INTERNAL MODEL IN LIFE INSURANCE: APPLICATION OF LEAST SQUARES MONTE CARLO IN RISK ASSESSMENT

Version 1.7

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Abstract

In this paper we show how prospective modelling of an economic balance sheet using the least squares Monte Carlo (LSMC) approach can be implemented in practice. The first aim is to review the convergence properties of the LSMC estimator in the context of life assurance. We pay particular attention to the practicalities of implementing such a technique in the real world. The paper also presents some examples of using the valuation function calibrated in this way.

KEYWORDS: Solvency II, Economic capital, Economic balance sheet, risk management, risk appetite, life insurance, stochastic models, least squares Monte Carlo

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1 INTRODUCTION

One of the key advances of Solvency II over Solvency I is that insurance companies' assets and liabilities must be valued at economic or "fair value" (see Solvency II, article 75). Fair value is the amount for which an asset could be exchanged or a liability settled between knowledgeable, willing parties in an arm's length transaction. The valuation principles in the insurance context are set out by WÜTHRICH et al. [2008].

The Solvency II standards thus take an economic view of the balance sheet and introduce a harmonized European view of economic capital that represents the minimum capital requirement to give an insurance or reinsurance company 99.5% confidence of surviving a situation of economic ruin on a one-year horizon. Economic capital can be estimated using either a modular approach (standard formula) or by (partial) internal modelling. The latter involves a finer analysis of the company's risks and requires the distribution of capital consumption to be defined over a one-year horizon.

Also, Solvency II encourages companies to develop a more detailed approach to risk management. Article 45 of the Solvency II Directive sets the rules for this internal risk management. The framework for personalised risk management is the *Own Risk and Solvency Assessment* (ORSA), which is based on the identification, definition and monitoring of the company's key risk indicators[1].

In addition, for financial reporting purposes, insurance companies value their business using the *Present Value of Future Profits* (PVFP). This is based on the portfolio of insurance policies written at the calculation date, taking into account all the contractual obligations that flow from them and including the value of any embedded options.

Finally, insurance companies must draw up a business strategy for a defined horizon. This strategy must project over the forecast horizon, based on a realistic set of assumptions:

- the economic balance sheet and solvency capital requirement,
- IFRS profit before tax and IFRS Balance Sheet.

There are, then, many and various issues relating to a better understanding of risk. Valuations at t=0 can be based on the Monte Carlo approach and generally pose no major technical problem. However, the forward-looking projection is much trickier and raises real challenges (see DEVINEAU and LOISEL [2009]). There is no closed formula for valuing options in insurance liabilities and the economic value of the balance sheet depends on the information available at the time of valuation: it is therefore random.

The purpose of this paper is to show how prospective modelling of an economic balance sheet using the least squares Monte Carlo (LSMC) approach is implemented in practice, making it possible to estimate the prospective value of its components. The LSMC technique is already used in the financial world to value exotic options (see LONGSTAFF and SCHWARZ [2001]). The first aim is to analyse the convergence properties of the LSMC estimator in the context of insurance as discussed by BAUER et al. [2010]. We pay particular attention to the practicalities of implementing such a technique in the real world. The paper also presents examples of the use of the evaluation function calibrated in this way. Section 2 reviews the difficulties of

[1] EIOPA Final Report on Public Consultation No. 13/009 on the Proposal for Guidelines on Forward Looking Assessment of Own Risks (ref.: EIOPA/13/414)

implementing nested scenarios and summarises possible solutions, including LSMC techniques. Section 3 describes the LSMC method, discusses the convergence properties of the approach and emphasises the issues with practical implementation. Section 4 presents an application of LSMC to the most common savings contract sold in France: the euro fund.

2 FROM NESTED SCENARIOS TO LEAST SQUARES MONTE CARLO

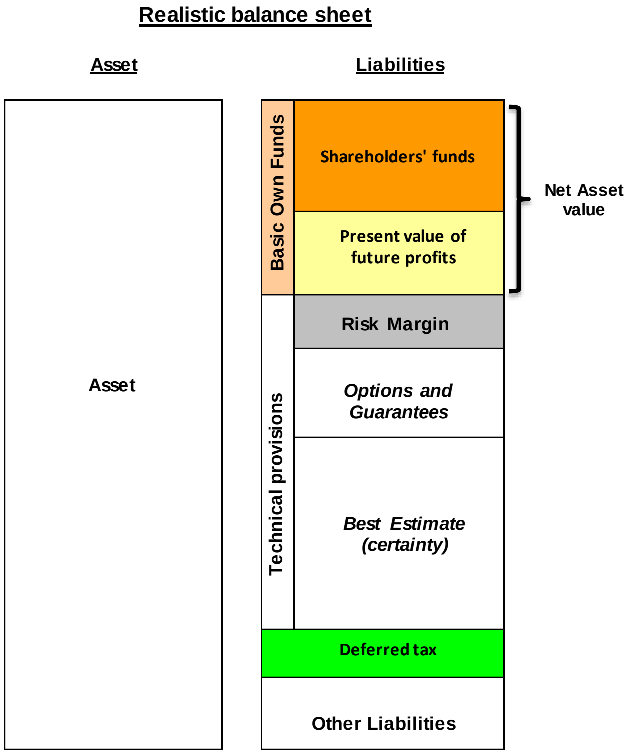
A better understanding of portfolio risk means being able to anticipate how the economic balance sheet will react to certain identified risk factors.

For instance, for any component C (net asset value, best estimate, etc.) of the economic balance sheet opposite, value at time *t* is written:

$$C(F_t) = E_{Q_t}\left[\sum_{u=t+1}^{T} f_u(C)\times DF(t,u)\middle| F_t\right]$$

where:

- F_t represents the vector of risk factors (yield curve, equity index, lapses rate, etc.) at time *t*,
- Q_t is a risk-neutral measure at time *t*.
- $f_u(C)$ is the cash flow associated with component C at the time $u \geq t$,
- $DF(t,u)$ is the discount factor for cash-flows over the period $u \geq t$.

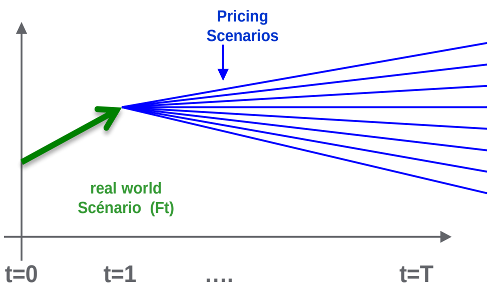


In practice: there is no analytic formula for this conditional expectation as the terms of insurance contracts contain a number of embedded options (rate guarantee, profit sharing constraint, surrender option, etc.) because the cash flows $f_u(C)$ are path-dependent and there are interactions between liabilities and assets.

$C(F_t)$ can be estimated using a Monte Carlo approach by:

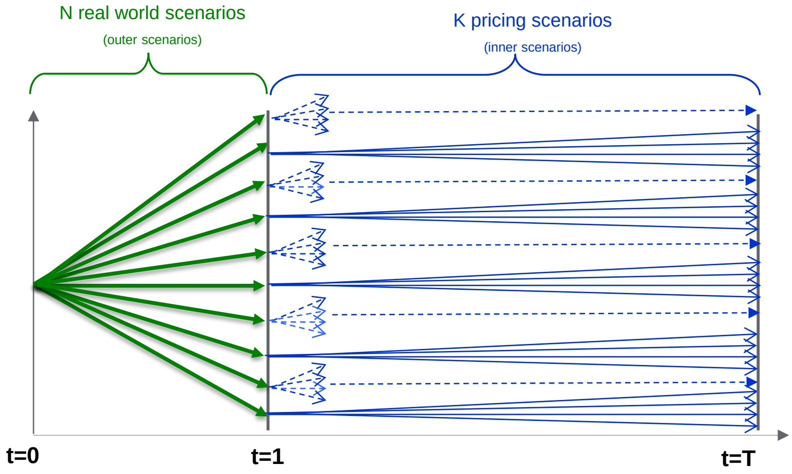
$$C(F_t) \approx \hat{C}_K(F_t) = \frac{1}{K}\sum_{k=1}^{K}\left[\sum_{u=t+1}^{T} f_u^k(C)\times DF^k(t,u)\middle| F_t\right]$$

IFERGAN [2013] shows that $\hat{C}_K(F_t)$ is a convergent estimator for calculation of the best estimate.



Usually, this method is used to value the balance sheet at t=0: the economic value of the assets is observed on the market and the liabilities are valued by Monte Carlo simulation using the valuation principles defined by EIOPA, IFRS or internal guidance. When $t \geq 1$, F_t and $C(F_t)$ become random variable. To determine the distribution of $C(F_t)$, we can use the "nested scenarios" approach:

When using Monte Carlo method to estimate a large number N of $C(F_t)$, $\{C(F_t^n), n = 1...N\}$ we talk about nested scenarios.



This approach has the advantage to give an initial estimate of the empirical distribution of the economic balance sheet and to give precise results for the real-world situations analysed. But, on the other side, it has several drawbacks:

- Heavy demands on processing resources (N x K simulations):
- calculating time: 1 month for 1,000 x 1,000 at 3 seconds per scenario,
- storage space: 250 GB for 1,000 x 1,000,
- Robustness of tail distribution: just 5 scenarios determine VAR 99.5% for 1,000 x 1,000 scenarios
- No information on points between the outer scenario.

Several solutions have been developed to get round these difficulties:

- REVELEN [2011] sets out the replication approaches. These techniques struggle with the complexity of life insurance contracts (long duration, redemption options, profits sharing constraint, etc.),
- DEVINEAU and LOISEL [2009] describe an acceleration algorithm that can be applied when using nested scenarios to calculate Solvency II economic capital. The algorithm works by reducing the number of outer scenarios. They are particularly interested in tail distribution. The focus of this method is on estimating economic capital and it is hard to apply to the mechanics of risk management and portfolio valuation.
- NTEUKAM and PLANCHET [2012] are interested in cutting the number of inner scenarios and show that valuation error can be cut to less than 5% when inner scenarios are replaced by a few well-chosen composite scenarios.
- BONNIN et al. [2014] try to approximate the market value of liabilities using analytical formulas,
- BAUER et al. [2010] are the first to set out a detailed and documented application of LSMC in life insurance. However, the authors encounter problems in showing the convergence of the LSMC estimator due to the change in probabilities at t=1. The

authors also measure the impacts on a fictional portfolio and fail to address the problems applying these techniques to a real-world portfolio (complexity of liabilities and assets, processing time, storage space, calculation tools, etc.). Finally, the mechanism for selecting the regression base is not explained in detail.

- Another approach that could overcome some of the problems with nested stochastics is to interpolate the results of the nested scenarios: so-called curve fitting.

Of all these techniques, the LSMC approach is emerging as the standard for internal modelling in the life insurance industry, for several reasons:

- it can estimate the value of economic capital,
- it can be used to manage risk (ORSA): risk hedging, calculation of risk appetite indicators,
- it is used in ALM studies: to determine optimal allocation, project portfolios, etc.

Below, we present the application of LSMC for a portfolio of life insurance contracts.

3 DESCRIPTION OF THE LSMC METHOD

The core idea of the least squares Monte Carlo (LSMC) method is to mimic the behaviour of the liabilities using a function that includes all targeted risk factors as inputs (economic and/or non-economic variables).

The precise behaviour (or pricing) function of the liabilities is unknown. It is approximated by an approach based on the Taylor series approximation. This technique approximates the behaviour of the function by a linear combination of basis functions applied to the targeted risk factors:

$$C(F_t) \cong \sum_{i=1}^{M} \alpha_i \times L_i(F_t)$$

- $M \in \aleph^*$ is the number of regressors,
- $F_t = \left(F_t^1, \ldots, F_t^k\right)$ represents the vector for the risk factors at time t,
- $k \in \aleph^*$, total number of targeted risk factors,
- $L(\ ) = \left(L_1(\cdot), L_2(\cdot), \cdots L_M(\cdot)\right)$ represents a series of functions (the “regression basis”),
- α_i represents the impact of the term $L_i(F_t)$ on the quantity $C(F_t)$.

The idea[2] beyond this approximation derives from the properties of conditional expectations in L^p-spaces, which are specific Hilbert spaces with countable orthonormal basis[3]. More precisely, because the conditional expectancy $E\left(Y \middle| Z_i, i \in I\right)$ minimize the Euclidian distance between the random variable $Y \in L^2$, it can be computed as the orthogonal projection of Y on the subspace generated by the random variables $Z_i, i \in I$. Moreover, because, in the applications considered here the factors Z_i are risk factors affecting the balance sheet of the

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[2] For a theoretical justification, see YOSIDA [1980] for the functional analysis part and NEVEU [1964] for the definition and properties of conditional expectation.

[3] See BECK et al. [2005] or YOSIDA [1980] p. 90.

insurer, one can assume, without restriction, that the set I is countable. For this reason, we will consider in the rest of this paper the particular case of conditional expectancy with respect to a subspace generated by a countable set of random variables.

Because of this assumption, $C(F_t) = E_{Q_t}\left[\sum_{u=t+1}^{T} f_u(C) \times DF(t,u) \middle| F_t\right]$, which is in L^2, can thus be expressed as a linear combination of a countable set of orthogonal functions measurable in L^2

$$C(F_t) = \sum_{i=1}^{\infty} \alpha_i \times L_i(F_t)^t$$

where $\alpha = (\alpha_i)_{i \in \{1,\cdots,\infty\}}, \alpha_i \in \Re$ are real numbers and $L(\ ) = (L_1(\cdot), L_2(\cdot), \cdots)$ an orthogonal basis[4] in L^2. If we choose $M \in \aleph^*$ a strictly positive natural integer, we can write:

$$\begin{aligned} C(F_t) &= \sum_{i=1}^{M} \alpha_i \times L_i(F_t) + \sum_{i=M+1}^{\infty} \alpha_i \times L_i(F_t) \\ &= C_M(F_t) + \varepsilon(M) \end{aligned}$$

with:

- $C_M(F_t) = \sum_{i=1}^{M} \alpha_i \times L_i(F_t) = L^M(F_t) \bullet \alpha_M{}^t$,
- $\varepsilon(M) = \sum_{i=M+1}^{\infty} \alpha_i \times L_i(F_t)$,
- $L^M(\cdot) = (L_1(\cdot), L_2(\cdot), \cdots, L_M(\cdot))$,
- $\alpha_M = (\alpha_i)_{i \in \{1,\cdots,M\}}, \alpha_i \in R$.

We have $\varepsilon(M) \to 0$ in L^2 when $M \to +\infty$, we arrive at the approximation $C_M(F_t) \cong C(F_t)$. The α_M coefficients are then estimated in two stages:

1. Using the *Monte Carlo* method we generate N realisations of the random variables[5] $\left\{\left(\mathbf{C}\left(F_t^n\right), F_t^n\right), n = 1\ldots N\right\}$,
2. we calculate $\overline{\alpha}_M^N$ by least squares regression of $\mathbf{C}\left(F_t^n\right)$ in $L^M\left(F_t^n\right)$

$$\overline{\alpha}_M^N :\equiv \underset{\alpha_M}{\mathbf{arg\,min}} \sum_{n=1}^{N} \left(\mathbf{C}\left(F_t^n\right) - L^M\left(F_t^n\right) \bullet \alpha_M^t\right)^2$$

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[4] For $i = 1,\ldots,k$ $L_i(\cdot)$ is defined by $\begin{aligned} L_i(\cdot) &: \Re^{\mathrm{k}} \to \Re \\ F_t &\mapsto L_i(F_t) \end{aligned}$

[5] The $\mathbf{C}\left(F_t^n\right)$ are realisations of the random variable $z(F_t) = \sum_{u=t+1}^{T} f_u(C) \times DF(t,u) \Big| F_t$ such that $C(F_t) = E_{Q_t}\left[z(F_t)\right]$

The result of this optimisation programme is the OLS estimator:

$$\overline{\alpha}_M^N = \left[L^{M,N}\left(F_t^N\right)^t \bullet L^{M,N}\left(F_t^N\right)\right]^{-1} \bullet L^{M,N}\left(F_t^N\right)^t \bullet \mathbf{C}^N\left(F_t^N\right)^t$$

with the M rows and N columns matrix:

- $L^{M,N}\left(F_t^N\right) = \left(L^M\left(F_t^1\right),\ldots,L^M\left(F_t^N\right)\right) = \left(L_k\left(F_t^n\right)\right)_{\substack{1\le k\le M\\ 1\le n\le N}}$ is an $M \times N$ matrix,
- $\mathbf{C}^N\left(F_t^N\right) = \left(C\left(F_t^1\right),\ldots,C\left(F_t^N\right)\right)$,

The LSMC function can be written
$$\begin{aligned} C_M^{LS,N}\left(F_t\right) &= \sum_{i=1}^{M} \overline{\alpha}_i^N \times L_i\left(F_t\right) \\ &= L^M\left(F_t\right) \bullet \overline{\alpha}_M^N \end{aligned}$$

3.1 EXAMPLE

Here, we consider an at-the-money European call option with maturity T=2. We assume the underlying is a geometrical Brownian motion, the risk-free rate is constant at 3.5% and the underlying's implied volatility is 30%. The price of this European option is derived by the Black-Scholes formula:

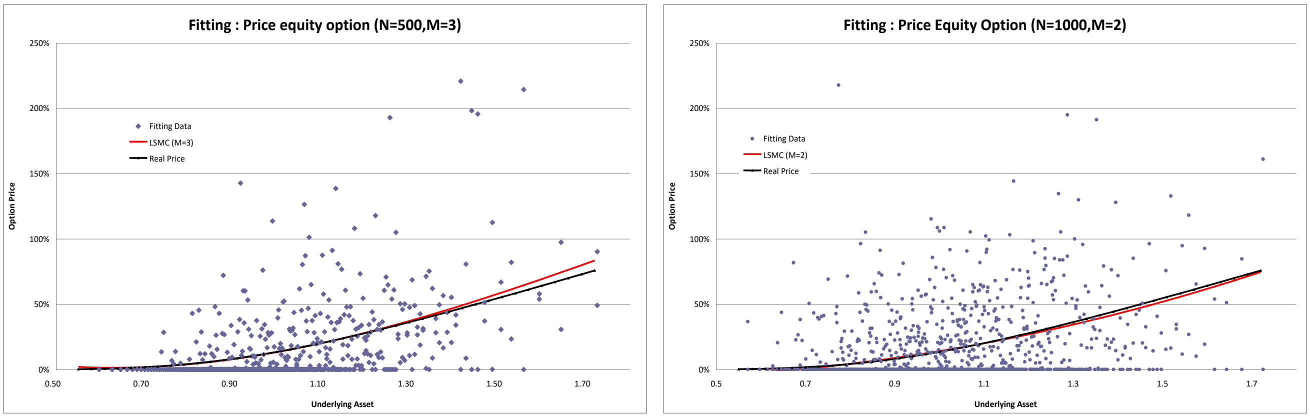
$$S_t\Phi\left(d_1\right) - K\,\mathbf{exp}\left(-r\left(T-t\right)\right)\Phi\left(d_2\right)$$
$$avec \begin{cases} d_1 = \dfrac{1}{\sigma\sqrt{T-t}}\left[\mathbf{ln}\left(\dfrac{S_t}{K}\right) + \left(r + \dfrac{1}{2}\sigma^2\right)\left(T-t\right)\right] \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{cases}$$

The table shows two examples of LSMC functions (defined with the stock price $x = S_t$) for the option price at t = 1:

i	Li(x)	LSMC (N=500,M=3)	LSMC (N=1000,M=2)
0	1	0.46	- 0.03
1.00	x^1	- 1.52	- 0.21
2.00	x^2	1.45	0.39
3.00	x^3	- 0.25	-

The charts below compare the LSMC results to the Black-Scholes calculation.

Fig. 1. B&S vs LSMC approximation



Note the greater precision of the LSMC estimator when the number of fitting data is large.

3.2 ISSUES WITH THE LEAST SQUARES MONTE CARLO METHOD

In practice, applying LSMC to life insurance raises a number of issues.

First, we need to identify characteristic risk factors F_t. Many risk factors may affect an insurance firm's economic balance sheet, both economic risks (rate risk, equity risk, property risk, credit risk, volatility, etc.) and non-economic risks (Lapse risk, mortality risk, expense risk, etc.). However, it is enough to target a limited number of risk factors. The LSMC function derived in this context gives an estimate of liabilities when only the targeted risk factors are random: it is therefore often called a ***partial internal model***.

The choice of regression basis function $L_i(F_t)$ can have non-negligible effects on tail distribution. In a one-dimensional environment, there are many functions that have the property of orthogonality: Chebyshev, Hermite, Laguerre, Legendre, etc. (see ABRAMOWITZ and STEGUN [1964]).

In general, we analyse the impact of k > 1 risk factors. To correctly measure the interaction of risk factors on the balance sheet it is important to specify the form of the orthogonal functions defined in multi-dimensional space. We have to choose the optimal dimension M to give the best approximation of $C(F_t)$? In practice, it may prove impossible to calibrate the LSMC function when the number of regressors is very high.

The choice of the methods can be used to determine optimal α_i coefficients touches on two issues:

- definition of fitting data or calibration: how many calibration scenarios N does it take to get a best approximation of $C(F_t)$ and how generate of the fitting data: Monte Carlo or quasi-Monte Carlo to accelerate convergence of the LSMC simulation,
- Optimisation of the regression function: backward, forward, stepwise, etc.

3.3 CONVERGENCE OF THE LSMC

3.3.1 CONVERGENCE OF THE LSMC UNDER THE RISK-NEUTRAL MEASURE Q_t

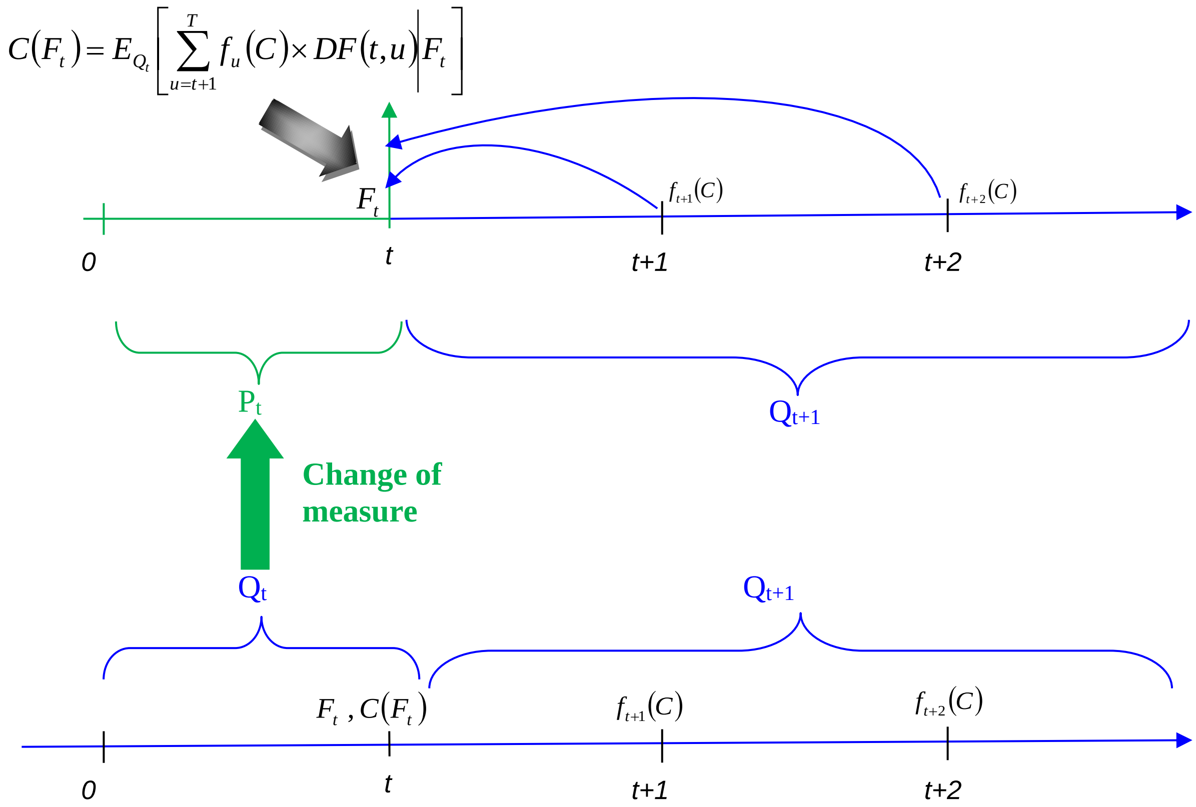
We show that $C_M^{LS,N}(F_t)$ converges in L^2 towards $C(F_t)$ under the risk-neutral measure Q_t. The convergence is demonstrated in two stages:

$$C_M^{LS,N}(F_t)=\sum_{i=1}^{M}\overline{\alpha}_i^N \times L_i(F_t)\xrightarrow[L^2]{N\to\infty}\sum_{i=1}^{M}\alpha_i \times L_i(F_t)\xrightarrow[L^2]{M\to\infty}\sum_{i=1}^{\infty}\alpha_i \times L_i(F_t)=C(F_t)$$

The first stage is based on the convergence of the Monte Carlo estimator. It can be simply demonstrated using the law of large numbers and properties of orthogonal functions (see appendix) and the second stage is obvious (see also Bauer et al [2010] section 5.3). As mentioned in the very beginning of this section, this result is thru only because we assumed that the set of risk factors is countable.

3.3.2 CONVERGENCE OF THE LSMC UNDER THE HISTORICAL MEASURE P_t

In insurance, the distribution of risk factors F_t is under the historical measure P_t, implying that the distribution of $C(F_t)$ is obtained under P_t.



We therefore need to establish the convergence properties under the historical measure. BAUER and al. [2010] specify that because of the change in measure at time *t*, the convergence of $C_M^{LS,N}(F_t)$ toward $C(F_t)$ under the measure P_t cannot be guaranteed.

In this section we will show that $C_M^{LS,N}(F_t)$ converges in probability toward $C(F_t)$ under the measure P_t. True, convergence in probability is weaker than L^2 convergence, but it is still stronger than convergence in distribution and enough to demonstrate the relevance of the LSMC approach to valuing economic capital.

Property: $C_M^{LS,N}(F_t)$ converges in probability toward $C(F_t)$ under the historical measure P_t

Proof:

In section 3.3.1, we showed that $C_M^{LS,N}(F_t) \xrightarrow[L^2]{N,M\to\infty} C(F_t)$ under the measure Q_t. This implies that $C_M^{LS,N}(F_t)$ converges in probability toward $C(F_t)$ under the measure Q_t.

$$C_M^{LS,N}(F_t) \xrightarrow[Q_t]{N,M\to\infty} C(F_t) \Leftrightarrow \lim Q_t(A_{N,M}) = 0$$

with $A_{N,M} = \left\{\omega; \left|C_M^{LS,N}(F_t)(\omega) - C(F_t)(\omega)\right| > \xi\right\}$ and $\xi > 0$.

The probability measures Q_t and P_t are equivalent $\Leftrightarrow \exists f \in \mathrm{L}^2$, $f > 0$, $dP_t = f \times dQ_t$, and f a random variable with $\int f \times dQ_t = 1$:

$$P_t(A_{N,M}) = \int f 1_{A_{N,M}} \times dQ_t.$$

We have $f 1_{A_{N,M}} \leq f$, Lebesgue's dominated convergence theorem[6] implies that:

$$\begin{aligned}\mathbf{lim}\ P_t(A_{N,M}) &= \mathbf{lim} \int f 1_{A_{N,M}} \times dQ_t \\ &= \int \mathbf{lim}\ f 1_{A_{N,M}} \times dQ_t \\ &= \int 0 \times dQ_t \\ &= 0\end{aligned}$$

$\mathbf{lim}\ f 1_{A_{N,M}} = 0$ because $C_M^{LS,N}(F_t)$ converges in probability toward $C(F_t)$ under the measure Q_t and the only possible values for $1_{A_{N,M}}$ are 0 and 1. Thus $C_M^{LS,N}(F_t)$ converges in probability toward $C(F_t)$ under the measure P_t □

3.4 CHOICE OF NUMBER OF SIMULATIONS AND NUMBER OF REGRESSORS

i.e. N et $\mathrm{M} \in \aleph^*$: the LSMC function that approximates the value of component C of the economic balance sheet at time *t* is:

$$C_M^{LS,N}(F_t) = L^M(F_t) \times \overline{\alpha}_M^{N\,t}$$

[6] If the sequence (f_n) converges pointwise to a function *f* and is dominated by some integrable function *g* in the sense that $|f_n| \leq g$ for all numbers *n* in the index set of the sequence, then *f* is integrable and the integral of f_n converge towards *f*.

We showed in the previous section that $C_M^{LS,N}(F_t)$ converges toward $C(F_t)$. Now, we are interested in fixed N et $M \in \aleph^*$, with an error between $C_M^{LS,N}(F_t)$ and $C(F_t)$:

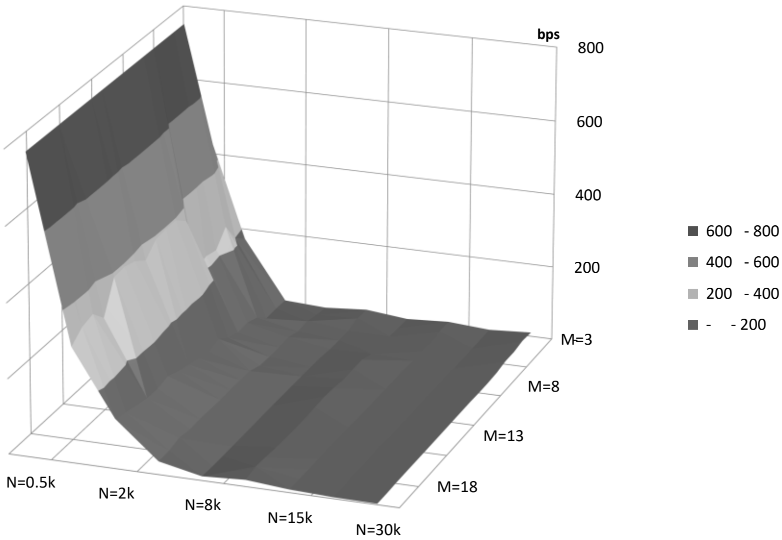
$$\varepsilon(N,M) = \left\| C_M^{LS,N}(F_t) - C(F_t) \right\|^2.$$

In absolute terms, this function does not have an optimum. Also, except in particular cases (financial assets) we do not know $C(F_t)$, so $\varepsilon(N,M)$ is hard to quantify[7]. We can estimate it by measuring the deviation between $C_M^{LS,N}(F_t)$ and the results of nested scenarios. But this approach suffers from the major disadvantages of the nested scenarios approach (see section 2).

In practice, we measure the deviation between the $C_M^{LS,N}(F_t)$ function and a series of values for $C(F_t)$: we call these validation scenarios. The validation scenarios are chosen from the distribution set of $C(F_t)$. In general, some twenty points are enough to measure the quality of the LSMC function.

The chart below shows the estimation error (sum of square of deviations) for the price of a European option from using the LSMC method:

Fig. 2. SSE on validation scenario



Note that the convergence is faster when N is very large:

- When $N = 10{,}000$, the sum of the square of pricing errors falls to 0.04% for polynomial degree $M = 3$.
- When $N = 2{,}000$, the sum of the square of errors is always more than 0.13% irrespective of the degree of the polynomial M.

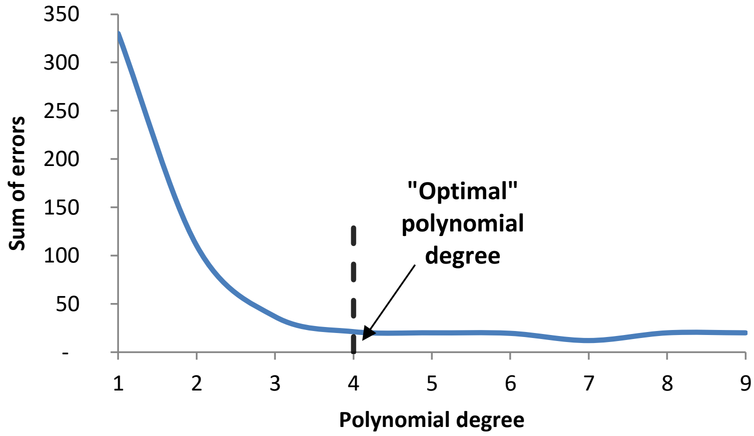
[7] The value $\min_{\alpha_M} \sum_{n=1}^{N} \left(\mathbf{C}\left(F_t^n\right) - L^M\left(F_t^n\right) * \alpha_M^t \right)^2$ is not an estimator of $\varepsilon(N,M)$ as $\mathbf{C}\left(F_t^n\right)$ are realisations of the random variable $z = \sum_{u=t+1}^{T} f_u(C) \times DF(t,u) \Big| F$ and not values $C(F_t) = E_{Q_t}[z]$

So, we fix a maximum acceptable error $\mathrm{E}_{\max}$, and try to determine $\left(N_{E_{\max}}, \mathrm{M}_{E_{\max}}\right)$ such that $\left\|C_{M_{E_{\max}}}^{LS,N_{E_{\max}}}(F_t) - C(F_t)\right\|^2 \leq \mathrm{E}_{\max}$. To achieve this error we must fix N as high as possible:

- Depending on calculation and storage capacity: in general[8] 100,000 scenarios provide good convergence of the LSMC result,
- We combine this with variance reduction techniques to achieve a better convergence property.

Having fixed the number of simulations, we choose the “optimal” polynomial degree by measuring the sum of the squares of errors observed in the validation scenarios:

Fig. 3. Choice of the optimal degree



3.5 REGRESSION BASIS FUNCTIONS

3.5.1 ONE-DIMENSIONAL ORTHOGONAL BASIS FUNCTIONS

LONGSTAFF and SCHWARTZ [2001] propose using orthogonal polynomials such as Laguerre, Legendre or Chebyshev polynomials, weighted with a falling exponential term (to prevent the polynomials exploding to infinity). A system of $L_n(x)$ functions is orthogonal over the interval $[a,b]$ with weighting function $w(x)$ if it verifies the following property:

$$\int_a^b w(x) \times L_n(x) \times L_m(x)\,dx = \begin{cases} 0, si\ n \neq m \\ C_n, si\ n = m \end{cases}$$

[8] This number strongly depends on the number of risk factors.

Fig. 4. Examples of orthogonal functions

Orthogonal basis	$[a,b]$	$w(x)$	$L_n(x)$	C
Hermite	$]-\infty,+\infty[$	$\exp(-x^2/2)$	$He_n(x) = -(1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{x^2/2}$	$\frac{2}{2n+1}$
Chebyshev	$]-1,1[$	$\frac{1}{\sqrt{1-x^2}}$	$T_n(\cos(\theta)) = \cos(n\theta)$	$\begin{cases} \pi : n = m = 0 \\ \frac{\pi}{2} : n = m \neq 0 \end{cases}$
Legendre	$]-1,1[$	1	$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$	$\sqrt{2\pi}n!$
Laguerre	$[0,+\infty[$	$\exp(-x)$	$L_n(x) = \frac{e^x}{n} \frac{d^n}{dx^n}\left(e^{-x}x^n\right)$	1

ABRAMOWITZ and STEGUN [1964] present other examples of orthogonal functions. The basis functions used in our study are weighted by their weighting function to prevent them exploding to infinity.

It is important for the regression function to be orthogonal, to make sure that the regression program detailed at the very beginning of this section have a solution. Thus $L^{M,N}\left(F_t^N\right)^t \bullet L^{M,N}\left(F_t^N\right)$ have to be invertible.

3.5.2 MULTI-DIMENSIONAL ORTHOGONAL BASIS FUNCTIONS

In general, we analyse the impact of k > 1 risk factors. In this context, it is important to specify the form of the orthogonal functions defined in $\Re^k$. In a Brownian environment where the risk factors being analysed are independent, a simple approach is to generalise the orthogonal functions defined in $\Re$. Given a system of orthogonal functions $L(x) = (L_1(x), \cdots L_n(x) \cdots)$ over $[a,b]$, for $n_1, n_2, ..., n_k \in \aleph^*$ we have:

$$P_{n_1,n_2,\dots n_k}(\cdot) : [a,b]^k \to \Re$$
$$F_t \mapsto P_{n_1,n_2,\dots n_k}(F_t) = \prod_{i=1}^{k} L_{n_i}\left(F_t^i\right)$$

In dimension 2, it is clear that: $\int_a^b P_{n,m} \times P_{n',m'}(x) \times w(x_1) w(x_2) dx_1 dx_2 = \begin{cases} 0, \text{ if } (n,m) \neq (n',m') \\ C_n, \text{ if not} \end{cases}$.

In the Brownian case the weighted Hermite function $wHe_n(x) = \frac{(2n+1) \times He_n(x)}{2}$ verifies the orthogonal property by virtue of the form of the weighting function $\exp(-x^2/2)$. It remains to be shown whether this result can be generalised, as the risks analysed are not always mutually independent nor Brownian ones.

In our study, we are however going to use the set of functions $\left\{ P_{n_1,n_2,\ldots n_k}(\cdot), \sum_{j=1}^{k} n_j \le n \right\}$ for $n \in \aleph^*$ and a fixed one-dimensional basis function $L(x) = (L_1(x), \ldots, L_n(x) \ldots)$. The table below shows an example application in dimension 3 for the first 3 terms in the univariate function.

Fig. 5. Orthonormal basis function of dimension 3

3 Risk factors			New orthonormal basis function, up to 3 degrees
x1	x2	x3	
0	0	1	P_0_0_1(x1, x2, x3)=L1(x3)
0	1	0	P_0_1_0(x1, x2, x3)=L1(x2)
1	0	0	P_1_0_0(x1, x2, x3)=L1(x1)
0	1	1	P_0_1_1(x1, x2, x3)=L1(x2)L1(x3)
1	0	1	P_1_0_1(x1, x2, x3)=L1(x1)L1(x3)
1	1	0	P_1_1_0(x1, x2, x3)=L1(x1)L1(x2)
0	0	2	P_0_0_2(x1, x2, x3)=L2(x3)
0	2	0	P_0_2_0(x1, x2, x3)=L2(x2)
2	0	0	P_2_0_0(x1, x2, x3)=L2(x1)
0	1	2	P_0_1_2(x1, x2, x3)=L1(x2)L2(x3)
0	2	1	P_0_2_1(x1, x2, x3)=L2(x2)L1(x3)
1	0	2	P_1_0_2(x1, x2, x3)=L1(x1)L2(x3)
2	0	1	P_2_0_1(x1, x2, x3)=L2(x1)L1(x3)
1	2	0	P_1_2_0(x1, x2, x3)=L1(x1)L2(x2)
2	1	0	P_2_1_0(x1, x2, x3)=L2(x1)L1(x2)
0	0	3	P_0_0_3(x1, x2, x3)=L3(x3)
0	3	0	P_0_3_0(x1, x2, x3)=L3(x2)
3	0	0	P_3_0_0(x1, x2, x3)=L3(x1)
1	1	1	P_1_1_1(x1, x2, x3)=L1(x1)L1(x2)L1(x3)

We find that there are 19 terms in this multi-dimensional function. The table below shows the number of terms in the multidimensional function as a function of the number of risk factors and degree of the one-dimensional function:

Fig. 6. Maximum number of regressors

Maximum number of regressors								
Risk Factor / Polynomial degree	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	4	5	6	7	8
2	2	5	9	14	20	27	35	44
3	3	9	19	34	55	83	119	164
4	4	14	34	69	125	209	329	494
5	5	20	55	125	251	461	791	1286
6	6	27	83	209	461	923	1715	3002
7	7	35	119	329	791	1715	3431	6434
8	8	44	164	494	1286	3002	6434	12869
9	9	54	219	714	2001	5004	11439	24309

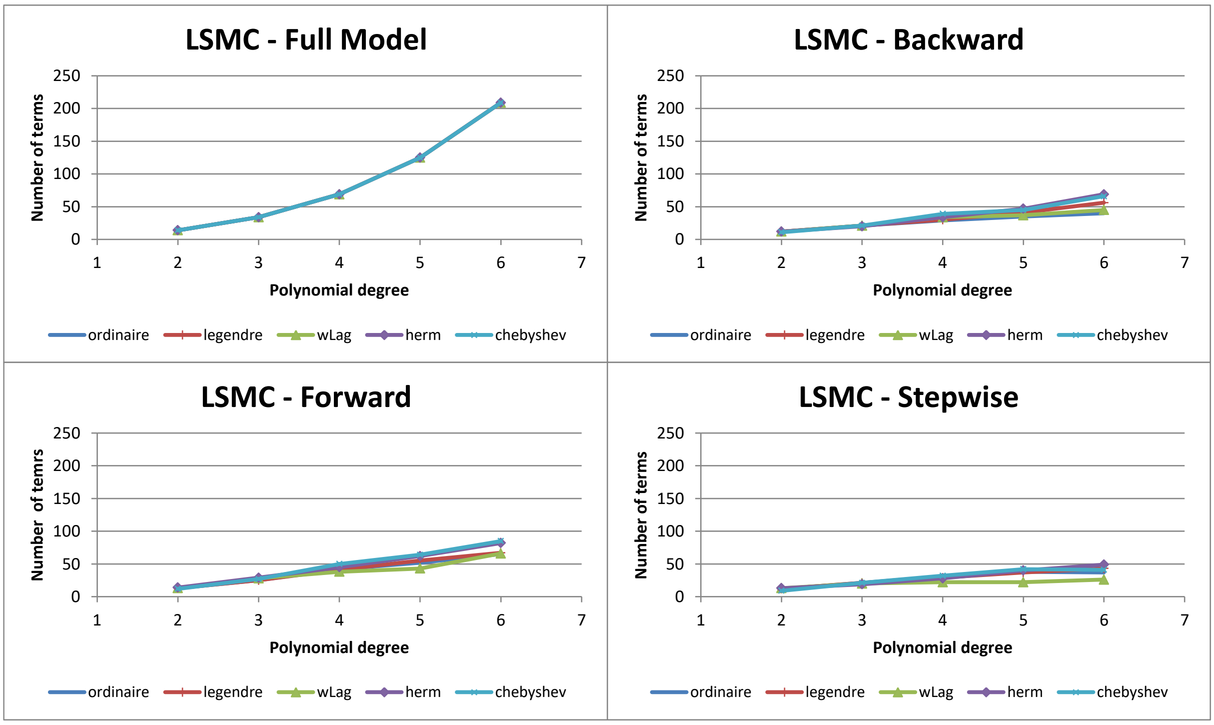
3.6 REGRESSION MODEL

In the previous section, we saw that the number of terms of the LSMC functions could be very high in the insurance context. There are various econometric techniques for selecting the best model from a set of possible candidates. For instance (see HOCKING [1976]):

- **Selection (forward):** Start with a model containing only the constant, then add one variable at each stage:
    - at each stage, select the most significant variable,
    - repeat until all the most significant variables have been selected.
- **Elimination (backward):** Start with a model containing all regressors and eliminate one at each stage.
    - at each stage, eliminate the least significant variable,
    - repeat until all the least significant variables have been eliminated.
- **bidirectional: a combination of forward/backward approaches (stepwise).** Start with a model containing only the constant.
    - Carry out a forward selection, leaving open the possibility of dropping any of the variables that becomes insignificant at each stage.
    - Repeat until all the variables selected are significant and all the eliminated variables are insignificant.

There are many criteria for significance (R^2, AIC, BIC, C_p, etc.). BAUER et al. [2010] show that *Mallow's* C_p criterion works well in an LSMC context as it gives the best results in the event of heteroskedasticity of residuals. The charts below show the number of regressors obtained using the different configurations analysed:

Fig. 7. Number of regressors of the fitting function based on the options analysed



For these four model selection methods, the number of regressors increases with the maximum degree of the one-dimensional basis function. The stepwise method results in the fewest regressors. The backward and full model regression models explode if the polynomial degree and number of risk factors are higher than 7.

4 APPLICATION

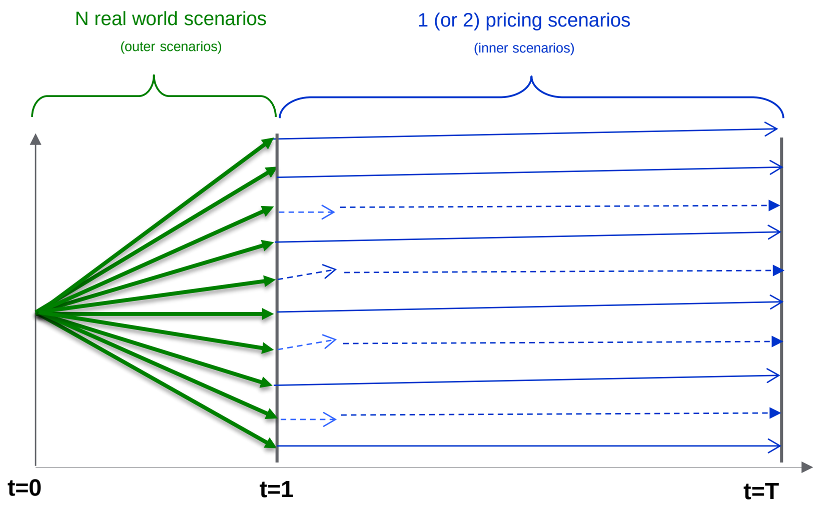
In this section we look at the practical implementation of LSMC method.

4.1 PRESENTATION

Calibration of the LSMC function is done by determining the coefficients: $\left(\overline{\alpha}_i\right)_{1\le i\le M}$ for a regression basis $\left(L_1(\cdot),L_2(\cdot),\ldots,L_M(\cdot)\right)$ fixed such that $C(F_t)\cong\sum_{i=1}^{M}\alpha_i\times L_i(F_t)$. In practice, this is a multi-stage process:

- **Stage 1:** simulate a number N of outer scenarios: **fitting scenarios**,
- **Stage 2:** simulate one inner scenario for each outer scenario (in practice, for faster convergence of LSMCs, we simulate 2 antithetical inner scenarios).

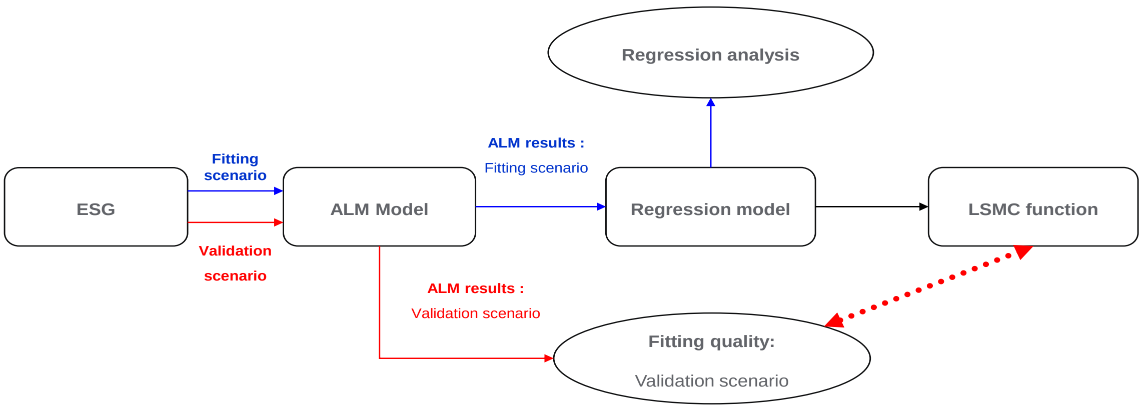
Fig. 8. ComputationAlgorithm



- **Stage 3:** using the ALM model, value each balance sheet item by DCF for each inner scenario,
- **Stage 4:** choose a regression basis and carry out a linear regression that gives least squares between the ALM results and the series of risk factors selected,
- **Stage 5:** test the function's validity against the **validation scenarios**.

Schematically, the process has the following architecture:

Fig. 9. Model structure



4.1.1 DESCRIPTION OF THE PORTFOLIO: CHOICE OF RISK FACTORS TO ANALYSE

In France the life insurance market generated revenue of €108.8 billion in 2012 making it the fourth-largest in the world and the second-largest in Europe[9]. The total value of life contracts outstanding in France was €1,458.3 billion at 31/12/2012. More than 85% of this is made up of euro funds. Euro contracts are savings contracts that contractually guarantee the capital invested. The sums paid in cannot fall in value and are increased each year by a return, the minimum guaranteed rate (Rate Guarantee) plus a profit sharing bonus (based on the technical and financial returns on the assets representing regulated commitments). The insurer must pay out at least 85% of financial gains and 90% of technical profits to policyholders (Profit-sharing

[9] See. www.ffsa.fr

option). In addition, income earned each year is definitively accrued. The insurer effectively guarantees the accrued value of capital at all times (Surrender option).

To cover these regulated commitments, French life insurers are invested in the following asset classes (Source: FFSA[10]): OECD sovereign debt: 32%, corporate bonds: 37%, equities, property, investment funds and other assets: 25% and money markets: 6%. Euro contracts are affected by market and technical risks. In this paper, we analyse the impact of the following market risks:

- Rate risk, rise or fall,
- Risk of a fall in equities markets,
- Risk of a fall in the property market.

Note that it is simple to extend the technique presented here to non-economic risks operationally.

4.1.2 ALM MODELLING

Results are based on the ALM model used by HSBC Assurances Vie. This software meets the insurer's aim of having a powerful stochastic modelling tool with easily auditable results. This model is the reference tool used in all ALM work which allows us to value the economic balance sheet and its various sensitivities. Besides stochastic simulations, it provides the following functionalities:

- compliance with insurance rules by carrying out accounting closes,
- reproducing the insurer's targets (including payments to policyholders),
- the option of generating stresses that impede the insurers' targets,
- the option of using stochastic scenarios to value options embedded in the contracts.

4.1.3 ESG: FITTING AND VALIDATION SCENARIOS

We determined 100,000 fitting scenarios. To accelerate convergence of LSMC functions we use Sobol's quasi-Monte Carlo (QMC) technique to simulate 50,000 outer scenarios coupled with antithetical variables for the inner scenarios[11]. Statistics for the fitting data are summarised below:

Fig. 10. Statistics for the fitting data

Statistics data fitting	Yield Curve shock (bps)		Capital Index shock (%)	
	Short rate	Long rate	Equity Index	Property Index
MIN	-30	-141	-100%	-100%
MAX	277	538	150%	150%
MEAN	31	42	20%	30%
STD	70	128	70%	70%

[10] https://www.ffsa.fr/sites/jcms/p1_1292377/fr/lassurance-francaise-en-2013?cc=fn_7345
[11] This is an efficient way of generating random numbers, because the Sobol is a small discrepancy sequence (see JUILLARD et al. [2011]) page 181.

The validation scenarios were chosen to measures the individual effects and interactions of the target variables on the economic balance sheet. They are shown below:

Fig. 11. Individual effects

Validation scenario	Yield Curve shock (bps)		Capital Index shock (%)	
	Short rate	Long rate	Equity Index	Property Index
1	-29	-133	0%	0%
2	0	0	-50%	-50%
3	0	0	80%	80%
4	0	0	-50%	80%
5	0	0	80%	-50%
6	115	281	0%	0%
7	90	-14	0%	0%
8	-25	46	0%	0%
9	112	228	-50%	50%
10	-21	32	50%	-50%
11	-27	-124	-70%	-70%
12	139	324	80%	80%
13	0	0	0%	80%
14	0	0	-70%	0%

4.1.4 REGRESSION TOOLS

The LSMC function was calibrated using 100,000 results taken from the ALM model. The regression tool used is the "**reg**" procedure in the SAS software package. We analysed 20 LSMC approaches, composed of the following combinations of regression basis functions and selection methods:

Fig. 12. Regression basis/selection method

Regression basis/selection method	**Ordinary**	**Laguerre**	**Legendre**	**Hermite**	**Chebyshev**
Full Model	x	x	x	x	x
Forward	x	x	x	x	x
Backward	x	x	x	x	x
Stepwise	x	x	x	x	x

4.2 REGRESSION QUALITY

4.2.1 DETAILED EXAMPLES

- **Laguerre function/full model**

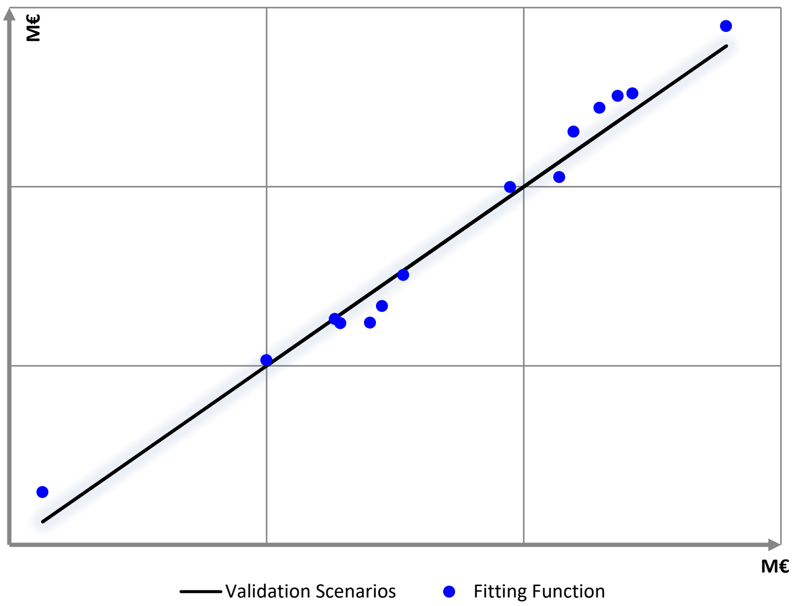
The table below shows an example of the LSMC function obtained. The one-dimensional basis for the regression is the Laguerre function. The maximum degree is 2 and the balance sheet item being modelled is net asset value (NAV).

Fig. 13. A sample of LSMC function

Term	Degree of Laguerre function				Coeff.
	Short rate	Long rate	Equity Index	Property Index	
Intercept	0	0	0	0	3.10
P1	0	0	0	1	-0.15
P2	0	0	0	2	0.02
P3	0	0	1	0	-0.71
P4	0	0	1	1	-0.51
P5	0	0	2	0	0.00
P6	0	1	0	0	0.40
P7	0	1	0	1	0.04
P8	0	1	1	0	0.62
P9	0	2	0	0	-1.06
P10	1	0	0	0	-0.20
P11	1	0	0	1	0.04
P12	1	0	1	0	0.00
P13	1	1	0	0	1.19
P14	2	0	0	0	-0.07

No selection method was used on the regressors and the LSMC function therefore has 14 terms. Note, however, that the coefficients of terms P5 and P12 are zero. The chart shows the quality of the NAV fitting to the validation scenarios.

Fig. 14. Test of the quality of the regression (degree 2)

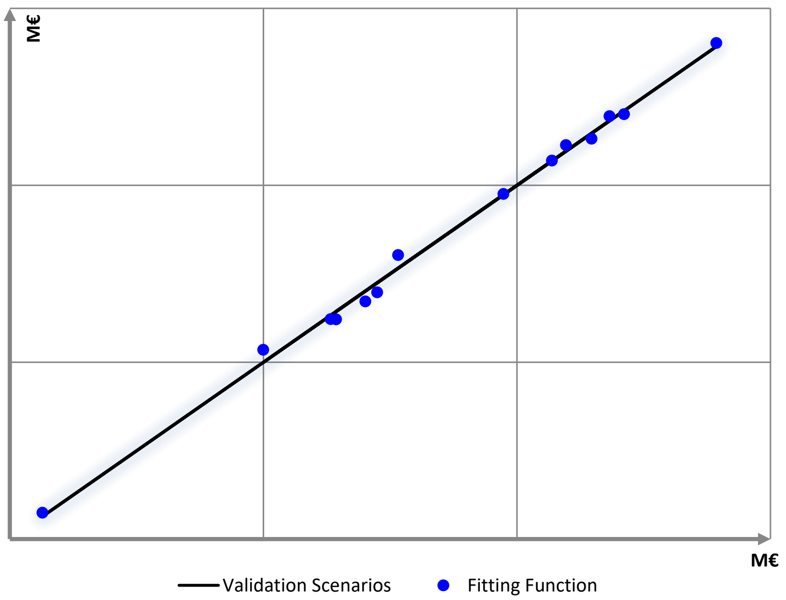


The average valuation error on NAV for this LSMC function is **-0.7%**. The maximum error is **6%**.

- **Ordinary polynomial/stepwise**

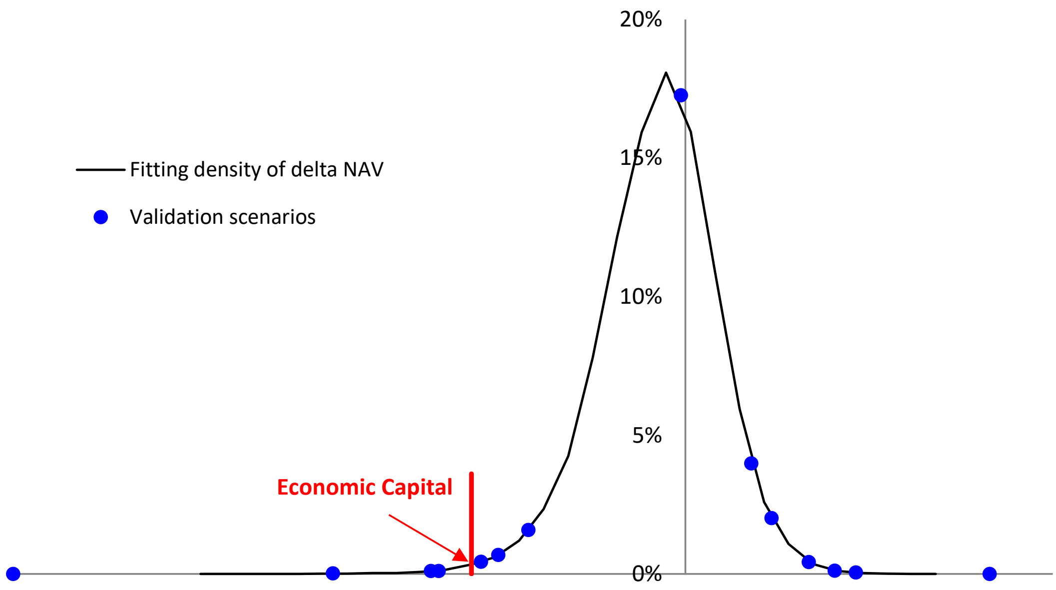
The chart below shows the quality of the regression for another LSMC function used to simulate NAV. Here, we used a stepwise selection method and the one-dimensional regression basis function was an ordinary polynomial with a maximum degree of 5:

Fig. 15. Test of the quality of the regression (degree 5)



We find this gives a better quality of regression. The average relative error is 0.02 %. The maximum relative error is 1.8% of the value of the validation scenario. The chart below shows how this LSMC function works for economic capital:

Fig. 16. LSMC function works for economic capital



Note that the validation scenarios are well spread out along the NAV distribution. Economic capital exposed to rate, equity and property risk is estimated by this LSMC function.

4.2.2 STATISTICS FOR THE TEST CASES

In this section, we present the statistics for the results of all the cases we tested. We tested 20 configurations of possible regression models (see section 3.6) with one-dimensional functions of degrees between 2 and 6. The charts below show average and maximum fitting errors for NAV for each option analysed:

Fig. 17. Average error of the fitting function from the options analysed

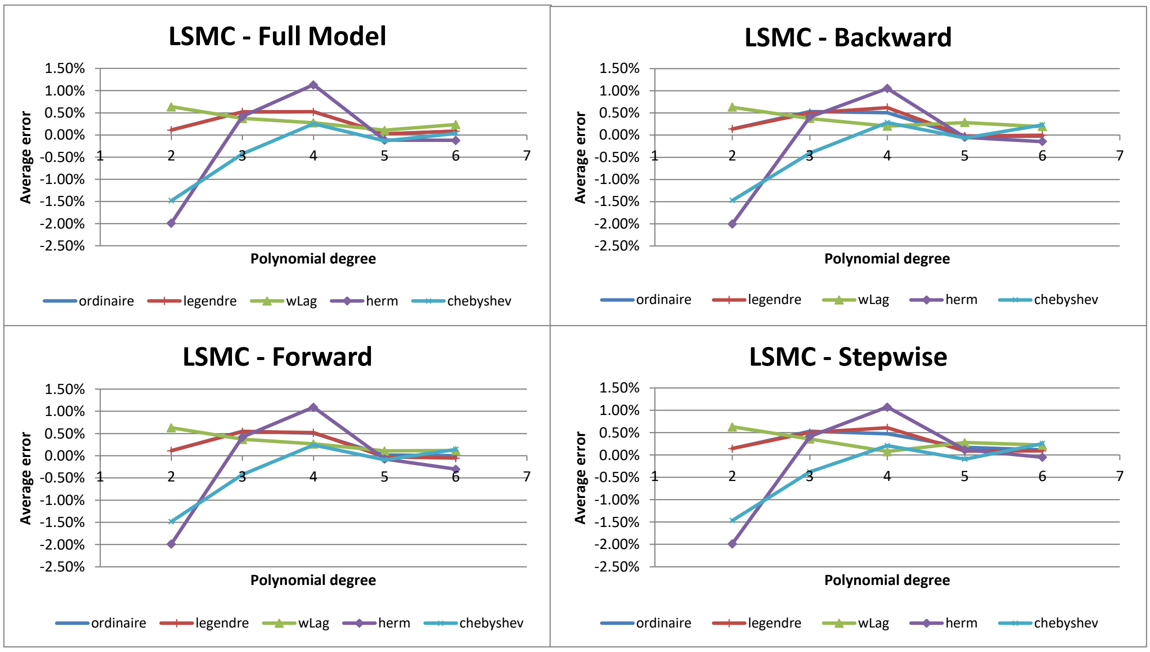
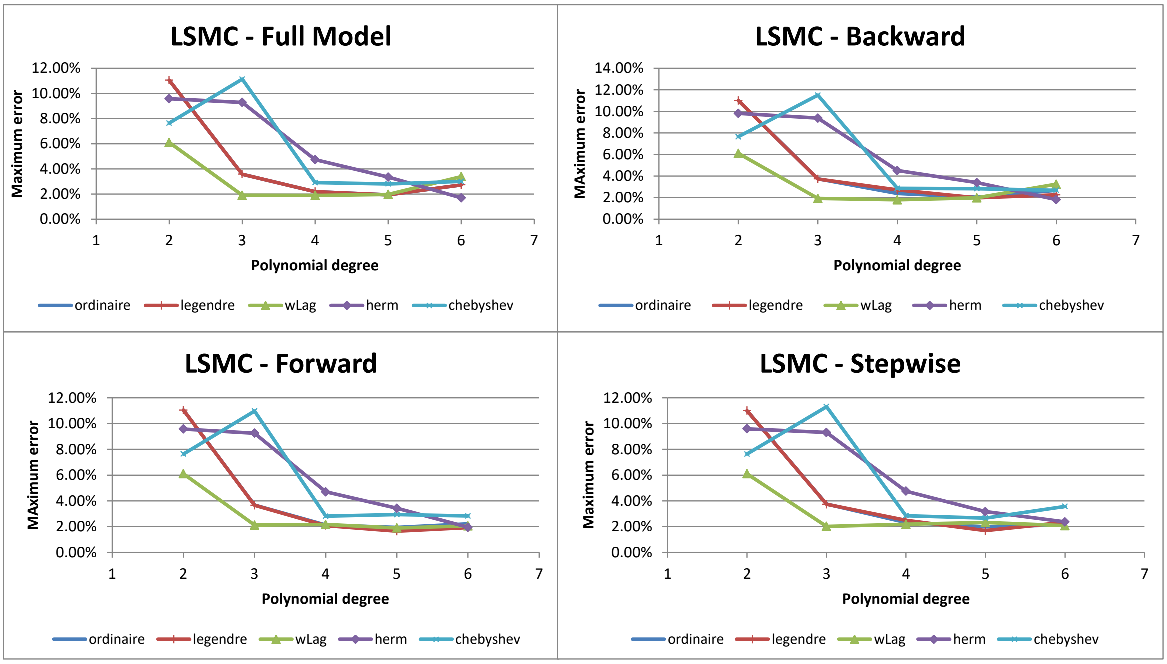


Fig. 18. Maximum error of the fitting function from the options analysed



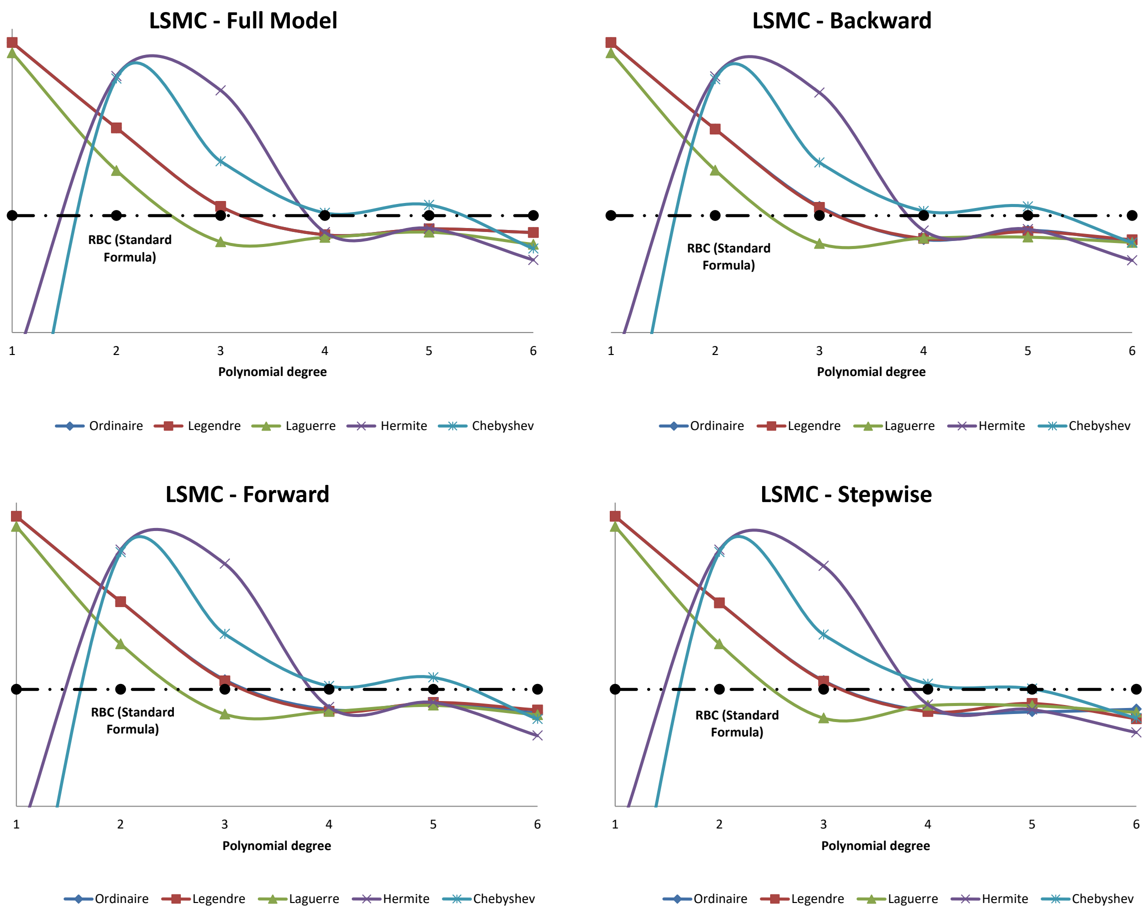
Note that:

- The shape of the curves is comparable whatever regression technique is used (although the backward, forward and stepwise methods result in fewer regressors than the full model, see section 3.6): valuation error falls with the degree of the polynomial. However, implementing LSMC technique in practice becomes impossible with a polynomial degree of more than 12.

- Note a substantial error margin (maximum error of over 2% when the basis polynomial function has degree less than 3, in all cases). Valuation error starts to stabilise at 4. The choice of optimum degree must therefore be either 4 or 5,
- The ordinary and Legendre polynomial both give very similar results because of the constant weighting function (see appendix). The Laguerre polynomial stabilises more quickly. The Hermite and Chebyshev polynomials are the least stable.

The charts below show the value of economic capital (RBC) estimated using the different options analysed:

Fig. 19. Value of risk based capital using the options analysed



Fitting quality is comparable whatever regression method is used. This is because the use of optimisation techniques for the LSMC functions does not reduce the precision of the results. The main difference is in the regression bases, with greater convergence from the Laguerre polynomial which stabilises more quickly. Note that the LSMC simulation gives a lower value of economic capital than the standard formula for all basis functions of degree 4 or more.

4.3 ANALYSIS OF THE LSMC FUNCTION

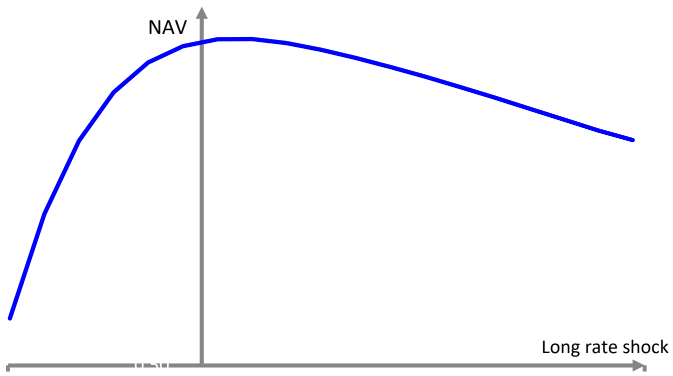
In this section, we examine the characteristics of the LSMC function. The function analysed here is obtained from the degree 4 Laguerre polynomial basis function using a stepwise selection method.

4.3.1 IMPACT OF RISK FACTORS

The purpose of this section is to check that the behaviour of the LSMC function is consistent with what we know about the portfolio.

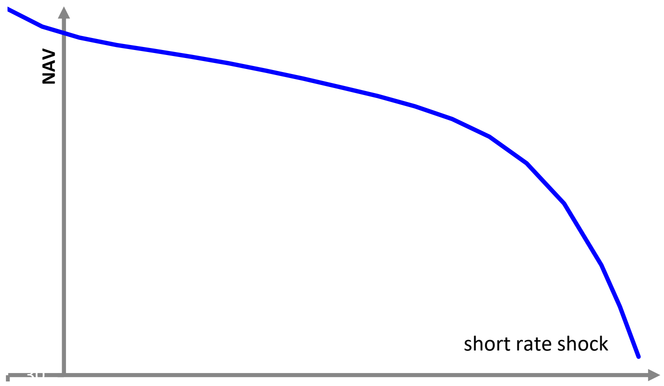
The chart below plots the value of the LSMC function as a function of long-term rates (all else being equal).

Fig. 20. NAV as a function of long-term yields



The long duration of their liabilities makes life insurance contracts highly sensitive to movements in long-term interest rates. The curve is a bell curve reflecting the opposing effects of the rate guarantee and surrender option on NAV. A strongly negative change in long rates means that the rise in the guarantee rate outweighs the fall in value of the redemption option. Vice-versa, a drastic rise in the long rate would drive sharply up the value of the surrender option outweighing the impact of the fall in the rate guarantee. The curve below plots the value of the LSMC function as a function of short-term interest rates (all else being equal).

Fig. 21. NAV as a function of short-term rates



The curve is falling. This reflects the impact of a progressive inversion of the rate curve.

The following charts show the value of the LSMC function as a function of equity and property indices (only an equity risk shock is modelled):

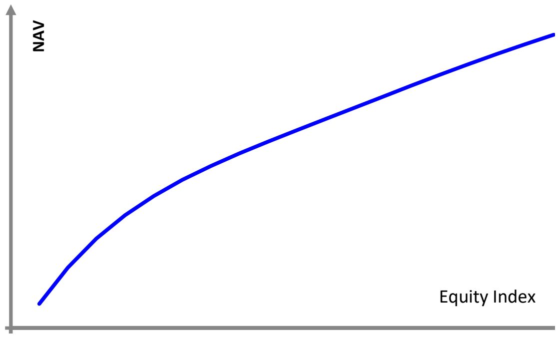


Fig. 22. NAV as a function of equity risk

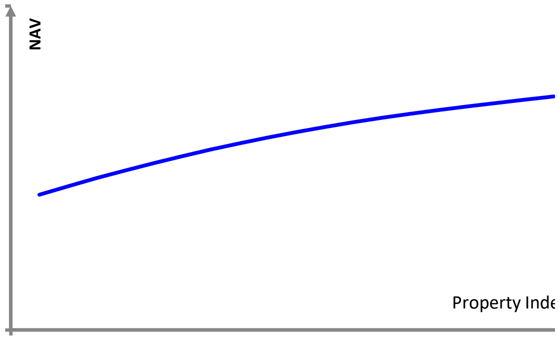
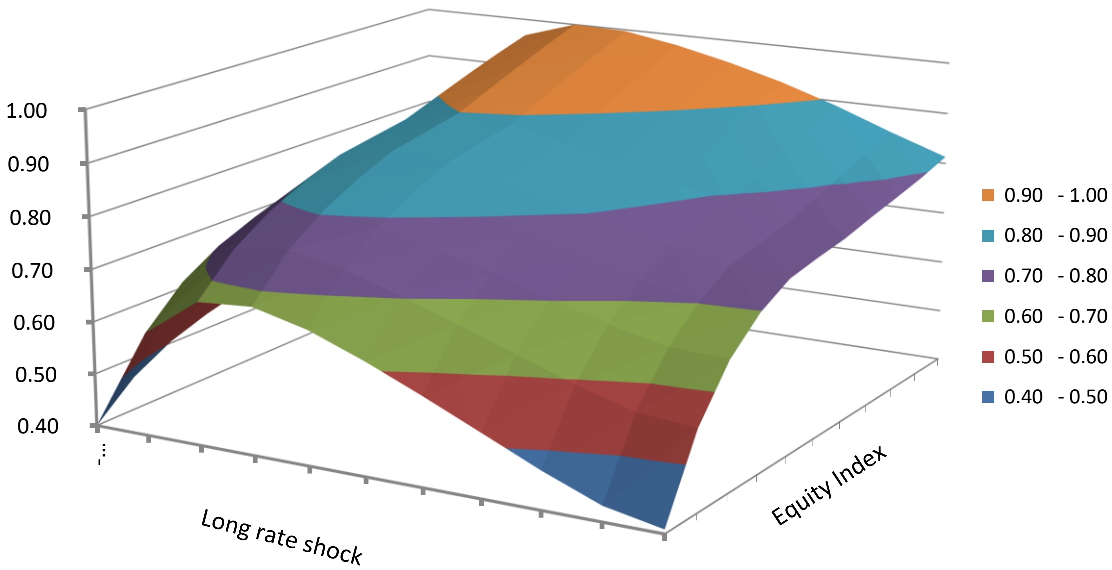


Fig. 23. NAV as a function of property risk

The curves are rising, reflecting the beneficial impact on NAV of rising equity markets and property prices. All the individual effects analysed above are consistent with what we know about portfolio risks.

The chart below shows the LSMC function as a function of long rates and the equity index.

Fig. 24. NAV as a function of long rate risk and equity risk

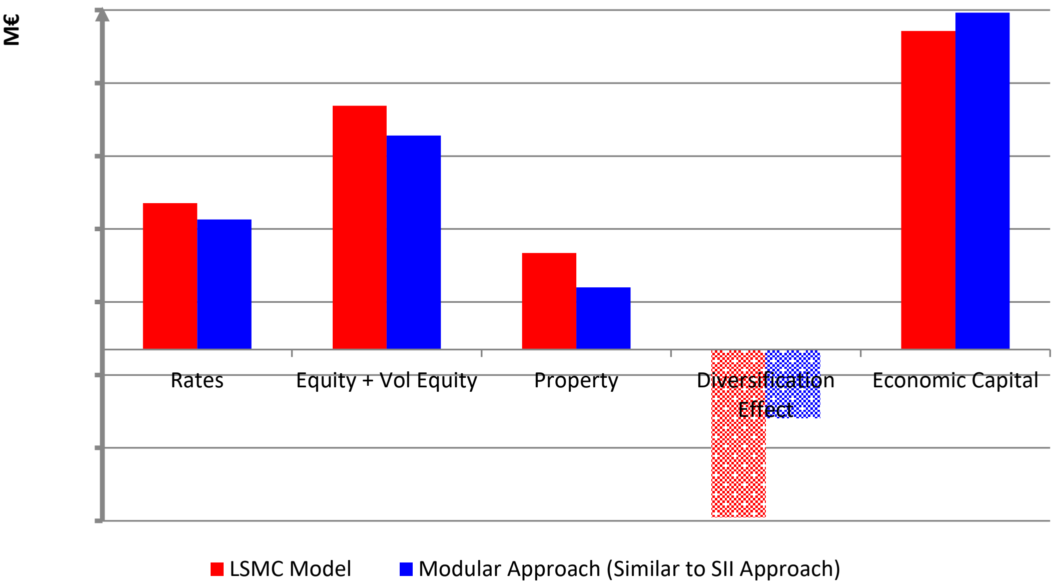


The chart shows the combined impact of long rates and equities on NAV.

4.3.2 CALCULATION OF ECONOMIC CAPITAL: MODULAR APPROACH VS LSMC MODELLING

The standard formula for valuing economic capital is to apply shocks to the balance sheet at t=0. The value of economic capital results from the combination of individual shocks and a matrix aggregating individual items' consumption of capital. The chart below compares the values of economic capital derived using the standard formula and the LSMC function:

Fig. 25. Economic capital: LSMC vs modular approach



Although the impact of stresses on the standard formula is lower, the value of economic capital derived is still comparable to that from the LSMC model, mainly due to greater diversification in the LSMC approach.

5 CONCLUSION

In this paper, we are interested in how least squares Monte Carlo technique (LSMC) can be used in the field of life insurance. The levels of convergence under the historical measure suggests that LSMC is effective in valuing economic capital. Results obtained show a good quality fit (average error in NAV of less than 0.02% and maximum error of 1.8%). Also, the LSMC function accurately reflects the behaviour of the liability being analysed.

Regarding the choice of regression technique, we saw that that stepwise selection method led to the simplest LSMC function without impairing the precision of the results. When the number of risk factors is higher than 7 and the degree of the basis polynomial function is higher than 7 only forward and stepwise selection give good results.

Regarding the basis functions analysed, we found that all functions examined gave good results. Laguerre functions stabilised fastest. However, for practical implementation of LSMC technique, the use of aggregation techniques (clustering, etc.) is essential due to the massive calculation times required.

Also, it is hard to interpret the parameters of the LSMC function. In some cases, where the number of terms in the function is very high, the LSMC function becomes unreadable. The number of risk factors that can be analysed thus quickly reaches a limit.

Also, although the empirical results are relevant, further work is needed on the multi-dimensional analysis of the orthogonality property of the functions analysed in this study.

Finally, in life insurance, it is essential to incorporate non-economic risk factors but here the convergence properties remain unproven, mainly because of the change in probability. The stability of the function over time makes it tempting to introduce initial wealth as a parameter as this reflects a capacity to absorb liability shocks. Calibration of the LSMC function over multiple periods should be the next step.

6 APPENDICES

Convergence Q_t of the LSMC estimator

Let $M \in \mathbb{N}^*$, we saw (see section 3) that we could approximate $C(F_t)$ by:

$$C(F_t) \cong C_M(F_t) = \sum_{i=1}^{M} \alpha_i \times L_i(F_t) = L^M(F_t).\alpha_M{}^t$$

Assuming that $F_t = \left(F_t^1, F_t^2, \cdots, F_t^k\right)$ is a Markov process vector.

Property: $\alpha_M = \left(A_M\right)^{-1} \times E_{Q_t}\left[L^M(F_t) \times C(F_t)\right]$

where $A_M = \left(A_{(k,l)}\right)_{1\le k,l \le M}$ et $A_{(k,l)} = E_{Q_t}\left[L_k(F_t) \times L_l(F_t)\right]$

Proof: for $k \in \{1,...,M\}$, using the property of orthogonality[12] of $L(\ ) = \left(L_1(\cdot), L_2(\cdot), \cdots\right)$ we can clearly show that $\left\langle L_k(F_t), C(F_t) - C_M(F_t)\right\rangle = 0$

$$\left\langle L_k(F_t), C(F_t) - C_M(F_t)\right\rangle = 0 \Rightarrow \begin{cases} E_{Q_t}\left[L_k(F_t) \times \left(C(F_t) - C_M(F_t)\right)\right] = 0 \\ E_{Q_t}\left[L_k(F_t) \times \left(C(F_t) - \sum_{i=1}^{M} \alpha_i \times L_i(F_t)\right)\right] = 0 \\ E_{Q_t}\left[L_k(F_t) \times C(F_t)\right] - \sum_{i=1}^{M} \alpha_i \times E\left[L_k(F_t) \times L_i(F_t)\right] = 0 \ (*) \end{cases}$$

$(*)$ In matrix form, this is rewritten:

$$\left.\begin{array}{ll} \text{for k} = 1, & \sum_{i=1}^{M} \alpha_i \times E_{Q_t}\left[L_1(F_t) \times L_i(F_t)\right] = E_{Q_t}\left[L_1(F_t) \times C(F_t)\right] \\ \ldots & \\ \text{for k}, & \sum_{i=1}^{M} \alpha_i \times E_{Q_t}\left[L_k(F_t) \times L_i(F_t)\right] = E_{Q_t}\left[L_k(F_t) \times C(F_t)\right] \\ \ldots & \\ \text{for k} = \text{M}, & \sum_{i=1}^{M} \alpha_i \times E_{Q_t}\left[L_M(F_t) \times L_i(F_t)\right] = E_{Q_t}\left[L_M(F_t) \times C(F_t)\right] \end{array}\right\}$$

$$\Leftrightarrow A_M \times \alpha_M = E_{Q_t}\left[L^M(F_t) \times C(F_t)\right]$$

[12] $E_{Q_t}\left[L_k(F_t) \times L_i(F_t)\right] = \begin{cases} = 0 \text{ si k} \ne \text{i} \\ > 0 \text{ si k} = \text{i} \end{cases}$

Because of the orthogonality property of $L(\ )=\left(L_1(\cdot),L_2(\cdot),\cdots\right)$, the matrix A_M is a diagonal matrix and all the elements on the diagonal must be greater than 0. It is therefore invertible. □

The LSMC estimator is written as $C_M^{LS,N}(F_t)=L^M(F_t)\times\overline{\alpha}_M^{N\,t}$ where $\overline{\alpha}_M^N$ is the OLS estimator:

$$\overline{\alpha}_M^N=\left[L^{M,N}\left(F_t^N\right)^t\times L^{M,N}\left(F_t^N\right)\right]^{-1}\times L^{M,N}\left(F_t^N\right)^t\times\mathbf{C}^N\left(F_t^N\right)$$

with:

- $L^{M,N}\left(F_t^N\right)=\left(L^M\left(F_t^1\right),\dots,L^M\left(F_t^N\right)\right)=\left(L_k\left(F_t^n\right)\right)_{\substack{1\le k\le M\\1\le n\le N}}$,
- $\mathbf{C}^N\left(F_t^N\right)=\left(C\left(F_t^1\right),\dots,C\left(F_t^N\right)\right)$,

Property: $C_M^{LS,N}(F_t)\xrightarrow{N\to\infty}C_M(F_t)$ ps.

Proof: We need only show that $\overline{\alpha}_M^N\xrightarrow{N\to\infty}\alpha_M$ ps.

Let $A_M^N=L^{M,N}\left(F_t^N\right)^t\times L^{M,N}\left(F_t^N\right)$, then:

$$\begin{aligned}\overline{\alpha}_M^N&=\left[A_M^N\right]^{-1}\times L^{M,N}\left(F_t^N\right)^t\times\mathbf{C}^N\left(F_t^N\right)^t\\&=\left[\frac{A_M^N}{N}\right]^{-1}\times\frac{L^{M,N}\left(F_t^N\right)^t\times\mathbf{C}^N\left(F_t^N\right)^t}{N}\end{aligned}$$

We have [13] $\left[\frac{A_M^N}{N}\right]=\left(\frac{1}{N}\sum_{n=1}^N L_k\left(F_t^n\right)\times L_l\left(F_t^n\right)\right)_{1\le k,l\le M}$

Applying the law of large numbers;

$\frac{1}{N}\sum_{n=1}^N L_k\left(F_t^n\right)\times L_l\left(F_t^n\right)\xrightarrow{N\to\infty}E_{Q_t}\left(L_k\left(F_t\right)\times L_l\left(F_t\right)\right)$ ps. $\Rightarrow\left[\frac{A_M^N}{N}\right]\xrightarrow{N\to\infty}A_M$ a.s.

In addition, $\frac{L^{M,N}\left(F_t^N\right)^t\times C^N\left(F_t^N\right)}{N}=\left(\frac{1}{N}\sum_{n=1}^N L_k\left(F_t^n\right)\times\mathbf{C}\left(F_t^n\right)\right)_{1\le k\le M}$

Applying the law of large numbers $\frac{1}{N}\sum_{n=1}^N L_k\left(F_t^n\right)\times\mathbf{C}\left(F_t^n\right)\xrightarrow{N\to\infty}E_{Q_t}\left[L_k\left(F_t\right)\times C\left(F_t\right)\right]$ ps.

We have

$$\overline{\alpha}_M^N=\left[\frac{A_M^N}{N}\right]^{-1}\times\frac{L^{M,N}\left(F_t^N\right)^t\times\mathbf{C}^N\left(F_t^N\right)^t}{N}\xrightarrow{N\to\infty}\left[A_M\right]^{-1}\times E_{Q_t}\left[L^M\left(F_t\right)\times C\left(F_t\right)\right]\quad\text{a.s.}$$

$\overline{\alpha}_M^N\xrightarrow{N\to\infty}\alpha_M$ a.s. □

[13] Without impairing the general proof, we need only show that this is true for the cases N=2 and M=2.

7 REFERENCES

ABRAMOWITZ M., STEGUN I.A. [1964]: « Handbook of mathematical functions with formulas, Graphs And mathematical Tables », *National Bureau of standard*, Applied Mathematic Series 55. June 1964.

BAUER D., BERGAMNN D.; REUSS A. [2010]: « Solvency II and Nested Simulations – a Least-Squares Monte Carlo Approach » Proceedings of the 2010 ICA congress.

BECK V., MALICK J., PEYRÉ G. [2005] *Objectif agrégation*. Paris : H & K.

BONNIN F., JUILLARD M., PLANCHET F. [2014] « Best Estimate Calculations of Savings Contracts by Closed Formulas - Application to the ORSA », *European Actuarial Journal*, http://dx.doi.org/10.1007/s13385-014-0086-z.

BOYLE P. [1977] « Option: A monte carlo approach » *Journal of Financial Economics* 4, 323-338

DEVINEAU L. ; LOISEL S. [2009]: « Construction d'un algorithme d'accélération de la méthode des «simulations dans la simulation » pour le calcul du capital économique Solvabilité II » *Bulletin Français d'Actuariat* 10, 17 (2009) 188-221

HOCKING, R. R. [1976] « The Analysis and Selection of Variables in Linear Regression » *Biometrics*, 32, 1-49

IFERGAN E. [2013] « Mise en œuvre d'un calcul de best estimate », Actuarial Thesis, Université Paris-Dauphine.

JUILLARD M., PLANCHET F., THÉROND P.E. [2011] *Modèles financiers en assurance. Analyses de risques dynamiques - seconde édition revue et augmentée*, Paris : Economica (première édition : 2005).

LONGSTAFF F.A.; SCHWARTZ E.S. [2001]: « Valuing American Options by Simulation: A Simple Least-Squares Approach » *The review of financial Studies*, Spring 2001 Vol. 14, No. 1 pp. 113-147

NEVEU J. [1964] *Bases mathématiques du calcul des probabilités*, Paris: Masson.

NTEUKAM O.; PLANCHET F. [2012] « Stochastic evaluation of life insurance contracts: Model point on asset trajectories & measurement of the error related to aggregation. » *Insurance: Mathematics and Economics*, / Volume (Year): 51 (2012) Issue (Month): 3, Pages: 624-631

REVELEN J. [2011] « Replicating Portfolio et capital économique en assurance vie », Actuarial Thesis, ISFA 04/01/2011.

WÜTHRICH M. V, BÜHLMANN H., FURRER H.[2008] *Market-Consistent Actuarial Valuation*, Springer, ISBN: 978-3-540-73642-4

YOSIDA K. [1980] *Functional Analysis*, Berlin : Springer