On the Control of Defined-Benefit Pension Plans

Hong-Chih Huang[1] and Andrew J.G. Cairns[2]

Abstract

Conventionally, contribution rates for defined-benefit pension plans have been set with reference to funding levels without making allowance for current market interest rates: for example, on one-year bonds where rates of return on fund assets are not independent from one year to the next. We consider how to make use of market information to reduce contribution rate volatility. The purpose of this paper is to provide a model for determining an appropriate contribution rate for defined benefit pension plans under a model where interest rates are stochastic and rates of return are random.

We extend previous work in two ways. First, we introduce a model for short-term interest rates, which can be used to help control contribution-rate volatility. Second, we model three assets rather than the usual one (cash, bonds and equities) to allow comparison of different asset strategies. We develop formulae for unconditional means and variances. We then discuss how variability can be controlled most efficiently by setting contribution rates with reference tocurrent funding levels and interest rates.

Keywords

Stochastic pension plan model; contribution rate; stability; minimum variance; efficient region; stochastic interest rates; asset-allocation strategy.

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1 Introduction

A variety of factors that influence the volatility of the funding level and the contribution rate of a define-benefit (DB) pension plan including: the amortization strategy (Cairns 1994, Dufresne 1989, Bowers, Hickman and Nesbitt 1979); the amortization period (Dufresne 1988, 1989, Haberman 1994, Cairns 1994, Cairns and Parker, 1997); frequency of valuation (Cairns 1994, Haberman 1993); and the delay period (Balzer and Benjamin 1980, and Zimbidis and Haberman 1993). The main purpose of this paper is to develop further the approach to setting contribution rates as a means of reducing the variance of the funding level and contribution rate under DB plans. The choice of spread period for surplus and deficit is one of the most important ways of control of the stability of the pension plan (see, for example, Dufresne, 1988, 1989, Haberman, 1994, and Cairns and Parker, 1997). In this paper, we aim to extend the spread period contribution model and take advantage of the current market information about interest rates to reduce further the variance of the funding level and contribution rate.

A pension plan's trustees are responsible for choosing long-term investment advice and the actuary is normally required to advise the trustees and/or the employers. Thus actuaries are essential for advising trustees on a variety of possible investment strategies and for making sensible comments and suggestions on the implementation of the distribution of assets for each plan in order to match its anticipated liabilities. The aggregate investment return rate of the pension fund has been investigated on a model with independent and identically distributed (i.i.d.) returns (Dufresne, 1988, 1989), an AR time-series model(Mandl and Mazurova, 1996, Haberman, 1994, Cairns and Parker, 1997), and an MA time-series model (Haberman, 1997, Bedard, 1999). The plausible term structure of AR and MA time series models was considered by Chang (2000). These aggregate-return models take the investment strategy as given exogenously and model the returns on the fund as a univariate times series. In an attempt to make the approach to investments more realistic we explicitly allow for several assets in the portfolio. Thus, instead of using an aggregate return rate of the pension plan, we consider a more general investment model where the pension plan's return is a combination of numbers of the return on the individual assets.

In this paper we extend previous work to include three assets rather than just one: cash, long bonds and equities. Their returns are underpinned in a coherent way by a model for the one-year, risk-free interest rate and with appropriate correlations between different asset classes. Section 2 describes the basic details of the model and proposes a simple method for setting the contribution rate which accounts for both the current funding level (as normal) and current interest rates (new). With this model we are able to derive formulae for unconditional (that is long-run) means and variances of the funding level and for the contribution rate. In section 3 we discuss how the contribution strategy can be used to control most effectively variability in the funding level and in the contribution rate itself. Here we reintroduce and extend the concept of efficient contribution strategies.

In Section 4, we build a super efficient region which minimizes the variance of contribution rate based upon specific funding constraints and discuss the optimal investment and contribution strategies.

2 A discrete-time model pension plan

We assume that we have three assets: a one-year bond (cash); a long-dated bond; and an equity asset. The log-return on cash between times $t-1$ and t is $y(t-1)$. The log-return rate on the bond is $\delta_b(t)$, and the log-return on the equity is $\delta_e(t)$. Thus, investments of 1 at time $t-1$ will grow to $e^{y(t-1)}$, $e^{\delta_b(t)}$ or $e^{\delta_e(t)}$ respectively. We will further assume that $y(t)$ follows the $AR(1)$ process

$$y(t) = y + \phi(y(t-1) - y) + \sigma_y Z_y(t) \tag{2.1}$$

where the $Z_y(t)$ are independent and identically distributed (i.i.d.) standard normal random variables. This is similar to a discrete-time version of the Vasicek (1977) model. Excess returns on the equity asset, $\Delta_e(t) = \delta_e(t) - y(t-1)$, are assumed to be i.i.d. and normally distributed with a mean greater than zero (that is, a positive risk premium). Similarly, the excess returns on a long-dated bond, $\Delta_b(t) = \delta_b(t) - y(t-1)$, are also assumed to be i.i.d. and normally distributed with mean greater than zero. Thus

$$\begin{aligned}
\Delta_b(t) &= \delta_b(t) - y(t-1) \\
&= \Delta_b + \sigma_{by} Z_y(t) + \sigma_b Z_b(t) && (2.2) \\
\Delta_e(t) &= \delta_e(t) - y(t-1) \\
&= \Delta_e + \sigma_{ey} Z_y(t) + \sigma_{eb} Z_b(t) + \sigma_e Z_{\delta e}(t) && (2.3)
\end{aligned}$$

where the $Z_\delta(t)$, $Z_y(t)$ and $Z_b(t)$ are $N(0,1)$ random variables that are independent of one another and i.i.d. through time. Both σ_{ey} and σ_{by} will normally be negative since if the short-term interest rate, $y(t)$, goes up, then the prices of long-term bonds or equities typically go down and vice versa. The $\sigma_b Z_b(t)$ term allows us to use, in effect, a two-factor interest-rate model since it allows for a degree of independence from one-year bonds.

Since we are considering a one-year bond, the return from $t-1$ up to t is known at time $t-1$ whereas the return on equities and bonds are only known at time t. This explains the use of $y(t-1)$ for the return on the one-year bond for $t-1$ to t rather than $y(t)$. In contrast, the unknown $\Delta_b(t)$ and $\Delta_e(t)$ are used to reflect the unknown elements of returns on the long-dated bonds and equities. In particular, bond prices at time t depend upon the new one-year rate of interest at t, $y(t)$, through their dependence on $\sigma_{by} Z_y(t)$. The extent to which unanticipated returns on equities $(\Delta_e(t))$ reflect unanticipated changes in $y(t)$ appears in the parameter σ_{ey} with further equity specific risk being reflected through $\sigma_{\delta e}$ and $Z_e(t)$. Further correlation with long bonds is reflected through the parameter σ_{eb}.

Suppose we invest a proportion p_1 of the pension fund in equities, p_2 in long-term bonds and the remaining assets in one-year bonds. Assuming constant rebalancing during the year and continuous sample paths for prices, the return on the fund from $t-1$ to t is:

$$1 + i(t) = e^{y(t-1) + p_1 \Delta_e(t) + p_2 \Delta_b(t)}.$$

(This is approximately equal to the total return on an annual buy-and-hold strategy.)

We use the following additional notation which assumes that we have a stable membership in the pension plan with no salary increases (or we use the total salary roll as the unit of currency):

$$\begin{aligned} F(t) &= \text{fund size at } t \\ C(t) &= \text{contribution rate at } t \\ B &= \text{benefit outgo at the start of each year (assumed constant)} \\ i_v &= \text{actuarial valuation interest rate} \\ AL &= \text{actuarial liability (assumed constant)} \\ NC &= \text{normal contribution rate consistent with } AL \text{ and } i_v \end{aligned}$$

Stability of the membership with no salary increases means that the actuarial liability does not change over time. Consistency between NC and AL thus means that

$$\begin{aligned} \Rightarrow AL &= (1+i_v)(AL + NC - B) \\ \Rightarrow NC &= B - d_v AL \qquad (2.4) \\ \text{where } d_v &= 1 - v_v \\ \text{and } v_v &= (1+i_v)^{-1}. \end{aligned}$$

Annual contributions, $C(t)$, are allowed to depend not just upon the current funding level (as is normal) but also on the current level of interest rates. The particular form we use is

$$C(t) = NC + k_1(AL - F(t)) + k_2 \frac{(e^{y'} - e^{y(t)})}{e^{y(t)}}$$

where k_1, k_2 and y' are the key control factors. If $k_2 = 0$ then we revert to the classical case (see, for example, Cairns & Parker, 1997, and Haberman, 1994, 1997). In a continuous-time model with $y(t)$ constant and only one asset class, Cairns (2000) proved that this contribution strategy using the spread method is superior (mathematically optimal) to other approaches (such as the amortization of losses method used in North America).

The purpose of introducing the k_2 term is to allow adjustment for future expected returns. For example, if $y(t)$ is currently high then we might feel that contributions could be lower than would otherwise be the case because of higher expected returns than normal in the next few years. We will see later if this term allows us to reduce variability.

Given $C(t)$ we have the usual dynamics for $F(t)$:

$$F(t) = (1 + i(t)) \left[F(t-1) + C(t-1) - B \right]$$

We now take into account the earlier expression for $(1 + i(t))$ and work backwards recursively to get (see, for example, Cairns & Parker, 1997):

Lemma 2.1

$$
\begin{aligned}
F(t) &= (\theta_v - k_2)\sum_{s=0}^{\infty}(1-k_1)^s \exp\left(S_y(t,s) + S_p(t,s)\right) \\
&\quad + k_2 e^{y'} \sum_{s=0}^{\infty}(1-k_1)^s \exp\left(S_y(t,s) - y(t-1-s) + S_p(t,s)\right) \\
\textit{where} \;\; \theta_v &= (k_1 - d_v)AL
\end{aligned}
$$

provided k_1 has been chosen so that this sum converges.

Within this expression, first,

$$
\begin{aligned}
S_y(t,s) &= \sum_{j=0}^{s} y(t-1-j) \\
&= (s+1)y + \sum_{j=1}^{s+1} \frac{(1-\phi^j)\sigma_y}{1-\phi} Z_y(t-j) \\
&\qquad + \sum_{j=s+2}^{\infty} \frac{\phi^{j-s-1}(1-\phi^{s+1})\sigma_y}{1-\phi} Z_y(t-j) \\
\Rightarrow S_y(t,0) - y(t-1) &= 0 \\
\text{and for } s \geq 1 & \\
S_y(t,s) - y(t-1-s) &= S_y(t,s-1) \\
&= sy + \sum_{j=1}^{s} \frac{(1-\phi^j)\sigma_y}{1-\phi} Z_y(t-j) \\
&\qquad + \sum_{j=s+1}^{\infty} \frac{\phi^{j-s}(1-\phi^{s})\sigma_y}{1-\phi} Z_y(t-j).
\end{aligned}
$$

(The latter equality is, of course, zero if we define $\sum_{j=1}^{s}(\cdot) \equiv 0$ when $s = 0$.) Second,

$$
\begin{aligned}
S_p(t,s) &= \sum_{j=0}^{s} p_1 \Delta_e(t-j) + \sum_{j=0}^{s} p_2 \Delta_b(t-j) \\
&= (s+1)\alpha_0 + \alpha_1 \sum_{j=0}^{s} Z_y(t-j) + \alpha_2 \sum_{j=0}^{s} Z_e(t-j) + \alpha_3 \sum_{j=0}^{s} Z_b(t-j) \\
\text{where } \alpha_0 &= p_1\Delta_e + p_2\Delta_b \\
\alpha_1 &= p_1\sigma_{ey} + p_2\sigma_{by} \\
\alpha_2 &= p_1\sigma_e \\
\text{and } \alpha_3 &= p_1\sigma_{eb} + p_2\sigma_b.
\end{aligned}
$$

Theorem 2.2

The unconditional expected values and the variances of the fund size and contribution rate are as follows:
(a)

$$\begin{aligned}
E[F(t)] &= (\theta_v - k_2)\Psi_1 + k_2\Psi_2 \\
E[C(t)] &= NC + k_1\left(AL - E[F(t)]\right) + k_2\left(e^{y'}e^{-y+\frac{1}{2}\frac{\sigma_y^2}{1-\phi^2}} - 1\right) \\
\textit{where}\ \ \Psi_1 &= \sum_{s=0}^{\infty}(1-k_1)^s \exp\left((s+1)(y+\alpha_0) + \frac{1}{2}V_1(s)\right) \\
\Psi_2 &= e^{y'}\sum_{s=0}^{\infty}(1-k_1)^s \exp\left((s+1)(y+\alpha_0) - y + \frac{1}{2}V_2(s)\right) \\
\textit{and}\ \ V_1(s) &= Var\left(S_y(t,s) + S_p(t,s)\right) \\
V_2(s) &= Var\left(S_y(t,s-1) + S_p(t,s)\right).
\end{aligned}$$

Thus $E[F(t)]$ and $E[C(t)]$ are both linear functions of k_2 but nonlinear functions of k_1.
(b)

$$Var[F(t)] = h_2k_2^2 + h_1k_2 + h_0 \tag{2.5}$$
$$Var[C(t)] = a_2k_2^2 + a_1k_2 + a_0 \tag{2.6}$$

$$\begin{aligned}
\textit{where}\ \ h_0 &= \sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_1(r,s) + e^{2y'}\sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_3(r,s) \\
&\quad -2e^{y'}\sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_2(r,s) \\
h_1 &= -2\theta_v\sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_1(r,s) + 2\theta_v e^{y'}\sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_2(r,s) \\
h_2 &= \theta_v^2\sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_1(r,s) \\
a_0 &= k_1^2h_0 + e^{2y'}Var(e^{-y(t)}) + 2k_1e^{y'}\sum_{r=0}^{\infty}(1-k_1)^rC_4(r) \\
&\quad -2k_1e^{2y'}\sum_{r=0}^{\infty}(1-k_1)^rC_5(r) \\
a_1 &= k_1^2h_1 - 2k_1e^{y'}\theta_v\sum_{r=0}^{\infty}(1-k_1)^rC_4(r) \\
a_2 &= k_1^2h_2
\end{aligned}$$

$$
\begin{aligned}
\textit{and} \;\; C_1(r,s) &= Cov\left(e^{S_y(t,r)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)}\right) \\
C_2(r,s) &= Cov\left(e^{S_y(t,r)-y(t-1-r)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)}\right) \\
C_3(r,s) &= Cov\left(e^{S_y(t,r)-y(t-1-r)+S_p(t,r)}, e^{S_y(t,s)-y(t-1-s)+S_p(t,s)}\right) \\
C_4(r) &= Cov\left(e^{S_y(t,r)+S_p(t,r)}, e^{-y(t)}\right) \\
C_5(r) &= Cov\left(e^{S_y(t,r)-y(t-1-r)+S_p(t,r)}, e^{-y(t)}\right).
\end{aligned}
$$

For a proof of this result and more detailed formulae for these functions, see Appendix A. In these expressions note that $\psi_1, \psi_2, h_0, h_1, h_2, a_0, a_1, a_2$ are all functions of k_1 but not of k_2.

For the actuarial liability we will assume a simple model (as in Cairns & Parker, 1997) where:

- there is one member at each of ages 25 to 64;
- each year one new member aged 25 joins the plan;
- no deaths or other decrements before age 65;
- on retirement at age 65 each member receives a benefit of $B = 40$ which accrues uniformly over the 40 years of service.

Thus the accrued or past-service liability, when the valuation rate of interest is i_v, is

$$
\begin{aligned}
AL = AL(i_v) &= \sum_{x=25}^{64} (x-25)(1+i_v)^{x-65} = \left(40 - \frac{(1-v_v^{40})}{i_v}\right)\frac{(1+i_v)}{i_v} \qquad (2.7) \\
\text{where} \;\; v_v &= \frac{1}{1+i_v}.
\end{aligned}
$$

Sample values for $AL(i_v)$ are given in Table 1.

i_v	$AL(i_v)$	$NC(i_v)$
0.02	644.87	27.36
0.03	579.73	23.11
0.04	525.39	19.79
0.05	479.66	17.16
0.06	440.85	15.05

Table 1: Values for $AL(i_v)$ (equation 2.7) for different values of i_v, with the corresponding normal contribution rates $NC(i_v)$ (equation 2.4).

3 Optimal strategies for the contribution rate

In this section, we will discuss how to make the best use of current market interest rates to control variability. Specifically, what are good values for k_2. Now we can note that, given k_1, the variances of both $F(t)$ and $C(t)$ are quadratic in k_2 (equations 2.5 and 2.6). It follows that the values

$$k_{2f} = \frac{-h_1}{2h_2}$$
$$k_{2c} = \frac{-a_1}{2a_2}$$

minimise, respectively, the variances of $F(t)$ and $C(t)$.

In Figure 1, we plot contours for $Var[F(t)]$ and $Var[C(t)]$ over a range of values for k_1 and k_2 in the case where $p_1 = 0.4$ and $p_2 = 0.3$. By superimposing one set of contours on the other we are able to compare simultaneously the effect of k_1 and k_2 on the two variances. First suppose that $k_2 = 0$ (the old method for determining $C(t)$). The minimum value for $Var[C(t)]$ is just under 500 when k_1 is around 0.16. Minimising over k_2 as well clearly delivers substantial reductions in the variances. For example, if the objective is to minimize $Var[C(t)]$, then by the k_1 approach (minimize $Var[C(t)]$ over k_1 with $k_2 = 0$) we have $Var[F(t)] \approx 24000$ and $Var[C(t)] \approx 500$. By the k_2 approach (minimize $Var[C(t)]$ over k_1 and k_2), we have $Var[F(t)] \approx 12000$ (a reduction of about 50%) and $Var[C(t)] \approx 400$ (a reduction of about 20%) when $k_1 = 0.16$ and $k_2 = 240$.

Depending on what the plan objectives and constraints are, we will have different strategies for k_1 and k_2. One example might be the imposition of a constraint that $Var[F(t)]$ is less than 6000. We then choose a k_1 which is larger than about 0.2. Then, given k_1 it is always optimal to choose k_2 between the lines for k_{2f} and k_{2c} (since there is always a value in this interval which can reduce both variances compared with values of k_2 outside). A second example might specify the value of k_1 (for example, an amortization factor based on the average future working lifetime) with minimisation over k_2 only. Then it will always be efficient to choose a value of k_2 between $k_{2c}(k_1)$ and $k_{2f}(k_1)$. Any value outside this range can be improved upon (that is both $Var[C(t)]$ and $Var[F(t)]$ can be reduced) by changing k_2 to a suitable point between $k_{2c}(k_1)$ and $k_{2f}(k_1)$. We define the region between the lines k_{2f} and k_{2c} as the **efficient region**.

To be more precise, for a fixed value of k_1, we define $k_2^* = \min(k_{2f}, 2k_{2c})$ and $k_2^\diamond = \min(k_{2c}, 2k_{2f})$. $Var[F(t)]$ and $Var[C(t)]$ are quadratic functions of k_2, achieving their minima at k_{2f} and k_{2c} respectively by definition.

If $k_{2f} > k_{2c}$, choose any $\hat{k}_2 \in [k_{2c}, k_2^*]$. Since $0 \le \hat{k}_2 \le k_{2f}$, we have

$$Var[F(t)]_{k_2=0} \ge Var[F(t)]_{\hat{k}_2} \ge Var[F(t)]_{k_{2f}}$$

and since $k_{2c} \le \hat{k}_2 \le 2k_{2c}$, we have

$$Var[C(t)]_{k_2=0} = Var[C(t)]_{2k_{2c}} \ge Var[C(t)]_{\hat{k}_2} \ge Var[C(t)]_{k_{2c}}.$$

Hence $k_2 = \hat{k}_2$ achieves a simultaneous reduction in both $Var[F(t)]$ and $Var[C(t)]$ from their values at $k_2 = 0$ in the case of $k_{2f} > k_{2c}$.

If $k_{2c} > k_{2f}$, choose any $\hat{k}_2 \in [k_{2f}, k_2^\diamond]$. Since $0 \le \hat{k}_2 \le k_{2c}$, we have

$$Var[C(t)]_{k_2=0} \ge Var[C(t)]_{\hat{k}_2} \ge Var[C(t)]_{k_{2c}}$$

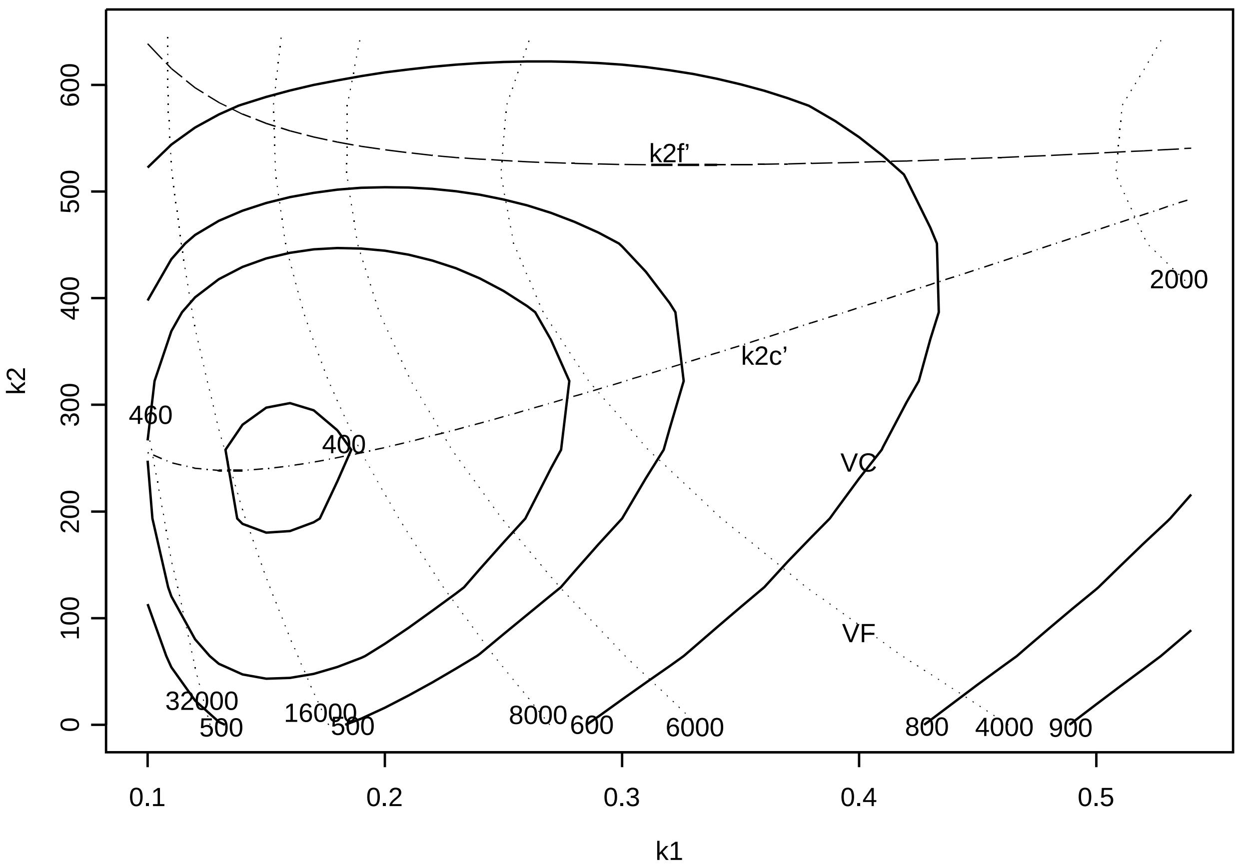

Figure 1: Contour plot of $Var[F(t)]$ (dotted lines, contours at the levels 2000, 4000, 6000, 8000, 16000, and 32000) and $Var[C(t)]$ (solid lines, contours at the levels 400, 500, 600, 800 and 900) for different k_2 and k_1 with $p_1 = 0.4$ (equities) and $p_2 = 0.3$ (bonds). Also plotted are $k_{2f}(k_1)$ (short dashed line) and $k_{2c}(k_1)$ (long dashed line). Parameter values are $y = 0.03, \Delta_e = 0.02, \Delta_b = 0.01, \phi = 0.7, \sigma_e = 0.12, \sigma_{by} = -0.05, \sigma_{ey} = -0.03, \sigma_y = 0.03, \sigma_{eb} = 0.02, \sigma_b = 0.03$, $y' = 0.0309$ and $i_v = 0.02$.

and since $k_{2f} \leq \hat{k}_2 \leq 2k_{2f}$, we have

$$Var[F(t)]_{k_2=0} = Var[F(t)]_{2k_{2f}} \geq Var[F(t)]_{\hat{k}_2} \geq Var[F(t)]_{k_{2f}}.$$

Hence $k_2 = \hat{k}_2$ achieves a simultaneous reduction in both $Var[F(t)]$ and $Var[C(t)]$ from their values at $k_2 = 0$ in the case of $k_{2c} > k_{2f}$.

If $k_{2c} = k_{2f}$, then $\hat{k}_2 = k_{2c} = k_{2f}$ is the best strategy for reducing both $Var[F(t)]$ and $Var[C(t)]$ simultaneously.

These ideas are illustrated in Figure 2. In the top graph (a) we have plotted k_{2c} and k_2^*. Given k_1, any value of k_2 between k_{2c} and k_2^* will reduce both $VarF(t)$ and $VarC(t)$ relative to $k_2 = 0$. However, in some cases ($k_1 < 0.24$) $VarF(t)$ can be reduced further by increasing k_2 from k_2^* to k_{2f} (Figure 2, bottom (b)).

Corresponding to Figure 2(a), Figure 2(b) gives us the graphs of $Var[F(t)]_{k_2=0}$, $Var[F(t)]$ subject to $Var[C(t)] \leq Var[C(t)]_{k_2=0}$ and optimal $Var[F(t)]$. We see from Figure 2(b) that VF^* ($VarF(t)$ at k^*) is not much different from VF' ($VarF(t)$ at k_{2f}) and that it can give us the rate of minimum $Var[F(t)]$ subject to $Var[C(t)] \leq Var[C(t)]_{k_2=0}$. Thus, from the *efficient region* we obtain a region which guarantees reductions in both $Var[F(t)]$ and $Var[C(t)]$. Within this efficient region, we then can choose an optimal adjustment of contribution rate according to different objective functions and constraints.

4 Optimal Investment and Contribution Strategies

In this section we will consider optimization when there are specific objectives and constraints put in place. In the previous discussion we were concerned only with minimisation of the Variance of $F(t)$ or $C(t)$. As the basis for what follows we will start by investigating the problem:

$$\text{minimize over } k_1 \text{ and } k_2\text{:} \quad Var[C(t)], \qquad \text{subject to} \quad Var[F(t)] = V_f$$

and for specified values of p_1 (equities) and p_2 (bonds).

In Figure 3 we have plotted contours for the optimal values of k_1 (left-hand plot) and k_2 (right-hand plot). In this plot we have restricted ourselves to asset strategies where $p_1 + p_2 = 1$ (that is, zero investment in cash). For example, when we require $VarF(t) = V_f = 8,000$ with $p_1 = 0.4$ and $p_2 = 0.6$, the optimal value for k_1 is just under 0.1, and the optimal value for k_2 is a bit less than 170. (The slightly lumpy nature of these curves is a consequence of the interpolation and the underlying lattice structures being used in the numerical procedures.)

In Figure 4 we show what the consequences are of using these optimal values for k_1 and k_2 for the chosen values of V_f, p_1 and p_2. For these inputs we have calculated the values of $Var[C(t)]$, $E[C(t)]$ and $E[F(t)]$. Contours for each of these variables are shown in Figure 4. First, (solid lines) we can see that $Var[C(t)]$ decreases as we move from left to right. This reflects the fact that we are investing more in bonds and less in equities. For the same reason, however, $E[C(t)]$ is increasing from left to right, since bonds are low return as well as low risk. The impact of this is less marked on $E[F(t)]$, which at first

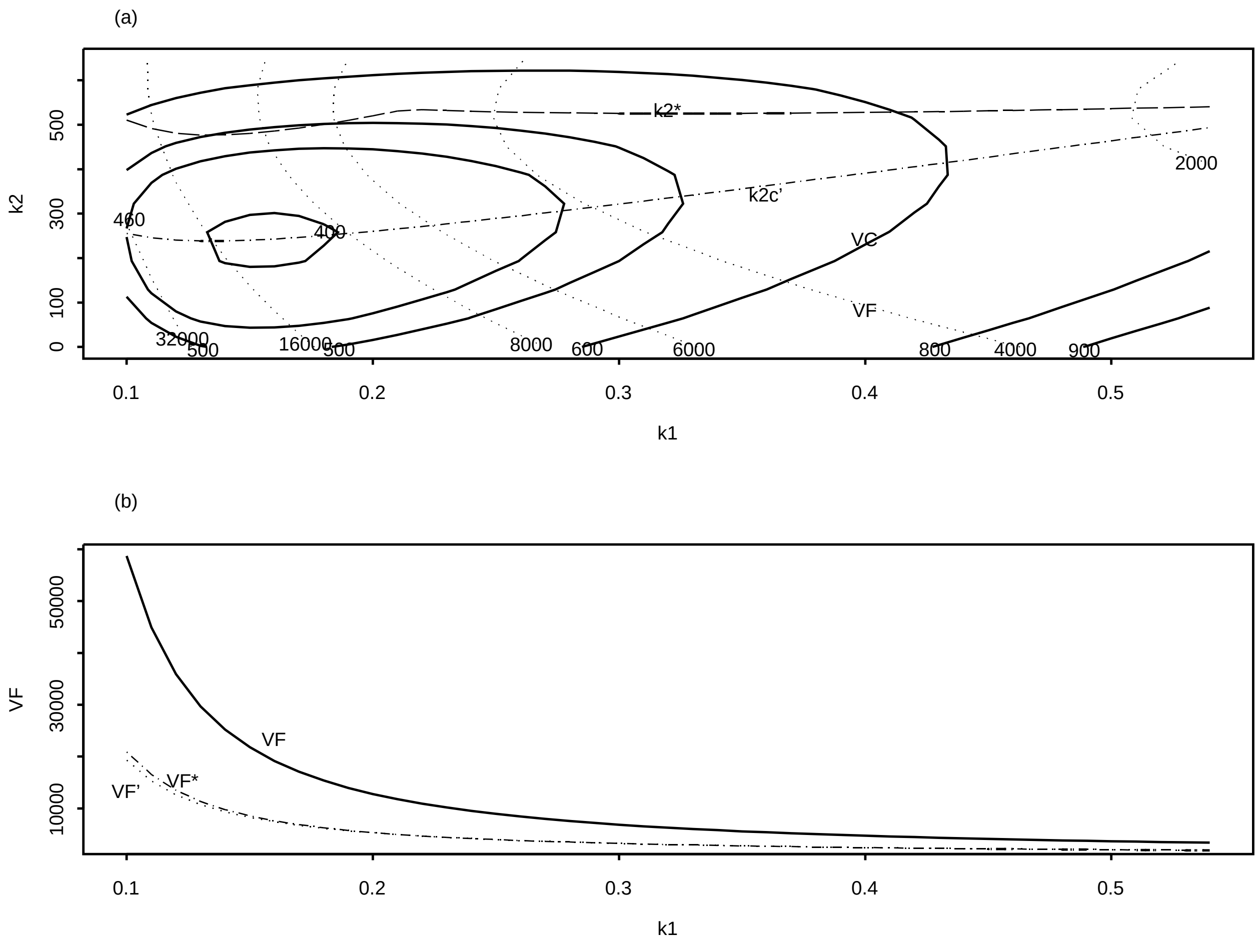

Figure 2: (a) (top) Contour plot of $Var[F(t)]$ (dotted lines) and $Var[C(t)]$ (solid lines) for different k_2 and k_1 when $p_1 = 0.4$ and $p_2 = 0.3$. Also plotted are k_2^* (long dashed line) and k_{2c} (short dashed line). (b) (bottom) Values of VF ($Var[F(t)]$ at $k_2 = 0$), VF^* ($Var[F(t)]$ at k_2^*) and VF' ($Var[F(t)]$ at k_{2f}) corresponding to the first graph. Parameter values: $y = 0.03, \Delta_e = 0.02, \Delta_b = 0.01, \phi = 0.7, \sigma_e = 0.12, \sigma_{by} = -0.05, \sigma_{ey} = -0.03, \sigma_y = 0.03, \sigma_{eb} = 0.02, \sigma_b = 0.03$, $y' = 0.0309$ and $i_v = 0.02$.

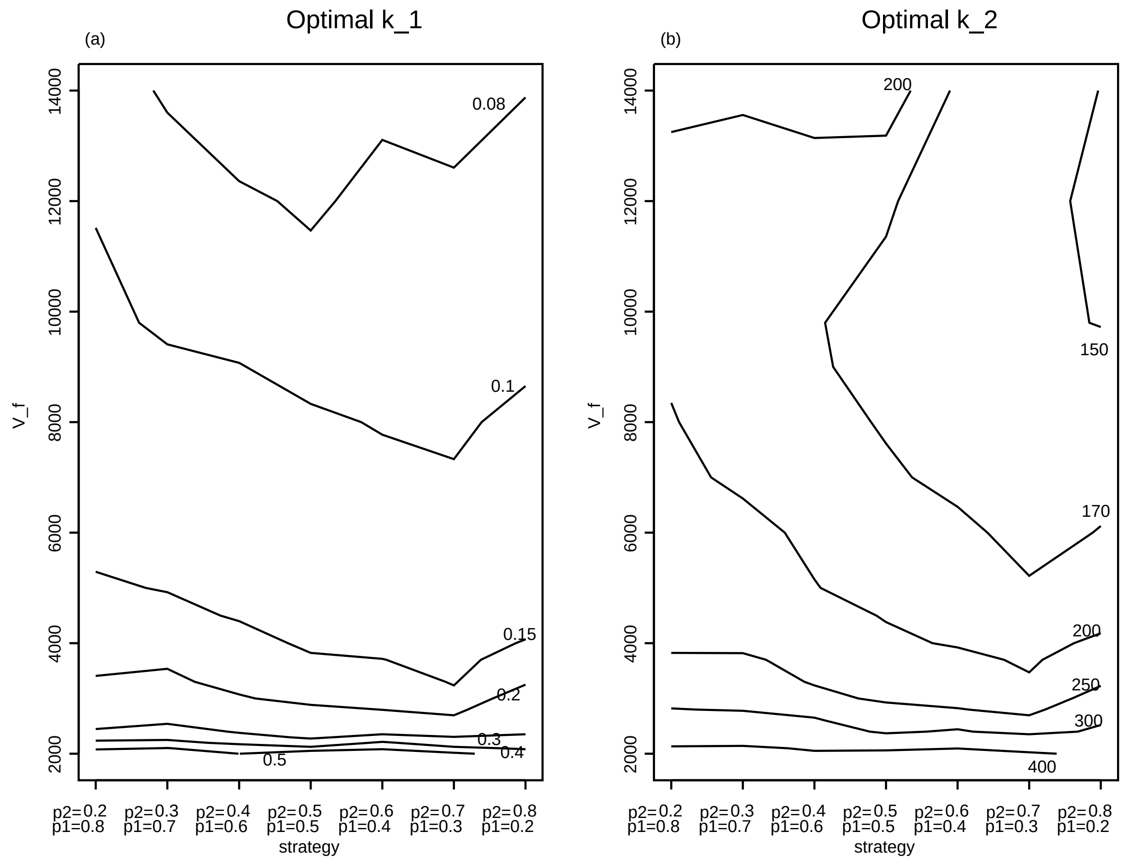

Figure 3: Contour plots for the optimal values of k_1 (left-hand plot (a)) and k_2 (right-hand plot (b)) for the problem *minimise* $VarC(t)$ subject to $VarF(t) = V_f$ and for specified asset strategies (p_1, p_2). p_1 = proportion in equities, p_2 = proportion in bonds. Parameter values are $y = 0.03, \Delta_e = 0.02, \Delta_b = 0.01, \phi = 0.7, \sigma_e = 0.12, \sigma_{by} = -0.05, \sigma_{ey} = -0.03, \sigma_y = 0.03, \sigma_{eb} = 0.02, \sigma_b = 0.03$, $y' = 0.0309$ and $i_v = 0.02$

is surprising. However, we can see from Figure 3 that k_1 is closely linked to the value of V_f: the lowest values of $Var[F(t)]$ can only be achieved by amortizing surplus or deficit as quickly as possible (that is, by having k_1 close to 1). The same high values of k_1 mean that $E[F(t)]$ will be close to the actuarial liability $AL = 644$ (Table 1, for $i_v = 0.02$).

Example 1: Suppose the objective function is to minimize $Var[C(T)]$ with the constraint that $Var[F(T)]$ is less than 8000. From Figure 3 we can see that k_1 must greater than around 0.1 (that is, the amortization period should be less than about 11 years). If the required $Var[C(T)]$ can not be more than 200, then Figure 4 indicates that the investment strategy cannot allocate more than 45% to equities. If we further require that $E[C(T)]$ can not be more than 5, then we become restricted to an approximately triangular region in Figure 4. This region indicates that we must invest between 35% and 45% in equities and k_1 should be between 0.1 and 0.15. For example: If we require that $Var[C(T)] = 150$ and $E[C(T)] = 5$, our optimal strategy is to invest 37 percent in equity and 63 percent in bonds and have the amortization period around 11 years with the corresponding values

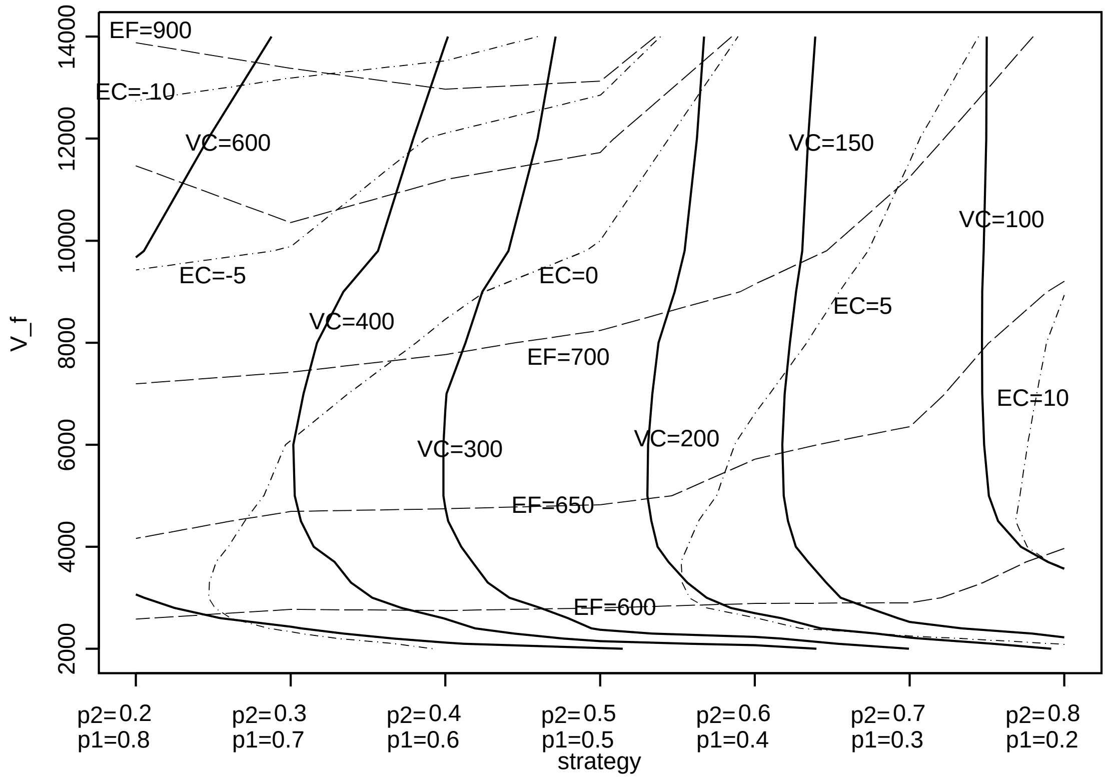

Figure 4: Contours for $Var[C(t)]$ (solid lines), $E[C(t)]$ (dot-dashed lines) and $E[F(t)]$ (long-dashed lines) as a function of V_f, p_1 and p_2 and assuming that the optimal values for k_1 and k_2 are being used for each (V_f, p_1, p_2). Parameter values are $y = 0.03, \Delta_e = 0.02, \Delta_b = 0.01, \phi = 0.7, \sigma_e = 0.12, \sigma_{by} = -0.05, \sigma_{ey} = -0.03, \sigma_y = 0.03, \sigma_{eb} = 0.02, \sigma_b = 0.03$, $y' = 0.0309$ and $i_v = 0.02$

of $E[F(T)] = 675$ and $k_2 = 165$.

Example 2: If we wish to obtain an optimal $Var[C(T)]$ under the control that $E[C(T)]$ is between 0 and 5, $Var[F(T)]$ is less than 8000, and $E[F(T)]$ is more than 650, the available region in Figure 4 would be a diamond shape, with investment strategy holding equities between 37 percent and 73 percent, and V_f between 4500 and 8000. The minimum $Var[C(t)]$ would be just under 150 at the top right corner of this diamond. Our optimal strategy then is to invest 36 percent in equity and the rest in bonds with the amortization period near to 12 years (k_1=0.095) and with $Var[F(T)] = 8000$ and $E[C(T)] = 5$. If, instead, we wish to restrict $Var[C(T)]$ to be not more than 300 and seek for the smallest $E[C(T)]$, we will obtain an optimal $E[C(T)] = 1$. Our optimal strategy is to invest 58 percent in equity and 42 percent in bonds with the amortization period near to 10 years ($k_1 = 0.11$) and $Var[F(T)] = 8000$.

Example 3: If our constraint is that the amortization period must not be more than 7 years (that is, we require k_1 to be larger than 0.15), and $E[C(T)]$ is less than 5, in order to minimize $Var[C(T)]$, our optimal strategy will be to invest 46 percent in equity and the rest in bonds with the optimal $Var[C(T)]$ equal to about 200.

5 Conclusions

In this paper we have investigated a model for defined-benefit pension plans which incorporates a Vasicek type of model for the short-term interest rate and three assets: cash, bonds and equities. We have proposed a simple method for adjusting the contribution rate to account for the current level of interest rates as well as the usual adjustment for the current funding level. Using this model we have derived formulae for the unconditional moments of the funding level and the contribution rate.

A number of illustrative examples have been given which demonstrate that the new adjustment, taking account of current interest rates, to the contribution rate does improve stability significantly, particularly where there is a strong degree of persistence in interest rates. The approach therefore indicates that the standard approach to liability valuation using an artificial valuation interest rate can be improved upon by making an adjustment for market conditions. What we have not done here is to look at direct methods for valuing liabilities using the current term-structure of interest rates. This is a topic for further investigation.

We have developed further the notion of efficient regions for various subsets of the control parameters k_1, k_2, i_v and (p_1, p_2) depending on different constraints and objectives. These are regions that we can move into to reduce the variances of both $F(t)$ and $C(t)$.

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Appendix: Proof of Theorem 2.2

(a)(i)
Recall that

$$\begin{aligned} F(t) &= (\theta_v - k_2)\sum_{s=0}^{\infty}(1-k_1)^s \exp\left(S_y(t,s)+S_p(t,s)\right) \\ &\quad + k_2 e^{y'}\sum_{s=0}^{\infty}(1-k_1)^s \exp\left(S_y(t,s)-y(t-1-s)+S_p(t,s)\right). \end{aligned} \tag{A.1}$$

For notational convenience write

$$\begin{aligned} X_s &= \exp\left(S_y(t,s)+S_p(t,s)\right) \\ \text{and } Y_s &= \exp\left(S_y(t,s)-y(t-1-s)+S_p(t,s)\right). \end{aligned}$$

Then

$$\begin{aligned} E[F(t)] &= (\theta_v - k_2)\sum_{s=0}^{\infty}(1-k_1)^s E[X_s] + k_2 e^{y'}\sum_{s=0}^{\infty}(1-k_1)^s E[Y_s] \\ \text{with } E[X_s] &= \exp\left[(s+1)(y+\alpha_0)+\frac{1}{2}V_1(s)\right] \\ E[Y_s] &= \exp\left[(s+1)(y+\alpha_0)-y+\frac{1}{2}V_s(s)\right] \\ \text{where } V_1(s) &= Var\left(S_y(t,s)+S_p(t,s)\right) \\ V_2(s) &= Var\left(S_y(t,s)-y(t-1-s)+S_p(t,s)\right) \end{aligned}$$

(a)(ii)
Next recall that

$$C(t) = NC + k_1(AL - F(t)) + k_2\left(e^{y'-y(t)}-1\right). \tag{A.2}$$

Hence,

$$\begin{aligned} E[C(t)] &= NC + k_1(AL - E[F(t)]) + k_2\left(E\left[e^{y'-y(t)}\right]-1\right) \\ &= NC + k_1(AL - E[F(t)]) + k_2\left(E\left[e^{y'-y+\frac{1}{2}\gamma_y(0)}\right]-1\right) \\ \text{where } \gamma_y(s) &= Cov[y(t), y(t-s)] \\ &= \frac{\sigma_y^2\phi^s}{1-\phi^2}. \end{aligned}$$

(b)(i)

From equation (A.1) we also have

$$\begin{aligned} Var[F(t)] &= (\theta_v - k_2)^2 \sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_1(r,s) + 2(\theta_v - k_2)k_2 e^{y'} \sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_2(r,s) \\ &\quad + k_2^2 e^{2y'} \sum_{r,s=0}^{\infty}(1-k_1)^{r+s}C_3(r,s) \end{aligned}$$

$$\begin{aligned} \text{where } C_1(r,s) &= Cov(X_r, X_s) \\ C_2(r,s) &= Cov(Y_r, X_s) \\ C_3(r,s) &= Cov(Y_r, Y_s). \end{aligned}$$

Expressions for C_1, C_2 and C_3 are given below. Finally we separate out terms involving k_2 and k_2^2 to get $Var[F(t)] = h_0 + h_1 k_2 + h_2 k_2^2$ as in the statement of the theorem.

(b)(ii)

From (A.2) we can deduce that

$$\begin{aligned} Var[C(t)] &= k_1^2 Var[F(t)] - 2k_1 k_2 e^{y'} Cov\left[F(t), e^{-y(t)}\right] + k_2^2 e^{2y'} Var\left[e^{-y(t)}\right] \\ &= k_1^2(h_0 + h_1 k_2 + h_2 k_2^2) \\ &\quad -2k_1 k_2 e^{y'}\left((\theta_v - k_2)\sum_{s=0}^{\infty}(1-k_1)^s C_4(s) + k_2 e^{y'}\sum_{s=0}^{\infty}(1-k_1)^s C_5(s)\right) \\ &\quad + k_2^2 e^{2y'} Var\left[e^{-y(t)}\right] \end{aligned}$$

$$\begin{aligned} \text{where } C_4(s) &= Cov\left[X_s, e^{-y(t)}\right] \\ C_5(s) &= Cov\left[Y_s, e^{-y(t)}\right] \end{aligned}$$

Rearranging this we get

$$\begin{aligned} Var[C(t)] &= a_0 + a_1 k_2 + a_2 k_2^2 \\ \text{where } a_0 &= k_1^2 h_0 \\ a_1 &= k_1^2 h_1 - 2k_1 e^{y'}\theta_v \sum_{s=0}^{\infty}(1-k_1)^s C_4(s) \\ a_2 &= k_1^2 h_2 + 2k_1 e^{y'}\sum_{s=0}^{\infty}(1-k_1)^s C_4(s) - 2k_1 e^{2y'}\sum_{s=0}^{\infty}(1-k_1)^s C_5(s) + k_2^2 e^{2y'} Var\left[e^{-y(t)}\right] \end{aligned}$$

To calculate these moments more explicitly we need to work out the V_i's and the C_i's.

$$\begin{aligned} C_1(r,s) &= Cov\left(e^{S_y(t,r)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)}\right) \\ &= E[e^{S_y(t,r)+S_p(t,r)+S_y(t,s)+S_p(t,s)}] - E[e^{S_y(t,r)+S_p(t,r)}] \times E[e^{S_y(t,s)+S_p(t,s)}] \\ &= \exp\left((r+s+2)(y+\alpha_0) + \frac{1}{2}W_1(r,s)\right) \\ &\quad - \exp\left((r+1)(y+\alpha_0) + \frac{1}{2}V_1(r)\right)\exp\left((s+1)(y+\alpha_0) + \frac{1}{2}V_1(s)\right) \end{aligned}$$

$$
\begin{aligned}
C_2(r,s) &= Cov\left(e^{S_y(t,r-1)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)}\right) \\
&= E[e^{S_y(t,r-1)+S_p(t,r)+S_y(t,s)+S_p(t,s)}] - E[e^{S_y(t,r-1)+S_p(t,r)}] \times E[e^{S_y(t,s)+S_p(t,s)}] \\
&= \exp\left((r+s+2)(y+\alpha_0) - y + \frac{1}{2}W_2(r,s)\right) \\
&\quad - \exp\left((r+1)(y+\alpha_0) - y + \frac{1}{2}V_2(r)\right)\exp\left((s+1)(y+\alpha_0) + \frac{1}{2}V_1(s)\right)
\end{aligned}
$$

$$
\begin{aligned}
C_3(r,s) &= Cov\left(e^{S_y(t,r-1)+S_p(t,r)}, e^{S_y(t,s-1)+S_p(t,s)}\right) \\
&= E[e^{S_y(t,r-1)+S_p(t,r)+S_y(t,s-1)+S_p(t,s)}] - E[e^{S_y(t,r-1)+S_p(t,r)}] \times E[e^{S_y(t,s-1)+S_p(t,s)}] \\
&= \exp\left((r+s+2)(y+\alpha_0) - 2y + \frac{1}{2}W_3(r,s)\right) \\
&\quad - \exp\left((r+1)(y+\alpha_0) - y + \frac{1}{2}V_2(r)\right)\exp\left((s+1)(y+\alpha_0) - y + \frac{1}{2}V_2(s)\right)
\end{aligned}
$$

$$
\begin{aligned}
C_4(r) &= Cov\left(e^{S_y(t,r)+S_p(t,r)}, e^{-y(t)}\right) \\
&= E[e^{S_y(t,r)+S_p(t,r)-y(t)}] - E[e^{S_y(t,r)+S_p(t,r)}] \times E[e^{-y(t)}] \\
&= \exp\left((r+1)(y+\alpha_0) - y + \frac{1}{2}W_4(r,s)\right) \\
&\quad - \exp\left((r+1)(y+\alpha_0) + \frac{1}{2}V_1(r)\right)\exp\left((-y + \frac{1}{2}\gamma_y(0)\right)
\end{aligned}
$$

$$
\begin{aligned}
C_5(r) &= Cov\left(e^{S_y(t,r-1)+S_p(t,r)}, e^{-y(t)}\right) \\
&= E[e^{S_y(t,r-1)+S_p(t,r)-y(t)}] - E[e^{S_y(t,r-1)+S_p(t,r)}] \times E[e^{-y(t)}] \\
&= \exp\left((r+1)(y+\alpha_0) - 2y + \frac{1}{2}W_5(r,s)\right) \\
&\quad - \exp\left((r+1)(y+\alpha_0) - y + \frac{1}{2}V_2(r)\right)\exp\left((-y + \frac{1}{2}\gamma_y(0)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
V_1(s) &= Var\left(S_y(t,s) + S_p(t,s)\right) \\
V_2(s) &= Var\left(S_y(t,s-1) + S_p(t,s)\right) \\
W_1(r,s) &= Var\left(S_y(t,r) + S_y(t,s) + S_p(t,r) + S_p(t,s)\right) \\
W_2(r,s) &= Var\left(S_y(t,r-1) + S_y(t,s) + S_p(t,r) + S_p(t,s)\right) \\
W_3(r,s) &= Var\left(S_y(t,r-1) + S_y(t,s-1) + S_p(t,r) + S_p(t,s)\right) \\
W_4(s) &= Var\left(S_y(t,s) + S_p(t,s) - y(t)\right) \\
W_5(s) &= Var\left(S_y(t,s-1) + S_p(t,s) - y(t)\right)
\end{aligned}
$$

These formulae for the C_k exploit the normality of $y(t)$, $S_y(t,r)$ etc.
We now derive each of these five functions:

$$
\begin{aligned}
V_1(s) &= Var\left(S_y(t,s)+S_p(t,s)\right) \\
&= (\alpha_2^2+\alpha_3^2)(s+1) \\
&\quad +\alpha_1^2(s+1)+\frac{2\alpha_1\sigma_y}{1-\phi}(s-\frac{\phi(1-\phi^s)}{1-\phi}) \\
&\quad +\frac{\sigma_y^2}{(1-\phi)^2}\left(s-\frac{2\phi(1-\phi^s)}{1-\phi}+\frac{\phi^2(1-\phi^{2s})}{1-\phi^2}\right) \\
&\quad +\left(\frac{(1-\phi^{s+1})\sigma_y}{1-\phi}\right)^2 \\
&\quad +\frac{\sigma_y^2}{(1-\phi)^2}(1-\phi^{s+1})^2\frac{\phi^2}{1-\phi^2}
\end{aligned}
$$

If $s=0$, then $V_2(s)=\alpha_1^2+\alpha_2^2+\alpha_3^2$.
Suppose $s\geq 1$. Then:

$$
\begin{aligned}
V_2(s) &= Var\left(S_y(t,s-1)+S_p(t,s)\right) \\
&= (\alpha_2^2+\alpha_3^2)(s+1) \\
&\quad +\alpha_1^2(s+1)+\frac{2\alpha_1\sigma_y}{1-\phi}(s-\frac{\phi(1-\phi^s)}{1-\phi}) \\
&\quad +\frac{\sigma_y^2}{(1-\phi)^2}\left(s-\frac{2\phi(1-\phi^s)}{1-\phi}+\frac{\phi^2(1-\phi^{2s})}{1-\phi^2}\right) \\
&\quad +\frac{\sigma_y^2}{(1-\phi)^2}(1-\phi^{s})^2\frac{\phi^2}{1-\phi^2}
\end{aligned}
$$

For $W_1(r,s)$, if $r=s$ then $W_1(r,s)=4V_1(s)$.
Suppose $s\geq r$. Let

$$Q_1(r,s)=S_y(t,r)+S_y(t,s)+S_p(t,r)+S_p(t,s)$$

Then:

$$
\begin{aligned}
Q_1(r,s) &= S_y(t,r)+S_y(t,s)+S_p(t,r)+S_p(t,s) \\
&= 2\alpha_1 Z_y(t) + \sum_{j=1}^{r}\left(\frac{2(1-\phi^j)\sigma_y}{1-\phi}+2\alpha_1\right) Z_y(t-j) \\
&\quad + \left(\frac{2(1-\phi^{r+1})\sigma_y}{1-\phi}+\alpha_1\right) Z_y(t-r-1) \\
&\quad + \sum_{j=r+2}^{s}\left(\frac{(\phi^{j-r-1}(1-\phi^{r+1})+1-\phi^j)\sigma_y}{1-\phi}+\alpha_1\right) Z_y(t-j) \\
&\quad + \frac{(\phi^{s-r}(1-\phi^{r+1})+1-\phi^{s+1})\sigma_y}{1-\phi} Z_y(t-s-1) \\
&\quad + \sum_{j=s+2}^{\infty}\left(\frac{\phi^{j-r-1}(1-\phi^{r+1})+\phi^{j-s-1}(1-\phi^{s+1})}{1-\phi}\sigma_y\right) Z_y(t-j) \\
&\quad + 2\alpha_2 \sum_{j=0}^{r} Z_e(t-j) + \alpha_2 \sum_{j=r+1}^{s} Z_e(t-j) \\
&\quad + 2\alpha_3 \sum_{j=0}^{r} Z_b(t-j) + \alpha_2 \sum_{j=r+1}^{s} Z_b(t-j)
\end{aligned}
$$

Thus:

$$
\begin{aligned}
& W_1(r,s) \\
=\; & Var(Q_1(r,s)) \\
=\; & (\alpha_2^2+\alpha_3^2)(4+s+3r) \\
& +4\alpha_1^2(r+1) + \frac{8\alpha_1\sigma_y}{1-\phi}(r-\frac{\phi(1-\phi^r)}{1-\phi}) \\
& +\frac{4\sigma_y^2}{(1-\phi)^2}\left(r-\frac{2\phi(1-\phi^r)}{1-\phi}+\frac{\phi^2(1-\phi^{2r})}{1-\phi^2}\right) \\
& +\left(\frac{2(1-\phi^{r+1})\sigma_y}{1-\phi}+\alpha_1\right)^2 \\
& +\alpha_1^2(s-r-1)+\frac{2\alpha_1\sigma_y}{1-\phi}\left(s-r-1+\frac{\phi(1-\phi^{s-r-1})}{1-\phi}-\frac{2\phi^{r+2}(1-\phi^{s-r-1})}{1-\phi}\right) \\
& +\frac{\sigma_y^2}{(1-\phi)^2}\left(s-r-1+2(1-2\phi^{r+1})\frac{\phi(1-\phi^{s-r-1})}{1-\phi}\right) \\
& +\frac{\sigma_y^2}{(1-\phi)^2}\left((1-4\phi^{r+1}+4\phi^{2r+2})\frac{\phi^2(1-\phi^{2(s-r-1)})}{1-\phi^2}\right) \\
& +\frac{\phi^2\sigma_y^2}{1-\phi^2}\frac{(1+\phi^{s-r}-2\phi^{s+1})^2}{(1-\phi)^2}
\end{aligned}
$$

Suppose $s \geq r$. Then:

$$\begin{aligned}
Q_2(r,s) &= S_y(t,r-1) + S_y(t,s) + S_p(t,r) + S_p(t,s) \\
&= 2\alpha_1 Z_y(t) + \sum_{j=1}^{r} \left(\frac{2(1-\phi^j)\sigma_y}{1-\phi} + 2\alpha_1 \right) Z_y(t-j) \\
&\quad + \sum_{j=r+1}^{s} \left(\frac{(\phi^{j-r}(1-\phi^r) + 1 - \phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) \\
&\quad + \frac{(\phi^{s-r+1}(1-\phi^r) + 1 - \phi^{s+1})\sigma_y}{1-\phi} Z_y(t-s_1) \\
&\quad + \sum_{j=s+2}^{\infty} \left(\frac{\phi^{j-r}(1-\phi^r) + \phi^{j-s-1}(1-\phi^{s+1})}{1-\phi} \sigma_y \right) Z_y(t-j) \\
&\quad + 2\alpha_2 \sum_{j=0}^{r} Z_e(t-j) + \alpha_2 \sum_{j=r+1}^{s} Z_e(t-j) \\
&\quad + 2\alpha_3 \sum_{j=0}^{r} Z_b(t-j) + \alpha_2 \sum_{j=r+1}^{s} Z_b(t-j)
\end{aligned}$$

$$\begin{aligned}
& W_2(r,s) \\
=\; & Var(Q_2(r,s)) \\
=\; & (\alpha_2^2 + \alpha_3^2)(4 + s + 3r) \\
& + 4\alpha_1^2(r+1) + \frac{8\alpha_1\sigma_y}{1-\phi}\left(r - \frac{\phi(1-\phi^r)}{1-\phi}\right) \\
& + \frac{4\sigma_y^2}{(1-\phi)^2} \left(r - \frac{2\phi(1-\phi^r)}{1-\phi} + \frac{\phi^2(1-\phi^{2r})}{1-\phi^2} \right) \\
& + \alpha_1^2(s-r) + \frac{2\alpha_1\sigma_y}{1-\phi} \left(s - r + (1-2\phi^r)\frac{\phi(1-\phi^{s-r})}{1-\phi} \right) \\
& + \frac{\sigma_y^2}{(1-\phi)^2} \left(s - r + 2(1-2\phi^r)\frac{\phi(1-\phi^{s-r})}{1-\phi} + (1 - 4\phi^r + 4\phi^{2r})\frac{\phi^2(1-\phi^{2(s-r)})}{1-\phi^2} \right) \\
& + \left(\frac{(\phi^{s-r+1}(1-\phi^r) + 1 - \phi^{s+1})\sigma_y}{1-\phi} \right)^2 \\
& + \frac{\sigma_y^2}{(1-\phi)^2}(1 + \phi^{s-r+1} - 2\phi^{s+1})^2 \frac{\phi^2}{1-\phi^2}
\end{aligned}$$

Suppose $s = r$. Then:

$$
\begin{aligned}
Q_2(r,s) &= S_y(t,s-1) + S_y(t,s) + 2S_p(t,s) \\
&= 2\alpha_1 Z_y(t) + \sum_{j=1}^{s} \left(\frac{2(1-\phi^{j+1})\sigma_y}{1-\phi} + 2\alpha_1 \right) Z_y(t-j) \\
&\quad + \left(\frac{(\phi(1-\phi^s) + 1 - \phi^{s+1})\sigma_y}{1-\phi} \right) Z_y(t-s-1) \\
&\quad + \sum_{j=s+2}^{\infty} \left(\frac{\phi^{j-s+1}(1-\phi^{s+1}) + \phi^{j-s}(1-\phi^s)}{1-\phi} \sigma_y \right) Z_y(t-j) \\
&\quad + 2\alpha_2 \sum_{j=0}^{s} Z_e(t-j) \\
&\quad + 2\alpha_3 \sum_{j=0}^{s} Z_b(t-j)
\end{aligned}
$$

$$
\begin{aligned}
W_2(r,s) &= Var(Q_2(r,s)) \\
&= (\alpha_2^2 + \alpha_3^2)(4+4s) \\
&\quad + 4\alpha_1^2(s+1) + \frac{8\alpha_1\sigma_y}{1-\phi}\left(s - \frac{\phi(1-\phi^s)}{1-\phi}\right) \\
&\quad + \frac{4\sigma_y^2}{(1-\phi)^2} \left(s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad + \left(\frac{(\phi(1-\phi^s) + 1 - \phi^{s+1})\sigma_y}{1-\phi} \right)^2 \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2}(1+\phi-2\phi^{s+1})^2 \frac{\phi^2}{(1-\phi^2)}
\end{aligned}
$$

Suppose $r \geq s$. Then:

$$\begin{aligned}
Q_2(r,s) &= S_y(t,r-1) + S_y(t,s) + S_p(t,r) + S_p(t,s) \\
&= 2\alpha_1 Z_y(t) + \sum_{j=1}^{s} \left(\frac{2(1-\phi^{j+1})\sigma_y}{1-\phi} + 2\alpha_1 \right) Z_y(t-j) \\
&\quad + \left(\frac{2(1-\phi^{s+1})\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-s_1) \\
&\quad + \sum_{j=s+2}^{r} \left(\frac{(\phi^{j-s-1}(1-\phi^{s+1}) + 1 - \phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) \\
&\quad + \sum_{j=r+1}^{\infty} \left(\frac{\phi^{j-r}(1-\phi^r) + \phi^{j-s-1}(1-\phi^{s+1})}{1-\phi} \sigma_y \right) Z_y(t-j) \\
&\quad + 2\alpha_2 \sum_{j=0}^{s} Z_e(t-j) + \alpha_2 \sum_{j=s+1}^{r} Z_e(t-j) \\
&\quad + 2\alpha_3 \sum_{j=0}^{s} Z_b(t-j) + \alpha_3 \sum_{j=s+1}^{r} Z_b(t-j)
\end{aligned}$$

$$\begin{aligned}
W_2(r,s) &= Var(Q_2(r,s)) \\
&= (\alpha_2^2 + \alpha_3^2)(4 + r + 3s) \\
&\quad + 4\alpha_1^2(s+1) + \frac{8\alpha_1\sigma_y}{1-\phi}\left(s - \frac{\phi(1-\phi^s)}{1-\phi}\right) \\
&\quad + \frac{4\sigma_y^2}{(1-\phi)^2} \left(s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad + \left(\frac{2(1-\phi^{s+1})\sigma_y}{1-\phi} + \alpha_1 \right)^2 \\
&\quad + \alpha_1^2(r-s-1) + \frac{2\alpha_1\sigma_y}{1-\phi} \left(r - s - 1 + (1 - 2\phi^{s+1}) \frac{\phi(1-\phi^{r-s-1})}{1-\phi} \right) \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} \left(r - s - 1 + 2(1 - 2\phi^{s+1}) \frac{\phi(1-\phi^{r-s-1})}{1-\phi} \right) \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} \left((1 - 4\phi^{s+1} + 4\phi^{2s+2}) \frac{\phi^2(1-\phi^{2(r-s-1)})}{1-\phi^2} \right) \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} (1 + \phi^{r-s-1} - 2\phi^r)^2 \frac{\phi^2}{(1-\phi^2)}
\end{aligned}$$

Suppose $r = 0$. Then:

$$\begin{aligned}
Q_2(r,s) &= S_y(t,s) + S_p(t,0) + S_p(t,s) \\
&= 2\alpha_1 Z_y(t) \\
&\quad + \sum_{j=1}^{s} \left(\frac{(1-\phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) \\
&\quad + \left(\frac{(1-\phi^{s+1})\sigma_y}{1-\phi} \right) Z_y(t-s-1) \\
&\quad + \sum_{j=s+2}^{\infty} \left(\frac{\phi^{j-s-1}(1-\phi^{s+1})}{1-\phi} \sigma_y \right) Z_y(t-j) \\
&\quad + 2\alpha_2 Z_e(t) + \alpha_2 \sum_{j=1}^{s} Z_e(t-j) \\
&\quad + 2\alpha_3 Z_b(t) + \alpha_3 \sum_{j=1}^{s} Z_b(t-j)
\end{aligned}$$

$$\begin{aligned}
W_2(r,s) &= Var(Q_2(r,s)) \\
&= (\alpha_2^2 + \alpha_3^2)(s+4) + 4\alpha_1^2 \\
&\quad + \alpha_1^2 s + \frac{2\alpha_1\sigma_y}{1-\phi}\left(s - \frac{\phi(1-\phi^s)}{1-\phi}\right) \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} \left(s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad + \left(\frac{(1-\phi^{s+1})\sigma_y}{1-\phi} \right)^2 \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2}(1-\phi^{s+1})^2 \frac{\phi^2}{(1-\phi^2)}
\end{aligned}$$

Now consider $W_3(r,s)$
Suppose $s \geq r$. Then:

$$\begin{aligned}
Q_3(r,s) &= S_y(t,r-1) + S_y(t,s-1) + S_p(t,r) + S_p(t,s) \\
&= 2\alpha_1 Z_y(t) + \sum_{j=1}^{r} \left(\frac{2(1-\phi^{j+1})\sigma_y}{1-\phi} + 2\alpha_1 \right) Z_y(t-j) \\
&\quad + \sum_{j=r+1}^{s} \left(\frac{(\phi^{j-r}(1-\phi^r) + 1 - \phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) \\
&\quad + \sum_{j=s+1}^{\infty} \left(\frac{\phi^{j-r}(1-\phi^r) + \phi^{j-s}(1-\phi^s)}{1-\phi} \sigma_y \right) Z_y(t-j) \\
&\quad + 2\alpha_2 \sum_{j=0}^{r} Z_e(t-j) + \alpha_2 \sum_{j=r+1}^{s} Z_e(t-j) \\
&\quad + 2\alpha_3 \sum_{j=0}^{r} Z_b(t-j) + \alpha_3 \sum_{j=r+1}^{s} Z_b(t-j)
\end{aligned}$$

$$\begin{aligned}
& W_3(r,s) \\
=\ & Var(Q_3(r,s)) \\
=\ & (\alpha_2^2 + \alpha_3^2)(4 + s + 3r) \\
& + 4\alpha_1^2(r+1) + \frac{8\alpha_1\sigma_y}{1-\phi}\left(r - \frac{\phi(1-\phi^r)}{1-\phi}\right) \\
& + \frac{4\sigma_y^2}{(1-\phi)^2}\left(r - \frac{2\phi(1-\phi^r)}{1-\phi} + \frac{\phi^2(1-\phi^{2r})}{1-\phi^2} \right) \\
& + \alpha_1^2(s-r) + \frac{2\alpha_1\sigma_y}{1-\phi}\left(s - r + (1-2\phi^r)\frac{\phi(1-\phi^{s-r})}{1-\phi} \right) \\
& + \frac{\sigma_y^2}{(1-\phi)^2}\left(s - r + 2(1-2\phi^r)\frac{\phi(1-\phi^{s-r})}{1-\phi} + (1 - 4\phi^r + 4\phi^{2r})\frac{\phi^2(1-\phi^{2(s-r)})}{1-\phi^2} \right) \\
& + \frac{\sigma_y^2}{(1-\phi)^2}(1 + \phi^{s-r} - 2\phi^s)^2 \frac{\phi^2}{(1-\phi^2)}
\end{aligned}$$

Suppose $r = s$. Then:

$$W_3(r,s) = 4V_2(s)$$

If $r = s = 0$, $W_3(r,s) = 4\alpha_1^2 + 4(\alpha_2^2 + \alpha_3^2)$.

Suppose $r = 0$. Then:

$$Q_3(r,s) = S_y(t,s-1) + S_p(t,0) + S_p(t,s)$$

$$\begin{aligned}
Q_3(r,s) &= S_y(t,s-1) + S_p(t,0) + S_p(t,s) \\
&= 2\alpha_1 Z_y(t) \\
&\quad + \sum_{j=1}^{s-1} \left(\frac{(1-\phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) \\
&\quad + \sum_{j=s+1}^{\infty} \left(\frac{\phi^{j-s}(1-\phi^s)}{1-\phi} \sigma_y \right) Z_y(t-j) \\
&\quad + 2\alpha_2 Z_e(t) + \alpha_2 \sum_{j=1}^{s} Z_e(t-j) \\
&\quad + 2\alpha_3 Z_b(t) + \alpha_3 \sum_{j=1}^{s} Z_b(t-j)
\end{aligned}$$

$$\begin{aligned}
W_3(r,s) &= Var(Q_3(r,s)) \\
&= (\alpha_2^2 + \alpha_3^2)(s+4) + 4\alpha_1^2 \\
&\quad + \alpha_1^2 s + \frac{2\alpha_1\sigma_y}{1-\phi}\left(s - \frac{\phi(1-\phi^s)}{1-\phi}\right) \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} \left(s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} (\phi - \phi^s)^2 \frac{\phi^2}{(1-\phi^2)}
\end{aligned}$$

$$y(t) = y + \sum_{j=0}^{\infty} \phi^j \sigma_y Z_y(t-j)$$

Consider:

$$
\begin{aligned}
Q_4(s) &= S_y(t,s) + S_p(t,s) - y(t) \\
&= sy + (s+1)\alpha_0 + (\alpha_1 - \sigma_y) Z_y(t) \\
&\quad + \sum_{j=1}^{s} \left(\frac{(1-\phi^j)\sigma_y}{1-\phi} - \phi^j \sigma_y + \alpha_1 \right) Z_y(t-j) \\
&\quad + \left(\frac{(1-\phi^{s+1})\sigma_y}{1-\phi} - \phi^{s+1}\sigma_y \right) Z_y(t-j) \\
&\quad + \sum_{j=s+2}^{\infty} \left(\frac{\phi^{j-s-1}(1-\phi^{s+1})}{1-\phi}\sigma_y - \phi^j \sigma_y \right) Z_y(t-j) \\
&\quad + \alpha_2 \sum_{j=0}^{s} Z_e(t-j) \\
&\quad + \alpha_3 \sum_{j=0}^{s} Z_b(t-j)
\end{aligned}
$$

$$
\begin{aligned}
W_4(s) &= Var(Q_4(s)) \\
&= (\alpha_1 - \sigma_y)^2 + \alpha_1^2 s + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2}\sigma_y^2 \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} \left(s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad - 2\alpha_1\sigma_y \frac{\phi(1-\phi^s)}{1-\phi} + \frac{2\alpha_1\sigma_y}{1-\phi}\left(s - \frac{\phi(1-\phi^s)}{1-\phi}\right) \\
&\quad - \frac{2\sigma_y^2}{1-\phi} \left(\frac{\phi(1-\phi^s)}{1-\phi} - \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad + \left(\frac{(1-\phi^{s+1})\sigma_y}{1-\phi} - \phi^{s+1}\sigma_y \right)^2 \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2}(1-\phi^{s+1})^2 \frac{\phi^2}{(1-\phi)^2} \\
&\quad + \sigma_y^2 \frac{\phi^{2s+4}}{1-\phi^2} - \frac{2\sigma_y^2}{1-\phi}(\phi^{s-1} - \phi^{2s}) \frac{\phi^4}{1-\phi^2} \\
&\quad + (\alpha_2^2 + \alpha_3^2)(s+1)
\end{aligned}
$$

Consider, for $s \geq 0$:

$$\begin{aligned}
Q_5(s) &= S_y(t, s-1) + S_p(t, s) - y(t) \\
&= (s-1)y + (s+1)\alpha_0 + (\alpha_1 - \sigma_y)Z_y(t) \\
&\quad + \sum_{j=1}^{s} \left(\frac{(1-\phi^j)\sigma_y}{1-\phi} - \phi^j \sigma_y + \alpha_1 \right) Z_y(t-j) \\
&\quad + \sum_{j=s+1}^{\infty} \left(\frac{\phi^{j-s}(1-\phi^s)}{1-\phi}\sigma_y - \phi^j \sigma_y \right) Z_y(t-j) \\
&\quad +\alpha_2 \sum_{j=0}^{s} Z_e(t-j) \\
&\quad +\alpha_3 \sum_{j=0}^{s} Z_b(t-j)
\end{aligned}$$

$$\begin{aligned}
W_5(s) &= Var(Q_5(s)) \\
&= (\alpha_1 - \sigma_y)^2 + \alpha_1^2 s + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2}\sigma_y^2 \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2} \left(s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad -2\alpha_1\sigma_y \frac{\phi(1-\phi^s)}{1-\phi} + \frac{2\alpha_1\sigma_y}{1-\phi}(s - \frac{\phi(1-\phi^s)}{1-\phi}) \\
&\quad - \frac{2\sigma_y^2}{1-\phi} \left(\frac{\phi(1-\phi^s)}{1-\phi} - \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) \\
&\quad + \frac{\sigma_y^2}{(1-\phi)^2}(1-\phi^s)^2 \frac{\phi^2}{(1-\phi)^2} \\
&\quad + \sigma_y^2 \frac{\phi^{2s+2}}{1-\phi^2} - \frac{2\sigma_y^2}{1-\phi}(\phi^s - \phi^{2s}) \frac{\phi^2}{1-\phi^2} \\
&\quad + (\alpha_2^2 + \alpha_3^2)(s+1)
\end{aligned}$$

If $s = 0$, then:

$$Q_5(s) = S_p(t, 0) - y(t+1)$$

$$\begin{aligned}
W_5(s) &= Var(S_p(t, 0) - y(t)) \\
&= (\alpha_1 - \sigma_y)^2 + \sigma_y^2 \frac{\phi^2}{1-\phi^2} + \alpha_2^2 + \alpha_3^2
\end{aligned}$$