 | \mathrm{AC}_0 = x\right) \overset{!}{\geq} 99.5\%.$$

The SCR would then be defined as the smallest amount x that satisfies this condition. This is an implicit definition of the SCR ensuring that if the Available Capital at $t = 0$ is greater or equal to the Solvency Capital Requirement, then the probability that the Available Capital at $t = 1$ is positive is at least 99.5%.

However, in practical applications, one usually relies on a simpler, but approximately equivalent notion of the SCR, which avoids the implicit nature of the definition given above.[3] For this purpose, we define the one-year loss function, evaluated at $t = 0$ as

$$L := \mathrm{AC}_0 - \frac{\mathrm{AC}_1}{1+i},$$

where i is the one-year risk-free rate at $t = 0$. The SCR is then defined as the α-quantile of L, where the security level α is set equal to 99.5%:

$$\begin{aligned} \mathrm{SCR} &:= \operatorname{argmin}_x \left\{ P\left(\mathrm{AC}_0 - \frac{\mathrm{AC}_1}{1+i} > x\right) \overset{!}{\leq} 1-\alpha \right\} \\ &= \operatorname{argmin}_x \left\{ P\left(\mathrm{MCEV}_0 - \frac{\mathrm{MCEV}_1 + X_1}{1+i} > x\right) \overset{!}{\leq} 1-\alpha \right\}. \end{aligned} \tag{3}$$

The probability that the loss over one year exceeds the SCR is less or equal to $1 - \alpha$, i.e. we need to calculate a one-year Value-at-Risk (VaR). The *Excess Capital* at $t = 0$, on the other hand, is defined as $\mathrm{AC}_0 - \mathrm{SCR}$ and satisfies the following requirement:

$$P\left(\frac{\mathrm{AC}_1}{1+i} \geq \mathrm{AC}_0 - \mathrm{SCR}\right) \geq \alpha,$$

so the probability (evaluated at $t = 0$) that the Available Capital at $t = 1$ is greater or equal to the Excess Capital is at least α (e.g. 99.5%).

Note that under this definition the SCR depends on the actual amount of capital held at $t = 0$ and may also include capital for covering losses arising from assets backing Excess Capital. Based on this definition, the solvency ratio can be calculated as $\mathrm{AC}_0/\mathrm{SCR}$.

2.1.3 SCR Aggregation Formula

Within *standard models*, the SCR is calculated via an aggregation formula in a *modular approach*. Under the assumptions that the aggregate one-year loss L is a linear combination of loss random variables L_i, $1 \leq i \leq d \in \mathbb{N}$ attributable to d risk modules, $L = \sum_{i=1}^{d} L_i$, and that the L_i are jointly normally distributed, we obtain for the SCR the so-called "square-root formula":

$$\mathrm{SCR} = \sum_{i=1}^{d} \mu_i + \sqrt{\sum_{i=1}^{d} (\mathrm{SCR}_i - \mu_i)^2 + 2 \sum_{1 \leq i < j \leq d} \rho_{ij} \, (\mathrm{SCR}_i - \mu_i) \, (\mathrm{SCR}_j - \mu_j)}, \quad (4)$$

where $\mu_i = E[L_i]$, SCR_i is the risk-charge for risk i (i.e. the 99.5% quantile of the loss function L_i) and ρ_{ij} is the linear correlation between the risk variables L_i and L_j, $1 \leq i \neq j \leq d$. The individual risk charges are calculated using either factor-based or scenario-based models (cf. [24]).

However, there obviously arise problems with this formula if the individual risks are not normally distributed. On one hand, skewness or excess kurtosis of the marginal distributions can lead to considerable erratic outcomes of Equation (4) (see [24]). On the other hand, dependence structures beyond linear correlation effects may yield situations where the square-root formula severely underestimates the true SCR (see [22]). Moreover, even if the influence of the different risk factors may be represented by Normal random variables as in some standard asset models (see e.g. Section 6), their influence on the aggregate loss in general will not be additive.

Hence, in order to obtain more accurate results regarding the solvency position of the company, in general it is necessary to rely on numerical methods for simultaneously assessing all risk factors in a multivariate approach. According to Equation (3), the MCEV can serve as a basis for the determination of risk-based funding requirements under Solvency II in such an approach. Therefore, in the next subsection, we provide a more precise definition of the MCEV which is based on the MCEV principles issued by the CFO Forum (see [1]).

2.2 Definition of the MCEV

According to the Market-Consistent Embedded Value Principles [1], the MCEV is defined as the sum of Adjusted Net Asset Value (ANAV) and the Present Value of Future Profits (PVFP) less a Cost-of-Capital charge (CoC):

$$\text{MCEV} \;\; := \;\; \text{ANAV} + \text{PVFP} - \text{CoC}. \tag{5}$$

The ANAV is derived from the (statutory) Net Asset Value (NAV)[4], and includes adjustments for intangible assets, unrealized gains and losses on assets etc. It consists of two parts, the free surplus and required capital (cf. Principles 4 and 5 in [1]). In most cases, the ANAV can be calculated from statutory balance sheet figures and the market value of assets; hence, the calculation does not require simulations.

The PVFP corresponds to the present value of the post-taxation shareholder cash flows from the in-force business and the assets backing the associated (statutory) liabilities. In particular, it also includes the time value of financial options and guarantees (cf. Principles 6 and 7 in [1]). The determination of the PVFP is quite challenging since it highly depends on the future development of the financial market, i.e. on the evolution of the yield curve, equity returns, credit spreads etc. Hence, the PVFP needs to be determined based on stochastic models, where, in general, risk-neutral valuation approaches are applied.

The CoC is the sum of the frictional cost of required capital and the cost of residual non-hedgeable risks (cf. Principles 8 and 9 in [1]). The calculation of the CoC can be based on a number of deterministic or stochastic (simulation-based) approaches, which are beyond the scope of this paper.

Based on these principles, the MCEV and, therefore, the solvency position of a life insurance company under Solvency II can be determined. For this purpose, we do not only need to calculate the MCEV at time $t = 0$, but we also need to assess the distribution of the MCEV at time $t = 1$ as seen from time $t = 0$. Risk measures such as Value-at-Risk (VaR) (or Tail-Value-at-Risk (TVaR)) are then derived based on this distribution in order to calculate the required risk capital.

3 Nested Simulations Approach

3.1 Mathematical Framework

We assume that investors can trade continuously in a frictionless financial market and we let T be the maturity of the longest-term policy in the life insurer's portfolio.[5] Let $(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]})$ be a complete filtered probability space on which all relevant quantities exist, where Ω denotes the space of all possible states in the financial market and $\mathcal{P}$ is the so-called real-world (physical) measure. $\mathcal{F}_t$ represents all information about the financial market up to time t, and $\mathbb{F}$ is assumed to satisfy the usual conditions.

The uncertainty with respect to the insurance company's future profits arises from the uncertain development of a number of influencing factors, such as equity returns, interest rates or credit spreads. We introduce the d-dimensional, sufficiently regular Markov process $Y = (Y_t)_{t\in[0,T]} = (Y_{t,1}, \dots, Y_{t,d})_{t\in[0,T]}$, the so-called *state process*, to model the uncertainty of the financial market, i.e. all risky assets in the market can be expressed in terms of Y. In particular, we suppose the existence of a locally risk-free process $(B_t)_{t\in[0,T]}$ (the bank account) with $B_t = \exp\{\int_0^t r_u\, du\}$, where $r_t = r(Y_t)$ is the instantaneous risk-free interest rate at time t.

In this market, we take for granted the existence of a risk-neutral probability measure $\mathcal{Q}$ equivalent to $\mathcal{P}$ under which payment streams can be valuated as expected discounted cash flows with respect to the numéraire process $(B_t)_{t\in[0,T]}$.[6]

Finally, we assume that there exists a cash flow projection model of the insurance company, i.e. there exist functionals $f_1, \dots, f_T$ that derive the future profits at time t from the development of the financial market up to time t ($t = 1, \dots, T$). This cash flow model reflects legal and regulatory requirements as well as management rules. Hence, we model the future profits due to the in-force business as a sequence of random variables $X = (X_1, \dots, X_T)$ where $X_t = f_t(Y_s, s \in [0, t])$.

In order to keep our presentation concise, we abstract by limiting our focus to market risk, i.e. non-hedgeable risk as well as the corresponding cost-of-capital charges are ignored. However, non-financial risk factors such as a mortality index could also be incorporated in the state process. The corresponding cost-of-capital charges as well as other frictional cost could then be considered by an appropriate choice of $\mathcal{Q}$ and f_i, $1 \le i \le T$.

3.2 Available Capital

3.2.1 Available Capital at $t = 0$

According to the risk-neutral valuation formula, we can determine the PVFP at time $t = 0$, V_0, as the expectation of the sum of the discounted future profits X_t, $t = 1, \dots, T$, under the risk-neutral measure $\mathcal{Q}$:

$$V_0 := E^{\mathcal{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_0^t r_u\, du\right) X_t\right].$$

Furthermore, we define

$$\sigma_0 := \sqrt{\mathrm{Var}^{\mathcal{Q}}\left[\sum_{t=1}^{T} \exp(-\int_0^t r_u\, du) X_t\right]}.$$

In most cases, V_0 cannot be computed analytically due to the complexity of the interaction between the development of financial market variables Y_t and the liability side, or, more precisely, the shareholders' profits X_t. Thus, in general, we have to rely on numerical methods to estimate V_0.

A common approach is to use Monte Carlo simulations of independent sample paths $(Y_t^{(k)})_{t\in[0,T]}$, $k = 1, \ldots, K_0$, of the underlying state process Y under the risk-neutral measure $\mathcal{Q}$. Based on these different scenarios for the financial market, we first derive the resulting cash flows $X_t^{(k)}$ $(t = 1, \ldots, T;\ k = 1, \ldots, K_0)$ using the cash flow projection model. Then, we discount the cash flows with the appropriate discount factor, and average over all K_0 sample paths, i.e.

$$\widetilde{V}_0(K_0) := \frac{1}{K_0} \sum_{k=1}^{K_0} \sum_{t=1}^{T} \exp\left(-\int_0^t r_u^{(k)}\, du\right) X_t^{(k)},$$

where $r_t^{(k)}$ denotes the instantaneous risk-free interest rate at time t in sample path k.

By Equation (8) and since the ANAV can be derived from the statutory balance sheet, an estimator for AC_0 (under the framework described in Section 3.1) is given by

$$\widetilde{\mathrm{AC}}_0 = \mathrm{ANAV}_0 + \widetilde{V}_0.$$

3.2.2 Available Capital at $t = 1$

For the calculation of the Solvency Capital Requirement, in addition to the Available Capital at $t = 0$, we need to assess the (physical) distribution of the Available Capital at $t = 1$. Assuming that the profit of the first year, X_1, has not been paid to shareholders yet, we need to determine the $\mathcal{P}$-distribution of the $\mathcal{F}_1$-measurable random variable (cf. Equations (2) and (5))

$$\mathrm{AC}_1 := \mathrm{ANAV}_1 + \underbrace{E^{\mathcal{Q}}\left[\sum_{t=2}^{T} \exp\left(-\int_1^t r_u\, du\right) X_t \,\middle|\, Y_s,\ s \in [0,1]\right]}_{=:V_1} + X_1.$$

The complexity of this task mainly arises from the structure of V_1. However, in practical applications, V_1 usually does not depend on the "entire" history of the financial market up to time 1: Aggregate asset-liability projection models rely on a simultaneous extrapolation of a finite number of items or *accounts* representing both market factors and liability positions; if, on the other hand, the company's financial situation is projected forward on a single or representative contract basis, each contract will again be represented by a finite number of entries within the insurer's bookkeeping system (see [2] for a more detailed discussion). Hence, all necessary information for the projection of the cash flows is contained in a finite collection of *Markov state variables* (Y_1, D_1), where $D_1 = (D_1^{(1)}, \ldots, D_1^{(m)})$, and we can write

$$V_1 = E^{\mathcal{Q}}\left[\sum_{t=2}^{T} \exp\left(-\int_1^t r_u\, du\right) X_t \,\middle|\, (Y_1, D_1)\right]. \tag{6}$$

We may now estimate the distribution of AC_1 by the corresponding empirical distribution function: Given $N \in \mathbb{N}$ sample paths $(Y_s^{(i)})_{s\in[0,1]}$ for the development of the financial market under the real-world measure $\mathcal{P}$ with corresponding state variables $(Y_1^{(i)}, D_1^{(i)})$, $i \in \{1, \ldots, N\}$, the PVFP at $t = 1$ conditional on the state of the financial market in scenario i can be described by

$$V_1^{(i)} := E^{\mathcal{Q}} \left[\underbrace{\sum_{t=2}^{T} \exp(-\int_1^t r_u \, du) X_t \,\middle|\, (Y_1, D_1) = (Y_1^{(i)}, D_1^{(i)})}_{=: PV_1^{(i)}} \right]. \tag{7}$$

Furthermore, we define

$$\sigma_1^{(i)} := \sqrt{\mathrm{Var}^{\mathcal{Q}} \left[\sum_{t=2}^{T} \exp(-\int_1^t r_u \, du) X_t \,\middle|\, (Y_1, D_1) = (Y_1^{(i)}, D_1^{(i)}) \right]}.$$

Note that $\sigma_1^{(i)}$ may differ significantly for different scenarios i, i.e. under different realizations of the state variables $(Y_1^{(i)}, D_1^{(i)})$, the discounted cash flows $\sum_{t=2}^{T} \exp(-\int_1^t r_u \, du) X_t$ are usually not identically distributed.

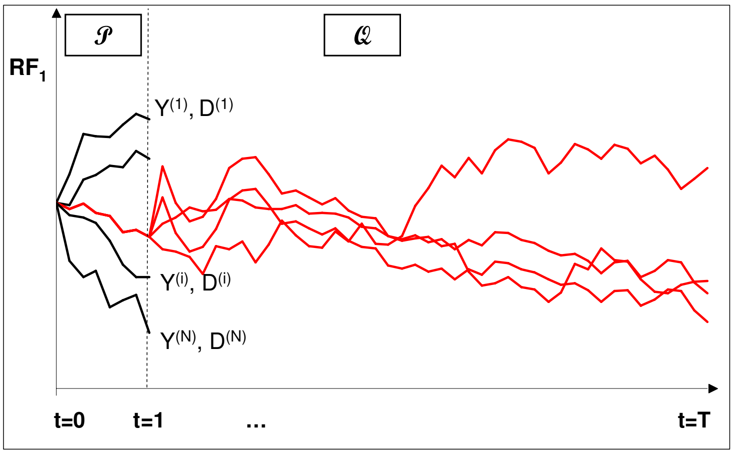

Figure 1: Illustration of the Nested Simulations Approach

In addition, realizations for the remaining components of AC_1, X_1 and ANAV_1, can easily be calculated for each of the N first-year paths. Therefore we would obtain N realizations of AC_1 by

$$\mathrm{AC}_1^{(i)} = \mathrm{ANAV}_1^{(i)} + V_1^{(i)} + X_1^{(i)}.$$

Note that these $\mathcal{F}_1$-measurable random variables $\mathrm{AC}_1^{(i)}$, $1 \leq i \leq N$, are independent and identically distributed as Monte Carlo realizations.

But just as at time zero, the valuation problem (6)/(7) in general cannot be solved analytically. Akin to Section 3.2.1, we may rely on Monte Carlo simulations. As illustrated in Figure 1, based on the state $(Y_1^{(i)}, D_1^{(i)})$ in (real-world) scenario $i \in \{1, \ldots, N\}$, we simulate $K_1^{(i)} \in \mathbb{N}$ risk-neutral scenarios and denote them by $(Y_s^{(i,k)})_{s\in(1,T]}$. Then, by determining the resulting future profits $X_t^{(i,k)}$ ($t = 2, \ldots, T$; $k = 1, \ldots, K_1^{(i)}$; $i = 1, \ldots, N$) and averaging over all $K_1^{(i)}$ sample paths for each first-year path $i \in \{1, \ldots, N\}$, we obtain Monte Carlo estimates for $V_1^{(i)}$:

$$\widetilde{V}_1^{(i)}(K_1^{(i)}) := \frac{1}{K_1^{(i)}} \sum_{k=1}^{K_1^{(i)}} \underbrace{\sum_{t=2}^{T} \exp\left(-\int_1^t r_u^{(i,k)}\, du\right) X_t^{(i,k)}}_{=:PV_1^{(i,k)}}, \quad i \in \{1, \ldots, N\}.$$

The number of simulations in the i^{th} real-world scenario may depend on i since for different standard deviations $\sigma_1^{(i)}$, a different number of simulations may be necessary to obtain acceptable results. We obtain the following sample standard deviation for $PV_1^{(i)}$:

$$\widetilde{\sigma}_1^{(i)}(K_1^{(i)}) := \sqrt{\frac{1}{K_1^{(i)} - 1} \sum_{k=1}^{K_1^{(i)}} \left(PV_1^{(i,k)} - \widetilde{V}_1^{(i)}(K_1^{(i)})\right)^2}.$$

Now, we can estimate N realizations of AC_1 by

$$\widetilde{\mathrm{AC}}_1^{(i)}(K_1^{(i)}) := \mathrm{ANAV}_1^{(i)} + \widetilde{V}_1^{(i)}(K_1^{(i)}) + X_1^{(i)}, \quad i = 1, \ldots, N.$$

3.3 Solvency Capital Requirement

From Equation (3), it follows that the SCR is the α-quantile of the random variable $L = \mathrm{AC}_0 - \frac{\mathrm{AC}_1}{1+i}$. Since AC_0 is approximated by the unbiased estimator $\widetilde{\mathrm{AC}}_0$ (see Section 3.2.1) and i is known at $t = 0$, the only remaining random component is AC_1 and the task is to estimate the α-quantile of $-\mathrm{AC}_1$.

Based on the Nested Simulations Approach described in the previous section, we obtain N estimated realizations of the random variable $Z = -\mathrm{AC}_1$, which we denote by $\tilde{z}_1, \ldots, \tilde{z}_N$. The corresponding order statistic is denoted by $\tilde{Z}_{(1)}, \ldots, \tilde{Z}_{(N)}$ with realizations $\tilde{z}_{(1)}, \ldots, \tilde{z}_{(N)}$.

A simple approach for estimating the α-quantile z_α is to rely on the corresponding empirical quantile, i.e.

$$\tilde{z}_\alpha = \tilde{z}_{(m)},$$

where $m = \lfloor N \cdot \alpha + 0.5 \rfloor$. The SCR can then be estimated as

$$\widetilde{\mathrm{SCR}} = \widetilde{\mathrm{AC}}_0 + \frac{\tilde{z}_{(m)}}{1+i}. \tag{8}$$

Alternatively, extreme value theory could be applied to derive a robust estimate of the quantile; see e.g. [9] for details.

3.4 Quality of the Resulting Estimator and Choice of K_0, K_1 and N

Within our estimation process, we have three sources of error. First, we estimate the Available Capital at $t = 0$ with the help of (only) K_0 sample paths. Second, we only use N real-world scenarios to estimate the distribution function and, third, the Available Capital at $t = 1$ is estimated with the help of (only) K_1 sample paths in every scenario.[7] As a consequence of the latter, Equation (8) does not necessarily present an estimate for the quantile of the distribution function of the "true" $\mathcal{F}_1$-measurable loss

$$L = L(Y_1, D_1) = \mathrm{AC}_0 - \frac{\mathrm{AC}_1}{1+i} = \mathrm{AC}_0 - \frac{\mathrm{ANAV}_1 + V_1 + X_1}{1+i},$$

but instead, we actually consider the distribution of the estimated loss

$$\tilde{L}(Y_1, D_1) = \widetilde{\mathrm{AC}}_0 - \frac{\mathrm{ANAV}_1 + \left(\frac{1}{K_1} \sum_{k=1}^{K_1} \sum_{t=2}^{T} \mathrm{e}^{-\int_1^t r_u^{(k)}\, du} X_t^{(k)} \middle| (Y_1, D_1) \right) + X_1}{1+i}.$$

In particular, $\tilde{L}(Y_1, D_1)$ is not $\mathcal{F}_1$-measurable due to the random sampling error resulting from the estimation of AC_0 and the inner simulation.

Obviously,

$$\tilde{L}(Y_1, D_1) \to L(Y_1, D_1) \quad \text{a.s. as } K_0, K_1 \to \infty$$

by the LLN. Nevertheless, we base our estimation of the SCR on distorted samples. To analyze the influence of this inaccuracy on our actual estimate $\widetilde{\mathrm{SCR}}$, we follow [12] and decompose the mean-square error (MSE) into the variance of our estimator and a bias:

$$\mathrm{MSE} \;=\; E\left[(\widetilde{\mathrm{SCR}} - \mathrm{SCR})^2\right] = \mathrm{Var}(\widetilde{\mathrm{SCR}}) + \left[\underbrace{E(\widetilde{\mathrm{SCR}}) - \mathrm{SCR}}_{\text{bias}}\right]^2. \tag{9}$$

Since $\widetilde{\mathrm{AC}}_0$ is an unbiased estimator of AC_0 and since it is independent of $\tilde{z}_{(m)}$, (9) simplifies to

$$\mathrm{MSE} = \mathrm{Var}\left(\widetilde{\mathrm{AC}}_0\right) + \mathrm{Var}\left(\frac{\tilde{z}_{(m)}}{1+i}\right) + \left[E\left(\frac{\tilde{z}_{(m)}}{1+i}\right) - \frac{z_\alpha}{1+i}\right]^2. \tag{10}$$

Obviously, $\mathrm{Var}\left(\widetilde{\mathrm{AC}}_0\right) = \frac{\sigma_0^2}{K_0}$, and we will now focus on the second and third term in (10). Again following [12], let

$$
\begin{aligned}
Z^{K_1}(Y_1, D_1) \;&=\; \frac{\mathrm{ANAV}_1 + \left(\frac{1}{K_1} \sum_{k=1}^{K_1} \sum_{t=2}^{T} \mathrm{e}^{-\int_1^t r_u^{(k)}\, du} X_t^{(k)} \Big| (Y_1, D_1) \right) + X_1}{1+i} \\
&\quad - \frac{\mathrm{ANAV}_1 + V_1 + X_1}{1+i}
\end{aligned}
$$

denote the difference between the estimated loss and its "true" value under the assumption that $\widetilde{\mathrm{AC}}_0$ is exact. Furthermore, define $g_{K_1}(\cdot,\cdot)$ to be the joint distribution function of L and $\tilde{Z}^{K_1} := Z^{K_1} \cdot \sqrt{K_1}$.

Then, with Proposition 2 from [12], under some regulatory conditions, we obtain

$$
\begin{aligned}
E\left[\frac{\tilde{z}_{(m)}}{1+i}\right] - \frac{z_\alpha}{1+i} \;&=\; \frac{\theta_\alpha}{K_1 \cdot f(\mathrm{SCR})} + o_{K_1}(1/K_1) + O_N(1/N) + o_{K_1}(1) O_N(1/N), \\
\mathrm{Var}\left(\frac{\tilde{z}_{(m)}}{1+i}\right) \;&=\; \frac{\alpha(1-\alpha)}{(N+2) f^2(\mathrm{SCR})} + O_N(1/N^2) + o_{K_1}(1) O_N(1/N),
\end{aligned}
$$

where $f(\cdot)$ denotes the density function of L and

$$
\begin{aligned}
\theta_\alpha \;&=\; -\frac{1}{2} \frac{\partial}{\partial u} \left[f(u) E\left[\mathrm{Var}(\tilde{Z}^{K_1} | Y_1, D_1) | L = u \right] \right] \Big|_{u=\mathrm{SCR}} \\
&=\; -\frac{1}{2} \int_{-\infty}^{\infty} z^2 \frac{\partial}{\partial u} g_{K_1}(u, z)\, dz \Big|_{u=\mathrm{SCR}}.
\end{aligned}
$$

The sign of θ_α – and, hence, the direction of the bias – will eventually be determined by the sign of $\frac{\partial}{\partial u} g_{K_1}(u,z)$. Since the SCR is located in the right-hand tail of the distribution and since $\frac{g_{K_1}(u,z)}{\int_{-\infty}^{\infty} g_{K_1}(l,z)\, dl}$ is a (conditional) density function, $\frac{\partial}{\partial u} g_{K_1}(u,z)\big|_{u=\mathrm{SCR}}$ will in general be negative and, hence, we expect to overestimate the SCR, i.e. the probability that the company is solvent after one year is in average higher than $\alpha = 99.5\%$.

To optimize our estimate, we would like to choose K_0, K_1 and N such that the MSE is as small as possible. Disregarding lower order terms, this yields the following optimization problem in K_0, K_1 and N

$$
\frac{\sigma_0^2}{K_0} + \frac{\theta_\alpha^2}{K_1^2 \cdot f^2(\mathrm{SCR})} + \frac{\alpha(1-\alpha)}{(N+2) f^2(\mathrm{SCR})} \to \min
$$

subject to the effort restriction $K_0 + N \cdot K_1 = \Gamma$.[8] Using Lagrangian multipliers, we obtain that for any choice of Γ,

$$
\begin{aligned}
N \;&\approx\; \frac{\alpha\,(1-\alpha) \cdot K_1^2}{2\theta_\alpha^2}, \\
K_0 \;&\approx\; \frac{\sigma_0 \cdot K_1 \cdot f(\mathrm{SCR})}{\theta_\alpha} \sqrt{\frac{N \cdot K_1}{2}}
\end{aligned}
$$

i.e. given any choice of K_1, we may determine θ_α and subsequently choose an optimal N and K_0.

In practical applications, f, σ_0 and θ_α are unknown, but may be estimated in a pilot simulation with only a small number of sample paths. However, the estimation of θ_α generally will be quite inaccurate for large α because it is necessary to estimate a derivative in the very tail of the distribution (see Section 6.2.1).

Furthermore, note that although there are many parallels between estimating VaR of a portfolio of financial derivatives and VaR of an insurance portfolio, there is at least one important difference. In a portfolio of financial derivatives, the single instruments can be valuated independently and hence the pricing errors diversify away when the portfolio is large (see [12]). This is in general not the case for an insurance portfolio. Due to management rules applied at company level (e.g. strategic asset allocation and profit participation) the cash flows of different insurance contracts may depend on each other. Therefore, we need to simulate the whole portfolio simultaneously based on the same stochastic scenarios. Thus, pricing errors in the inner simulation will in general not diversify away when the portfolio is large and hence, the required number of inner simulations will not necessarily decrease for large portfolios.

3.5 Alternative Estimation of the SCR

So far in the present section, we have specified the Available Capital – and, consequently, the SCR – based on cash flows from the shareholders' perspective. As already noted in Section 2.1, an alternative approach is to calculate the Available Capital as the difference of the market value of assets and the market value of liabilities, i.e. by considering cash flows from the policyholders' perspective.

While of course both approaches are equivalent in the sense that the quantity to be estimated is the same, the two methods may well yield different estimates for the SCR. In particular, the quality of the resulting estimate can differ considerably (cf. Section 6), where it is primarily dependent on the model specification which estimator is superior.

The quality of the alternative estimator may be assessed in an analogous fashion to Section 3.4, so we omit the presentation for the sake of brevity. We continue to limit our exposition to the specification presented in the beginning of this section as it is more in line with the MCEV principles. However, we will rely on both approaches in our applications.

The primary problem with the approaches presented in this section is the nested simulation structure: In order to obtain accurate results, a large number of total simulations is required. Possibilities to increase the efficiency are variance reduction techniques such as control variates (see [11]) or bias reduction techniques such as jackknife procedures (see [12]). However, it is questionable if these techniques can lead to the necessary efficiency gain in an insurance context in view of the rather complex – and hence computationally intensive – projection of the liability side. Thus, carrying out this Nested Simulations Approach is often not feasible within practical applications.

4 Analytic Approximations of the SCR

In Section 2.1.3, we pointed out that there are severe problems with the SCR Aggregation formula as applied in *standard models*. In order to find a "pragmatic" alternative, let us more generally than in Section 2.1.3 assume that the aggregate loss L can be represented as a continuous, componentwise strictly monotonic increasing function g of the underlying risk factors $(Y_1, \ldots, Y_d)' = (Y_{1,1}, \ldots, Y_{1,d})'$, i.e.

$$L = g(Y_1, \ldots, Y_d).$$

Moreover, assume that we are given the joint distribution function F of $(Y_1, \ldots, Y_d)'$, which, for instance, may be represented via the marginal distributions $F_{Y_i}(\cdot)$, $1 \leq i \leq d$, and the corresponding copula function $C : [0,1]^d \rightarrow [0,1]$ by Sklar's Theorem.[9] Then

$$\text{SCR} = \inf \{ x | \mathcal{P} (g(Y_1, \ldots, Y_d) \leq x) \geq \alpha \},$$

and we have the following relationship:

Proposition 1.

$$\min \{ g(y_1, \ldots, y_d) | F(y_1, \ldots, y_d) \geq \alpha \} \geq SCR. \tag{11}$$

Proof.
If $(Y_1, \ldots, Y_d) \leq (y_1, \ldots, y_d)$ componentwise, then $g(Y_1, \ldots, Y_d) \leq g(y_1, \ldots, y_d)$ since g is componentwise increasing. Therefore,

$$F(y_1, \ldots, y_d) = \mathcal{P} (Y_1 \leq y_1, \ldots, Y_d \leq y_d) \leq \mathcal{P} (g(Y_1, \ldots, Y_d) \leq g(y_1, \ldots, y_d)),$$

and hence,

$$\min_{y_1, \ldots, y_d : F(y_1, \ldots, y_d) \geq \alpha} \mathcal{P} (g(Y_1, \ldots, Y_d) \leq g(y_1, \ldots, y_d)) \geq \alpha.$$

Now, since $\mathcal{P} (g(Y_1, \ldots, Y_d) \leq \cdot)$ is increasing,

$$\min_{y_1, \ldots, y_d : F(y_1, \ldots, y_d) \geq \alpha} \{ g(y_1, \ldots, y_d) \} \geq \inf \{ x | \mathcal{P} (g(Y_1, \ldots, Y_d) \leq x) \geq \alpha \} = \text{SCR}.$$

$\square$

Note that in case of a componentwise strictly increasing, continuous distribution function, due to the monotonicity of g, (11) reads as

$$\min \{ g(y_1, \ldots, y_d) | F(y_1, \ldots, y_d) = \alpha \} \geq \text{SCR}. \tag{12}$$

Therefore, in case the manifold $\{(y_1, \ldots, y_d) | F(y_1, \ldots, y_d) = \alpha\}$ can be expressed in an explicit form – e.g. if the risk factors are normally distributed – the solution of (12) may yield a *pragmatic* and *conservative* approximation for the SCR. However, the approximation may not be very close.

5 Least-Squares Monte Carlo Approach

As was pointed out in Section 3, in order to determine the Solvency Capital Requirement, we need to determine the distribution of

$$\begin{aligned} \mathrm{AC}_1 &= \mathrm{ANAV}_1 + V_1 + X_1 \\ &= \mathrm{ANAV}_1 + E^{\mathcal{Q}}\left[\sum_{t=2}^{T} \exp\left(-\int_1^t r_u\,du\right) X_t \middle| (Y_1, D_1)\right] + X_1. \end{aligned}$$

Here, the conditional expectation causes the primary difficulty for developing a suitable Monte Carlo technique. This is analogous to the pricing of Bermudan options, where "*the conditional expectations involved in the iterations of dynamic programming cause the main difficulty for the development of Monte-Carlo techniques*" (cf. [6]). A suitable solution to this problem was proposed by [20], who use least-squares regression on a suitable finite set of functions in order to approximate the conditional expectation.

As pointed out by [6], the algorithm more precisely consists of two different types of approximations. Within the first approximation step, the conditional expectation is replaced by a finite linear combination of "basis" functions. As the second approximation, Monte Carlo simulations and least-squares regression are employed to approximate the linear combination given in step one. They show that under certain completeness assumptions on the basis functions, the algorithm converges, i.e. it presents a valid and in comparison to Nested Simulations considerably more efficient approach to the pricing problem.

In what follows, we exploit this analogy by transferring their ideas to our problem.

5.1 Least-Squares-Algorithm

As the first approximation, we replace the conditional expectation, V_1, by a finite combination of basis functions $(e_k(Y_1, D_1))_{k \in \{1,\dots,M\}}$,

$$V_1 \approx \hat{V}_1^{(M)}(Y_1, D_1) = \sum_{k=1}^{M} \alpha_k \cdot e_k(Y_1, D_1),$$

assuming that the sequence $(e_k(Y_1, D_1))_{k \geq 1}$ is linearly independent and complete in the Hilbert space $L^2\left(\Omega, \sigma(Y_1, D_1), \mathcal{P}\right)$.

Subsequently, we determine approximate $\mathcal{P}$-realizations of V_1 using Monte Carlo simulations. We generate N independent paths $\left(Y_t^{(1)}, D_t^{(1)}\right)$, $\left(Y_t^{(2)}, D_t^{(2)}\right)$,..., $\left(Y_t^{(N)}, D_t^{(N)}\right)$ for $t \in (0, T]$, where we generate the Markovian increments under the physical measure for the first year and under the risk-neutral measure for the remaining periods. Subsequently, we calculate the realized cumulative discounted cash flows

$$\mathrm{PV}_1^{(i)} = \sum_{t=2}^{T} \exp\left(- \int_1^t r_u^{(i)}\, du \right) X_t^{(i)},\ 1 \leq i \leq N.$$

Here, clearly $r_t^{(i)}$ and $X_t^{(i)}$ denote the interest rate and cash flow at time t under path $\left(Y_t^{(i)}, D_t^{(i)}\right)_{t\in(0,T]}$, respectively, for $i = 1, \ldots, N$.

Subsequently, we use these realizations in order to determine the coefficients $\alpha = (\alpha_1, \ldots, \alpha_M)$ in the approximation $\hat{V}_1^{(M)}$ by least-squares regression:

$$\hat{\alpha}^{(N)} = \operatorname{argmin}_{\alpha\in\mathbb{R}^M} \left\{ \sum_{i=1}^{N} \left[\mathrm{PV}_1^{(i)} - \sum_{k=1}^{M} \alpha_k \cdot e_k \left(Y_1^{(i)}, D_1^{(i)} \right) \right]^2 \right\}.$$

Replacing α by $\hat{\alpha}^{(N)}$, we obtain the second approximation

$$V_1 \approx \hat{V}_1^{(M)}(Y_1, D_1) \approx \hat{V}_1^{(M,N)}(Y_1, D_1) = \sum_{k=1}^{M} \hat{\alpha}_k^{(N)} \cdot e_k(Y_1, D_1).$$

By means of this approximation, we can calculate realizations for AC_1 resorting to the previously generated paths $\left(Y_t^{(i)}, D_t^{(i)}\right)$, $i = 1, \ldots, N$, or, more precisely, to the sub-paths for the first year, by evaluating

$$\widehat{\mathrm{AC}}_1^{(i)} = \mathrm{ANAV}_1^{(i)} + \hat{V}_1^{(M,N)}(Y_1^{(i)}, D_1^{(i)}) + X_1^{(i)},$$

where clearly $\mathrm{ANAV}_1^{(i)}$ and $X_1^{(i)}$ denote ANAV_1 and X_1 for paths $i \in \{1, \ldots, N\}$.

Based on these realizations, we may now determine a corresponding empirical distribution function and, consequently, the Solvency Capital Requirement. We denote the estimated SCR resulting from the Least-Squares Monte Carlo (LSM) Approach by $\widehat{\mathrm{SCR}}$.

5.2 Choice of the Regression Function

While several simple methods for the variable selection in regression models are available in statistical and econometrical literature, common criteria such as Mallows' complexity parameter (C_p), the Akaike information criterion (AIC), or simple versions of the Schwarz information criterion (SIC) rely on the rather restrictive assumptions of homoscedasticity and/or normally distributed errors (see e.g. [17] for details). However, these assumptions are likely to be violated in the current setting; for example, for many asset models the conditional variance of the residuals in the regression for V_1 will depend on the assets' first-year path.

In order to obtain a generalized selection criterion, note that

$$\begin{aligned}
& E_1\left[\sum_{i=1}^{N}\left(\mathrm{PV}_1^{(i)}-\hat{V}_1^{(M,N)}\left(Y_1^{(i)},D_1^{(i)}\right)\right)^2\right] \\
= \; & \operatorname{tr}\Big[\operatorname{Cov}_1\Big(\left(\mathrm{PV}_1^{(1)},\ldots,\mathrm{PV}_1^{(N)}\right)'-\Big(\hat{V}_1^{(M,N)}\left(Y_1^{(1)},D_1^{(1)}\right),\ldots, \\
& \hat{V}_1^{(M,N)}\left(Y_1^{(N)},D_1^{(N)}\right)\Big)'\Big)\Big] \\
& +\sum_{i=1}^{N}\left(E_1\left[\mathrm{PV}_1^{(i)}\right]-E_1\left[\hat{V}_1^{(M,N)}\left(Y_1^{(i)},D_1^{(i)}\right)\right]\right)^2 \\
= \; & \operatorname{tr}\left[\left(I-\mathcal{E}\left(\mathcal{E}'\mathcal{E}\right)^{-1}\mathcal{E}'\right)\operatorname{Cov}_1\left(\left(X^{(1)},\ldots,X^{(N)}\right)'\right)\right] \\
& +\sum_{i=1}^{N}\left(V_1^{(i)}-E_1\left[\hat{V}_1^{(M,N)}\left(Y_1^{(i)},D_1^{(i)}\right)\right]\right)^2 \\
= \; & \underbrace{\sum_{i=1}^{N}E_1\left[V_1^{(i)}-\hat{V}_1^{(M,N)}\left(Y_1^{(i)},D_1^{(i)}\right)\right]^2}_{=\mathrm{SMSE}}+\sum_{i=1}^{N}\sigma_1^{(i)} \\
& -2\operatorname{tr}\left(\mathcal{E}\left(\mathcal{E}'\mathcal{E}\right)^{-1}\mathcal{E}'\operatorname{diag}\left(\sigma_1^{(1)},\ldots,\sigma_1^{N}\right)\right),
\end{aligned}$$

where E_1 and Cov_1 denote the conditional expectation and covariance at $t=1$, respectively, $e_i=\left(e_i(Y_1^{(1)},D_1^{(1)}),\ldots,e_i(Y_1^{(N)},D_1^{(N)})\right)'$, $1\le i\le M$, and $\mathcal{E}=(e_1,\ldots,e_M)$ is the matrix of explanatory variables. Moreover, I is the identity and $\operatorname{diag}\left(\sigma_1^{(1)},\ldots,\sigma_1^{(N)}\right)$ is a diagonal matrix with entries $\sigma_1^{(1)},\ldots,\sigma_1^{(N)}$. A generalized complexity parameter may now be defined via the empirical version of the Sum of Squared Errors (SMSE),

$$\begin{aligned}
\widehat{SMSE} \; = \; & \sum_{i=1}^{N}\left(\mathrm{PV}_1^{(i)}-\hat{V}_1^{(M,N)}\left(Y_1^{(i)},D_1^{(i)}\right)\right)^2-\sum_{i=1}^{N}\tilde{\sigma}_1^{(i)} \\
& +2\operatorname{tr}\left(\mathcal{E}\left(\mathcal{E}'\mathcal{E}\right)^{-1}\mathcal{E}'\operatorname{diag}\left(\tilde{\sigma}_1^{(1)},\ldots,\tilde{\sigma}_1^{N}\right)\right)
\end{aligned}$$

since the different $\mathrm{PV}_1^{(i)}$ are independent as Monte Carlo realizations. The primary problem with this criterion is that it requires the knowledge or estimation of the conditional variance, where the latter again would require nested simulations. One potential solution to this problem is given in [4]. Here, Baek et al. propose a generalized version of Mallow's C_p for heteroscedastic data. They split the data into smaller groups such that homoscedasticity can be assumed within one group. Then the variances are estimated for each group and the generalized version of Mallow's C_p (GC_p) is derived from the resulting weighted least-squares-estimators. They "show by means of simulation study that GC_p selects the correct model more often than C_p for data with significant heteroscedasticity."[10] However, for roughly homoscedastic data, C_p gives better results, i.e. it depends on the degree of heteroscedasticity, whether the use of Mallow's C_p is still appropriate or whether more sophisticated criteria need to be applied.

5.3 Convergence

From the assumptions on the sequence of basis functions, we automatically obtain the mean-square convergence of

$$\hat{V}_1^{(M)} \longrightarrow \sum_{k=1}^{\infty} \alpha_k \cdot e_k(Y_1, D_1) = V_1,\ M \to \infty,$$

and, hence, convergence in distribution. Therefore, it suffices to show that

$$\hat{V}_1^{(M,N)} \longrightarrow \hat{V}_1^{(M)},\ N \to \infty$$

in distribution. The only issue keeping us from applying results from econometric literature are the change of measure at time $t = 1$ and the structural implications for the considered probability space. However, a potential back door would be the construction of an alternative probability measure, say $\widetilde{\mathcal{P}}$, on an identical copy of our filtered measurable space, say $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}})$, such that

$$E^{\mathcal{Q}}[Y(\omega)|\mathcal{F}_1] = E^{\widetilde{\mathcal{P}}}[Y(\widetilde{\omega})|\widetilde{\mathcal{F}}_1]$$

for all random variables Y, and

$$\mathcal{P}(Z(\omega) \leq z) = \widetilde{\mathcal{P}}(Z(\widetilde{\omega}) \leq z)\ \forall z \in \mathbb{R}$$

for all $\mathcal{F}_t$-measurable random variables Z, $0 \leq t \leq 1$. Then, since the realizations of the basis functions are iid across paths, we can proceed analogously to [20], Section 2.2, and quote Theorem 3.5 of [25], which states that under weak regularity conditions,

$$\hat{V}_1^{(M,N)} \longrightarrow \hat{V}_1^{(M)},\ N \to \infty,$$

in $L^2\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{P}}\right)$, and, hence, in distribution.

For the generalized Black-Scholes setup considered in Section 6, such a measure $\widetilde{\mathcal{P}}$ can be easily constructed by appropriately manipulating the drift terms. For example, in the classical Black-Scholes market, "the stock" S evolves according to the stochastic differential equations

$$\begin{aligned} dS_t &= S_t\left(\mu\, dt + \sigma\, dW_t\right),\ S_0 > 0, \\ dS_t &= S_t\left(r\, dt + \sigma\, dZ_t\right),\ S_0 > 0, \end{aligned}$$

where $\mu \in \mathbb{R}$ is the stock's drift, $r \in \mathbb{R}$ is the interest rate, $\sigma > 0$ is the volatility, and W and Z are Brownian motions under $\mathcal{P}$ and $\mathcal{Q}$, respectively. Now, if we let $\widetilde{W}$ be a Brownian motion on $(\widetilde{\Omega}, \widetilde{F}, \widetilde{\mathcal{P}})$, and S evolves according to

$$dS_t = S_t \left(\tilde{\mu}_t \, dt + \sigma \, d\widetilde{W}_t \right), \; S_0 > 0,$$

where $\tilde{\mu}_t = \mu \cdot 1_{\{0 \leq t \leq 1\}} + r \cdot 1_{\{1 < t < \infty\}}$, then $\widetilde{\mathcal{P}}$ satisfies the required properties and, hence, convergence is shown in this special case.

While it seems feasible to construct a canonical probability measure $\widetilde{\mathcal{P}}$ for far more general setups by separating events into an $\mathcal{F}_1$-measurable and an orthogonal part measured by $\mathcal{P}$ and $\mathcal{Q}$, respectively, we leave the further exploration of this idea as well as an assessment of convergence rates for our future work. After all, as was pointed out by [20], the ultimate test of such an algorithm is "*how well it performs using a realistic number of paths and basis functions*" in a somewhat realistic framework.

6 Application

6.1 Asset and Liability Model

As an example framework for our considerations, we use the model for a single participating term-fix contract introduced in [3]. As was pointed out in [16], under certain assumptions, this framework may serve as a simplified model for the overall financial evolution of a life insurance company offering participating contracts.

6.1.1 General Setup

A simplified balance sheet is employed to represent the insurance company's financial situation (see Table 1). Here, A_t denotes the market value of the in-

Assets	**Liabilities**
A_t	L_t
	R_t
A_t	A_t

Table 1: Simplified balance sheet

surer's asset portfolio, L_t is the policyholder's statutory account balance, and $R_t = A_t - L_t$ are the free funds (also referred to as "reserve") at time t.

Disregarding debt financing, the total assets A_0 at time zero derive from two components, the policyholder's account balance ("liabilities") and the shareholders' capital contribution ("equity"). Ignoring charges as well as unrealized gains or losses, these components are equal to the single up-front premium L_0 and the reserve at time zero, R_0, respectively. In particular, the shareholders' funds are available to cover potential losses, i.e. they are exposed to risk. Thus, as compensation for the adopted risk, we assume that dividends d_t may be paid out to shareholders each period. Moreover, shareholders may benefit from a favorable evolution of the company in that the market value of their capital contribution increases. More specifically, they may realize $\mathrm{ROI}_T := R_T - \exp\left(\int_0^T r_u\,du\right) R_0$ as their (time value-adjusted) return on investment at the end of the projection period (also referred to as "maturity") T.

For the bonus distribution scheme, i.e. for modeling the evolution of the liabilities, we rely on the so-called MUST-case from [3]. This distribution mechanism describes what insurers are obligated to pass on to policyholders according to German regulatory and legal requirements: On one hand, companies are obligated to guarantee a minimum rate of interest g on the policyholder's account; on the other hand, according to the regulation about minimum premium refunds in German life insurance, a minimum participation rate δ of the earnings on book values has to be credited to the policyholder's account.[11] Since earnings on book values usually do not coincide with earnings on market values due to accounting rules, we assume that earnings on book values amount to a portion y of the latter.

In case the asset returns are so poor that crediting the guaranteed rate g to the policyholder's account will result in a negative reserve R_t, the insurer will default due to the shareholders' limited liability (cf. the notion of a "shortfall" in [16]). However, as was pointed out in Section 2.1.1, the MCEV should not reflect the shareholders' put option, i.e. the MCEV should be calculated under the supposition that shareholders cover any deficit. In accordance with this hypothesis, we assume that the company obtains an additional contribution c_t from its shareholders in case of such a shortfall.

Therefore, the earnings on market values equal to $A_t^- - A_{t-1}^+$, where A_t^- and $A_t^+ = A_t^- - d_t + c_t$ describe the market value of the asset portfolio shortly before and after the dividend payments d_t and capital contributions c_t at time t, respectively. In particular, we have

$$L_t = (1+g)\,L_{t-1} + \left[\delta y\left(A_t^- - A_{t-1}^+\right) - gL_{t-1}\right]^+, \quad 1 \le t \le T.$$

Assuming that the remaining part of earnings on book values is paid out as dividends, we obtain

$$\begin{aligned} d_t &= (1-\delta) y\left(A_t^- - A_{t-1}^+\right) \mathbb{1}_{\left\{\delta y\left(A_t^- - A_{t-1}^+\right) > gL_{t-1}\right\}} \\ &\quad + \left[y\left(A_t^- - A_{t-1}^+\right) - gL_{t-1}\right] \mathbb{1}_{\left\{\delta y\left(A_t^- - A_{t-1}^+\right) \le gL_{t-1} \le y\left(A_t^- - A_{t-1}^+\right)\right\}}. \end{aligned}$$

Obviously, dividend payments equal zero whenever a capital contribution is required. Therefore, the capital contribution at time t can be described as

$$c_t = \max\{L_t - A_t^-, 0\}.$$

For more details on the contract model we refer to [3].

6.1.2 Relevant Quantities

Since we ignore unrealized gains and losses on assets as well as other adjustments, we have $\mathrm{ANAV}_0 = \mathrm{NAV}_0 = R_0$. Therefore, the Available Capital at time $t = 0$ can be described as follows:

$$
\begin{aligned}
& \mathrm{AC}_0 = \mathrm{ANAV}_0 + V_0 \\
= \; & R_0 + E^{\mathcal{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_0^t r_u\,du\right)(d_t - c_t) + \exp\left(-\int_0^T r_u\,du\right)\mathrm{ROI}_T\right] \\
= \; & R_0 + E^{\mathcal{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_0^t r_u\,du\right)(d_t - c_t) + \exp\left(-\int_0^T r_u\,du\right)R_T - R_0\right] \\
= \; & E^{\mathcal{Q}}\left[\sum_{t=1}^{T} \exp\left(-\int_0^t r_u\,du\right)X_t\right]
\end{aligned}
$$

where

$$
X_t = \begin{cases} d_t - c_t & , \text{if } t \in \{1, \ldots, T-1\} \\ d_T - c_T + R_T & , \text{if } t = T \end{cases} .
$$

So far, we described AC_0 based on cash flows from the shareholders' point of view. But as already mentioned in Section 3.5, we can also express AC_0 based on cash flows from the policyholders' perspective, i.e.

$$
\mathrm{AC}_0 = A_0 - E^{\mathcal{Q}}\left[\exp\left(-\int_0^T r_u\,du\right)L_T\right].
$$

As we will see in Section 6.2, the quality of the two different estimation approaches differs considerably.

Similarly, we obtain

$$
\mathrm{AC}_1 \;=\; \mathrm{ANAV}_1 + V_1 + X_1 = E^{\mathcal{Q}}\left[\sum_{t=2}^{T} \exp\left(-\int_1^t r_u\,du\right)X_t \,\middle|\, \mathcal{F}_1\right] + X_1
$$

and

$$
\mathrm{AC}_1 \;=\; A_1^+ - E^{\mathcal{Q}}\left[\exp\left(-\int_1^T r_u\,du\right)L_T \,\middle|\, \mathcal{F}_1\right] + X_1.
$$

6.1.3 Asset Model

For the evolution of the financial market, similarly to [26], we assume a generalized Black-Scholes model with stochastic interest rates. The asset process and the short rate process evolve according to the stochastic differential equations

$$\begin{aligned} dA_t &= \mu A_t\, dt + \rho\, \sigma_A\, A_t\, dW_t + \sqrt{1-\rho^2}\, \sigma_A\, A_t\, dZ_t, \quad A_0 > 0, \\ dr_t &= \kappa\,(\xi - r_t)\; dt + \sigma_r\, dW_t, \quad r_0 > 0, \end{aligned}$$

respectively, where $\rho \in [-1, 1]$ describes their correlation, $\mu \in \mathbb{R}$, $\sigma_A, \kappa, \xi, \sigma_r > 0$, and W and Z are two independent Brownian motions under the real-world measure $\mathcal{P}$. Hence, the market value of the assets at $t = 1$ can be expressed as

$$A_1^- = A_0 \exp\left(\mu - \frac{\sigma_A^2}{2} + \rho\, \sigma_A\, W_1 + \sqrt{1-\rho^2}\, \sigma_A\, Z_1\right),$$

and for the short rate process, we have

$$r_1 = e^{-\kappa} r_0 + \xi\left(1 - e^{-\kappa}\right) + \int_0^1 \sigma_r e^{-\kappa(t-s)}\, dW_s.$$

Moreover, we assume that the market price of interest rate risk is constant and denote it by λ. Then, we obtain the following dynamics under the risk-neutral measure $\mathcal{Q}$:

$$\begin{aligned} dA_t &= r_t A_t\, dt + \rho\, \sigma_A\, A_t\, d\tilde{W}_t + \sqrt{1-\rho^2}\, \sigma_A\, A_t\, d\tilde{Z}_t, \\ dr_t &= \kappa\left(\tilde{\xi} - r_t\right)\, dt + \sigma_r\, d\tilde{W}_t, \end{aligned}$$

where $\tilde{\xi} = \xi - \frac{\lambda \sigma_r}{\kappa}$, and $\tilde{W}$ and $\tilde{Z}$ are two independent Brownian motions under $\mathcal{Q}$. Hence, under $\mathcal{Q}$, we have

$$\begin{aligned} A_t^- &= A_{t-1}^+ \exp\left(\int_{t-1}^t r_s ds - \frac{\sigma_A^2}{2} + \rho\, \sigma_A\, (\tilde{W}_t - \tilde{W}_{t-1}) + \sqrt{1-\rho^2}\sigma_A(\tilde{Z}_t - \tilde{Z}_{t-1})\right), \\ r_t &= e^{-\kappa} r_{t-1} + \tilde{\xi}\left(1 - e^{-\kappa}\right) + \int_{t-1}^t \sigma_r e^{-\kappa(t-s)}\, d\tilde{W}_s, \end{aligned}$$

and

$$\int_{t-1}^t r_s\, ds = \frac{r_{t-1} - \tilde{\xi}}{\kappa}\left(1 - e^{-\kappa}\right) + \tilde{\xi} + \frac{\sigma_r}{\kappa}\int_{t-1}^t \left(1 - e^{-\kappa(t-s)}\right)\, d\tilde{W}_s,$$

which can be conveniently used in Monte Carlo algorithms (cf. [26]).

We estimated the parameters for our asset model from German data from June 1998 to June 2008 using a kalman filter. The parameters for the asset portfolio are calibrated to an index consisting of 80% REXP[12] and 20% DAX[13] . For the short rate process we use interest rates for government bonds with maturities of

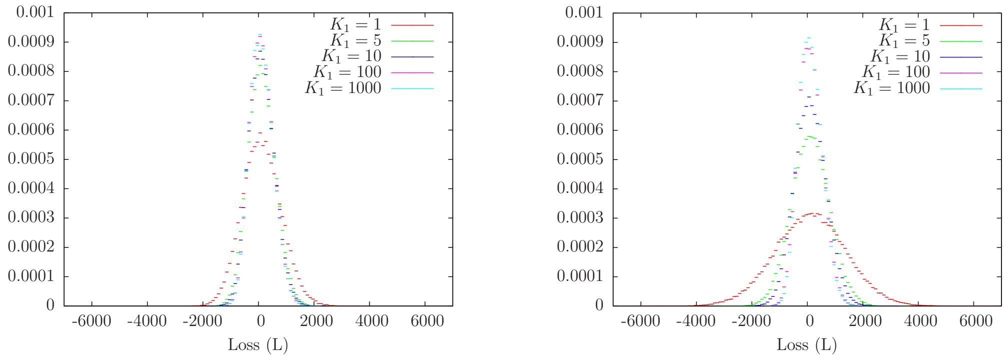

Figure 2: Empirical density function for different choices of K_1 for the estimator based on the policyholders' cash flows (left) and the shareholders' cash flows (right), $N = 100,000$, $K_0 = 250,000$

3 months, 1, 3, 5 and 10 years. We obtain the following results: The drift of the asset process is $\mu = 4.25\%$, and its volatility is $\sigma_A = 4.28\%$. For the short rate process we have $\kappa = 14.49\%$, $\xi = 3.64\%$ and $\sigma_r = 0.6\%$. The initial value of the short rate is $r_0 = 4.19\%$. The estimated correlation is $\rho = -0.0597$ and the market price of risk is $\lambda = -0.5061$.

For the insurance contract, similarly to [3], we assume a guaranteed minimum interest rate of $g = 3.5\%$, a minimum participation rate of $\delta = 90\%$, an initial premium of $L_0 = 10,000$ and a maturity of $T = 10$. Moreover, we assume that $y = 50\%$ of earnings on market values are declared as earnings on book values and that the initial reserve quota equals $x_0 = R_0/L_0 = 10\%$, i.e. $R_0 = x_0 \cdot L_0 = 1,000$.

6.2 Results

In Sections 3 and 5, we introduced different methods on how to estimate the SCR in our framework. In what follows, we implement them in the setup described in Section 6.1. In particular, we focus on contemplating pitfalls, drawbacks, as well as advantages of the different methods.

6.2.1 Nested Simulations Approach

As indicated in Section 3.4, the estimation of the SCR using Nested Simulations is biased. This bias mainly depends on the choice of the estimator and the number of inner simulations. Hence, in order to develop an idea for the magnitude of this bias, we analyze the results for the estimator based on cash flows from the policyholders' and from the shareholders' perspective (see Section 6.1.2) and choose different numbers of inner simulations. Initially, we fix $K_0 = 250,000$ sample paths for the estimation of V_0, $N = 100,000$ realizations for the simulation over the first year, and choose $K_1^{(i)} = K_1 \; \forall 1 \leq i \leq N$.

In Figure 2, the empirical density functions for both estimators and different choices of K_1 are plotted. As expected, for both estimators the distribution is more dispersed for small K_1, which has a tremendous impact on our problem of estimating a quantile in the tail: We significantly overestimate the SCR for small choices of K_1. This can also be noticed in Table 2, where the estimated SCR for different choices of K_1 is displayed. Moreover, we observe that the distribution given by the estimator based on shareholders' cash flows is more dispersed than the estimator for the policyholders' cashflows for the same K_1. Since the bias mainly depends on the variance of $\widetilde{V}_1^{(i)}(K_1^{(i)})$, $1 \leq i \leq N$, this indicates that this estimator has higher variances and thus, we need more inner simulations to obtain reliable results. This can also be seen in Table 2, where the SCR estimated via shareholder cash flows always exceeds the SCR derived from policyholders' cash flows. Further analyses show that in our setting, the estimator based on cash flows from the policyholders' perspective is always superior to that based on shareholders' cash flows except for some very extreme (and unrealistic) parameter choices in the contract model. Therefore, we will rely on the estimator based on cash flows from the policyholders' perspective in the remainder of this paper.

K_1	policyholders' cash flows		shareholders' cash flows	
	$\widetilde{\text{SCR}}$	$\widetilde{\text{AC}_0}/\widetilde{\text{SCR}}$	$\widetilde{\text{SCR}}$	$\widetilde{\text{AC}_0}/\widetilde{\text{SCR}}$
1	1994.0	94%	3432.5	55%
5	1404.7	134%	1874.6	100%
10	1332.7	141%	1606.5	117%
100	1261.2	149%	1279.1	147%
1000	1246.3	151%	1254.6	149%

Table 2: Estimated SCR and estimated solvency ratio for different choices of K_1, $K_0 = 250,000$, $N = 100,000$, Nested Simulations Approach

The above results show that a proper allocation of resources, i.e. a careful choice of K_0, K_1 and N, is inevitable in order to obtain accurate results. In order to find (approximately) optimal combinations of K_0, K_1 and N, we estimate the unknown quantities σ_0, f and θ_α from a pilot simulation with $\tilde{K}_0 = 250,000$ sample paths for the estimation of AC_0, $\tilde{N} = 100,000$ real-world scenarios and $\tilde{K}_1 = 200$ inner simulations. Based on these scenarios, we calculate the empirical variances $\tilde{\sigma}_1^{(i)}$ for each real-world scenario i, $i = 1, \ldots, \tilde{N}$ and estimate the expected conditional variance via a regression analysis. More specifically, we assume

$$E^{\mathbb{Q}}\left[\text{Var}\left(\tilde{Z}^{K_1}|Y_1, D_1\right)|L\right] \approx \beta_0 + \beta_1 L + \beta_2 L^2$$

and estimate β_0, β_1 and β_2 from our results. Sensitivity analyses show that the optimal choice of K_0, K_1 and N is rather insensitive to different choices of the regression function. In a second step, we derive the empirical density function and approximate its derivative by the average of left and right-hand sided finite differences. In this case, sensitivity analyses indicate that the obtained results are not very exact due to the rather small number of observations in the tail. Nevertheless, our estimates provide a rough idea of the optimal ratio. The resulting estimate for θ_α is given by $\hat{\theta}_\alpha \approx 0.027$. σ_0 is approximated by the empirical standard deviation.

In order to obtain an accurate estimate of the 99.5% quantile based on the empirical distribution function, we choose a relatively large number of inner simulations, namely $K_1 = 300$. Then, we find that a choice of approximately $N = 320,000$ and $K_0 = 1,500,000$ is optimal, which results in a total budget of $\Gamma = 97,500,000$ simulations. In this setting, we obtain $\widetilde{\text{SCR}} = 1249.7$ and a solvency ratio of 150%. At first sight, it might be surprising that K_0 should be chosen that large compared to the two other parameters. But reducing the variance of AC_0 is relatively "cheap" compared to reducing the variance of $z_{(m)}$ because whenever we increase N we automatically have to perform K_1 inner simulations for every additional real-world scenario. Therefore, it is reasonable to allocate a rather large budget to K_0.

To demonstrate that, given a total budget of $\Gamma = 97,500,000$, this choice is roughly adequate, we estimate the SCR 150 times for fixed K_0 and different combinations of N and K_1, where each combination corresponds to a total budget of 97,500,000 simulations. We estimate the bias by $\frac{\bar{\theta}_\alpha}{K_1 \cdot \bar{f}(\widetilde{\text{SCR}})}$, where $\bar{\theta}_\alpha$ and $\bar{f}$ denote the average of the estimates resulting from the 150 estimation procedures as explained above. The MSE is then estimated by the sum of the empirical variance and the squared estimated bias. This allows us to correct the mean by the estimated bias. Figure 3 and Table 3 show our results.

N	K_1	Mean ($\widetilde{\text{SCR}}$)	Empirical Variance	Estimated Bias	Estimated MSE	Corrected Mean
160,000	600	1247.7	24.6	1.4	26.6	1246.3
320,000	**300**	**1249.3**	**15.8**	**2.9**	**24.0**	**1246.4**
640,000	150	1251.3	7.9	5.7	40.6	1245.6
1,280,000	75	1257.4	4.2	11.4	133.1	1246.1

Table 3: Choice of N and K_1 for the Nested Simulations Approach Approach, $K_0 = 1,500,000$

As expected, the mean of the estimated SCRs increases as K_1 decreases due to the increased bias. In contrast to this, the empirical variance obviously decreases as N increases. Furthermore, we find that our choice of N and K_1 yields the smallest estimated MSE from the combinations given in Table 3. Therefore, our choice appears reasonable within our framework. Moreover, it is remarkable

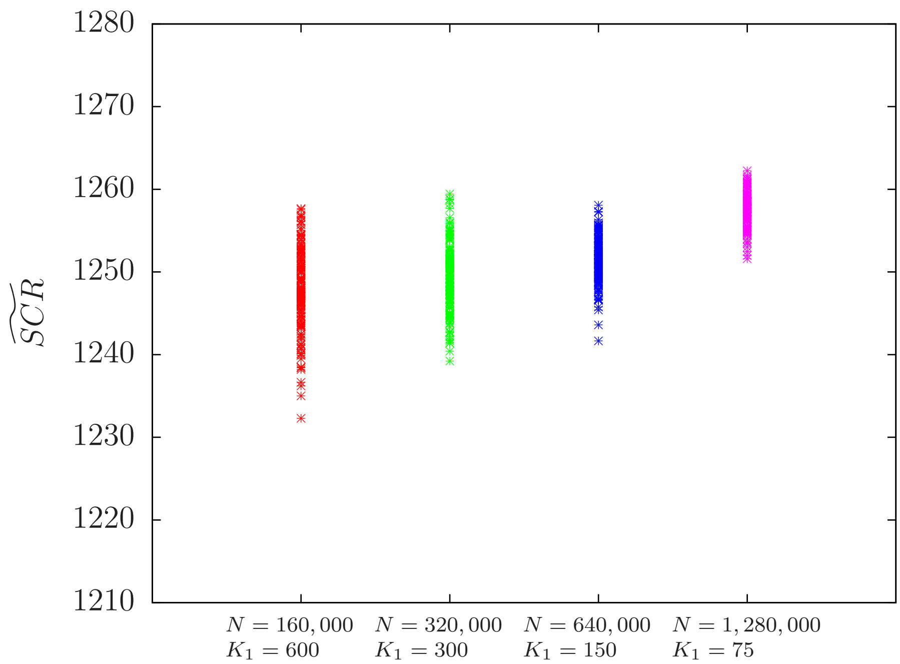

Figure 3: 150 simulations for different choices of N and K_1, $K_0 = 1,500,000$, Nested Simulations Approach

that if we correct the means in Table 3 by the corresponding bias, the difference between the results for the different combinations is almost negligible.

Therefore, we will use $N = 320,000$ and $K_1 = 300$ in the remaining part of this paper if not stated otherwise. With this parameter combination it takes about 16 minutes to carry out one run with our C++ implementation.[14] The bias corrected estimator $\widetilde{\mathrm{SCR}}_{\mathrm{cor}} = 1246.4$ shown above is the basis for comparisons with the LSM Approach.

6.2.2 Least-Squares Monte Carlo Approach

As we have illustrated in the previous paragraph, in order to obtain accurate results, the Nested Simulations Approach requires a large number of simulations and is hence very time-consuming. As a consequence, this approach may not be feasible for more complex specifications. For the Least-Squares Monte Carlo Approach, on the other hand, considerably less simulations are needed to obtain accurate results. However, the drawback of this method lies in the choice of the regression function.

Due to the construction of our contract and the asset model, the following variables are natural choices for the regressors:[15] A_1^+, r_1, L_1 and $x_1 = R_1/L_1$. Since we already have a good approximation of the desired distribution from the Nested Simulations Approach, we first choose the regression function on the basis of this knowledge. We use a bottom-up scheme starting with only one regressor; by analyzing the residuals, we successively add more regressors. Since clearly, lower variances $\sigma_1^{(i)}$, $1 \le i \le N$, result in a better least-squares estimate, we again use the estimator based on cash flows from the policyholders's perspective. Furthermore, we use $N = 320,000$ real-world scenarios and $K_0 = 1,500,000$. We perform 150 estimates of the SCR for each regression function. Subsequently, we compute the average of the 150 estimates. Table 4 shows our results for different regression functions.

#	Regression Function	Mean $(\widehat{\mathrm{SCR}})$
1	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1$	1007.3
2	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2$	1165.5
3	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1$	1272.6
4	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2$	1276.5
5	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1$	1233.2
6	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1$	1233.9
7	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1$ $+\hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$	1241.3
8	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1$ $+\hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1} + \hat{\alpha}_8^{(N)} \cdot L_1 \cdot e^{r_1}$	1244.5
9	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1$ $+\hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1} + \hat{\alpha}_8^{(N)} \cdot L_1 \cdot e^{r_1} + \hat{\alpha}_9^{(N)} \cdot e^{A_1/10000}$	**1245.9**

Table 4: Estimated SCR for different choices of the regression function, $K_0 = 1,500,000$, $N = 320,000$, LSM Approach

We find that the last two choices for the regression functions in Table 4 (8 and 9) approximate the value obtained via Nested Simulations quite well. In comparison to the result from the previous section, the differences are 1.9 and 0.5, respectively.

However, it is important to note that this insight in part is based on the Nested Simulations carried out previously. Alternatively, we may rely on the criteria introduced in Section 5.2. Even though underlying assumptions are not satisfied, we use Mallow's C_p to choose an appropriate model. The corresponding results and choices are displayed in Table 6 in the Appendix. We find that the lowest C_p is obtained when we choose 5 regressors. In this case, the average estimated SCR for 150 runs is 1245.9, i.e. although we have heteroscedasticity Mallow's C_p leads to a reasonable choice of the regression function. Thus, our results show that, on one hand, the choice of regressors appears to be of great importance since results deviate considerably when applying an arbitrary regression function. For example, regression function 4 in Table 4 yields an estimate considerably above the desired level, whereas the result of function 5 is significantly below. On the other hand, several variables appear to be highly correlated so that there does not seem to be a unique optimal choice, i.e. regressors may be substituted without losing accuracy. Therefore, we conclude that in order to obtain accurate results, it is important not to employ an arbitrary regression function, but it appears sufficient to rely on a roughly coherent method to determine a suitable choice.

In what follows, we use regression function 9 from Table 4 for further computations.

The major advantage of this method is that, on the same computer, it takes only approximately 25 seconds to estimate the SCR based on 320,000 real-world scenarios with the LSM Approach.

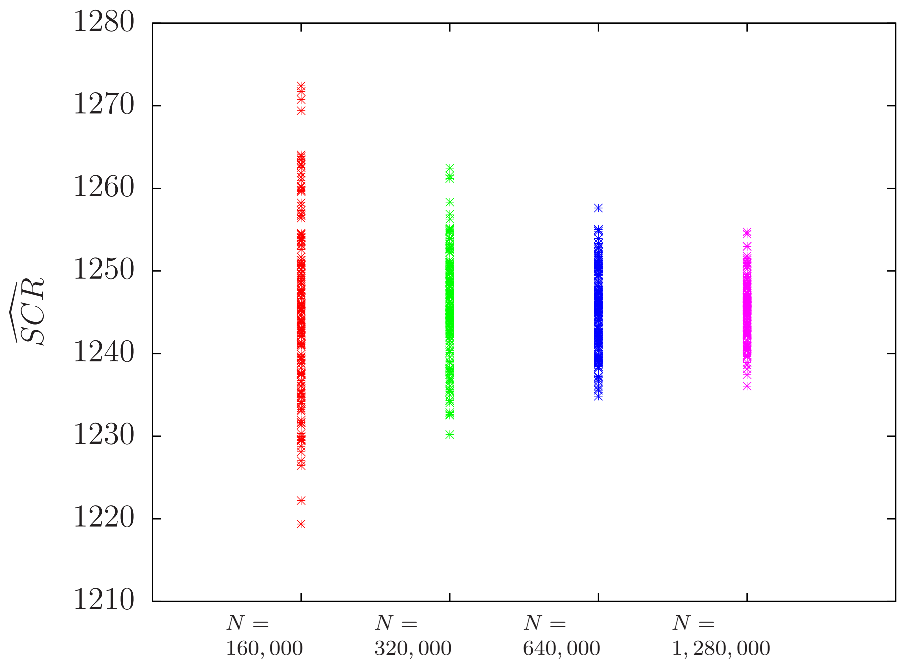

Figure 4: 150 simulations for different choices of N in the LSM Approach

In order to analyze the stability of the LSM estimator with respect to N, we carry out the simulation procedure 150 times for different numbers of real-world scenarios and again calculate the average of the estimated SCR. Figure 4 illustrates our results. Table 5 displays that the mean is quite stable and very close to the result from the Nested Simulations Approach. The empirical variance, on the other hand, is considerably higher than in the Nested Simulations Approach. However, one needs to keep in mind that we only need N sample paths for the time interval $(1, T]$ in the LSM Approach, whereas the Nested Simulations Approach requires $N \cdot K_1$ paths. Therefore, given the same computational constraint, we could employ far more real-world scenarios eventually yielding a significantly lower empirical variance.

Since we might also be interested in other quantiles or further information about the distribution such as alternative risk measures, we now analyze the

N	Mean $(\widehat{\text{SCR}})$	Empirical Variance	Solvency Ratio
160,000	1245.4	110.9	151%
320,000	**1245.9**	**39.1**	**151%**
640,000	1245.3	24.0	151%
1,280,000	1245.4	12.1	151%

Table 5: Results for the LSM estimator

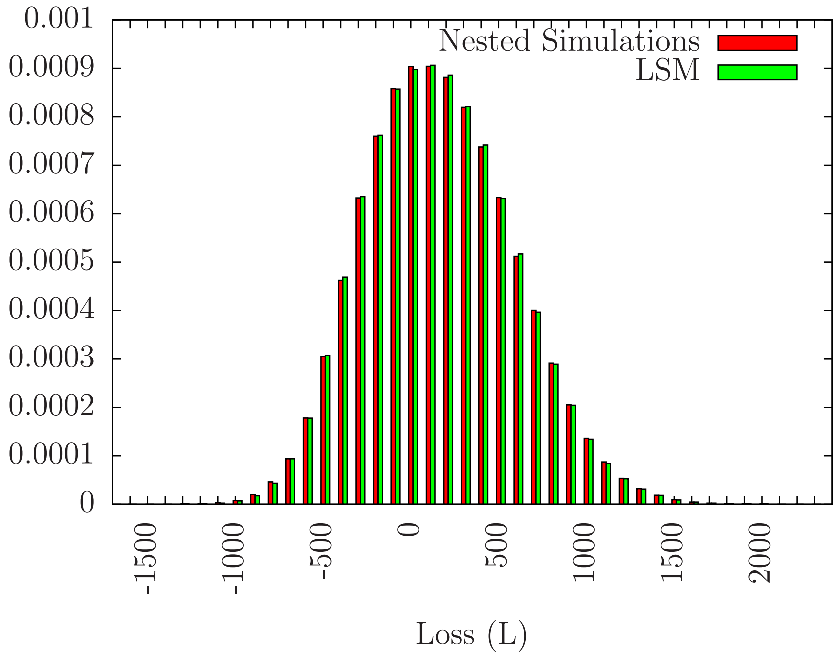

Figure 5: Empirical density functions for $N = 320,000$ and $K_1 = 300$

quality of the approximation of the whole distribution. Figure 5 shows the empirical density functions for the Nested Simulations Approach and the LSM Approach for one run with a fixed seed. We find that the two distributions are very similar and hence, the LSM Approach provides an efficient alternative to Nested Simulations.

Furthermore, in practice, the SCR needs to be calculated on a quarterly, monthly or even weekly basis for risk management purposes. In this case, one would like to avoid determining new regressors, but use the same regressors as in the preceding period instead. Therefore, it is interesting to analyze how small changes in the parameters influence the quality of the LSM estimate when using the same regressors as before.

One of the most important influencing factors in this model is the volatility

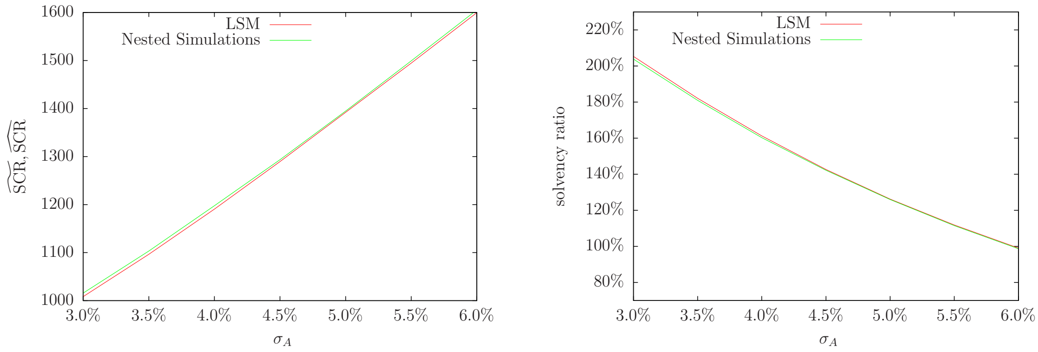

Figure 6: Variation of σ_A

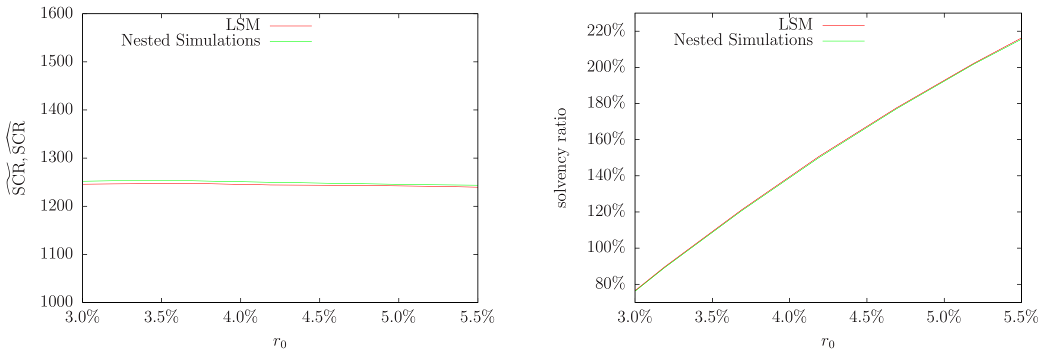

Figure 7: Shifts of the yield curve

σ_A of the asset process. Figure 6 shows the estimates for the two approaches for different choices of this volatility. Of course, the SCR increases in σ_A since a higher volatility imposes more risk on the insurance company. Moreover, we find that σ_A has a very strong impact on the estimated solvency ratio. Overall, we find that the LSM Approach is still quite close to the value resulting from Nested Simulations.[16]

Furthermore, the level of the yield curve has an impact on our estimates. Therefore, we shifted the whole yield curve, i.e. we increased or decreased the initial interest rate r_0 and the mean reversion level ξ by the same amount. In Figure 7, we observe that the SCR is almost constant when the yield curve is shifted. Obviously, both AC_0 and AC_1 increase for higher interest rates because the value of the guarantees decreases. But when we subtract the discounted AC_1 from AC_0 the absolute value of SCR is almost the same. However, an upwards shift of the yield curve has a positive impact on the insurance company's solvency ratio because AC_0 increases. Hence, the solvency ratio is significantly higher when the yield curve is shifted upwards. Again, we find that the LSM provides a good approximation.

7 Conclusion

In this paper, we give a detailed description how to determine the Solvency Capital Requirement within the framework of Solvency II. We present two different approaches how to numerically tackle the problem: a Nested Simulations Approach and a Least-Squares Monte Carlo (LSM) Approach. Based on numerical experiments, we find that the Nested Simulations Approach is very time-consuming and, moreover, the resulting estimator is biased. In contrast, the LSM Approach is more efficient and provides good approximations of the SCR, even though the significant impact of the choice of the regression function can be seen as a drawback for this method.

Another promising direction for future research is the combination of both approaches. By carrying out Nested Simulations with a small $K_1 > 1$ and, subsequently, applying a regression to estimate the loss function, we should be able to reduce the variance of the regressands and therefore, we expect to improve the LSM estimate. Furthermore, we intend to put a stronger focus on the relevant part of the distribution by employing an iterative scheme: A possible approach may be to sort the real-world scenarios with the help of very rough estimates and, then, to improve the estimates for the relevant scenarios in the tail. Hereby, we expect to obtain better estimates with the same (or even a smaller) number of simulations. Similar screening procedures have been used in [18] and [19] to estimate tail conditional expectation and expected shortfall, respectively. Moreover, we will try to derive confidence intervals for the SCR and we will analyze how variance reduction techniques can improve our results.

Finally, in future research, we intend to further explore pragmatic approaches as introduced in Section 4 to offer a valid alternative to current, suboptimal solutions. In the long run, however, we believe that advanced numerical approaches as presented here should allow for a computationally feasible and sufficiently accurate assessment of a life insurer's solvency position.

Notes

1 More specifically, if a company uses an *internal model*, the market value of liabilities is usually calculated using a Monte Carlo simulation approach. In some countries, so-called *standard models* are available, which estimate the market-consistent value of liabilities from some rough closed-form approximations.

2 More specifically, differences between the MCEV cost-of-capital (sum of frictional costs of required capital and cost of residual non-hedgeable risks) and the risk margin under Solvency II are ignored, and the eligibility of certain capital components (e.g. subordinated loans) is not considered here.

3 This simplification is equivalent to the definition used for the Swiss Solvency Test, see [21].

4 For an insurance company, the NAV is defined as the value of its assets less the value of its liabilities based on the statutory balance sheet, and therefore roughly coincides with the statutory shareholders' equity.

5 Since insurance contracts are usually long-term investments, T will in most cases be in the range of 30-40 years or even longer.

6 Under certain regularity conditions, there exists an risk-neutral probability measure if and only if the condition "No Free Lunch With Vanishing Risk" holds (see e.g. [5], Theorem 6.1.2).

7 For the sake of simplicity, in what follows we let $K_1^{(i)} = K_1$ for all $i \in \{1, \ldots, N\}$.

8 We disregard the cost for the generation of the N sample paths in the first period, since this effort is small compared to the effort for the Nested Simulations. Furthermore, we do not consider the fact that the sample paths for the estimation of AC_0 are one period longer than those for the estimation of AC_1 since in general T is relatively large.

9 Alternatively, given the marginal distributions, a risk manager may impose a certain dependence structure by choosing a copula function.

10 [4], p. 568

11 These earnings reflect the investment income on all assets, including the assets backing shareholders' equity R_t; this reduces the shareholders' ROI.

12 The REXP is a total return index of the German bond market.

13 The DAX is a total return index of the German stock market.

14 The simulations were carried out on a Windows machine with Intel Core 2 Duo CPU T7500, 2.20GHz and 2048 MB RAM. Of course, the computational time depends on our particular implementation; optimizations of the code may be possible.

15 While at time t=1, the state of the contract is entirely described by $Y_t = (A_t^+, r_t)$, this is not the case for later dates, where $D_t = (L_t)$ is necessary to represent the state of the contract. However, our first analyses show that the Least Squares algorithm performs similarly well in these situations.

16 Note that we only perform one run with a fixed seed for every parameter combination. Thus, due to the random sampling error it may happen that the LSM approach gives higher values than the Nested Simulations Approach. Also note that we did not correct the bias in the Nested Simulations Approach because the estimates for the bias resulting from only one run are not very exact.

A Appendix

# of reg	A_1	L_1	x_1	r_1	A_1^2	A_1^3	L_1^2	L_1^3	x_1^2	x_1^3	r_1^2	r_1^3	$A_1 \cdot L_1$	$A_1 \cdot x_1$	$A_1 \cdot e^{r_1}$	$L_1 \cdot e^{r_1}$	$x_1 \cdot e^{r_1}$	$A_1 L_1 \cdot x_1$	$A_1 \cdot x_1^2$	$L_1^2 \cdot x_1$	$A_1 \cdot e^{2r_1}$	$A_1^2 \cdot e^{r_1}$	$L_1 \cdot e^{2r_1}$	$L_1^2 \cdot e^{r_1}$	$A_1 x_1 \cdot e^{r_1}$	$A_1^2 x_1$	C_p
1	-	-	-	x	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	11822.99
2	-	-	-	-	-	x	-	-	-	-	-	-	-	-	-	-	-	-	-	-	x	-	-	-	-	-	513.91
3	-	-	-	x	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	x	-	x	52.13
4	-	-	-	x	-	-	-	-	-	-	x	-	-	-	-	-	-	-	-	x	-	x	-	-	-	-	7.90
5	-	-	x	-	-	x	-	-	-	-	x	-	-	-	-	-	-	-	-	-	x	x	-	-	-	-	3.72
6	x	-	-	-	-	x	-	-	-	-	x	-	x	-	-	-	-	-	-	-	x	x	-	-	-	-	4.33
7	x	-	-	-	-	x	-	-	x	-	x	-	-	-	-	-	x	-	-	-	-	-	-	x	x	-	4.54
8	-	-	-	x	-	x	-	-	-	-	x	-	-	x	-	x	-	x	-	-	x	x	-	-	-	-	5.48
9	x	x	-	-	-	x	x	-	-	-	x	-	x	-	-	-	-	-	-	x	x	x	-	-	-	-	6.44
10	x	x	-	-	-	-	x	x	-	-	-	-	x	-	-	x	x	-	x	x	x	-	-	-	-	-	5.35
11	x	x	-	-	x	-	x	x	-	-	-	-	x	-	-	x	x	-	x	-	x	-	-	-	-	x	3.90
12	x	x	-	-	x	-	x	x	-	-	-	-	x	x	-	x	x	-	x	-	x	-	-	-	-	x	4.98
13	x	x	-	-	x	-	x	x	-	x	-	-	x	-	-	x	x	x	x	-	x	-	-	-	-	x	6.17
14	x	x	x	-	x	-	x	x	-	-	-	-	x	x	-	-	x	x	x	-	-	x	x	x	-	-	6.91
15	x	x	-	-	x	-	x	x	-	x	-	-	x	x	x	-	-	-	-	x	x	x	x	x	x	-	7.54
16	x	x	-	-	x	-	x	x	-	x	-	-	x	x	x	-	x	-	-	x	x	x	x	x	x	-	9.03
17	x	x	-	x	x	x	x	x	-	x	x	-	x	-	-	x	-	-	x	x	x	x	-	x	x	-	10.51
18	x	x	-	x	x	x	x	x	-	x	x	-	x	-	-	x	-	-	-	x	x	x	x	x	x	x	12.05
19	x	x	-	-	x	-	x	x	x	x	x	x	x	x	-	-	-	-	x	x	x	x	x	x	x	x	13.92
20	x	x	-	-	x	x	x	x	x	x	x	x	x	x	-	-	-	-	x	x	x	x	x	x	x	x	15.86
21	x	x	x	x	x	x	x	x	x	-	x	x	x	x	x	x	x	x	-	x	-	x	x	-	-	x	17.59
22	x	x	x	x	x	x	x	x	-	x	-	-	x	x	x	x	x	x	x	x	x	x	x	-	x	x	19.37
23	x	x	x	x	x	x	x	x	-	x	-	x	x	x	x	x	x	x	x	x	x	x	x	-	x	x	21.24
24	x	x	x	x	x	x	x	x	x	-	x	x	x	x	x	x	x	x	x	x	x	-	x	x	x	x	23.13
25	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	-	x	x	x	x	x	x	x	x	25.04

Table 6: Choice of the regression function via Mallow's C_p

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